

AutoCorrelation

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Abstract

Autocorrelation is the cross-correlation of a signal with itself. Informally, it is the similarity between observations as a function of the time lag between them.

1 Definition and physical nature

The autocorrelation is the cross-correlation of a signal with itself. The cross-correlation is a measure of similarity of two waveforms as a function of a time-lag applied to one of them. It is the “sliding inner product”. For continuous functions f and g , the cross-correlation is defined¹ as

$$(f \star g)(t) \equiv \int_{-\infty}^{\infty} \bar{f}(\tau) g(t + \tau) d\tau, \quad (1)$$

where \bar{f} denotes the complex conjugate of f . Similarly, for discrete functions, the cross-correlation is defined as

$$(f \star g)[n] \equiv \sum_{m=-\infty}^{\infty} \bar{f}[m] g[m + n]. \quad (2)$$

The cross-correlation is similar in nature to the convolution ($f \circ g$) of two functions f and g :

$$(f \circ g)(t) \equiv \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau, \quad (3)$$

where one sees that convolution and cross-correlation are equivalent for symmetrical functions $g(\tau) = g(-\tau)$. One can also think of the convolution as equivalent to the cross-correlation of $f(t)$ and $g(-t)$.

As an illustration, consider two real functions f and g differing only by an unknown shift along the t axis. The cross-correlation slides the g function along the t axis, calculating the integral of their product at each position. When the functions match, the value of $(f \star g)$ is maximized.

¹it may differ in other fields

2 Properties

- The cross-correlation of functions $f(t)$ and $g(t)$ is equivalent to the convolution of $\bar{f}(-t)$ and $g(t)$.
- If f is Hermitian², then convolutions and cross-correlation are equivalent.
- $f \star f$ is maximum at the origin.
- The cross-correlation satisfies

$$\mathcal{F}\{f \star g\} = \overline{\mathcal{F}\{f\}} \cdot \mathcal{F}\{g\}, \quad (4)$$

where \mathcal{F} denotes the Fourier transform.

This last property will be used to compute efficiently our autocorrelation functions.

3 Efficient computation

Brute force algorithm from Eq. 2 is order n^2 . We use property of Eq. 4 to have the much better order $n \log(n)$. The algorithm is:

$$(f \star g) = \text{FFT}^{-1} \left\{ \text{FFT}\{f\} \overline{\text{FFT}\{g\}} \right\}. \quad (5)$$

Since f is a real function in our problem, Eq. 5 may be rewritten as

$$(f \star g) = \text{FFT}^{-1} \left\{ |\text{FFT}\{f\}|^2 \right\}, \quad (6)$$

where $|z|^2$ denotes the absolute square³ of x .

4 In Mathematica

Mathematica has a built-in function to compute autocorrelations: Correlation-Function. Nevertheless, in Mathematica, it is defined as the normalized covariance function, which is different from definition 1.

5 How to use our Fortran code

5.1 Input file

Input files must look like the following, e.g., in dimension 3:

$$f_x^{(i)}[t] \quad f_y^{(i)}[t] \quad f_z^{(i)}[t]$$

²The real part of f is an even function and the imaginary part of f is an odd function

³ $|z|^2 = z\bar{z}$

$$\begin{array}{ccccc}
f_x^{(i+1)}[t] & f_y^{(i+1)}[t] & f_z^{(i+1)}[t] & & \\
& \dots & & & \\
f_x^{(\dots)}[t] & f_y^{(\dots)}[t] & f_z^{(\dots)}[t] & & \\
f_x^{(N)}[t] & f_y^{(N)}[t] & f_z^{(N)}[t] & & \\
f_x^{(1)}[t+1] & f_y^{(1)}[t+1] & f_z^{(1)}[t+1] & & \\
f_x^{(2)}[t+1] & f_y^{(2)}[t+1] & f_z^{(2)}[t+1] & & \\
& \dots & \dots & \dots &
\end{array}$$

where N is the number of atoms in the supercell, and t is the discrete index of timesteps. For a given timestep, each atom is considered. The timestep is then incremented.

References

- [1] <http://en.wikipedia.org/wiki/Autocorrelation>
- [2] <http://en.wikipedia.org/wiki/Cross-correlation>
- [3] <http://mathworld.wolfram.com/Autocorrelation.html>