Resummation of divergent series expansions and the Stokes phenomenon

Josh Shelton
School of Mathematics and Statistics, University of St Andrews, St Andrews, KY16 9SS, UK
josh.shelton@st-andrews.ac.uk

Course outline

Thursday 24th: Lecture 1; Lecture 2; Lecture 3; Problems class 1.
Friday 25th: Lecture 4; Problems class 2.

Lecture 1: Convergent and divergent series expansions.
Consider a function $y(z)$, where $z \in \mathbb{C}$, which is represented by the following series expansion centred about $z = 0$,

$$y(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1.1}$$

Here, the series coefficients a_n are constant. Convergence of the series (1.1) can be established by the ratio test. In examining the ratio of successive terms for large n, given by $\lim_{n\to\infty} |(a_{n+1}z^{n+1})/(a_n z^n)|$, the series converges provided that this quantity is less than one. Rearranging this then yields the condition $|z| < \lim_{n\to\infty} |a_{n+1}/a_n|$ for convergence. For a given set of series coefficients, $\{a_n\}$, the right-hand side of this constraint is known. The series (1.1) then converges for all values of $|z|$ that satisfy this constraint. Thus, we define the radius of convergence of this series, ρ, by

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|. \tag{1.2}$$

We have therefore established that the radius of convergence of series expansion (1.1) can depend on the value of z. Three interesting cases emerge:

(i) $\rho = \infty$. The series (1.1) converges for all values of z.

(ii) $0 < \rho < \infty$. The series converges for $0 \leq |z| < \rho$, and diverges for $\rho < |z|$[†]. The equality in (1.1) therefore holds only for values of z where the series converges, and should be replaced with a $\neq$ sign otherwise.

(iii) $\rho = 0$. The series (1.1) diverges for all values of z[‡].

1.1. An example differential equation.
Consider Airy's equation,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - zy = 0. \tag{1.3}$$

This is a linear second-order ODE for the solution $y(z)$. We will solve this equation by using a convergent series expansion. By assuming that the solution of (1.3) takes the form $y(z) = \sum_{n=0}^{\infty} a_n z^n$, we can substitute this into the governing ODE to obtain

$$2a_2 z^0 + \sum_{n=1}^{\infty} \big[(n+2)(n+1)a_{n+2} - a_{n-1}\big] z^n = 0. \tag{1.4}$$

[†] It is significantly more difficult to establish whether the series converges or diverges when $|z| = \rho$.
[‡] The case with $z = 0$ trivially converges since $y(z) = a_0$.

The aim is now to solve for the constant coefficients, a_n, of the series solution. Since (1.4) holds for some finite region of z (we are assuming that $\rho > 0$), it decouples into a set of conditions obtained from each order of z. These are given by

$$2a_2 = 0, \quad (n+2)(n+1)a_{n+2} - a_{n-1} = 0, \tag{1.5}$$

for $n \geq 1$. The first of these yields $a_2 = 0$, and the second produces a linear recurrence relation for a_n in terms of a_{n-3}. Both of a_0 and a_1 are undetermined in this recurrence relation, which is to be expected for a second-order ODE. The resultant recurrence relations can be solved analytically to find

$$y(z) = a_0 \sum_{k=0}^{\infty} \frac{3^k \Gamma(k+1/3)}{\Gamma(3k+1)\Gamma(1/3)} z^{3k} + a_1 \sum_{k=0}^{\infty} \frac{3^k \Gamma(k+2/3)}{\Gamma(3k+2)\Gamma(2/3)} z^{3k+1}, \tag{1.6}$$

where $\Gamma(k)$ is the gamma function.

To determine the radius of convergence of solution (1.6), we need to examine the ratio of successive terms in each of the a_0 and a_1 expansions. From

$$\rho^3 = \lim_{k\to\infty} \left| \frac{b_k}{b_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{(3k+4)(3k+3)(3k+2)}{3(k+4/3)} \right|, \tag{1.7}$$

we find that $\rho = \infty$ and thus the series solution (1.6) converges for all values of z.

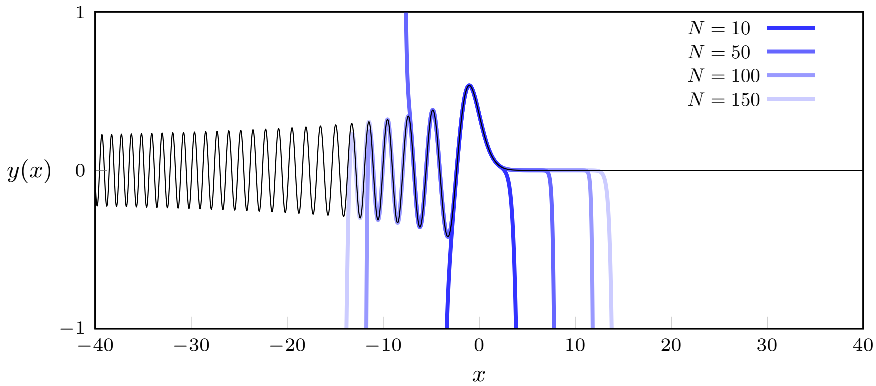

Fig. 1.1. *Comparison between a numerical solution to Airy's equation* (1.3) *(black line) and the convergent series solution* (1.6)*,* $y = \sum_{n=0}^{N} a_n z^n$ *for the different values of* $N = 10$*,* $N = 50$*,* $N = 100$*, and* $N = 150$*. Here, we have plotted with respect to the real axis,* $x = \mathrm{Re}[z]$*.*

We may now implement this convergent series expansion numerically in order to compare with a numerical solution of the same problem. For the purposes of this illustration we will take z to be real-valued and then plot the series expansion for an increasing number of retained coefficients. The series is shown in figure 1.1 when 10, 50, 100, and 150 terms. It is seen that the series solution converges towards the numerical solution (as expected) as the number of terms calculated is increased. However, this convergence is very slow. Even with 150 terms, the series is only a good approximation for $-10 < x < 10$. Further, there are also severe numerical issues associated with including further terms in the series because the coefficients become very small, such as $a_{150} = 3.7233 \times 10^{-177}$.

1.2. Series expansion about $z = \infty$.

In the 1850's and 1860's, Sir George Gabriel Stokes developed an alternative method to solve Airy's equation [1]. Motivated by the poor convergence of the series centred about $z = 0$, which we derived in § 1.1, he instead considered a series expansion centred about $z = \infty$ (essentially a series expansion in powers of $1/z$). The method used to solve for the series centred about $z = \infty$ is very similar to that presented in § 1.1, except that the solution is of Liouville-Green form (i.e. a normal series $\times$ e^{z^α}), and yields

$$y(z) = \frac{C\mathrm{e}^{\frac{2}{3}z^{3/2}}}{z^{1/4}} \sum_{n=0}^{\infty} \frac{a_n}{[4z^{3/2}/3]^n} + \frac{D\mathrm{e}^{-\frac{2}{3}z^{3/2}}}{z^{1/4}} \sum_{n=0}^{\infty} \frac{a_n}{[-4z^{3/2}/3]^n}, \tag{1.8a}$$

where

$$a_n = \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{\Gamma(n+1)}. \tag{1.8b}$$

Recall that the convergent solution to this problem from § 1.1 is single valued. Therefore, the series expansion (1.8a) must also be single valued and take the same value for both $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $z = r\mathrm{e}^{\mathrm{i}(\theta+2\pi)}$ (since $\mathrm{e}^{2\pi\mathrm{i}} = 1$). In substituting these two values for z into the series (1.8a), we can use $\mathrm{e}^{3\pi\mathrm{i}} = -1$ and $\mathrm{e}^{\pi\mathrm{i}/2} = \mathrm{i}$ to obtain the two constraints $C = \mathrm{i}D$ and $D = \mathrm{i}C$. Since we require at least one of these constants to be nonzero, this is a contradiction.

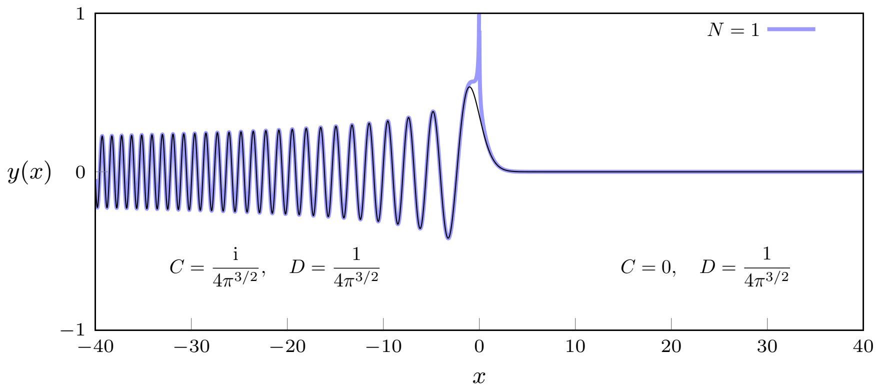

FIG. 1.2. *Comparison between a numerical solution to Airy's equation* (1.3) *(black line) and the series solution* (1.8a) *for $N = 1$. Here, we have plotted with respect to the real axis, $x = Re[z]$.*

By plotting the asymptotic solution (1.8a) against the numerical solution of the problem, shown dashed in figure 1.2, the resolution to this issue can be obtained. We see that these constants take different values for $x < 0$ and $x > 0$, and are given by

$$C = \begin{cases} \dfrac{\mathrm{i}}{4\pi^{3/2}} & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \qquad D = \begin{cases} \dfrac{1}{4\pi^{3/2}} & \text{for } x < 0, \\ \dfrac{1}{4\pi^{3/2}} & \text{for } x > 0. \end{cases} \tag{1.9}$$

This discontinuous change of the constants in solution (1.8a) at $z = 0$ is known as the *Stokes phenomenon* [2]. More generally, it occurs across contours within the complex plane, $z \in \mathbb{C}$, known as *Stokes lines*. Stokes lines emanate from singularities or branch points of the asymptotic series. In this example, there is a singularity in the series solution at $z = 0$.

There are three important features of this solution to remark upon:

(i) The radius of convergence of the series solution is zero, and it diverges for all values of z. While this contradicts our initial assumption that the series converges for some region of $z \in \mathbb{C}$, the series is an *asymptotic series* under the limit of $z \to \infty$, which means that we should have written $\sim$ rather than $=$ in (1.8a). Using methods in asymptotic expansions would have yielded the same series coefficients.

(ii) Unlike the convergent series (1.6), which required a large number of terms to approximate the numerical solution well (figure 1.1), the divergent series solution only requires one term to form an excellent approximation (figure 1.2). Since the series is divergent, comparison will becomes worse as the number of terms retained increases.

REFERENCES

[1] G. G. Stokes. On the theory of oscillatory waves. *Trans. Cam. Phil. Soc.*, 8:411–455, 1847.
[2] G. G. Stokes. On the discontinuity of arbitrary constants which appear in divergent developments. *Trans. Cam. Phil. Soc.*, 10:105, 1864.

Lecture 2: Divergent asymptotic expansions and the Stokes phenomenon.
Consider the solution expansion

$$y(z;\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n y_n(z) \tag{2.1}$$

under the limit of $\epsilon \to 0$. By definition, this series is said to be *asymptotic* if consecutive terms are smaller than previous terms under the limit of $\epsilon \to 0$. That is,

$$\lim_{\epsilon\to 0}\left[\frac{\epsilon^{n+1}y_{n+1}}{\epsilon^n y_n}\right] = 0, \tag{2.2}$$

or equivalently $y_n \gg \epsilon y_{n+1}$.

Condition (2.2) is the key assumption made when deriving solutions to differential equations in the form of an asymptotic series, and most of the difficulty emerges when it is violated. For instance, if $y_0 = 1/z$ and $y_1 = 1/z^2$ then $y_0 \gg \epsilon y_1$ is violated as $z \to 0$. The asymptotic expansion subsequently reorders as $z \to 0$, and requires the use of boundary-layer theory and matched asymptotics expansions to resolve. Further, divergent asymptotic series also reorder. Rather than this reordering occurring at certain locations for $z \in \mathbb{C}$, it occurs as $n \to \infty$. There are many different methods used to resolve this issue, including optimal truncation, series resummation, and resurgence. The unifying feature across all of these methods is that the expansion divergence is directly linked to exponentially-small (often called non-perturbative) terms in the asymptotic expansion that are subdominant to the algebraic expansion (2.1).

Asymptotic series to singular perturbative problems typically take the form of an *asymptotic transseries*,

$$y(z;\epsilon) \sim \sum_{i=1}^{N} \frac{\sigma_i}{\epsilon^{\alpha_i}}\left[A_0^{(i)}(z) + \epsilon A_1^{(i)}(z) + \cdots\right]\mathrm{e}^{-\chi_i(z)/\epsilon}. \tag{2.3}$$

Here, $\chi_i(z)$ are called the singulant functions, and $A_n^{(i)}(z)$ are the amplitude functions. Both α_i and σ_i are constant, the latter of which are denoted *Stokes multipliers* as they change value across Stokes lines. The number of possible exponentials, N, in the series expansion is problem dependent. For Airy's equation (1.3), a linear second-order ODE, there are $N = 2$ exponentials. For nonlinear problems we often have $N = \infty$. The algebraic expansion (2.1) is a special case with $\alpha_1 = 0$ and $\chi_1(z) = 0$.

2.1. The Stokes phenomenon.

Stokes phenomenon occurs whenever one exponential in the asymptotic transseries (2.3) reaches peak exponential dominance over another. If this occurs with the two exponentials $\mathrm{e}^{-\chi_i/\epsilon}$ and $\mathrm{e}^{-\chi_j/\epsilon}$, the result is that the Stokes multiplier of the subdominant exponential changes value. This change in value may be regarded as an automorphism of the transseries parameters $\{\sigma_i\}$, and it occurs across the Stokes line defined by $l_{i>j}$,

$$(\text{Stokes line } l_{i>j}): \quad \mathrm{Im}[\chi_j - \chi_i] = 0 \quad \text{and} \quad \mathrm{Re}[\chi_j - \chi_i] > 0, \tag{2.4a}$$

$$(\text{Stokes automorphism } \mathcal{S}_{i>j}): \quad \sigma_j \mapsto \sigma_j + 2\pi\mathrm{i}\tau_{i,j}\sigma_i. \tag{2.4b}$$

In (4.13b) above, the constant $\tau_{i,j}$ is known. Note that the automorphism (4.13b) occurs from $\mathrm{Im}[\chi_j - \chi_i] < 0$ to $\mathrm{Im}[\chi_j - \chi_i] > 0$. Thus, to fully resolve the Stokes phenomenon for a given problem, we:

(i) Calculate the amplitude functions, $A_0^{(i)}(z)$, and the corresponding singulant functions $\chi_i(z)$. Plot all possible Stokes lines for the problem from condition (4.13a). If there are N singulants, then the total number of Stokes lines is given by

$$2\sum_{i=1}^{N} i = N(N-1),$$

since for each Stokes line $l_{i>j}$, there is also the Stokes line $l_{j>i}$.

(ii) Determine the values of $\tau_{i,j}$. These capture how much of the jth exponential is seen in the late terms of the ith expansion, and are defined in (2.1) below. In lecture 3 we obtain $\tau_{i,j}$ by solving for the divergence of $A_n^{(i)}(z)$ exactly (which works for some linear problems), and in lecture 4 we obtain $\tau_{i,j}$ by solving for the divergence of $A_n^{(i)}(z)$ under the limit of $n \to \infty$ (which is more general and works for nonlinear problems).

(iii) Pick values for the transseries parameters $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ in some sector of $z \in \mathbb{C}$. These values come from far-field or behavioural conditions of the problem.

(iv) Compute every Stokes automorphism to obtain the values that σ_i takes for $z \in \mathbb{C}$.[†]

The divergence of each expansion in transseries (2.3) typically takes the *factorial-over-power* form of

$$A_n^{(i)}(z) \sim \tau_{i,j} A_0^{(j)}(z) \frac{\Gamma(n + \alpha_j - \alpha_i)}{(\chi_j(z) - \chi_i(z))^{n+\alpha_j-\alpha_i}} \quad \text{as } n \to \infty,$$

and the Stokes phenomenon is simply a mathematical consequence of this divergence. In lecture 3 we will find $\tau_{i,j}$ by first obtaining an exact solution to $A_n^{(i)}(z)$, and then expanding this as $n \to \infty$. In lecture 4 we will consider a nonlinear problem where this approach fails, and the equation governing $A_n^{(i)}(z)$ must be approximated under the limit of $n \to \infty$, yielding a differential-difference equation. In this latter case, $\tau_{i,j}$ is obtained by matching with a boundary layer.

2.2. Derivation from the Borel transform.

Consider the divergent series expansion $A^{(i)}(z,\epsilon) \sim A_0^{(i)}(z) + \epsilon A_1^{(i)}(z) + \cdots$. The Borel transform may be used to express this as a convergent series. We define the forward and inverse Borel transforms by

$$\text{(Borel transform)}: \quad y_B^{(i)}(z,w) = \sum_{n=0}^{\infty} w^n \frac{A_n^{(i)}(z)}{\Gamma(n+1)}, \tag{2.5a}$$

$$\text{(Inverse transform)}: \quad A^{(i)}(z;\epsilon) \sim \frac{1}{\epsilon}\int_0^{\infty} y_B^{(i)}(z,w)\mathrm{e}^{-w/\epsilon}\mathrm{d}w. \tag{2.5b}$$

[†]There are actually several complications that can emerge which require additional steps to resolve. Firstly, Stokes lines can coincide with one another (which is problem specific) resulting in a different change for σ_j in the automorphism. Secondly, there is a concept of higher-order Stokes phenomenon which is an automorphism acting on the values of $\tau_{i,j}$ in (4.13b), rather than the transseries coefficients σ_i. This can be generalised to n levels of Stokes phenomenon, where only the level-1 Stokes phenomenon is resolved above.

The Borel transform (2.5a) may be regarded as a convergent generating function for each order of the $\epsilon \to 0$ asymptotic expansion of $A^{(i)}(z;\epsilon)$, and the Stokes phenomenon will be obtained by studying the asymptotics of the inverse transform (2.5b).

The radius of convergence of the Borel transform (2.5a) will be finite due to singularities at certain values of w.[†] By using Darboux's method [1], the behaviour of $y_B^{(i)}(z,w)$ local to each singularity can be obtained. We focus here only on the singularity that will induce the $l_{i>j}$ Stokes line, which is given by

$$y_B^{(i)}(z,w) \sim \frac{\tau_{i,j}\Gamma(\alpha_j-\alpha_i)A_0^{(j)}(z)}{\left[\chi_j(z)-\chi_i(z)-w\right]^{\alpha_j-\alpha_i}}. \tag{2.6}$$

Thus, there is a singularity at $w=\chi_j(z)-\chi_i(z)$ in the integrand of the inverse transform (2.5b), the position of which is complex-valued and dependent on z. Consider the case when $\arg[\chi_j-\chi_i]<0$. In this case, the dominant asymptotics to $A^{(i)}(z;\epsilon)$ may be obtained from the inverse transform (2.5b) by substituting for the series (2.5a) and integrating by parts, which yields

$$A^{(i)}(z;\epsilon) \sim A_0^{(i)}(z)+\epsilon A_1^{(i)}(z)+\cdots \tag{2.7a}$$

for $\arg[\chi_j(z)-\chi_i(z)]<0$. However, if $\arg[\chi_j(z)-\chi_i(z)]$ moves through zero and becomes positive, then the contour from 0 to ∞ must be deformed for its value to be continuous. The deformation of the integration contour is depicted in figure 2.1(*b*). The result of this is that when $\arg[\chi_j(z)-\chi_i(z)]>0$, there is a Hankel contour contribution to the integral, yielding

$$A^{(i)}(z;\epsilon) \sim A_0^{(i)}(z)+\epsilon A_1^{(i)}(z)+\cdots+\frac{\tau_{i,j}\Gamma(\alpha_j-\alpha_i)A_0^{(j)}(z)}{\epsilon}\oint\frac{\mathrm{e}^{-w/\epsilon}}{(\chi_j-\chi_i-w)^{\alpha_j-\alpha_i}}\mathrm{d}w \tag{2.7b}$$

for $\arg[\chi_j(z)-\chi_i(z)]>0$.

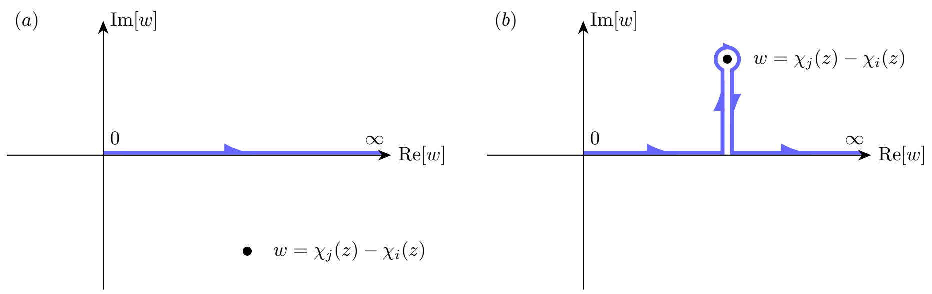

FIG. 2.1. (*a*) *The integration contour for the inverse Borel transform* (2.5b) *is shown in blue. As the singularity in the integrand crosses the positive real axis, the integration contour must be deformed, resulting in an additional Hankel contour. This deformation is shown in* (*b*).

By using the integral formula for the reciprocal gamma function,

$$\frac{1}{\Gamma(z)}=\frac{\mathrm{i}}{2\pi}\oint(-t)^{-z}\mathrm{e}^{-t}\mathrm{d}t,$$

[†]As an analogy, think about the Taylor expansion of $(1-w)^{-1}$ about $w=0$. This series has a radius of convergence equal to one due to the singularity at $w=1$ of the resummed function.

and substituting for $-\epsilon t = \chi_j - \chi_i - w$, the Hankel integral in (2.7b) can be directly evaluated, and yields

$$A^{(i)}(z;\epsilon) \sim A_0^{(i)}(z) + \epsilon A_1^{(i)}(z) + \cdots + 2\pi \mathrm{i}\tau_{i,j}\frac{A_0^{(j)}(z)}{\epsilon^{\alpha_j-\alpha_i}}\mathrm{e}^{-(\chi_j-\chi_i)/\epsilon}. \tag{2.8}$$

This extra contribution to the asymptotics of $A^{(i)}(z;\epsilon)$ is gained as the Stokes line is crossed, which we see now is the contour $\mathrm{Im}[\chi_j - \chi_i] = 0$ and $\mathrm{Re}[\chi_j - \chi_i] > 0$ previously given in (4.13a). In multiplying (2.8) by $\epsilon^{-\alpha_i}\sigma_i \mathrm{e}^{-\chi_i/\epsilon}$ to obtain the extra contribution to the asymptotics of $y(z;\epsilon)$, we see that this extra term is of the form $2\pi\mathrm{i}\tau_{i,j}\sigma_i\epsilon^{-\alpha_j}A_0^{(j)}(z)\mathrm{e}^{-\chi_j/\epsilon}$. Thus, we have gained an extra component to the jth term in the asymptotic transseries (2.3), and found that across the Stokes line

$$\sigma_j\frac{A_0^{(j)}(z)}{\epsilon}\mathrm{e}^{-\chi_j(z)/\epsilon^{\alpha_j}} \mapsto \left[\sigma_j + 2\pi\mathrm{i}\tau_{i,j}\sigma_i\right]\frac{A_0^{(j)}(z)}{\epsilon^{\alpha_j}}\mathrm{e}^{-\chi_j(z)/\epsilon}, \tag{2.9}$$

which is the Stokes phenomenon.

The above derivation only informs us of the component gained via Stokes phenomenon, i.e. the automorphism (4.13b), rather than the smooth manner in which it occurs across the Stokes line †. In fact, the behaviour is smooth and takes the form of an error function [2],

$$\left[\sigma_j + \sqrt{2\pi}\mathrm{i}\tau_{i,j}\sigma_i\int_{-\infty}^{\frac{\arg[\chi_j-\chi_i]}{\sqrt{|\chi_j-\chi_i|\epsilon}}}\mathrm{e}^{-t^2/2}\mathrm{d}t\right]\frac{A_0^{(j)}(z)}{\epsilon^{\alpha_j}}\mathrm{e}^{-\chi_j(z)/\epsilon}, \tag{2.10}$$

such that the width of the Stokes line is $\arg[\chi_j - \chi_i] = O(\epsilon^{1/2})$. This means that when $\arg[\chi_j - \chi_i] = 0$ and we are on the Stokes line itself, half the total contribution, $\pi\mathrm{i}\tau_{i,j}\sigma_i$, has been obtained.

REFERENCES

[1] P. Henrici. *Applied and Computational Complex Analysis, Volume 2: Special Functions, Integral Transforms, Asymptotics, Continued Fractions*, volume 2. John Wiley & Sons, 1991.

[2] M. V. Berry. Uniform asymptotic smoothing of Stokes's discontinuities. *Proc. R. Soc. of Lond. A.*, 422(1862):7–21, 1989.

†And while it relied upon $\alpha_j - \alpha_i > 0$, the same result emerges for $\alpha_j - \alpha_i \leq 0$.

Lecture 3: Linear ODE example (Airy's equation).
Consider Airy's equation,

$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - zy = 0, \tag{3.1a}$$

which is obtained from (1.3) by taking $z \mapsto \epsilon^{-2/3} z$. Further, we enforce the far-field behavioural condition

$$y(z;\epsilon) \sim \frac{\epsilon^{1/6}}{2\pi^{1/2} z^{1/4}} \mathrm{e}^{-\frac{2z^{3/2}}{3\epsilon}} \tag{3.1b}$$

as $|z| \to \infty$ and $\arg[z] = 0$ (i.e. far along the real axis).

In posing the solution in the form of an asymptotic transseries, we have from (1.8a) that

$$\begin{aligned} y(z;\epsilon) \sim{}& \frac{\sigma_1}{\epsilon^{-1/6}} \left[\sum_{n=0}^{\infty} \frac{\epsilon^n}{z^{1/4}} \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{\Gamma(n+1)(-4z^{3/2}/3)^n} \right] \mathrm{e}^{-\frac{2z^{3/2}}{3\epsilon}} \\ &+ \frac{\sigma_2}{\epsilon^{-1/6}} \left[\sum_{n=0}^{\infty} \frac{\epsilon^n}{z^{1/4}} \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{\Gamma(n+1)(4z^{3/2}/3)^n} \right] \mathrm{e}^{\frac{2z^{3/2}}{3\epsilon}} \quad \text{as } \epsilon \to 0. \end{aligned} \tag{3.2}$$

Thus, $\alpha_1 = \alpha_2 = -1/6$[†] and with $N = 2$, we have the two singulants $\chi_1(z) = 2z^{3/2}/3$ and $\chi_2(z) = -2z^{3/2}/3$. There will be two Stokes lines in total generated from these, $l_{1>2}$ and $l_{2>1}$, defined by

$$l_{1>2}: \quad \mathrm{Im}[-4z^{3/2}/3] = 0 \quad \text{and} \quad \mathrm{Re}[-4z^{3/2}/3] \geq 0, \tag{3.3a}$$

$$l_{2>1}: \quad \mathrm{Im}[4z^{3/2}/3] = 0 \quad \text{and} \quad \mathrm{Re}[4z^{3/2}/3] \geq 0. \tag{3.3b}$$

Across each of these, we have the respective Stokes automorphisms

$$\begin{aligned} \mathcal{S}_{1>2}: &\quad \sigma_2 \mapsto \sigma_2 + 2\pi\mathrm{i}\tau_{1,2}\sigma_1, \\ \mathcal{S}_{2>1}: &\quad \sigma_1 \mapsto \sigma_1 + 2\pi\mathrm{i}\tau_{2,1}\sigma_2. \end{aligned}$$

It remains to find values for $\tau_{1,2}$, $\tau_{2,1}$, as well as the Stokes multipliers σ_1 and σ_2 in a certain sector of $z \in \mathbb{C}$. We begin with $\tau_{1,2}$ and $\tau_{2,1}$. First, we note that since $\gamma(5/6)\gamma(1/6) = 2\pi$, the leading-order amplitude functions from (3.2) are

$$A_0^{(1)}(z) = \frac{2\pi}{z^{1/4}} \quad \text{and} \quad A_0^{(2)}(z) = \frac{2\pi}{z^{1/4}}. \tag{3.4}$$

Then, we use the large-n behaviour $\Gamma(n+\alpha) \sim \Gamma(n)n^{\alpha}$ to calculate the $n \to \infty$ limit of $A_n^{(1)}(z)$ and $A_n^{(2)}(z)$, yielding

$$A_n^{(1)}(z) \sim \underbrace{\frac{1}{2\pi}}_{\tau_{1,2}} \cdot \underbrace{\frac{2\pi}{z^{1/4}}}_{A_0^{(2)}(z)} \cdot \frac{\Gamma(n)}{(\chi_2 - \chi_1)^n} \quad \text{and} \quad A_n^{(2)}(z) \sim \underbrace{\frac{1}{2\pi}}_{\tau_{2,1}} \cdot \underbrace{\frac{2\pi}{z^{1/4}}}_{A_0^{(1)}(z)} \cdot \frac{\Gamma(n)}{(\chi_1 - \chi_2)^n}. \tag{3.5}$$

Thus, we have that $\tau_{1,2} = \tau_{2,1} = (2\pi)^{-1}$. The respective Stokes automorphisms are then

$$\mathcal{S}_{1>2}: \ \sigma_2 \mapsto \sigma_2 + \mathrm{i}\sigma_1, \qquad \mathcal{S}_{2>1}: \ \sigma_1 \mapsto \sigma_1 + \mathrm{i}\sigma_2. \tag{3.6}$$

[†] I accidentally used an example with $\alpha_j - \alpha_i = 0$. Don't look at what happens with the Hankel integral in this case (I think that the Borel singularity is logarithmic here).

Next, we use the far-field condition (3.1b) to obtain values for σ_1 and σ_2 when $|z| \to \infty$ and $\arg[z] = 0$. By taking the same limit in the transseries expansion (3.2), we have that

$$y(z;\epsilon) \sim \sigma_2 2\pi\epsilon^{1/6} z^{-1/4} \mathrm{e}^{\frac{2z^{3/2}}{3\epsilon}} + \sigma_1 2\pi\epsilon^{1/6} z^{-1/4} \mathrm{e}^{-\frac{2z^{3/2}}{3\epsilon}}.$$

Thus, $\sigma_1 = 1/(4\pi^{3/2})$ and $\sigma_2 = 0$. The Stokes line structure for this problem is shown in figure 3.1. While there is a Stokes line, $l_{2>1}$, along the positive real axis, the

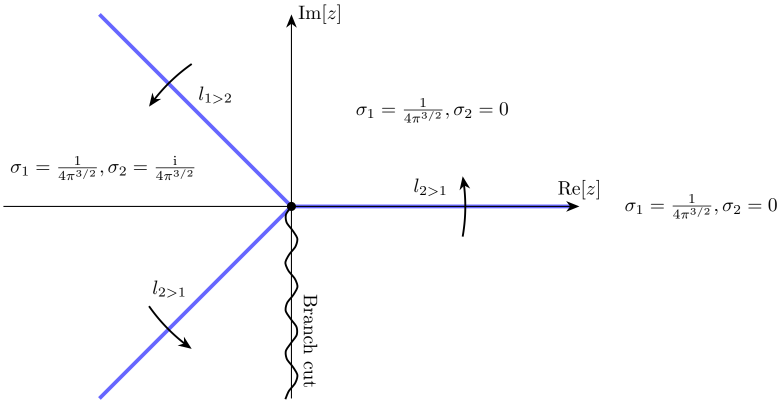

FIG. 3.1. *Stokes lines for the Airy equation are shown in blue.*

respective automorphism $\mathcal{S}_{2>1}$ does not change any Stokes multipliers because $\sigma_2 = 0$ here. It is the $l_{2>1}$ Stokes line that results in oscillatory behaviour of $y(z;\epsilon)$ along the negative real axis. Across $l_{1>2}$, the Stokes multiplier σ_2 changes value from zero to $\mathrm{i}/(4\pi^{3/2})$.

3.1. The Airy functions of the first and second kind.

Airy's equation permits two linearly independent solutions. These are the Airy functions of the first kind, $\mathrm{Ai}(z)$, and the second kind $\mathrm{Bi}(z)$, defined for $z \in \mathbb{C}$ by

$$\mathrm{Ai}(z) = \frac{1}{2\pi\mathrm{i}} \int_{\infty\mathrm{e}^{-\mathrm{i}\pi/3}}^{\infty\mathrm{e}^{\mathrm{i}\pi/3}} \exp\left(\frac{1}{3}t^3 - zt\right)\mathrm{d}t, \tag{3.7a}$$

$$\mathrm{Bi}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty\mathrm{e}^{\mathrm{i}\pi/3}} \exp\left(\frac{1}{3}t^3 - zt\right)\mathrm{d}t + \frac{1}{2\pi} \int_{-\infty}^{\infty\mathrm{e}^{-\mathrm{i}\pi/3}} \exp\left(\frac{1}{3}t^3 - zt\right)\mathrm{d}t. \tag{3.7b}$$

Note that the limits of integration indicate the line in the complex plane along which the t-integration occurs. We will now also revert back to the $|z| \to \infty$ limit of lecture 1, rather than the $\epsilon \to 0$ limit used in this lecture.

The Airy function of the first kind, $\mathrm{Ai}(z)$, satisfies the far-field condition (3.1b) as so we have already derived the asymptotic behaviour of this function. The following asymptotic formula for $\mathrm{Ai}(z)$ are exponentially-accurate, and differ depending on the sector of $z \in \mathbb{C}$.

$$\mathrm{Ai}(z) \sim \begin{cases} \dfrac{1}{4\pi^{3/2}z^{1/4}}\left[\displaystyle\sum_{n=0}^{\infty}\frac{a_n}{(-4z^{3/2}/3)^n}\right]\mathrm{e}^{-\frac{2z^{3/2}}{3}} & 0 \leq \arg[z] < 2\pi/3, \\ \dfrac{1}{4\pi^{3/2}z^{1/4}}\left[\displaystyle\sum_{n=0}^{\infty}\frac{a_n}{(-4z^{3/2}/3)^n}\right]\mathrm{e}^{-\frac{2z^{3/2}}{3}} & \\ \quad + \dfrac{\mathrm{i}}{8\pi^{3/2}z^{1/4}}\left[\displaystyle\sum_{n=0}^{\infty}\frac{a_n}{(4z^{3/2}/3)^n}\right]\mathrm{e}^{\frac{2z^{3/2}}{3}} & \arg[z] = 2\pi/3, \\ \dfrac{1}{4\pi^{3/2}z^{1/4}}\left[\displaystyle\sum_{n=0}^{\infty}\frac{a_n}{(-4z^{3/2}/3)^n}\right]\mathrm{e}^{-\frac{2z^{3/2}}{3}} & \\ \quad + \dfrac{\mathrm{i}}{4\pi^{3/2}z^{1/4}}\left[\displaystyle\sum_{n=0}^{\infty}\frac{a_n}{(4z^{3/2}/3)^n}\right]\mathrm{e}^{\frac{2z^{3/2}}{3}} & 2\pi/3 \leq \arg[z] \leq \pi, \end{cases} \tag{3.8}$$

where $a_n = \Gamma(n + 5/6)\Gamma(n + 1/6)/\Gamma(n + 1)$. These types of asymptotic formulas are often seen in special function theory [1]. Often, the behaviour is split up into additional sectors which corresponds to a switching in dominance of two exponentials, rather than the Stokes automorphism of the transseries coefficients.

REFERENCES

[1] *NIST Digital Library of Mathematical Functions.* https://dlmf.nist.gov/, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

Lecture 4: Nonlinear differential equations.
In lecture 3 we considered Airy's equation, which was a linear ODE with two different exponentials in the asymptotic transseries. We began by determining the singulant functions, $\chi_i(z)$, which were used to isolate the Stokes lines across which the Stokes phenomenon occurs.

For nonlinear problems, the asymptotic solution still typically takes the form of a transseries expansion that displays the Stokes phenomenon. However, deriving each component of this (the singulants $\chi_i(z)$ and corresponding amplitude functions $A_n^{(i)}(z)$) becomes more difficult. For the example studied in this lecture, the solution along the real axis will be dominated by the algebraic expansion $y(z;\epsilon) \sim y_0(z) + \epsilon y_1(z) + \cdots$. Only by first studying the singular points of $y_0(z)$ can the divergence of $y_n(z)$ for $n \to \infty$ be obtained. The Stokes phenomenon can then be studied from this divergence through the methods introduced in lecture 2.

Consider the fifth-order KdV equation,

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \epsilon\frac{\partial^5 u}{\partial x^5} = 0, \tag{4.1}$$

for the solution $u(x,t;\epsilon)$. In fluid dynamics, this equation models the depth, u, of free-surface water waves over topography, when the ratio of surface wavelength to fluid depth is large. The fifth-order term arises due to the inclusion of surface tension at the fluid-air interface. Solutions which travel without a change in speed (such as travelling periodic or solitary surface waves) may by studied by considering the solution to be a function of $x - ct$, where c is the speed of the travelling wave. This yields the fifth-order ODE

$$(6u - c)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \epsilon\frac{\partial^5 u}{\partial x^5} = 0, \tag{4.2}$$

for the solution $u(x;\epsilon)$. We will now study solutions of this under the limit of $\epsilon \to 0$. As $\epsilon \to 0$, the solution along the real-axis is dominated by an $O(\epsilon^0)$ component. Thus, we begin by taking[†]

$$u(x;\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n u_n(x). \tag{4.3}$$

The leading-order solution satisfies the equation $(6u_0 - c)u_0' + u_0''' = 0$, and is given by the well-known KdV soliton,

$$u_0(x) = \frac{c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}x\right). \tag{4.4}$$

An equation governing $u_n(x)$ for $n \geq 1$ is obtained at $O(\epsilon^n)$ in (4.2), and is given by

$$u_n''' + u_{n-1}''''' - cu_n' + \sum_{m=0}^{n} 6u_m u_{n-m}' = 0. \tag{4.5}$$

The solution to this will diverge as $n \to \infty$ on account of singularities in the leading-order solution (4.4). These singularities occur in the analytic continuation of the real

[†]Technically, c should also have been expanded as $\epsilon \to 0$, but we're ignoring this subtlety here.

variable x, so we now consdier the problem to be a function of $z \in \mathbb{C}$. Note that $\mathrm{sech}(z)$ is singular whenever $\cosh(z^*) = 0$, which occurs along the imaginary axis at $z^* = \pm \mathrm{i}\pi/2$, $z^* = \pm 3\mathrm{i}\pi/2$, and so forth. In taking $c = 1$, we then have the local singular behaviour

$$u_0(z) \sim -\big(z - z_k^*\big)^{-2} \quad \text{as } z \to z_k^* = (2k+1)\frac{\pi \mathrm{i}}{2} \tag{4.6}$$

for integer k. Note that the equation governing $u_1(z)$ at $O(\epsilon)$ involves the dominant balance $u_1''' \sim -u_0'''''$ near this singular point, which results in the singular behaviour $u_1(z) \sim -6(z - z_k^*)^{-4}$ as $z \to z_k^*$.

4.1. Expansion divergence.

This pattern will continue, such that the singular behaviour of the nth term in the asymptotic series is given by $u_n(z) = O([z - z_k^*]^{-2n-2})$ as $z \to z_k^*$. Additionally, the nth term in the series will diverge factorially due to differentiation of the growing singularity. This divergence will take the factorial-over-power form of

$$u_n(z) \sim \sum_{k=-\infty}^{\infty} A_k(z) \frac{\Gamma(2n + \alpha_k)}{\chi_k(z)^{2n+\alpha_k}} \quad \text{as } n \to \infty, \tag{4.7}$$

with each singularity of $u_0(z)$ at $z - z_k^*$ generating a separate component in the sum above. Note that the divergence is of the form $\Gamma(2n)$ rather than $\Gamma(n)$ in order to satisfy the dominant balance of $u_n''' \sim -u_{n-1}'''''$ as $n \to \infty$.

To obtain solutions for the singulants $\chi_k(z)$ and amplitude functions $A_k(z)$, we substitute (4.7) into the $O(\epsilon^n)$ equation and solve under the limit of $n \to \infty$. By differentiating ansatz (4.7), we find that

$$u_n'''(z) \sim \sum_{k=-\infty}^{\infty} \left[(-\chi_k')^3 A_k \frac{\Gamma(2n+\alpha_k+3)}{\chi_k^{2n+\alpha_k+3}} + \Big[3\chi_k'\chi_k'' A_k + 3(\chi_k')^2 A_k' \Big] \frac{\Gamma(2n+\alpha_k+2)}{\chi_k^{2n+\alpha_k+2}} + \cdots \right],$$

$$u_{n-1}'''''(z) \sim \sum_{k=-\infty}^{\infty} \left[(-\chi_k')^5 A_k \frac{\Gamma(2n+\alpha_k+3)}{\chi_k^{2n+\alpha_k+3}} + \Big[10(\chi_k')^3\chi_k'' A_k + 5(\chi_k')^4 A_k' \Big] \frac{\Gamma(2n+\alpha_k+2)}{\chi_k^{2n+\alpha_k+2}} + \cdots \right],$$

where the omitted terms are lower-order in the limit of $n \to \infty$. In the $O(\epsilon^n)$ equation, the singulant functions $\chi_k(z)$ will be obtained from $O(\Gamma(2n+\alpha_k+3)/\chi_k^{2n+\alpha_k+3})$ terms and the amplitude functions $A_k(z)$ from $O(\Gamma(2n+\alpha_k+2)/\chi_k^{2n+\alpha_k+2})$ terms. Since $u_n' = O(\Gamma(2n+\alpha_k+1)/\chi_k^{2n+\alpha_k+1})$, only $u_n'''(z)$ and $u_{n-1}'''''(z)$ feed into the two dominant orders as $n \to \infty$. We therefore obtain the equations

$$\chi_k'(z) = \pm \mathrm{i}, \quad A_k'(z) = 0, \tag{4.8}$$

which when integrated along with the boundary condition $\chi_k(z_k^*) = 0$ yield

$$\chi_k(z) = \pm \mathrm{i}(z - z_k^*), \quad A_k(z) = \Lambda_k, \tag{4.9}$$

where Λ_k is an unknown constant.

4.2. Inner solution.

To obtain values for Λ_k and α_k, we need to match the divergent solution (4.7) with an inner solution at a boundary layer centred about $z = z_k^*$. The constant α_k may be obtained quickly by noting that the singularity of $u_n(z)$ must be of the order $(z - z_k^*)^{-2n-2}$ as $z - z_k^*$, yielding

$$\alpha_k = 2. \tag{4.10}$$

The process to obtain Λ_k is more complicated, and requires matching all orders of the outer ϵ expansion with a boundary layer solution at $z = z_k^*$. This work is omitted here.

4.3. Stokes phenomenon.

Now that the singulant functions are known from (4.9), the Stokes lines may be obtained. The two singularities closest to the real axis† are

$$z_0^* = \frac{\pi \mathrm{i}}{2} \quad \text{and} \quad z_{-1}^* = -\frac{\pi \mathrm{i}}{2}. \tag{4.11}$$

We will focus only on the four singulant functions generated by these two singular points:

$$\chi_{0_\pm}(z) = \pm \mathrm{i}\left(z - \frac{\pi \mathrm{i}}{2}\right) \quad \text{and} \quad \chi_{-1_\pm}(z) = \pm \mathrm{i}\left(z - \frac{\pi \mathrm{i}}{2}\right). \tag{4.12}$$

Note that we have introduced the $\chi_{k_\pm}$ notation above. The four singulants are then χ_{0_+}, χ_{0_-}, χ_{-1_+}, and χ_{-1_-}. Since the divergent asymptotic expansion (4.3) in this example had $\chi_i = 0$, the Stokes line conditions become

$$\text{(Stokes line } l_{i>j}) : \quad \mathrm{Im}[\chi_j] = 0 \quad \text{and} \quad \mathrm{Re}[\chi_j] > 0, \tag{4.13a}$$

$$\text{(Stokes automorphism } \mathcal{S}_{i>j}) : \quad \sigma_j \mapsto \sigma_j + 2\pi \mathrm{i} \tau_{i,j} \sigma_i. \tag{4.13b}$$

In the example class, we will interpret these results further and plot possible solution profiles for the fifth-order KdV equation. The key message is that nonlinear problems can have a countably infinite set of singulant functions, and that determining the value of $\tau_{i,j}$ is significantly harder and requires inner-outer matching.

†The other singularities will generate Stokes lines that intersect the real-axis, but will produce a subdominant exponential.