Free-boundary problems — Exercises 2

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1. (Injection flow in porous media)

   Consider an infinite 3D porous medium. At the origin, fluid is injected at constant flux rate Q [$\mathrm{m}^3\mathrm{s}^{-1}$], and the far-field pressure is p_0.

   The fluid flux $\boldsymbol{U}$ and pressure p satisfy:

   $$\boldsymbol{U} = -\frac{k}{\mu}\nabla p, \qquad \nabla \cdot \boldsymbol{U} = 0.$$

   Assume the injection flow is radially symmetric, so that $p = p(r)$ and $\boldsymbol{U} = U\boldsymbol{e}_r$. In radially symmetric spherical polars,

   $$\nabla f = \frac{\mathrm{d}}{\mathrm{d}r}, \qquad \nabla \cdot (f\boldsymbol{e}_r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right).$$

   (a) Explain why for every $r > 0$,

   $$Q = 4\pi r^2 U(r).$$

   (b) Solve the flow equations to show that the pressure and flux are

   $$p = p_0 - \frac{\mu Q}{4\pi k r}, \qquad U = \frac{Q}{4\pi r^2}.$$

2. (Regions of different permeability)

   Consider a 1D Darcy flow through a porous material with regions of different permeability. Specifically, we consider the two configurations illustrated in Figure 1. In both there is an applied pressure drop of ΔP across a distance L in the x direction.

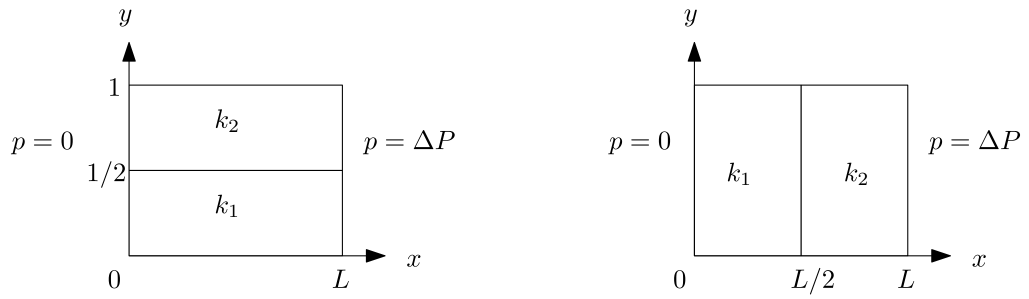

Figure 1: Porous media with permeability variations parallel to (left) and perpendicular to (right) the applied pressure gradient.

   (a) First consider the case on the left of Figure 1, in which there are two equal thickness layers of different permeabilities, k_1 and k_2. Assuming that the flow is purely horizontal, show that the horizontal Darcy velocity is

   $$U = \begin{cases} -\frac{k_1 \Delta P}{\mu L} & \text{for } y < 1/2, \\ -\frac{k_2 \Delta P}{\mu L} & \text{for } y > 1/2. \end{cases}$$

   Compute the average Darcy flow across the two layers, and hence show that the average permeability of the medium is

   $$k_{\parallel} = \frac{1}{2}(k_1 + k_2).$$

(b) Next consider the case on the right of Figure 1, in which the two permeability layers are perpendicular to the flow. Again assuming a purely horizontal flow, with U and p both continuous at $x = L/2$. By integrating Darcy's law over x, show that for this case the average permeability of the material is

$$k_\perp = \frac{2}{1/k_1 + 1/k_2}.$$

[*Note: These average permeabilities are the arithmetic and harmonic means. The flow of fluid through porous media is analogous to the conduction of heat through a material, or the conduction of electric current through a conductor: both are modelled the same way mathematically, with heat flux or current (analogous to the Darcy velocity) linearly proportional to the gradient in temperature or the electric field (analogous to the pressure gradient). The above analysis therefore also tells us how to compute the average thermal or electrical conductivity of a material. You might recognise these laws for electrical circuits, when computing the average resistance (the inverse of the electrical conductance) for circuit elements connected in parallel or in series.*]

3. (Carbon sequestration: the injection problem)

As in the lecture, during carbon sequestration the boundary of the CO_2 plume is $y = h(x,t)$ where h satisfies

$$\begin{aligned} \alpha \frac{\partial h}{\partial t} &= \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) && \text{for } x < a(t), \\ h &= 0 && \text{at } x = a(t), \\ \frac{\partial h}{\partial x} &= 0 && \text{at } x = 0. \end{aligned}$$

(a) Assume that CO_2 is injected at constant rate Q. Explain why

$$Qt = 2\phi \int_0^{a(t)} h(x,t)\, \mathrm{d}x.$$

(b) Look for a similarity solution of the form

$$h(x,t) = t^{1/3} f(\eta), \qquad a(t) = \zeta t^{2/3}, \qquad \text{where} \qquad \eta = \frac{x}{t^{2/3}},$$

and so show that

$$\begin{aligned} \alpha\left(f - 2\eta \frac{\mathrm{d}f}{\mathrm{d}\eta}\right) &= 3\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f \frac{\mathrm{d}f}{\mathrm{d}\eta}\right) && \text{for } \eta < \zeta, \\ f(\zeta) &= 0, \\ \frac{\mathrm{d}f}{\mathrm{d}\eta}(0) &= 0, \\ \frac{Q}{2\phi} &= \int_0^\zeta f(\eta)\, \mathrm{d}\eta. \end{aligned}$$

This system must be solved numerically, but the solution is sketched in figure 2. Sketch $h(x,t)$ against x at several times t.

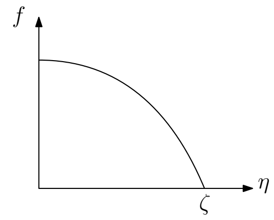

Figure 2: Sketch solution $f(\eta)$.