Free-boundary problems — Exercises 1

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1. (Work through the nondimensionalisation from the lecture)

   The two-phase Stefan problem on the rectangular domain $[0, d_1] \times [0, d_2]$ is

$$
\begin{aligned}
\rho c_p T^l_t &= k\nabla^2 T^l && \text{in } \Omega_l \\
\rho c_p T^s_t &= k\nabla^2 T^s && \text{in } \Omega_s \\
T^l &= T^{\text{hot}} && \text{on } x = 0, \\
\nabla T^{l \text{ or } s} \cdot \boldsymbol{n} &= 0 && \text{on all other fixed boundaries,} \\
T^l = T^s &= T_m && \text{on } \Gamma, \\
\rho L V &= k\nabla T^s \cdot \boldsymbol{n} - k\nabla T^l \cdot \boldsymbol{n} && \text{on } \Gamma,
\end{aligned}
$$

   where V is the normal velocity to the free boundary Γ.

   Nondimensionalise this problem using the scalings (change of variables)

$$
\boldsymbol{x} = d_1 \hat{\boldsymbol{x}}, \qquad t = \frac{\rho L d_1^2}{k(T^{\text{hot}} - T_m)}\hat{t}, \qquad T^{l \text{ or } s} = T_m + (T^{\text{hot}} - T_m)\hat{T}^{l \text{ or } s},
$$

   and show that the dimensionless problem depends on just two dimensionless groups:

$$
\text{St} = \frac{c_p(T^{\text{hot}} - T_m)}{L} \qquad \text{and} \qquad a = \frac{d_2}{d_1}.
$$

2. (A different freezing problem)

   Consider the 1D Stefan problem in the limit of $\text{St} \ll 1$ (like the lecture) but where we cool the solid rather than heating the liquid, so that we have freezing:

$$
\begin{aligned}
0 &= T^l_{xx} && \text{for } x \in (0, d(t)), \\
0 &= T^s_{xx} && \text{for } x \in (d(t), 1), \\
T^s &= -1 && \text{at } x = 1, \\
T^l_x &= 0 && \text{at } x = 0, \\
T^l = T^s &= 0 \quad \text{and} \quad d_t = T^s_x - T^l_x && \text{at } x = d(t).
\end{aligned}
$$

   Solve this free-boundary problem to find the temperatures $T^s(x,t)$ and $T^l(x,t)$, and the position of the freezing front $x = d(t)$. Sketch d against t and the temperature profile T against x at 3 different times.

3. (Is freezing always unstable?)

   Consider a similar freezing problem to q2, but in 2D and on an infinite domain:

$$
\begin{aligned}
0 &= T^l_{xx} + T^l_{yy} && \text{for } x < \gamma(y,t), \\
0 &= T^s_{xx} + T^s_{yy} && \text{for } x > \gamma(y,t), \\
T^s_x &\to -v && \text{as } x \to \infty, \\
T^l_x &\to 0 && \text{as } x \to -\infty, \\
T^l = T^s &= 0 \quad \text{and} \quad V = \nabla T^s \cdot \boldsymbol{n} - \nabla T^l \cdot \boldsymbol{n} && \text{at } x = \gamma(y,t).
\end{aligned}
$$

   Show that a base-state solution is

$$
T^l_B = 0, \qquad T^s_B = -v(x - vt), \qquad \gamma_B = -vt.
$$

   (if we assume $\gamma = 0$ at $t = 0$).

   Perform a linear stability analysis to determine the stability of this base-state solution:

(a) Make a small perturbation about the base state:

$$T^l = T^l_B + \theta^l, \qquad T^s = T^s_B + \theta^s, \qquad \gamma(y,t) = -vt + \xi(y,t),$$

where all of θ^l, θ^s, ξ are small. Substitute in and linearise to find a linear system (PDEs and BCs) for θ^l, θ^s and ξ.

(b) Look for Fourier mode solutions of the form

$$\theta^l = A(x)e^{iky+\sigma t}, \qquad \theta^s = B(x)e^{iky+\sigma t}, \qquad \xi = e^{iky+\sigma t},$$

and solve for $A(x)$ and $B(x)$. Hence show that the growth rate σ of the perturbation is

$$\sigma = -vk.$$

(c) Is this freezing problem stable or unstable?

4. (What if St is not small?)

So far we've used $\mathrm{St} \ll 1$ to simplify the PDEs, but it's possible to solve the full 1D problem when St is not negligible, using a *similarity solution.* (You may have seen these in your degree. If not, here's a first example, and we'll look at another in the lecture later...)

Consider the 1D Stefan problem

$$\begin{aligned}
\mathrm{St}T^l_t &= T^l_{xx} && \text{in } x \in (0, d(t))\\
\mathrm{St}T^s_t &= T^s_{xx} && \text{in } x \in (d(t), 1)\\
T^l &= 1 && \text{on } x = 0,\\
T^s_x &= 0 && \text{on } x = 1,\\
T^l = T^s &= 0 && \text{on } x = d(t),\\
d_t &= T^s_x - T^l_x && \text{on } x = d(t),\\
d = 0 \quad &\text{and} \quad T^s = T^s = 0 && \text{at } t = 0.
\end{aligned}$$

(a) Show that $T^s = 0$ everywhere satisfies the PDE for T^s and the Dirichlet boundary conditions. Hence reduce the problem to just a system for T^l and d.

(b) We'll now take advantage of the inherent symmetry of the problem to reduce the PDE system to an ODE system. Look for a similarity solution of the form:

$$T^l = f(\eta), \quad d(t) = \zeta\sqrt{t} \quad \text{where} \quad \eta = \frac{x}{\sqrt{t}} \text{ and } \zeta \text{ is a constant,} \tag{1}$$

ie: using the chain rule, show that the change of variables means derivatives are replaced by

$$\frac{\partial}{\partial x} = \frac{1}{\sqrt{t}}\frac{\mathrm{d}}{\mathrm{d}\eta}, \qquad \frac{\partial}{\partial t} = -\frac{\eta}{2t}\frac{\mathrm{d}}{\mathrm{d}\eta}.$$

Thus show that the PDE system reduces to the ODE system

$$\begin{aligned}
\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \mathrm{St}\frac{\eta}{2}\frac{\mathrm{d}f}{\mathrm{d}\eta} &= 0, && \text{for } \eta \in (0, \zeta),\\
f(0) &= 1,\\
f(\zeta) &= 0,\\
\frac{\mathrm{d}f}{\mathrm{d}\eta}(\zeta) &= -\frac{\zeta}{2}.
\end{aligned}$$

(c) Use an integrating factor to take a first integral of the ODE for f. Hence find an integral expression for $f(\eta)$. Apply the boundary conditions to fix the integration constants and to determine an integral equation for ζ (this must be solved numerically).

(d) Look up "error function, erf" online, and use this to simplify your expression for $f(\eta)$.

(e) Convert back to the original variables, to find $T^l(x,t)$ and $d(t)$, in terms of ζ.

(f) Show that in the limit $\mathrm{St} \to 0$ we regain the linear-in-x solution from lectures.

(g) Use an online solver, or a programming language like python or matlab to solve the equation for ζ numerically.

(h) Sketch $f(\eta)$, perhaps using an online plotting tool or your favourite programming language to help.

(i) Use your sketch of f to sketch $T^l(x,t)$.

(j) How could you have worked out the form (1) for the similarity solution if I hadn't told you? Why is there a similarity solution for this problem? Can you come up with other boundary conditions for a freezing or melting Stefan problem for which a similarity solution also works? Can you find choices of boundary conditions for which it doesn't work?