Knot theory, knot practice – Problems 2

LMS Summer School, Bath, July 2025 **Ailsa Keating**

N.B. There are more questions than your group will be about to talk about in an hour! The idea is to have a choice, depending on which topics you found most interesting so far. You're welcome to think about the left-over questions at a later time (but please don't feel obliged to). Questions marked ($\star$) are more exploratory.

1. Sketch a proof that the product operation on oriented knots is associative and commutative.

2. Give some examples to show that the product operation is not well-defined for unoriented knots.

3. Sketch a proof that the unknot is prime (without assuming the factorisation theorem from lectures). ($\star$) Do you think that all torus knots are prime?

4. ($\star$) A Brunnian link is a non-trivial link L such that removing any of the components of L gives the trivial link.

   (a) Find an n-component Brunnian link for every integer $n \geq 2$.

   (b) Do you think all Brunnian links are prime?

5. For $p > q \geq 2$, show that the torus link $T(p, q)$ has a Seifert surface of genus $\frac{1}{2}(p-1)(q-1)$.

   See below for a definition of genus. Fact: This minimises the genus of orientable surfaces in $\mathbb{R}^3$ with boundary $T(p, q)$! This is proved using an invariant called the Alexander polynomial.

6. Find an example which shows that for a given a link diagram, changing the orientation of a component can change the genus of the Seifert surface.

7. Draw Seifert surfaces for pretzel knots. What genera can they have? *[You may want to organise cases depending on the parities of p, q and r.]*

8. Suppose that L is a link with an odd number of components. Prove that the normalised bracket polynomial $\widetilde{V}_L(A)$ is a polynomial in A^4.

   ($\star$) What about links with an even number of components?

9. Compute the normalised bracket polynomial of the torus knot $T(2, p)$.

10. Show that the normalised bracket polynomial has the following properties:

    (a) $\widetilde{V}_L = \widetilde{V}_{-L}$

    (b) $\widetilde{V}_{L^*}(A) = \widetilde{V}_L(A^{-1})$

    (c) If a link can be factorised as $L_1 \# L_2$, then

    $$\widetilde{V}_{L_1 \# L_2} = \widetilde{V}_{L_1} \widetilde{V}_{L_2}$$

    (d) ($\star$) For a split link $L_1 \sqcup L_2$, we have

    $$\widetilde{V}_{L_1 \sqcup L_2}(A) = (-A^{-2} - A^2) \widetilde{V}_{L_1}(A) \widetilde{V}_{L_2}(A)$$

Note: If you haven't met genus yet, the genus of the Seifert surface F for a diagram D can be defined as $g(F) = \frac{1}{2}(2 - v(D) + c(D))$, where $c(D)$ is the number of crossings of D; and $v(D)$ is the number of embedded circles we obtained after resolving crossings in Seifert's algorithm.

Comments, corrections or suggestions are all welcome! You can either talk to me in person or write to `amk50@cam.ac.uk`.