Gaussian Analysis

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Outline

Gaussian measures on $\mathbb{R}$

Definition

γ is a **Gaussian** measure on $\mathbb{R}$ if it is either the *Dirac measure*, δ_a, or has density given by

$$p(x; a, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right).$$

The Fourier transform

Proposition

If γ is a Gaussian measure on $\mathbb{R}$, then its Fourier transform is of the form

$$\widehat{\gamma}(y) = \exp\left(iay - \frac{1}{2}\sigma^2 y^2\right).$$

Gaussian measures on arbitrary Banach spaces

Definition ([1])

Let X be a Banach space, with continuous dual X^*.
Then γ is a Gaussian measure on $\mathcal{E}(X, X^*)$ if for any $f \in X^*$, the induced measure $\gamma \circ f^{-1}$ on $\mathbb{R}$ is Gaussian.

The Fourier transform

Theorem

A measure γ on a Banach space X is Gaussian if and only if it has fourier transform of the form

$$\widehat{\gamma}(f) = \exp\left(i a_\gamma(f) - \frac{1}{2} R_\gamma(f)(f) \right).$$

The Wiener space

Example

- $C[0,1]$

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- for a given Brownian motion $B = \{B_t\}_{t\in[0,1]}$, and probability space (Ω, Σ, P):

$$\begin{array}{cccll} \phi & : & \Omega & \longrightarrow & C[0,1] \\ & & \omega & \longmapsto & B_* := (t \mapsto B_t(\omega)) \end{array}$$

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- the **Wiener measure** is $P^W = P \circ \phi^{-1}$

Cameron-Martin space

Definition

For $h \in X$

$$|h|_{H(\gamma)} := \sup \left\{ f(h) : f \in X^*, \, R_\gamma(f)(f) \leq 1 \right\}.$$

The **Cameron-Martin space** is then

$$H(\gamma) := \left\{ h \in X : |h|_{H(\gamma)} < \infty \right\}.$$

Cameron-Martin theorem

Theorem

Let γ be a Gaussian measure on X.

1. *If $h \notin H(\gamma)$, then γ and $\gamma_h := \gamma(\cdot - h)$ are mutually singular.*
2. *If $h \in H(\gamma)$, then γ and γ_h are equivalent.*

Outline of proof

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   - Show that the **total variation distance** $\|\gamma - \gamma_h\| = 1$.
2. If $h \in H(\gamma)$
   - Show that the measure with density with respect to γ

$$\rho_h(x) := \exp\left(g(x) - \frac{1}{2}|h|^2_{H(\gamma)}\right)$$

is γ_h.

Examples

Example

1. If γ is a nondegenerate measure on $\mathbb{R}^n$, then $H(\gamma) = \mathbb{R}^n$;
2. If γ is a degenerate measure, then $H(\gamma)$ is the support of γ.

Fernique's theorem

Theorem

If γ is a centred Gaussian measure on X, and q a $\mathcal{E}(X)$-measurable norm. Then there exists $\alpha > 0$ such that

$$\int_X \exp\left(\alpha q^2\right) \, \mathrm{d}\gamma < \infty.$$

Application of Fernique's theorem

- Recall the Wiener space $\left(C\left[0,1\right], \mathcal{C}, P^{W}\right)$

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Application of Fernique's theorem

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- Via a theorem by Kolmogorov [2], for $p \in \left(0, \frac{1}{2}\right)$ then P^W gives full measure to the subspace of p-Hölder continuous functions
- By Fernique's theorem there exists $\alpha > 0$ such that

$$\mathbb{E}\exp\left(\alpha\|w\|_{p\text{-Höl}}^2\right) < \infty.$$

Borel's Isoperimetric inequality

Theorem

Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$, and U be the closed unit ball. Then for any measurable A, $\varepsilon > 0$

$$\Phi^{-1}\left\{\gamma_n\left(A+\varepsilon U\right)\right\} \geq \Phi^{-1}\left\{\gamma_n(A)\right\} + \varepsilon.$$

Outline of proof

- From Borell [3] we have **Ehrhard's inequality**

$$\Phi^{-1}\left\{\gamma_n\left(\lambda A+(1-\lambda)B\right)\right\}\geq\lambda\Phi^{-1}\left\{\gamma_n(A)\right\}$$
$$+(1-\lambda)\Phi^{-1}\left\{\gamma_n(B)\right\}$$

Outline of proof

- From Borell [3] we have **Ehrhard's inequality**

$$\Phi^{-1}\left\{\gamma_n\left(\lambda A+(1-\lambda)B\right)\right\} \geq \lambda\Phi^{-1}\left\{\gamma_n(A)\right\} + (1-\lambda)\Phi^{-1}\left\{\gamma_n(B)\right\}$$

- Apply the above to $\lambda^{-1}A$ and $(1-\lambda)^{-1}\varepsilon U$

Wiener integral

- Take $H = L^2\left([0,1]\,;\mathbb{R}\right)$, with orthonormal basis $\left\{e_n\right\}_{n\in\mathbb{N}}$

Wiener integral

- Take $H = L^2([0,1];\mathbb{R})$, with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$
- Define $W : H \to L^2(\Omega)$ by

$$W(e_n) = \xi_n \sim \mathcal{N}(0,1)$$

The Derivative operator

Definition

- Define

$$\mathcal{S} := \left\{ f\left(W(h_1), \cdots, W(h_n)\right) : f \in C^{\infty}\left(\mathbb{R}^n\right), \, h_i \in H \right\}$$

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- For $F \in \mathcal{S}$ define

$$\mathscr{D}_t F := \sum_{1 \leq i \leq n} \frac{\partial}{\partial x_i} f\left(W(h_1), \cdots, W(h_n)\right) h_i(t)$$

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Example

- $\mathscr{D} W(h) = h$

The Divergence operator

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Definition

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- For $u \in \mathcal{S}_H$ define

$$\delta u := \sum_{1\leq j\leq n} \Big(F_j W(h_j) - \langle \mathscr{D} F_j, h_j \rangle_H \Big)$$

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Example

- $\delta h = W(h)$

Derivative & Divergence

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Not in general!

However, δ and $\mathscr{D}$ are **adjoint** in the sense that

$$\mathbb{E}\left(\langle \mathscr{D} F, u \rangle_H\right) = \mathbb{E}\left(F \delta u\right).$$

The Ornstein-Uhlenbeck operator

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Definition

We define the **Ornstein-Uhlenbeck operator**, $\mathcal{L} : L^2(\Omega) \to L^2(\Omega)$, by

$$\mathcal{L}F := -\delta\mathscr{D}F.$$

Wiener Chaos decomposition

Definition

Define the ***n*-th Wiener Chaos**, $\mathscr{H}_n$, by

$$\mathscr{H}_n = \overline{\mathrm{span}\left\{ H_n\left(W(h)\right) : \|h\|_H = 1 \right\}}.$$

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Proposition

If $G_n \in \mathscr{H}_n$ then

$$\mathcal{L}G_n = -nG_n.$$

Conclusion

Gaussian Measures

Structure of a Gaussian measure space

Main results

Malliavin Calculus

[1] Vladimir Bogachev. *Gaussian Measures*. Vol. 62. American Mathematical Society, Sept. 1998. DOI: 10.1090/surv/062. URL: http://www.ams.org/surv/062.

[2] Olav Kallenberg. *Foundations of Modern Probability*. eng. 3rd ed. 2021. Vol. 99. Probability Theory and Stochastic Modelling. Cham: Springer International Publishing. ISBN: 3030618706.

[3] Christer Borell. "The Ehrhard inequality". In: *C. R. Math. Acad. Sci. Paris* 337.10 (2003), pp. 663–666. ISSN: 1631-073X. DOI: 10.1016/j.crma.2003.09.031. URL: https://doi.org/10.1016/j.crma.2003.09.031.