The Prize Collecting Traveling Salesman Problem

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The following is a valid model for an important class of scheduling and routing problems. A salesman who travels between pairs of cities at a cost depending only on the pair, gets a prize in every city that he vitis and pays a penalty to every city that he fails to visit, wishes to minimize his travel costs and net penalties, while visiting enough cities to collect a prescribed amount of prize money. We call this problem the Prize Collecting Traveling Salesman Problem (PCTSP). This paper discusses structural properties of the PCTS polytope, the convex hull of solutions to the PCTSP. In particular, it identifies several families of facet defining inequalities for this polytope. Some of these inequalities are related to facets of the ordinary TS polytope, others to facets of the knapsack polytope. They can be used in algorithms for the PCTSP either as cutting planes or as ingredients of a Lagrangean optimand.

1. INTRODUCTION

A number of scheduling and routing problems can be formulated as the following generalization of the traveling salesman problem.

A traveling salesman who gets a prize w_k in every city k that he visits and pays a penalty p_l for every city l that he fails to visit, and who travels between cities i and j at cost c_{ij}, wants to minimize the sum of his travel costs and net penalties, while including in his tour enough cities to collect a prescribed amount w_0 of prize money.

If we let y_i be 1 if city i is included in the tour and 0 otherwise, and let x be the incidence vector of the tour, then our problem can be formulated on a complete directed graph $G' = (N, A)$ as

$$\min \sum_{i \in N} \sum_{j \in N - \{i\}} c_{ij} x_{ij} + \sum_{i \in N} p_i (1 - y_i) \tag{1.1'}$$

subject to

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$$\sum_{j \in N-\{i\}} x_{ij} - y_i = 0 \qquad i = 1, \ldots, n \tag{1.2'}$$

$$\sum_{i \in N-\{j\}} x_{ij} - y_j = 0 \qquad j = 1, \ldots, n$$

$$\sum_{i \in N} w_i y_i \geq w_0 \tag{1.3'}$$

$$y_i \in \{0, 1\},\ i \in N;\ x_{ij} \in \{0, 1\},\ (i, j) \in A \tag{1.4'}$$

$$G'(y, x) \text{ is a cycle.} \tag{1.5'}$$

Here $G'(y, x)$ is the subgraph of G' whose nodes and arcs are those defined by y and x, respectively, and by cycle we mean a closed directed path.

It is convenient to complement the variables y_i, $i \in N$, i.e. introduce n new variables $x_{ii} = 1 - y_i$, $i \in N$, to be interpreted as representing the loops (arcs whose head and tail are identical) of a graph $G = (N, A \cup O)$ obtained from G' by endowing every node with a loop (O is the set of loops). The incidence vector $(y, x) \in \{0, 1\}^{n^2}$ of nodes and arcs of G' is then replaced by the incidence vector $x \in \{0, 1\}^{n^2}$ of loops and arcs of G. If we define $c_{ii} := p_i$, $i \in N$, and $U := \Sigma_{i \in N} w_i - w_0$, the problem can be restated as

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1.1}$$

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= 1 \qquad i = 1, \ldots, n \\ \sum_{i=1}^{n} x_{ij} &= 1 \qquad j = 1, \ldots, n \end{aligned} \tag{1.2}$$

$$\sum_{i=1}^{n} w_i x_{ii} \leq U \tag{1.3}$$

$$x_{ij} \in \{0, 1\} \qquad i, j = 1, \ldots, n \tag{1.4}$$

$$G(x) \text{ has exactly one cycle of length} \geq 2. \tag{1.5}$$

Here $G(x)$ is the subgraph of G with node set N and loop-and-arc-set defined by x. Notice that the lower bounding constraint on the weighted sum of nodes to be included into the cycle of $G'(y, x)$ has now become an upper bounding constraint on the weighted sum of loops of $G(x)$. Notice also that for $U < \min w_i$ our problem becomes a TSP.

A typical solution to a PCTSP on 5 nodes is shown in Figure 1.

We formulated this problem in the spring of 1985 as a model for scheduling the daily operation of a steel rolling mill, and called it the Prize Collecting Traveling Salesman Problem (PCTSP). A rolling mill produces steel sheet from slabs by hot or cold rolling. For reasons that have to do with the wear and tear of the rolls as well as other factors, the sequence in which various orders are processed is essential. Scheduling a round consists of choosing from an inventory of slabs assigned to orders, a collection that satisfies a lower bound on total weight, and ordering it into an appropriate sequence, e.g., one that minimizes some function of the sequence. Since the choice of slabs for the round limits the options available for their sequencing, the two tasks must be solved jointly. The PCTSP

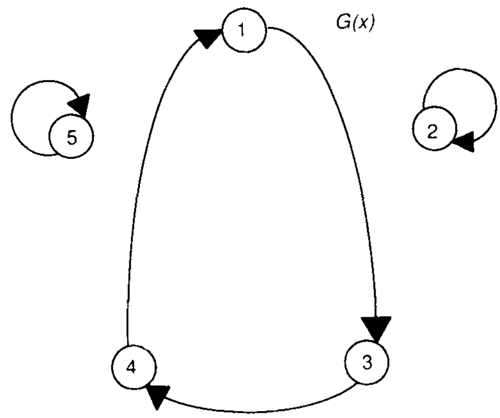

FIG. 1.

as a model captures the essential features of this problem. It served as the basis for an approach that was implemented in 1985–1986 by Balas and Martin [12] into a software package for scheduling steel rolling mills. The package uses a combination of several heuristics to find near-optimal solutions to a PCTSP and organize them into rounds, i.e., daily schedules.

In this paper we study the structural properties of PCTSP; in particular, we identify several families of facet inducing inequalities for the PCTS polytope, the convex hull of solutions to PCTSP. Some of these facets are related to facets of the common TS polytope, others to facets of the knapsack polytope. The facets that we describe can be used in an algorithm for the PCTSP either as cutting planes appended to a linear programming relaxation to be solved by the simplex method, as done for the symmetric TSP by Padberg and Hong [11]; or as inequalities taken into the objective function of a Lagrangean dual with appropriate multipliers, as done for the asymmetric TSP by Balas and Christofides [3]. Either approach requires of course some procedure for identifying inequalities violated by a given solution to the relaxed problem, and although we see no major difficulties in adapting to our case the corresponding procedures developed for the TSP, this task is not addressed in the present paper.

An early version of our results was presented at the April 1986 ORSA/TIMS meeting in Los Angeles [2].

As a way to investigate the structure of the PCTS polytope, we will examine a family of interrelated polytopes that form a hierarchy:

$$\begin{aligned}
AP &:= \text{conv}\,\{x \in \{0, 1\}^{n^2} \mid x \text{ satisfies } (1.2)\}\\
KP &:= \text{conv}\,\{x \in \{0, 1\}^{n^2} \mid x \text{ satisfies } (1.3)\}\\
P_0 &:= \text{conv}\,\{x \in AP \mid x \text{ satisfies } (1.5)\}\\
KAP &:= AP \cap KP\\
&= \text{conv}\,\{x \in AP \mid x \text{ satisfies } (1.3)\}\\
P^* &:= P_0 \cap KAP\\
&= \text{conv}\,\{x \in AP \mid x \text{ satisfies } (1.5) \text{ and } (1.3)\}
\end{aligned}$$

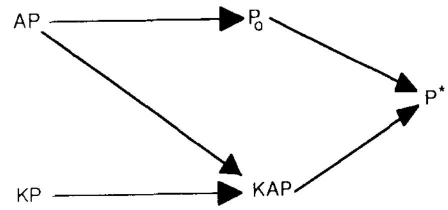

FIG. 2.

The first member of the family, AP, is the assignment polytope, i.e., the convex hull of incidence vectors of all spanning unions of (directed) cycles (where loops are considered cycles of length one). This is known to be a "friendly" polytope, in that

$$AP = \{x \in \mathbb{R}^{n^2} \mid x \geq 0 \text{ and } x \text{ satisfies } (1.2)\}$$

The next polytope, KP, is the 0–1 knapsack polytope defined by the constraint (1.3) on the loop variables x_{ii}, $i \in N$, with the arc variables x_{ij} unconstrained (except for the 0–1 condition). Although this polytope has exponentially many facets, a lot is known about its structure.

The polytope P_0 is obtained from AP by restricting the set of assignments (spanning unions of cycles) to those having exactly one cycle of length greater than one. This polytope will be our starting point for investigating the family of facet defining inequalities related to the well known subtour elimination constraints of the TS polytope. Its study is the subject of Section 2.

The polytope KAP is obtained from the assignment polytope by imposing the knapsack constraint (1.3). The resulting structure, which we call the Knapsack-Constrained Assignment Polytope, provides us with a starting point for examining the family of facets related to those of the 0–1 knapsack polytope. This is the subject of Section 3.

The next polytope, P^*, is the PCTS polytope itself. Its study, based on the results of Sections 2 and 3, is the subject of Section 4.

The hierarchical relations between these polytopes are illustrated in Figure 2, where $S \rightarrow T$ represents the inclusion $S \supset T$.

We will sometimes be interested in looking at what is known as the monotonized version of the above polytopes. For instance, $\overline{KAP}$, the monotonization of KAP, is obtained from KAP by replacing the $=$ in (1.2) by $\leq$. The monotonization of the other polytopes is defined in the same manner.

We recall that given an arbitrary polytope P, a face of P is the intersection of P with some of its supporting hyperplanes. If dim P denotes the dimension of P, a *facet* of P is a face of dimension dim $P - 1$. An inequality $\alpha x \leq \alpha_0$ is said to be *valid* for P if it is satisfied by all $x \in P$; and *facet defining* for P if $P \cap \{x \mid \alpha x = \alpha_0\}$ is a facet of P.

2. THE POLYTOPE P_0

We now turn to the polytope

$$P_0 := \text{conv}\,\{x \in \{0, 1\}^{n^2} \mid x \text{ satisfies } (1.2) \text{ and } (1.5)\}.$$

For every $x \in P_0$, $G(x)$ has exactly one cycle of length ≥ 2. To distinguish this cycle from the loops (which are cycles of length 1) we will refer to it as the long cycle of $G(x)$.

Proposition 2.1. $\dim P_0 = (n - 1)^2$.

Proof. The constraint set defining P_0 has n^2 variables and $2n$ explicitly given equations that form a system of rank $2n - 1$. Thus $\dim P_0 \leq n^2 - 2n + 1 = (n - 1)^2$. We will show that this bound is tight be exhibiting $(n - 1)^2 + 1$ affinely independent points $x \in P_0$.

For $x_{ii} = 0$, $i \in N$, P_0 becomes the traveling salesman polytope on G, whose dimension is known (see Grötschel and Padberg [6]) to be the same as that of the corresponding assignment polytope, i.e., $n(n - 1) - 2n + 1 = n^2 - 3n + 1$; hence there exists a set of $n^2 - 3n + 2$ affinely independent points $x^r \in P_0$ such that $x^r_{ii} = 0$ for all $i \in N$. Take an additional n points $x^s \in P_0$, one for each $s \in N$, such that $x^s_{ii} = 1$ for $i = s$ and $x^s_{ii} = 0$ for $i \in N - \{s\}$. Such points obviously exist (for each such x^s, the long cycle of $G(x^s)$ contains all nodes except s), and together with the points x^r they form an affinely independent set of cardinality $n^2 - 3n + 2 + n = (n - 1)^2 + 1$. ∎

Next we identify a class of facet defining inequalities for P_0 which are related to the subtour elimination constraints for the traveling salesman polytope. If not otherwise stated, we assume that $n \geq 3$.

Theorem 2.2. For all $S \subset N$, $2 \leq |S| \leq n - 1$, and all $k \in S$, $l \in N \setminus S$, the inequality

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S - \{k\}} x_{ii} - x_{ll} \leq |S| - 1 \tag{2.1}$$

is valid for P_0. Further, for $|S| \leq n - 2$, (2.1) defines a facet of P_0.

Proof. If x violates (2.1), then the long cycle of $G(x)$ has its node set in S and contains node k, while node $l \in N \setminus S$ has no loop in $G(x)$; hence $x \notin P_0$. This proves the validity of (2.1) for P_0.

To prove that for $|S| \leq n - 2$, (2.1) is facet defining, let $n \geq 4$ and let $\tilde{P}_0$ be the traveling salesman polytope associated with P_0, i.e., $\tilde{P}_0 : = P_0 \cap \{x \mid x_{ii} = 0, i \in N\}$. The inequality obtained from (2.1) by setting $x_{ii} = 0$, $i \in N$, i.e., the subtour elimination inequality associated with S, is known (see Grötschel [5]) to be facet defining for $\tilde{P}_0$ if $2 \leq |S| \leq n - 2$. Since $\dim \tilde{P}_0 = n(n - 1) - 2n + 1$, there are $n^2 - 3n + 1$ linearly independent vectors $x^r \in P_0$, with $x^r_{ii} = 0$ for all $i \in N$, satisfying (2.1) as equality.

Next consider $|S| - 1$ points $x^s \in P_0$, one for each $s \in S - \{k\}$, such that the only loop of $G(x^s)$ is at node s, and all remaining nodes belong to the long cycle. Then $x^s_{ss} = 1$, $x^s_{ii} = 0$ for all $i \in N - \{s\}$, and $x_{ij} = 1$ for exactly $|S| - 2$ pairs $i, j \in S - \{s\}$, $i \neq j$, i.e., x^s satisfies (2.1) with equality. Such x^s obviously exists for each $s \in S$.

Further, consider $n - 1 - |S|$ points $x^t \in P_0$, one for each $t \in N \setminus (S \cup \{l\})$, such that the only loop of $G(x^t)$ is at node t and all remaining nodes belong to the long cycle. Then $x_{tt}^t = 1$, $x_{ii}^t = 0$ for all $i \in N - \{t\}$, and $x_{ij} = 1$ for exactly $|S| - 1$ pairs $i, j \in S$, $i \neq j$, i.e., x^t satisfies (2.1) with equality. Again, such x^t clearly exists for each $t \in N\setminus(S \cup \{l\}$.

Finally, we need two more points, for the indices k and l. Let x^k be such that $x_{ii}^k = 1$ for $i \in$, $x_{ii}^k = 0$ for $i \in N\setminus S$, and $N\setminus S$ is the node set of the long cycle of $G(x^k)$; and let x^l be such that $x_{ii}^l = 1$ for $i \in N\setminus S$, $x_{ii}^l = 0$ for $i \in S$, and S is the node set of the long cycle of $G(x^l)$. Clearly, x^k, $x^l \in P_0$ and both vectors satisfy (2.1) as equality.

It is now easy to see that the matrix whose rows are the vectors x^s, $s \in S - \{k\}$, x^t, $t \in N\setminus(S \cup \{l\})$, x^k and x^l, is of the form (V, T), where T is $n \times n$, with columns corresponding to the variables x_{ii}, $i \in N$, and T is lower triangular up to row and column permutations. If W is the matrix whose rows are the $n^2 - 3n + 1$ linearly independent vectors of $\bar{P}_0$ considered at the beginning of this proof, then the matrix whose rows are the extensions x^r of these vectors to P_0, plus the vectors x^s, x^t, x^k, x^l, is of the form

$$X = \begin{pmatrix} W & O \\ V & T \end{pmatrix},$$

where W is of rank $n^2 - 3n + 1$ and T is of rank n. Clearly X is of rank $(n - 1)^2$, which proves that (2.1) defines a facet of P_0. ■

Corollary 2.3. For all $S \subset N$, $2 \leqslant |S| \leqslant n - 1$ and all $k \in S$, $l \in N\setminus S$, the inequality

$$\sum_{i \in S} \sum_{j \in N\setminus S} x_{ij} + x_{kk} + x_{ll} \geqslant 1 \tag{2.2}$$

is valid for P_0. Further, for $|S| \leqslant n - 2$, (2.2) defines a facet of P_0.

Proof. Equation (2.2) can be obtained by subtracting from (2.1) the equations

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i \in S. \quad ■$$

From the above Theorem and Corollary, we now have an equivalent expression for P_0:

Corollary 2.4.

$$P_0 = \left\{ x \in AP \;\middle|\; \begin{array}{l} x \text{ satisfies (2.1) for all } S \subset N,\ 2 \leqslant |S| \leqslant n - 1, \\ \text{and all } k \in S,\ l \in N\setminus S \end{array} \right\}$$

Proof. From Theorem 2.2 and Corollary 2.3, condition (1.5) implies (2.1) and (2.2) for all $S \subset N$, $2 \leqslant |S| \leqslant n - 1$. To see the converse, suppose $x \in AP$ violates (1.5), i.e., $G(x)$ has at least two cycles of length $\geqslant 2$, with node sets S_1 and S_2, respectively. Then x violates (2.1) and (2.2) for $S = S_1$ and $l \in S_2$. ■

Sometimes the constraints of PCTSP are amended with the requirement that a certain node, say 1, be included into the long cycle of $G(x)$. This is equivalent to adding the condition $x_{11} = 0$ to the constraint set. In this case the above inequalities can be strengthened.

Theorem 2.5. For all $S \subset N$, $2 \leqslant |S| \leqslant n - 1$, such that $1 \in S$, and all $l \in N \setminus S$, the inequality

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S} x_{ii} - x_{ll} \leqslant |S| - 1 \tag{2.3}$$

is valid for $P_0 \cap \{x \mid x_{11} = 0\}$ and, if $|S| \leqslant n - 2$, facet defining for $P_0 \cap \{x \mid x_{11} = 0\}$.

For all $S \subset N$, $2 \leqslant |S| \leqslant n - 1$, such that $1 \in N \setminus S$, and all $k \in S$, the inequality

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S - \{k\}} x_{ii} \leqslant |S| - 1 \tag{2.4}$$

is valid for $P_0 \cap \{x \mid x_{11} = 0\}$ and, if $|S| \leqslant n - 2$, facet defining for $P_0 \cap \{x \mid x_{11} = 0\}$.

Proof. Let $x \in P_0 \cap \{x \mid x_{11} = 0\}$ and let $1 \in S$. If S contains the long cycle of $G(x)$, then $x_{ll} = 1$ and (2.3) is satisfied. Otherwise at most $|S| - 1$ of the variables x_{ij}, $i, j \in S$, can be positive, and again (2.3) is satisfied.

Now let $1 \in N \setminus S$. Then the node set of the long cycle of $G(x)$ is not contained in S, hence the left hand side of (2.4) is at most $|S| - 1$. This proves the validity of (2.3) and (2.4) under the stated conditions.

To show that (2.3) and (2.4), when valid, are facet defining if $|S| \leqslant n - 2$, we assume $|S| \leqslant n - 2$ and proceed as in the proof of Theorem 2.2. In that proof, we exhibited $(n - 1)^2$ affinely independent points $x \in P_0$ that satisfy (2.1) with equality; let their set be Z. If we leave aside for the moment the last two points exhibited, x^k and x^l, all but one of the points in $Z - \{x^k, x^l\}$ lie in $P_0 \cap \{x \mid x_{11} = 0\}$ and satisfy with equality both (2.3) and (2.4), the one exception being x^s for $s = 1$. Discarding x^1 leaves a set $Z - \{x^1, x^k, x^l\}$ of $(n - 1)^2 - 3$ affinely independent points $x \in P_0 \cap \{x \mid x_{11} = 0\}$, two less than needed (since $\dim P_0 \cap \{x \mid x_{11} = 0\} = (n - 1)^2 - 1$).

Now let $1 \in S$. Then the point $x^l \in Z$ lies in $P_0 \cap \{x \mid x_{11} = 0\}$ and satisfies (2.3) with equality. To obtain the last missing point, consider $\bar{x}^k \in P_0$ such that the only loop of $G(\bar{x}^k)$ is at node k and all remaining nodes belong to the long cycle (note that $k \neq 1$ in the definition of Z). Then $\bar{x}^k \in P_0 \cap \{x \mid x_{11} = 0\}$ and $\bar{x}^k$ satisfies (2.3) with equality. Furthermore, the $(n - 1)^2 - 1$ points in $(Z - \{x^1, x^k\}) \cup \{\bar{x}^k\}$ are affinely independent. Thus (2.3) defines a facet of $P_0 \cap \{x \mid x_{11} = 0\}$.

Similarly, if $1 \in N \setminus S$, the point $x^k \in Z$ lies in $P_0 \cap \{x \mid x_{11} = 0\}$ and satisfies (2.4) with equality. Let $\bar{x}^l \in P_0$ be such that the only loop of $G(\bar{x}^l)$ is at node l and all the other nodes belong to the long cycle (note that $l \neq 1$ in the definition of Z). Then $\bar{x}^l \in P_0 \cap \{x \mid x_{11} = 0\}$ and $\bar{x}^l$ satisfies (2.4) with equality. Again, the points in $Z - \{x^1, x^l\} \cup \{\bar{x}^l\}$ are affinely independent, hence (2.4) defines a facet of $P_0 \cap \{x \mid x_{11} = 0\}$. ■

As in the case of (2.1), the inequalities (2.3) and (2.4) have their cutset-related alternative form:

Corollary 2.6. The inequalities (2.3) and (2.4) are equivalent to

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} + x_{ll} \geq 1 \tag{2.5}$$

and

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} + x_{kk} \geq 1 \tag{2.6}$$

respectively.

3. THE POLYTOPE KAP

Next we turn to the Knapsack-Constrained Assignment Polytope

$$KAP := \{x \in \{0, 1\}^{n^2} \mid x \text{ satisfies (1.2) and (1.3)}\}.$$

Define

$$R_U := \{i \in N \mid w_i > U\},$$

the set of those nodes that cannot have their loop in $G(x)$ if x is to satisfy (1.3). Clearly, for any $x \in KAP$, $x_{ii} = 0$ for all $i \in R_U$, hence $\dim KAP \leq \dim AP - |R_U|$.

Proposition 3.1. $\dim KAP = (n - 1)^2 - |R_U|$.

Proof. One can exhibit the required number of affinely independent points in KAP in the same way as in the proof of Proposition 2.1. ■

Since $KAP \subset KP$, all valid inequalities for the knapsack polytope KP are valid for KAP. The question we address next, is whether the inequalities that define facets of the knapsack polytope also define facets of the corresponding Knapsack-Constrained Assignment Polytope or its monotonization.

Facets of the 0–1 knapsack polytope have been extensively studied (we refer the reader to Balas [1], Hammer, Johnson and Peled [7], Wolsey [13], and Balas and Zemel [4]). The best known family of facets of the knapsack polytope KP is defined by inequalities of the form

$$\sum_{i \in S} x_{ii} + \sum_{i \in N \setminus S} \alpha_i x_{ii} \leq |S| - 1. \tag{3.1}$$

where $0 \leq \alpha_i \leq |S| - 1$ is integer, $i \in N$, and where $S \subseteq N$ is a minimal cover for the knapsack inequality (1.3), i.e.

$$\sum_{i \in T} w_i > U \tag{3.2}$$

holds for $T = S$ but fails to hold for $T = S - \{i\}$ for any $i \in S$.

Let $\overline{KAP}$ denote the monotonization of KAP, as defined in Section 1.

Theorem 3.2. Suppose the inequality (3.1) defines a facet of KP. Then (3.1) defines a facet of $\overline{KAP}$ if and only if $\alpha_i > 0$ for at least one $i \in N \setminus S$.

Proof. It is well known (see Padberg [10], Nemhauser and Trotter [9]) that, given the assumptions, there exist nonnegative integers β_{ij}, $(i, j) \in A$, called *lifting coefficients,* such that the inequality

$$\sum_{i \in S} x_{ii} + \sum_{i \in N \setminus S} \alpha_i x_{ii} + \sum_{i \in N} \sum_{j \in N - \{i\}} \beta_{ij} x_{ij} \leq |S| - 1 \tag{3.3}$$

defines a facet of $\overline{KAP}$. The lifting coefficients can be calculated by solving a sequence of integer programs, one for each coefficient. Different lifting sequences may give rise to different coefficient values, but a given coefficient attains its highest value when it is first in the lifting sequence. Let β^1_{ij} denote the value obtained for β_{ij} when β_{ij} is calculated before all other coefficients. We will prove the Theorem by showing that $\beta^1_{ij} = 0$ for all $(i, j) \in A$ if and only if $\alpha_i > 0$ for at least one $i \in N \setminus S$.

According to the lifting Theorem outlined above, $\beta^1_{ij} = |S| - 1 - z_{ij}$, where

$$z_{ij} = \max \left\{ \sum_{k \in S} x_{kk} + \sum_{k \in N \setminus S} \alpha_k x_{kk} \;\middle|\; \begin{array}{l} \sum_{k \in N} w_k x_{kk} \leq U \\ x_{ii} = x_{jj} = 0 \\ x_{kk} \in \{0,1\},\ k \in N - \{i,j\} \end{array} \right\}$$

If $\alpha_k = 0$ for all $k \in N \setminus S$, then for any pair $i, j \in S$, $z_{ij} = |S| - 2$, hence, $\beta^1_{ij} > 0$. On the other hand, if $\alpha_k \geq 1$, $k \in T$, for some nonempty set $T \subseteq N \setminus S$, then for any pair $i, j \in N$, $x_{kk} = 1$ for some $k \in T$ in any optimal solution and thus $z_{ij} = |S| - 1$ and $\beta^1_{ij} = 0$. ■

In the next section we will prove a stronger result (Theorem 4.8) which implies the following:

Theorem 3.3. Suppose (3.1) defines a facet of *KP* and $\alpha_i = 0$ for at least one $i \in N \setminus S$. Then (3.1) defines a facet of *KAP* if and only if $\alpha_i > 0$ for at least one $i \in N \setminus S$.

4. THE POLYTOPE P^*

Next we turn to P^*, the PCTS polytope itself. Since P^* is contained in each of the polytopes discussed in Sections 2 and 3, the inequalities of those sections are all valid for P^*. As we shall see, however, some of these inequalities can be strengthened.

First we establish the dimension of P^*. Since $P^* \subset KAP$, $\dim P^* \leq (n - 1)^2 - |R_U|$, where, as in Section 3,

$$|R_U| = \{i \in N \mid w_i > U\}$$

Proposition 4.1. $\dim P^* = (n - 1)^2 - |R_U|$.

Proof. One can exhibit the required number of affinely independent points $x \in P^*$ the same way as in the proof of Proposition 2.1 ■

In the sequel we will assume that $|R_U| = 0$, i.e., $\dim P^* = (n - 1)^2$.

Theorem 4.2. For $S \subset N$, $2 \leq |S| \leq n - 1$, the inequality

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S} x_{ii} \leq |S| - 1 \tag{4.1}$$

is valid for P^* if and only if

$$\sum_{i \in T} w_i > U \tag{4.2}$$

holds for both $T = S$ and $T = N \setminus S$.

Furthermore, if $|S| \leq n - 2$ and (4.1) is valid for P^*, then (4.1) is facet defining for P^*.

Proof. $\bar{x} \in P^*$ violates (4.1) if and only if S either contains the node set of the long cycle of $G(\bar{x})$, or contains no node of that cycle (i.e., $\bar{x}_{ii} = 1$, $i \in S$). In the first case, (4.2) is violated for $T = N \setminus S$, while in the second (4.2) is violated for $T = S$. This proves the first statement.

Now suppose $|S| \leq n - 2$ and (4.1) is valid. Then, as argued in the proof of Theorem 2.2, there are $(n - 1)^2 - n = n^2 - 3n + 1$ affinely independent points $x^r \in P^*$, with $x^r_{ii} = 0$, $i = 1, \ldots, n$, which satisfy (4.1) with equality. Also, for each $k \in N$, there exists $x^k \in P^*$ with $x^k_{kk} = 1$, $x^k_{ii} = 0$, $i \in N - \{k\}$, and such that x^k satisfies (4.1) with equality. These two sets of points together clearly form a set of $(n = 1)^2$ affinely independent points in P^*. ■

Theorem 4.3. For $S \subset N$, $2 \leq |S| \leq n - 1$, and for all $k \in S$, the inequality

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S - \{k\}} x_{ii} \leq |S| - 1 \tag{4.3}$$

is valid for P^* if and only if (4.2) holds for $T = N \setminus S$.

Furthermore, if $|S| \leq n - 2$ and (4.3) is valid for P^*, then (4.3) is facet defining for P^* if and only if (4.2) does not hold for $T = S$.

Proof. If $x \in P^*$ violates (4.3) for some S and $k \in S$, then S contains the long cycle of $G(x)$, and that cycle contains node k; i.e., $x_{ii} = 1$, $i \in N \setminus S$, and $x_{kk} = 0$. But then (4.2) does not hold for $T = N \setminus S$. Conversely, if (4.2) does not hold for $T = N \setminus S$, then any $x \in P^*$ such that $x_{ii} = 1$, $i \in N \setminus S$, and $x_{kk} = 0$, violates (4.3) for the given S and $k \in S$. This proves the first statement.

Suppose now that $|S| \leq n - 2$ and (4.3) is valid for P^*. If (4.2) holds for $T = S$, then (4.1) is also valid and hence (4.3) cannot be facet defining. On the other hand, if (4.2) does not hold for $T = S$, one can exhibit $(n - 1)^2$ affinely independent points $x \in P^*$ that satisfy (4.3) as equality. The first $(n - 1)^2 - n$ such points $x^r \in P^*$, all of which have $x^r_{ii} = 0$, $i \in N$, exist as a consequence of the fact that the traveling salesman polytope on n nodes is $(n^2 - 3n + 1)$-dimensional. For the remaining n points, one for each $s \in N$, one can proceed as follows. For $s \in S - \{k\}$ and $s \in N \setminus S$, choose $x^s \in P^*$ such that $x^s_{ss} = 1$, $x^s_i = 0$ for all $i \neq s$; and for $s = k$, choose $x^s \in P^*$ such that $x^s_{ii} = 1$ for $i \in S$, $x^s_{ii} = 0$ *for* $i \in N \setminus S$. These n vectors x^s are the rows of a matrix whose last n columns form a lower triangular submatrix, up to row and column permutations; and so together with the $n^2 - 3n + 1$ vectors x^r they form a $(n - 1)^2 \times n^2$ matrix of full row rank. ■

Corollary 4.4. For $S \subset N$, $2 \leqslant |S| \leqslant n - 2$, if (4.2) does not hold with $T = N \setminus S$, then for every $k \in S$ and $l \in N \setminus S$, the inequality

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S - \{k\}} x_{ii} - x_{ll} \leqslant |S| - 1 \tag{2.1}$$

defines a facet of P^*.

Proof. If (4.2) does not hold with $T = N \setminus S$, the argument used to prove that (2.1) defines a facet of P_0 (Theorem 2.2) carries over to the case of P^*. ∎

From Corollaries 4.4 and 2.4, condition (1.5) in the definition of P^* can be replaced by the system (2.1), i.e., an equivalent expression for P^* is

$$P^* = \operatorname{conv} \left\{ x \in \{0, 1\}^{n^2} \;\middle|\; \begin{array}{l} x \text{ satisfies (1.2), (1.3) and} \\ \text{(2.1) for all } S \subset N,\ 2 \leqslant |S| \leqslant n - 1 \\ \text{and all } k \in S,\ l \in N \setminus S \end{array} \right\}$$

The inequalities, (2.1), (4.3), and (4.1) can be viewed as extensions of the subtour elimination inequalities for the traveling salesman polytope to the PCTS polytope. The conditions under which these extensions are valid (and facet defining) are increasingly stringent as we move from (2.1) to (4.3) and (4.1), as each inequality in the sequence strictly dominates its predecessors.

As in the case of the TS polytope, where the subtour elimination inequalities have an equivalent form related to cutsets, the inequalities (4.1), (4.3), and (2.1) for the PCTS polytope have an alternative cutset-related form:

Corollary 4.5. The inequalities

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geqslant 1, \tag{4.4}$$

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} + x_{kk} \geqslant 1 \tag{4.5}$$

and

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} + x_{kk} + x_{ll} \geqslant 1 \tag{2.2}$$

are equivalent to (4.1), (4.3), and (2.1), respectively: for given S (and possibly $k \in S$, $l \in N \setminus S$), a member of the triplet (4.4), (4.5), (2.2) is valid (facet defining) for P^* if and only if the corresponding member of the triplet (4.1), (4.3), (2.1) is valid (facet defining).

Proof. For any $S \subset N$, each of the inequalities (4.4), (4.5), and (2.2) can be obtained by subtracting the equations $\Sigma_j x_{ij} = 1$, $i \in S$, from the corresponding inequality (4.1), (4.3), or (2.1). ∎

If the constraints of P^* include the condition $x_{11} = 0$, then the above inequalities can be strengthened, as in the case of P_0. Note that, as in the case of P_0, dim $P^* \cap \{x \mid x_{11} = 0\} = (n - 1)^2 - 1$.

Corollary 4.6. For all $S \subset N$, $2 \leqslant |S| \leqslant n - 1$, such that $1 \in S$, the inequalities

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S} x_{ii} \leqslant |S| - 1 \tag{4.1}$$

and

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geqslant 1 \tag{4.4}$$

are valid for $P^* \cap \{x \mid x_{11} = 0\}$ if and only if

$$\sum_{i \in T} w_i > U \tag{4.2}$$

holds with $T = N \setminus S$.

Further, if $|S| \leqslant n - 2$ and the inequalities (4.1), (4.4) are valid, then they are facet defining for $P^* \cap \{x \mid x_{11} = 0\}$.

Proof. Analogous to the proof of Theorem 4.1. Since $x_{11} = 0$ is now a constraint, (4.2) need not hold for $T = S$ in order for (4.1) to be valid. The argument used to show that (4.1) is facet defining for P^* when valid (and when $|S| \leqslant n - 2$), carries over to $P^* \cap \{x \mid x_{11} = 0\}$, since all but one of the $n^2 - 3n + 1$ affinely independent points exhibited satisfy $x_{11} = 0$. ∎

Corollary 4.7. For all $S \subset N$, $2 \leqslant |S| \leqslant n - 1$, such that $1 \in S$, and all $l \in N \setminus S$, the inequalities

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S} x_{ii} - x_{ll} \leqslant |S| - 1 \tag{2.3}$$

and

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} + x_{ll} \geqslant 1 \tag{2.5}$$

are valid for $P^* \cap \{x \mid x_{11} = 0\}$. Further, if $|S| \leqslant n - 2$ and (4.2) does not hold with $T = N \setminus S$, then (2.3) and (2.5) are facet defining for $P^* \cap \{x \mid x_{11} = 0\}$.

For all $S \subset N$, $2 \leqslant |S| \leqslant n - 1$, such that $1 \in N \setminus S$, and all $k \in S$, the inequalities

$$\sum_{i \in S} \sum_{j \in S - \{i\}} x_{ij} + \sum_{i \in S - \{k\}} x_{ii} \leqslant |S| - 1 \tag{2.4}$$

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} + x_{kk} \geqslant 1 \tag{2.6}$$

are valid for $P^* \cap (x \mid x_{11} = 0\}$, and if $|S| \leqslant n - 2$, facet defining for $P^* \cap \{x \mid x_{11} = 0\}$.

Proof. The validity of (2.3), (2.5) and (2.4), (2.6) follows from Theorem 2.5 and Corollary 2.6, respectively, in view of $P^* \subseteq P_0$. The fact that, when valid, these inequalities are facet defining if $|S| \leqslant n - 2$ (and, in the case of (2.3) and (2.5), if (4.2) does not hold with $T = N \setminus S$), follows from the proof of Theorem 2.5 which carries over to this case with only one change: the point $z^j \in Z$ used in that proof lies in $P^* \cap \{x \mid x_{11} = 0\}$ only if (4.2) does not hold for $T = N \setminus S$. Hence the condition concerning (4.2) in the above Corollary. ■

Next we turn to inequalities that come from the Knapsack-Constrained Assignment polytope.

Theorem 4.8. Suppose the inequality

$$\sum_{i \in S} x_{ii} + \sum_{i \in T} \alpha_i x_{ii} \leq |S| - 1, \tag{3.1}$$

where $\alpha_i > 0$ is integer, $i \in T$, defines a facet of KP, and $N \setminus (S \cup T) \neq \emptyset$. Then (3.1) defines a facet of P^* if and only if $T \neq \emptyset$.

Proof. From Theorem 3.2, (3.1) defines a facet of $\overline{KAP}$ if and only if $T \neq \emptyset$. The argument used in the proof of that Theorem carries over without change to the case of $\bar{P}^*$. Hence (3.1) is facet defining for $\bar{P}^*$ if and only if $T \neq \emptyset$.

Assume now that $T \neq \emptyset$. Consider the equation

$$\sum_{i \in S} x_{ii} + \sum_{i \in T} \alpha_i x_{ii} = |S| - 1 \tag{3.1'}$$

and define

$$F := P^* \cap \{x \mid x \text{ satisfies } (3.1')\}.$$

We will prove that F is a facet of P^*, i.e., that $\dim F = \dim P^* - 1$, by showing that for any inequality $\gamma x \leq \gamma_0$ valid for P^* and such that $\gamma x = \gamma_0$ for all $x \in F$, the equation $\gamma x = \gamma_0$ is a linear combination of the Eqs. (1.2) and (3.1'); i.e., that there exists multipliers λ_i, μ_i, $i \in N$, and π_0 such that

$$\gamma_{ij} = \begin{cases} \lambda_i + \mu_j & \text{if } i \neq j \text{ or } i = j \in N \setminus (S \cup T) \\ \lambda_i + \mu_j + \pi_0 & \text{if } i = j \in S \cup T \end{cases} \tag{4.6}$$

and

$$\gamma_o = \sum_{i \in N} (\lambda_i + \mu_i) + (|S| - 1)\pi_0. \tag{4.7}$$

Consider first $(i, j) \in A$, i.e., $i \neq j$. Let $n \in N \setminus (S \cup T)$ and define

$$\begin{aligned} \lambda_i &:= \gamma_{in} - \gamma_{nn}, & i &= 1, \ldots, n \\ \mu_j &:= \gamma_{nj}, & j &= 1, \ldots, n \end{aligned} \tag{4.8}$$

We claim that for all $i \neq j$ or $i = j \in N \setminus (S \cup T)$,

$$\begin{aligned} \gamma_{ij} &= \lambda_i + \mu_j \\ &= \gamma_{in} + \gamma_{nj} - \gamma_{nn}. \end{aligned} \tag{4.9}$$

For $i = n, j = 1, \ldots, n$ and for $j = n, i = 1, \ldots, n$, this is clearly true. Now let $i \neq n \neq j \neq i$, and consider $x \in F$ such that $x_{ij} = x_{kk} = 1$ for a given $k \in N\setminus(S \cup T)$. Such x exists for every $k \in N\setminus(S \cup T)$. Define x' by $x'_{ij} = x'_{kk} = 0$, $x'_{ik} = x'_{kj} = 1$, and $x'_{pq} = x_{pq}$ for all other p, q. If C is the arc set of the long cycle of $G(x)$, then $(C - \{(i, j)\} \cup \{(i, k), (k, j)\}$ is again the arc set of a cycle; thus $G(x')$ has exactly one long cycle, and since the set of its loops is strictly contained in that of $G(x)$, x' also satisfies the knapsack constraint (1.3). Hence $x' \in P^*$. Further since x satisfies (3.1′), so does x', i.e., $x' \in F$. By assumption, we then have $\gamma x = \gamma_0 = \gamma x'$, and hence $\gamma_{ij} + \gamma_{kk} = \gamma_{ik} + \gamma_{kj}$. Since this argument is valid for every $k \in N\setminus(S \cup T)$, we have

$$\gamma_{ij} = \gamma_{ik} + \gamma_{kj} - \gamma_{kk} \text{ for all } k \in N\setminus(S \cup T). \tag{4.10}$$

In particular, for $k = n$, (4.10) becomes (4.9) and thus proves our claim.

Now let $i = j \in N\setminus(S \cup T)$. If $i = n$, we are done; otherwise consider $x \in F$ such that $x_{ii} = x_{hl} = 1$ for some $h \neq i \neq l$. Such x clearly exists. Define x' by $x'_{ii} = x'_{hl} = 0$, $x'_{il} = 1$, and $x'_{pq} = x_{pq}$ for all other p, q.

Then $x' \in F$(for the same reasons as above), hence $\gamma x = \gamma x'$ and $\gamma_{ii} + \gamma_{hl} = \gamma_{hi} + \gamma_{il}$. Substituting from (4.10) for γ_{hl}, γ_{hi}, and γ_{il} with $k = n$ then yields

$$\begin{aligned}\gamma_{ii} &= \gamma_{hi} + \gamma_{il} - \gamma_{hl}\\ &= (\gamma_{hn} + \gamma_{ni} - \gamma_{nn}) + (\gamma_{in} + \gamma_{nl} - \gamma_{nn})\\ &\qquad - (\gamma_{hn} + \gamma_{nl} - \gamma_{nn})\\ &= \gamma_{in} + \gamma_{ni} - \gamma_{nn}\end{aligned}$$

which proves (4.9) for this case.

To prove that (4.6) also holds for $i = j \in S \cup T$, we define for all $i \in S \cup T$

$$\pi_i = (\gamma_{ii} - \lambda_i - \mu_i)/\alpha_i, \tag{4.11}$$

where for $i \in S$, $\alpha_i = 1$. We claim that all π_i are equal, i.e., $\pi_i = \pi_0$ for all $i \in S \cup T$.

Consider first two distinct nodes $i, k \in S \cup T$ such that $\alpha_i = \alpha_k = 1$. Let $x \in F$ be such that $x_{ii} = x_{hk} = x_{kl} = 1$, with $i \neq j \neq k \neq l \neq i$ for some $h \neq l$. Such x clearly exists. Define x' by $x'_{ii} = x'_{hk} = x'_{kl} = 0$, $x'_{kk} = x'_{hi} = x'_{il} = 1$ and $x'_{pq} = x_{pq}$ for all other p, q. Then $x' \in P$ and from what we know about facets of the knapsack polytope (see, for instance, [1]), if x satisfies (1.3) then so does x'. Thus $x' \in F$, and since this implies $\gamma x = \gamma x'$, we have

$$\gamma_{ii} + \gamma_{hk} + \gamma_{kl} = \gamma_{kk} + \gamma_{hi} + \gamma_{il}.$$

Substituting for γ_{ii} and γ_{kk} from (4.11) and for γ_{hk}, γ_{kl}, γ_{hi}, γ_{il} from (4.9) then yields $\pi_i = \pi_k$.

Consider now two nodes $i \in S$, $k \in T$, such that $\alpha_k = m$ for some positive integer $m \geqslant 2$. The existence of such α_k implies that $|S| \geqslant m + 1$. Let $x \in F$ be such that $x_{ii} = 1$ for m distinct indices $i \in S$, namely $i = i_1, \ldots, i_m$, and $x_{hk} = x_{kl} = 1$ for some $h \neq l$. Such x clearly exists. Define x' by $x'_{i_1 i_1} = \cdots = x'_{i_m i_m} = x'_{hk} = x'_{kl} = 0$, $x'_{kk} = x'_{h i_1} = x'_{i_1 i_2} = \cdots = x'_{i_{m-1} i_m} = x'_{i_m l} = 1$, and $x'_{pq} = x_{pq}$ for all other

p, q. If C is the long cycle of $G(x)$, then $(C - \{(h, k), (k, l)\} \cup \{(h, i), (i_1, i_2), \ldots, (i_m, l)\}$ is again a cycle, and so $x' \in P_0$. Also, from the theory of knapsack polytopes, if x satisfies (1.3) so does x', hence $x' \in P^*$. Finally, since $\alpha_{i_1} + \cdots + \alpha_{i_m} = \alpha_k = m$, if x satisfies (3.1') so does x'. Thus $x' \in F$, and from $\gamma x = \gamma x'$ we have

$$\gamma_{i_1 i_1} + \cdots + \gamma_{i_m i_m} + \gamma_{hk} + \gamma_{kl} = \gamma_{kk} + \gamma_{hi_1} + \gamma_{i_1 i_2} + \cdots + \gamma_{i_m l}$$

Now substituting for γ_{pq} from (4.11) if $p = q$ (since all such p belong to $S \cup T$) and from (4.9) if $p \neq q$ we obtain

$$\begin{aligned}(\pi_{i_1} + \lambda_{i_1} + \mu_{i_1}) + \cdots + (\pi_{i_m} + \lambda_{i_m} + \mu_{i_m}) + (\lambda_h + \mu_k) + (\lambda_k + \mu_l) \\ = (\alpha_k \pi_k + \lambda_k + \mu_k) + (\lambda_h + \mu_{i_1}) + (\lambda_{i_1} + \mu_{i_2}) + \cdots + (\lambda_{i_m} + \mu_l)\end{aligned}$$

Since $\pi_{i_1} = \cdots = \pi_{i_m}$, after simplifying this yields $m\pi_{i_1} = \alpha_k \pi_k$ or, since $\alpha_k = m$ and all π_i with $i \in S$ are equal, whereas $k \in T$ was chosen arbitrarily,

$$\pi_i = \pi_k = \pi_0 \qquad \text{for all } i \in S,\ k \in T.$$

This completes the proof of (4.6).

To prove (4.7), consider $\bar{x} \in F$ such that $x_{ii} = 1$ for all $i \in S - \{k\}$ for some $k \in S$, $\bar{x}_{ii} = 0$ for all $i \in (N \setminus S) \cup \{k\}$, and $x_{i_1 i_2} = \cdots = x_{i_t i_1}$ for some cycle $C = \{(i_1, i_2), \ldots, (i_{t-1}, i_t), (i_t, i_1)\}$ whose node set is $\{i_1, \ldots, i_t\} = (N \setminus S) \cup \{k\}$. Then

$$\begin{aligned}\gamma_o &= \gamma \bar{x} \\ &= \gamma_{i_1 i_2} + \cdots + \gamma_{i_t i_1} + \sum_{i \in S - \{k\}} \gamma_{ii} \\ &= \sum_{i \in (N \setminus S) \cup \{k\}} (\lambda_i + \mu_i) + \sum_{i \in S - \{k\}} (\lambda_i + \mu_i) + (|S| - 1)\pi_0 \\ &= \sum_{i \in N} (\lambda_i + \mu_i) + (|S| - 1)\pi_0. \quad \blacksquare\end{aligned}$$

The last theorem of Section 3 (Theorem 3.3) is now a direct consequence of Theorem 4.8.

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