不同螺旋 撕裂模的环耦合

Toroidal coupling of disparate helical tearing modes

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An analytical model has been developed for toroidal coupling of tearing modes with different helicities in the low-β and large aspects ratio tokamaks. A standard characteristic value problem has been naturally composed according to the structure of magnetohydrodynamic (MHD) solutions. The explicit general dispersion relation has been obtained by the asymptotic matching. The growth rates (characteristic values) and corresponding flux perturbations (characteristic functions) of the toroidal tearing mode have been derived. The toroidal coupling plays a role mainly through the correction for the ideal MHD solutions. Without loss of generality, variation tendency of growth rates has been analyzed for a toroidal tearing mode with cylindrical components m/n and $(m+1)/n$, the results indicate that (1) The toroidal coupling has a destabilizing effect on the tearing modes; (2) the "beating" of the growth rates of two components leads to strong coupling, even if the coupling parameter C is quite small, and the coupling effect does not explicitly rely on magnitude of $\Delta_m'^{(0)}$ of cylindrical component, so that $\Delta_m'^{(0)} \sim \epsilon$ is neither a necessary nor sufficient condition for strong coupling.

I. INTRODUCTION

The linear tearing modes were analyzed in slab geometry by Furth, Killeen, and Rosenbluth.[1] The boundary layer method was employed to solve the resistive magnetohydrodynamic (MHD) equations, since the resistivity only plays an important role in the vicinity of resonant surfaces. The asymptotic matching between resistive layer and outer region can determine the dispersion relation of tearing modes, namely, $\Delta'=\Delta$, where Δ' is the jump of logarithmic derivative of ideal MHD solution and Δ can be derived from resistive layer equations. The instability criterion is $\Delta'>0$. This approach can be generalized to the cylindrical and toroidal geometries. For example, Furth, Rutherford, and Selberg[2] carried out a detailed investigation of instability, depending on the current profile shape. Glasser, Greene, and Johnson[3] indicated the stabilizing effect of toroidal curvature on the tearing modes.

Since the 1970s, some analytical models and numerical codes have been developed for the toroidal coupling effect on tearing modes with different helicities.[4-7] Later, Connor *et al.*[8] constructed a set of $(M+N)$ basic functions as solutions of ideal MHD equations and used their linear combination to describe a toroidal tearing mode with M poloidal harmonics and N resonant surfaces. A solubility condition $|E-\Delta|=0$ was obtained, where the E matrix depends on basic function values at resonant surfaces and edge. Recently, Connor, Hastie, and Taylor[9] defined a "triplet" $Y_m=[y^{(1)}_{m,m-1},(y^{(0)}_{mm}+y^{(2)}_{mm}),y^{(1)}_{m,m+1}]$ as a particular ideal MHD solution and the superposition $Y=\Sigma\,\alpha_m Y_m$ as the description of a toroidal tearing mode under the limit of low-β and large aspect ratio. A similar solubility condition for $\{\alpha_m\}\neq 0$ and the explicit expressions of E matrix elements were derived for $\Delta_m^{(0)}\gg\epsilon$. A formal "strong-coupling" theory was constructed for $\Delta_m^{(0)}\sim\epsilon$.

The proposal of this paper is to present a new theoretical model for the toroidal tearing mode in the low-β and large aspect ratio tokamaks. For each toroidal tearing eigenmode, many individual cylindrical components with different helicities are coupled together as a global structure due to toroidicity. The reduced resistive MHD equations derived in an axisymmetric torus should reflect such an intrinsic property of toroidal coupling for tearing modes so it would not need to construct any artificial expression to describe a toroidal tearing mode. Actually, a general MHD solution $\Psi_m(\rho)$ for a toroidal tearing mode can be expressed as a general solution of cylindrical component $\Psi_m^{(0)}(\rho)$ plus the sum of all the particular solutions, which is a function of $\{\Psi_{m'}^{(0)}(\rho), m'\neq m\}$, caused by toroidal coupling. The relations among $[\Delta_m'\Psi_m(\rho_m)]$ $\{[\Delta_m\Psi_m(\rho_m)]\}$ for toroidal tearing modes and $[\Delta_m'^{(0)}\Psi_m^{(0)}(\rho_m)]$ $\{[\Delta_m^{(0)}\times\Psi_m^{(0)}(\rho_m)]\}$ for cylindrical components can be naturally derived in terms of the structure of solutions, which composes a standard characteristic value problem. The explicit general dispersion relation therefore can be resulted from the solubility condition for $\{\Psi_m^{(0)}(\rho)\}\neq 0$ and asymptotic matching; the growth rates (characteristic values) and corresponding magnetic flux perturbations (characteristic functions) of the toroidal tearing mode can be consequently obtained as well. Without loss of generality, variation tendency of growth rates for the toroidal tearing mode with cylindrical components m/n and $(m+1)/n$ has been analyzed as an example from which some significant conclusions have been resulted.

In Sec. II of this paper, the set of reduced resistive MHD equations is presented as our starting point (for the expression of some operators appearing in MHD equations, see Appendix A).

The toroidal coupling of m and $(m+1)$ tearing modes due to the correction for the ideal MHD solutions in outer regions is analyzed in Secs. III–V. The structure of ideal MHD solutions and the Δ' system for the toroidal tearing mode are analyzed in Sec. III. The explicit general dispersion relation, therefore the growth rates and corresponding flux perturbations of the toroidal tearing mode, are derived in Sec. IV. The variation tendency of growth rates for the

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toroidal tearing mode is investigated in Sec. V.

In Sec. VI, the toroidal coupling of m and $(m+1)$ tearing modes due to the correction for the resistive MHD solutions in singular layers is discussed. (For the derivation of the particular solutions induced by toroidicity, see Appendix B.)

The toroidal coupling of m and $(m\pm1)$ tearing modes due to the correction for the ideal MHD solutions is discussed in Sec. VII (for the toroidal coupling of many tearing modes due to that, see Appendix C).

Finally, conclusions and some discussion are given in Sec. VIII.

II. SET OF REDUCED RESISTIVE MHD EQUATIONS IN TORUS

The set of full resistive MHD equations in rational cgs with $c=1$ units are

$$\rho_0 \frac{d\mathbf{v}}{dt}=\nabla p+\mathbf{J}\times\mathbf{B}, \tag{1}$$

$$\mathbf{J}=\nabla\times\mathbf{B}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t}=-\nabla\times\mathbf{E}, \tag{3}$$

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{J}, \tag{4}$$

where the mass density ρ_0 and resistivity η are assumed as constants, d/dt is the convective time derivative, $\mathbf{v}$ is the fluid velocity, p is the pressure, $\mathbf{J}$ is the current density, and $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields, respectively.

We consider a plasma confined by the magnetic field:

$$\mathbf{B}=\nabla\zeta\times\nabla\Psi+B_\zeta\hat{\zeta}, \tag{5}$$

where B_ζ is the toroidal component and Ψ is the poloidal flux function. Equations (1)–(4) can be reduced to the scalar form[10,11] in the large aspects ratio ($\epsilon \ll 1$) and low-β ($\beta\sim\epsilon^2$) limit, here ϵ is the ratio of the minor radius a_0 to the major radius R_0 of tokamaks and β is the ratio of the plasma pressure to the total magnetic pressure. The velocity $\mathbf{v}$ can be expressed in terms of the streamfunction Φ as

$$\mathbf{v}_\perp=B_\zeta^{-1}\nabla\Phi\times\hat{\xi}+O(\epsilon^2). \tag{6}$$

Eqs. (1)–(4) can be reduced by the style similar to that in Ref. 7 to the following dimensionless form up to ϵ order in "PEST"[12] coordinates:

$$\frac{\partial\Psi}{\partial t}+\frac{1}{\rho}\left(\frac{\partial\Psi}{\partial\rho}\frac{\partial\Phi}{\partial\theta}-\frac{\partial\Psi}{\partial\theta}\frac{\partial\Phi}{\partial\rho}\right)=\eta R J_\zeta-\frac{\partial\Phi}{\partial\zeta}, \tag{7}$$

$$\begin{aligned}\frac{\partial U}{\partial t}&+\frac{1}{\rho}\left(\frac{\partial U}{\partial\rho}\frac{\partial\Phi}{\partial\theta}-\frac{\partial U}{\partial\theta}\frac{\partial\Phi}{\partial\rho}\right)\\&=-\frac{S^2}{R^2}\left\{\frac{1}{\rho}\left[\frac{\partial\Psi}{\partial\rho}\frac{\partial}{\partial\theta}\left(\frac{J_\zeta}{R}\right)-\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial\rho}\left(\frac{J_\zeta}{R}\right)\right]+\frac{1}{R}\frac{\partial J_\zeta}{\partial\zeta}\right.\\&\left.\quad+\frac{2\epsilon}{\rho R^2}\left(\frac{\partial Z}{\partial\rho}\frac{\partial^2\Psi}{\partial\theta\,\partial\zeta}-\frac{\partial Z}{\partial\theta}\frac{\partial^2\Psi}{\partial\rho\,\partial\zeta}\right)\right\},\end{aligned} \tag{8}$$

where $J_\zeta=R\nabla_\perp\cdot(R^{-2}\nabla_\perp\Psi)$ and $U=(B_\zeta R)^{-1}\nabla_\perp^2\Phi$ are the toroidal current density and vorticity, respectively. The parameter $S=\tau_r/\tau_h$ is the ratio of the skin time $\tau_r=a_0^2/\eta$ to the "slow" MHD time $\tau_h=\rho_0^{1/2}R_0/B_0$.

Let $\Psi=\Psi_{eq}(\rho)+\tilde{\Psi}(\rho,\theta,\zeta,t)$, $\Phi=\tilde{\Phi}(\rho,\theta,\zeta,t)$, then Eqs. (7) and (8) can be linearized as the following version:

$$\frac{\partial\tilde{\Psi}}{\partial t}=\eta R\tilde{J}_\zeta-\frac{1}{q}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\zeta}\right)\tilde{\Phi}, \tag{9}$$

$$\begin{aligned}\frac{\partial\tilde{U}}{\partial t}=-\frac{S^2}{R^2}&\left[\frac{1}{q}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\zeta}\right)\frac{\tilde{J}_\zeta}{R}-\frac{1}{q}\frac{\partial\tilde{\Psi}}{\partial\theta}\frac{d}{d\rho}\left(\frac{J_{eq}}{R}\right)\right.\\&\left.+\frac{2\epsilon}{\rho R^2}\left(\frac{\partial Z}{\partial\rho}\frac{\partial^2\tilde{\Psi}}{\partial\theta\,\partial\zeta}-\frac{\partial Z}{\partial\theta}\frac{\partial^2\tilde{\Psi}}{\partial\rho\,\partial\zeta}\right)\right],\end{aligned} \tag{10}$$

where $q(\rho)=\rho/\Psi'_{eq}$ is the safety factor and $J_{eq}=R\nabla_\perp^2\Psi_{eq}$ is the equilibrium current density.

The perturbation functions can be expanded in Fourier series in the toroidal and poloidal angles, namely,

$$\tilde{\Psi}=\sum_{nm}\Psi_{nm}(\rho)\cos(m\theta-n\zeta)\exp(\gamma t),$$

$$\tilde{\Phi}=\sum_{nm}\Phi_{nm}(\rho)\sin(m\theta-n\zeta)\exp(\gamma t),$$

where γ is the linear growth rate of toroidal tearing mode and $m>1$, $n>0$. The expressions of Fourier components have no priority for θ and ζ. However, substituting them into Eqs. (9) and (10), it can be found that each Fourier component has to satisfy such equations:

$$\begin{aligned}\gamma\Psi_m=\eta[L_m\Psi_m+\epsilon(K_m^{m+1}\Psi_{m+1}+K_m^{m-1}\Psi_{m-1})]\\+\left(n-\frac{m}{q}\right)\Phi_m,\end{aligned} \tag{11}$$

$$\begin{aligned}\gamma[L_m\Phi_m&+\epsilon(\tilde{K}_m^{m+1}\Phi_{m+1}+\tilde{K}_m^{m-1}\Phi_{m-1})]\\&=-S^2\left[\frac{m}{\rho}J_{eq}^{\prime(0)}\Psi_m+\left(n-\frac{m}{q}\right)[L_m\Psi_m+\epsilon(\bar{K}_m^{m+1}\Psi_{m+1}\right.\\&\quad+\bar{K}_m^{m-1}\Psi_{m-1})]-n\epsilon\left[\left(\frac{d}{d\rho}+\frac{m+1}{\rho}\right)\Psi_{m+1}\right.\\&\quad\left.\left.+\left(\frac{d}{d\rho}-\frac{m-1}{\rho}\right)\Psi_{m-1}\right]\right],\end{aligned} \tag{12}$$

and boundary conditions $\Psi_m(0)=\Psi_m(1)=0$, where $L_m=d^2/d\rho^2+d/\rho\, d\rho-m^2/\rho^2$; $K_m^{m\pm1}$, $\tilde{K}_m^{m\pm1}$, and $\bar{K}_m^{m\pm1}$ are the second-order linear differential operators which are expressed in Appendix A. Actually, Eqs. (11) and (12) are recurrence relations for the Fourier components with the same toroidal mode number n and different poloidal mode number, so that the subscript n has been omitted for convenience. It is reasonable to think that each toroidal eigenmode is expanded in Fourier series in the poloidal angle θ, i.e., $F_n(\rho,\theta)=\Sigma_m F_{nm}(\rho)\exp(im\theta)$, if we simply rewrite the perturbation function as $\tilde{F}=\Sigma_n F_n(\rho,\theta)\exp(-in\zeta)\exp(\gamma t)$. In other words, each

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two components, i.e., $\gamma_+(0, \kappa)=\eta^{1/3}A_m^{4/3}$ and $\gamma_-(0, \kappa)=\eta^{1/3}A_{m+1}^{4/3}$ if $A_{m+1}>0$. When $0<C<1$, $A_+(C, \kappa)>A_+(0, \kappa)=A_m$ and $A_-(C, \kappa)<A_-(0, \kappa)=A_{m+1}$, hence the branch $\gamma_+(C, \kappa)$ becomes more unstable, while the branch corresponding to $A_-(C, \kappa)$ becomes less unstable if $A_{m+1}>0$, or more stable if $A_{m+1}<0$. When $C=1$, $A_+(1, \kappa)\to\infty$, so that $C=1$ is an asymptote of $\gamma_+(C, \kappa)$, whereas $A_-(1, \kappa)=A_m A_{m+1}/(A_m+A_{m+1})$ is a finite value.

The above analyses indicate that the branch $\gamma_+(C,\kappa)$ is "dominant," since there always is $\gamma_+(C,\kappa)>\gamma_-(C,\kappa)$ and the toroidal coupling has a destabilizing effect on the tearing mode, since $\gamma_+(C, \kappa)>\gamma_\pm(0, \kappa)$ when $C>0$.

Next, the partial derivative of $A_\pm(C, \kappa)$ with respect to κ is

$$\frac{\partial A_\pm}{\partial\kappa}=\frac{A_m}{2(1-C)}\left(1\pm\frac{2C-(1-\kappa)}{\sqrt{(1-\kappa)^2+4\kappa C}}\right). \tag{44}$$

It can be proved that $\partial A_\pm/\partial\kappa>0$, so $A_\pm(C, \kappa)$, therefore $\gamma_\pm(C, \kappa)$ are monotone increasing functions of κ. For a fixed C value, we could regard the toroidal coupling effect as strong when the coupling term is larger than another term in the radical in Eq. (41), i.e., $C>C_0=(1-\kappa)^2/4|\kappa|$. Obviously, this condition is easier to be satisfied as the value of κ is larger, in other words, the toroidal coupling effect becomes stronger when κ is increasing. When $\kappa\leqslant 0$, this condition cannot be satisfied, so that the coupling effect is weak.

When $\kappa=-1$, $A_\pm(C,-1)=\pm A_m(1-C)^{-1/2}$, hence the toroidal coupling exists even if only one component is unstable. When $-1<\kappa<0$, $A_\pm(C, \kappa)>A_\pm(C,-1)$ so the coupling effect is increasing as κ increasing but still weak. When $\kappa=0$, $A_+(C, 0)=A_m/(1-C)>A_+(C,-1)$, $A_-(C, 0)=0>A_-(C,-1)$, which means that the toroidal coupling exists when one component is unstable, whereas another one is marginal stable. When $0<\kappa<1$, $A_\pm(C, \kappa)>A_\pm(C, 0)$, the strong coupling condition can be easily satisfied for large κ and C; when κ is near 1, the coupling effect becomes strong, even if C is quite small. When $\kappa=1$, $A_+(C, 1)=A_m/(1-C^{1/2})=(1+C^{1/2})A_+(C, 0)$ and $A_-(C, 1)=A_m/(1+C^{1/2})$, so that the coupling effect is always strong for $0\leqslant C\leqslant 1$.

These analyses show that the toroidal coupling effect does explicitly depend on the ratio of growth rates of two cylindrical components and "beating" of these two growth rates leads to strong coupling, even if C is quite small.

Now, let us check what occurs for $\{\Delta_m'^{(0)}\}\sim\epsilon$.[9] It is easy to see from above that the coupling effect does not explicitly rely on the magnitude of $\{\Delta_m'^{(0)}\}$ of cylindrical components so $\{\Delta_m'^{(0)}\}\sim\epsilon$ is not a necessary condition for strong coupling. Substituting $A_m=\epsilon/\alpha_m$, $A_{m+1}=\epsilon/\alpha_{m+1}$ and $1/\alpha_{m+1}=\kappa/\alpha_m$ into Eq. (40), then it still comes to Eq. (41). The coupling effect is very weak if both of κ and C are quite small, so that $\{\Delta_m'^{(0)}\}\sim\epsilon$ is also not a sufficient condition for strong coupling.

VI. COUPLING EFFECT DUE TO RESISTIVE MHD SOLUTIONS

In this section the toroidal coupling effect due to the correction for the resistive MHD solutions in singular layers is discussed.

Inside the resistive layer near $\rho=\rho_m$, only the second-order derivative in the operator L_m in Eqs. (12) and (13) should be kept, all the coupling terms can be approximated by the "constant-Ψ." Introducing the new variable $x\equiv\rho-\rho_m$, then $n-m/q\simeq\mu_m(\rho-\rho_m)$, here $\mu_m=mq'(\rho_m)/q^2(\rho_m)$, Eqs. (11) and (12) become

$$\gamma\Psi_m(x)=\eta\frac{d^2\Psi_m(x)}{dx^2}+\epsilon\eta K_m^{m+1}\Psi_{m+1}(\rho_m)$$
$$+\mu_m(\rho_m)x\Phi_m(x), \tag{45}$$

$$\gamma\frac{d^2\Phi_m(x)}{dx^2}+\epsilon\gamma\bar{K}_m^{m+1}\Phi_{m+1}(\rho_m)$$
$$=\epsilon nS^2\left(\frac{d}{d\rho}+\frac{m+1}{\rho}\right)\Psi_{m+1}(\rho_m)$$
$$-S^2\mu_m x\left(\frac{d^2\Psi_m(x)}{dx^2}+\epsilon\bar{K}_m^{m+1}\Psi_m(\rho_m)\right). \tag{46}$$

If we abbreviate $K_m=K_m^{m+1}\Psi_{m+1}(\rho_m)$, $\bar{K}_m=\bar{K}_m^{m+1}\Psi_{m+1}(\rho_m)$, $\tilde{K}_m=\mu_m^{-1}S^{-2}\bar{K}_m^{m+1}\Phi_{m+1}(\rho_m)$, and $\hat{K}_m=(n/\mu_m)[d/d\rho+(m+1)/\rho]\Psi_{m+1}(\rho_m)$ and let $x=\delta_m X$, $\delta_m=(\eta\gamma/\mu_m^2S^2)^{1/4}$, $\lambda=\gamma\delta_m^2/\eta$, and $\Phi_m=(\mu_m\delta_m S^2/\gamma)\Phi$, Eqs. (45) and (46) read as

$$\Psi_m''(X)=\lambda\Psi_m(X)-X\Phi(X)-\epsilon\,\delta_m^2\,K_m, \tag{47}$$

$$\Phi''(X)+\epsilon\delta_m\gamma\tilde{K}_m=-X[\Psi_m''(X)+\epsilon\delta_m^2\bar{K}_m]+\epsilon\delta_m\hat{K}_m. \tag{48}$$

When the coupling terms are neglected, since they are quite small comparing with other terms, the homogeneous differential equations corresponding to Eqs. (47) and (48) can be combined as the following:

$$\Phi^{(0)''}(X)-X^2\Phi^{(0)}(X)=-\lambda X\Psi_m^{(0)}(X). \tag{49}$$

Replacing $\Psi_m^{(0)}(X)$ by $\Psi_m^{(0)}(0)$, substituting $\Phi^{(0)}=\exp(-X^2/2)\Sigma_n C_n H_n(X)$ into Eq. (49), one can obtain a solution by utilizing the properties of Hermite polynomial:

$$\Phi^{(0)}(X)=\sqrt{2}\Psi_m^{(0)}(0)\lambda\exp\left(-\frac{X^2}{2}\right)\sum_{n=0}^{\infty}\frac{H_{2n+1}(X)}{4^n n!(4n+3)} \tag{50}$$

and

$$\Delta_m^{(0)}=\frac{1}{\Psi_m^{(0)}(0)\delta_m}\int_{-\infty}^{+\infty}dX\frac{d^2\Psi_m^{(0)}}{dX^2}=\frac{\sqrt{2}\pi\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{\gamma\delta_m}{\eta} \tag{51}$$

If we keep the coupling terms, Eqs. (47) and (48) can be combined as

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$$\Psi^{(1)}_{m\pm1}(\rho)=\Psi^{(0)}_m(\rho_m)\begin{cases}F^{m<}_{m\pm1}(\rho), & \rho<\rho_m,\\ F^{m>}_{m\pm1}(\rho), & \rho>\rho_m,\end{cases} \tag{64}$$

where the expressions of $F_i^{j<>}(\rho)$ are the same as those below Eq. (25), but i and j are taken as different values among m and $m\pm1$. The general solutions of Eqs. (59) and (60) can be expressed as

$$\begin{pmatrix}\Psi_{m-1}(\rho)\\ \Psi_m(\rho)\\ \Psi_{m+1}(\rho)\end{pmatrix}=\begin{pmatrix}Z^{<>}_{m-1}(\rho) & \epsilon F^{m<>}_{m-1}(\rho) & 0\\ \epsilon F^{m-1<>}_m(\rho) & Z^{<>}_m(\rho) & \epsilon F^{m+1<>}_m(\rho)\\ 0 & \epsilon F^{m<>}_{m+1}(\rho) & Z^{<>}_{m+1}(\rho)\end{pmatrix}\begin{pmatrix}\Psi^{(0)}_{m-1}(\rho_{m-1})\\ \Psi^{(0)}_m(\rho_m)\\ \Psi^{(0)}_{m+1}(\rho_{m+1})\end{pmatrix}, \tag{65}$$

where the signs "<" and ">" depend on the relative position of the singular layers. The $\{\Delta_i'\}$ for the toroidal tearing mode can be derived from the solution (65):

$$\Delta_m'=\frac{\Delta_m'^{(0)}\Psi^{(0)}_m(\rho_m)}{\Psi^{(0)}_m(\rho_m)+\epsilon\Psi^{(0)}_{m-1}(\rho_{m-1})F^{m-1>}_m(\rho_m)+\epsilon\Psi^{(0)}_{m+1}(\rho_{m+1})F^{m+1<}_m(\rho_m)}, \tag{66}$$

$$\Delta_{m\pm1}'=\frac{\Delta'^{(0)}_{m\pm1}\Psi^{(0)}_{m\pm1}(\rho_{m\pm1})}{\Psi^{(0)}_{m\pm1}(\rho_{m\pm1})+\epsilon\Psi^{(0)}_m(\rho_m)F^{m<>}_{m\pm1}(\rho_{m\pm1})}, \tag{67}$$

the signs "<" and ">" in Eq. (67) are chosen relevantly for $m-1$ and $m+1$, respectively.

The relations (66) and (67) can be rewritten as the following matrix form as well since $\{\Psi_i^{(0)}(\rho_i)\}$ have not yet been determined:

$$\begin{pmatrix}\Delta'_{m-1}-\Delta'^{(0)}_{m-1} & \epsilon F^{m<}_{m-1}(\rho_{m-1})\Delta'_{m-1} & 0\\ \epsilon F^{m-1>}_{m+1}(\rho_m)\Delta'_m & \Delta'_m-\Delta'^{(0)}_m & \epsilon F^{m+1<}_m(\rho_m)\Delta'_m\\ 0 & \epsilon F^{m>}_{m+1}(\rho_{m+1})\Delta'_{m+1} & \Delta'_{m+1}-\Delta'^{(0)}_{m+1}\end{pmatrix}\begin{pmatrix}\Psi^{(0)}_{m-1}(\rho_{m-1})\\ \Psi^{(0)}_m(\rho_m)\\ \Psi^{(0)}_{m+1}(\rho_{m+1})\end{pmatrix}=0. \tag{68}$$

An explicit general dispersion relation for the toroidal tearing mode can be derived from the solubility condition for $\{\Psi_i^{(0)}(\rho_i)\}\neq0$ and asymptotic matching condition $\{\Delta_i(\gamma)\}=\{\Delta_i'\}$ with the help of Eq. (33):

$$\begin{vmatrix}\alpha_{m-1}/\Delta'^{(0)}_{m-1}-y & \epsilon F^{m<}_{m-1}(\rho_{m-1})\alpha_{m-1}/\Delta'^{(0)}_{m-1} & 0\\ \epsilon F^{m-1>}_m(\rho_m)\alpha_m/\Delta'^{(0)}_m & \alpha_m/\Delta'^{(0)}_m-y & \epsilon F^{m+1<}_m(\rho_m)\alpha_m/\Delta'^{(0)}_m\\ 0 & \epsilon F^{m>}_{m+1}(\rho_{m+1})\alpha_{m+1}/\Delta'^{(0)}_{m+1} & \alpha_{m+1}/\Delta'^{(0)}_{m+1}-y\end{vmatrix}=0, \tag{69}$$

or

$$y^3+by^2+cy+d=0; \tag{70}$$

here

$$b=-(\alpha_{m-1}/\Delta'^{(0)}_{m-1}+\alpha_m/\Delta'^{(0)}_m+\alpha_{m+1}/\Delta'^{(0)}_{m+1}),$$

$$c=-(1-C_{m-1,m})\frac{\alpha_{m-1}\alpha_m}{\Delta'^{(0)}_{m-1}\Delta'^{(0)}_m}+(1-C_{m,m+1})\frac{\alpha_m\alpha_{m+1}}{\Delta'^{(0)}_m\Delta'^{(0)}_{m+1}}+\frac{\alpha_{m-1}\alpha_{m+1}}{\Delta'^{(0)}_{m-1}\Delta'^{(0)}_{m+1}},$$

$$d=-(1-C_{m-1,m}-C_{m,m+1})\alpha_{m-1}\alpha_m\alpha_{m+1}/\Delta'^{(0)}_{m-1}\Delta'^{(0)}_m\Delta'^{(0)}_{m+1},$$

in which $C_{m-1,m}=\epsilon^2F^{m<}_{m-1}(\rho_{m-1})F^{m-1>}_m(\rho_m)$ and $C_{m,m+1}=\epsilon^2F^{m+1<}_m(\rho_m)F^{m>}_{m+1}(\rho_{m+1})$ are the coupling parameters of components $m-1$ and m, respectively, components m and $m+1$. It seems that the toroidal coupling of three components are a superposition of the toroidal coupling of two nearby components.

Let $p=c-b^2/3$, $q=2(b/3)^3-bc/3+d$, when $\Delta=(q/2)^2+(p/3)^3\leqslant0$; three eigenvalues $y_{1,2,3}$ can be expressed as

$$y_1=A+B-b/3;\quad y_2=\omega A+\omega^2B-b/3;\quad y_3=\omega^2A+\omega B-b/3, \tag{71}$$

where $A=\sqrt[3]{-q/2+\sqrt{\Delta}}$, $B=\sqrt[3]{-q/2-\sqrt{\Delta}}$, $\omega=(-1+i\sqrt{3})/2$.

Similar to Sec. IV, the characteristic functions corresponding to $y_{1,2,3}$, respectively, can be resulted if $\{\Psi_i^{(0)}(\rho_i)\}$ is assumed to be normalized to 1:

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