COMP4130 Linear and Discrete Optimization

Lecture 2: Fundamentals of Linear Programming

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Overview

- **Linear Programming (LP) Problems**
- **Formulating LP Models**
- **Graphical Methods for Solving LP Models**
- **Special Cases in LP Models**

Additional Reading: Chapters 3 of *Introduction to Operations Research*

Introduction to Linear Programming Problems

- **Definition:** A *Linear Programming (LP) Problem* is an optimization problem in which the objective function and all constraints are **linear**.
- **Objective Function:** A linear function of decision variables to be maximized or minimized.
  - Example: maximize $3x + 5y$ (profit), or minimize $2x + 4y$ (cost).
- **Constraints:** Restrictions expressed as linear equations or inequalities.
  - Example: $x + y = 50, \quad 2x + y \leq 100, \quad x, y \geq 0.$

Scenario. A nutritionist designs a daily diet using two foods, C and D.

- Each unit of C costs \$2 and provides 200 calories and 10 g of protein.
- Each unit of D costs \$3 and provides 100 calories and 30 g of protein.
- The diet must contain at most 1000 calories and at least 120 g of protein per day.

The goal is to determine the daily quantities of C and D that minimize the total cost of the diet.

Concepts in Linear Programming Models

- **Data (parameters):**
  - Food C: cost \$2, provides 200 calories and 10 g of protein
  - Food D: cost \$3, provides 100 calories and 30 g of protein
  - Daily requirement: at most 1000 calories and at least 120 g of protein
- **Decision variables (competing activities):**

$$x = \text{units of Food C per day}$$
$$y = \text{units of Food D per day}$$

- **Objective function:**

$$\min \ Z = 2x + 3y \text{ (daily cost)}$$

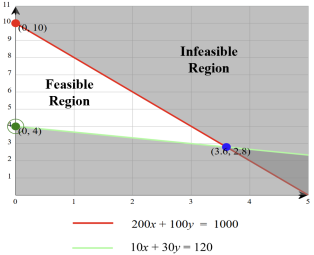
- **Constraints:**
  - *Functional (nutrition requirements):*

$$\begin{aligned} \text{Calories:} \quad 200x + 100y &\leq 1000, \\ \text{Protein:} \quad 10x + 30y &\geq 120. \end{aligned}$$

  - *Non-negativity:* $x \geq 0, \ y \geq 0.$

Concepts in Linear Programming Models

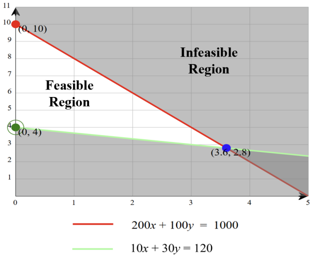
- **Search space:** the entire (x, y) plane, including both feasible and infeasible regions
- **Feasible region:** the white polygon where all constraints are satisfied
- **Infeasible region:** the shaded area where at least one constraint is violated



Vertex	Lines through vertex	Value of objective
(3.6, 2.8)	$200x + 100y = 1000$ $10x + 30y = 120$	15.6
(0, 10)	$200x + 100y = 1000$ $x = 0$	30
(0, 4)	$10x + 30y = 120$ $x = 0$	12 Minimum

Concepts in Linear Programming Models

- **Feasible solutions:** any point inside or on the boundary of the feasible region (e.g., $(3.6, 2.8)$, $(0, 4)$, $(1, 4.5)$)
- **Infeasible solutions:** points outside the feasible region (e.g., $(1, 1)$)
- **Corner-point feasible (CPF) solutions:** The feasible region is a polygon, and its vertices are candidate solutions. In this problem, the CPF solutions are: $(0, 10), (3.6, 2.8), (0, 4)$.
- **Optimal solution:** By evaluating the objective function at each CPF solution, the minimum cost occurs at: $(0, 4), Z = 12$.



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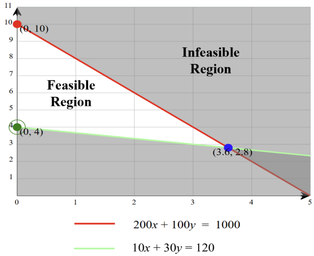
● (0, 10)	$200x + 100y = 1000$ $x = 0$	30
● (0, 4)	$10x + 30y = 120$ $x = 0$	12 Minimum

Concepts in Linear Programming Models

- **Binding constraints:** A constraint is binding if it is satisfied as an equality at the optimal solution and directly restricts the feasible region. In this problem, at the optimal solution $(0, 4)$, the binding constraints are: $10x + 30y \geq 120$ (protein requirement) and $x \geq 0$ (non-negativity).
- **Non-binding constraints:** A constraint is non-binding if it is not tight at the optimal solution; slack remains. In this problem, at the optimal solution $(0, 4)$, the non-binding constraints are: $200x + 100y \leq 1000$ (calorie limit) and $y \geq 0$ (non-negativity).



Vertex	Lines through vertex	Value of objective
(3.6, 2.8)	$200x + 100y = 1000$ $10x + 30y = 120$	15.6
(0, 10)	$200x + 100y = 1000$ $x = 0$	30
(0, 4)	$10x + 30y = 120$ $x = 0$	12 Minimum

Estimating the Size of an LP Model

The size of an LP model can be estimated by:

- The number of decision variables
- The number of constraints
- The size of the search space

Example: Estimating the Size of an LP Model

$$
\begin{aligned}
\text{maximize:}\quad & Z = 120X_1 + 80X_2 & \\
\text{subject to:}\quad & X_1 \leq 40 & (1)\\
& X_2 \leq 10 & (2)\\
& 20X_1 + 10X_2 \leq 500 & (3)\\
& X_1 - X_2 \leq 5 & (4)\\
& X_2 - X_1 \leq 5 & (5)\\
& X_1, X_2 \geq 0 & (6)
\end{aligned}
$$

- The number of decision variables:
- The number of constraints:
- The size of the search space:

Example: Estimating the Size of an LP Model

$$
\begin{aligned}
\text{maximize:}\quad & Z = \sum_{i=1}^{15} (P_i - C_i) X_i \\
\text{subject to:}\quad & X_i \geq \text{Min}_i \quad \text{for } i = 1, \ldots, 15 && (1) \\
& X_i \leq \text{Max}_i \quad \text{for } i = 1, \ldots, 15 && (2) \\
& \sum_{i=1}^{15} 1.25 X_i \leq 18{,}000 && (3) \\
& \sum_{i=1}^{15} C_i X_i \leq 30{,}000 && (4) \\
& X_i \geq 0 \quad \text{for } i = 1, \ldots, 15 && (5) \\
& P_i \text{ and } C_i \text{ are the price and cost of item } i \text{ respectively.} && (6)
\end{aligned}
$$

- The number of decision variables:
- The number of constraints:
- The size of the search space:

Example: Estimating the Size of an IP Model

$$\text{minimize:} \quad Z = \sum_{i=A}^{F} \sum_{j=Mo}^{Fr} W_i X_{i,j}$$

$$\text{subject to:} \quad \sum_{i=A}^{F} X_{i,j} \geq 14 \quad \text{for } j = Mo, \ldots, Fr \tag{1}$$

$$\sum_{j=Mo}^{Fr} X_{i,j} \geq 8 \quad \text{for } i = A, \ldots, D \tag{2}$$

$$\sum_{j=Mo}^{Fr} X_{i,j} \geq 7 \quad \text{for } i = E, F \tag{3}$$

$$X_{i,j} \leq V_{i,j} \quad \text{for } i = A, \ldots, F; \; j = Mo, \ldots, Fr \tag{4}$$

$$X_{i,j} \geq 0 \;\; \text{integer} \quad \text{for } i = A, \ldots, F; \; j = Mo, \ldots, Fr \tag{5}$$

$$X_{i,j} \;\; \text{is the number of hours assigned to operator } i \text{ in weekday } j \tag{6}$$

$$W_i \;\; \text{is the wage rate of operator } i \tag{7}$$

$$V_{i,j} \;\; \text{is the availability of operator } i \text{ in weekday } j \tag{8}$$

- The number of decision variables:
- The number of constraints:
- The size of the search space:

Example: Estimating the Size of an BIP Model

$$
\begin{aligned}
\text{maximize:} \quad & Z = \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} x_{ij} \\
\text{subject to:} \quad & \sum_{j=1}^{m} x_{ij} = 1 \quad i = 1, \ldots, n && (1) \\
& \sum_{i=1}^{n} x_{ij} \geq 2 \quad j = 1, \ldots, m && (2) \\
& x_{ij} \in \{0, 1\} \quad i = 1, \ldots, n; \ j = 1, \ldots, m && (3)
\end{aligned}
$$

- The number of decision variables:
- The number of constraints:
- The size of the search space:

Formulating Linear Programming Models

General LP Formulation:

- An LP formulation represents the problem of optimally allocating limited resources to competing activities.
- Constraints may be expressed as inequalities ($\geq$, $\leq$, $>$, $<$) or equalities ($=$).
- Both the objective function and the constraints are linear algebraic expressions.

Formulating Linear Programming Models

Standard Form of an LP Model:

$$
\begin{aligned}
\text{maximize}: \quad & Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to}: \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
& x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0
\end{aligned}
$$

- Linear objective function (maximize or minimize)
- Linear constraints, including non-negativity
- Decision variables are continuous (can take any real values ≥ 0).
- Problem parameters (c_i, a_{ij}, b_j) are assumed known and certain

Formulating Linear Programming Models

Recommended Steps:

1. Identify parameters (numerical data)
2. Define decision variables (competing activities)
3. State the objective function (maximize or minimize Z)
4. Formulate linear constraints
5. Define non-negativity (LP) or integrality constraints (IP)

Example: Formulating an LP Model

ATLAS Problem:

Company ATLAS produces two products, A and B, using raw materials M1 and M2. The objective is to determine the production quantities of A and B to maximize profit. The production of B cannot exceed that of A by more than one unit, and the demand for product B is limited to at most two units. The resource requirements for each product and the availability of raw materials are summarized in the table below.

	Product A	**Product B**	**Availability (Units)**
M1 (per unit)	6	4	24
M2 (per unit)	1	2	6
Profit (per unit)	5	4	

Example: Formulating an LP Model

Step 1: Identify Parameters (numerical data)

Company ATLAS produces two products, A and B, using raw materials M1 and M2. The objective is to determine the production quantities of A and B to maximize profit. The production of B cannot exceed that of A by more than one unit, and the demand for product B is limited to at most two units. The resource requirements for each product and the availability of raw materials are summarized in the table below.

	Product A	Product B	Availability (Units)
M1 (per unit)	6	4	24
M2 (per unit)	1	2	6
Profit (per unit)	5	4	

Example: Formulating an LP Model

Step 1: Identify Parameters (numerical data)

- Profit per unit of A = 5
- Profit per unit of B = 4
- Material M1 availability = 24 units
- Material M2 availability = 6 units
- Material requirement per unit of A: (M1=6, M2=1)
- Material requirement per unit of B: (M1=4, M2=2)
- Production of B cannot exceed A by more than 1
- Production of B is limited to at most 2

Example: Formulating an LP Model

Step 2: Define Decision Variables (competing activities)

Company ATLAS produces two products, A and B, using raw materials M1 and M2. The objective is to determine the production quantities of A and B to maximize profit. The production of B cannot exceed that of A by more than one unit, and the demand for product B is limited to at most two units. The resource requirements for each product and the availability of raw materials are summarized in the table below.

	Product A	**Product B**	**Availability (Units)**
M1 (per unit)	6	4	24
M2 (per unit)	1	2	6
Profit (per unit)	5	4	

- Let x_1 = number of units of Product A to produce
- Let x_2 = number of units of Product B to produce

Example: Formulating an LP Model

Step 3: State the Objective Function

Company ATLAS produces two products, A and B, using raw materials M1 and M2. The objective is to determine the production quantities of A and B to maximize profit. The production of B cannot exceed that of A by more than one unit, and the demand for product B is limited to at most two units. The resource requirements for each product and the availability of raw materials are summarized in the table below.

	Product A	**Product B**	**Availability (Units)**
M1 (per unit)	6	4	24
M2 (per unit)	1	2	6
Profit (per unit)	5	4	

Maximize profit: $Z = 5x_1 + 4x_2$

Example: Formulating an LP Model

Step 4: Formulate Linear Constraints

Company ATLAS produces two products, A and B, using raw materials M1 and M2. The objective is to determine the production quantities of A and B to maximize profit. The production of B cannot exceed that of A by more than one unit, and the demand for product B is limited to at most two units. The resource requirements for each product and the availability of raw materials are summarized in the table below.

	Product A	**Product B**	**Availability (Units)**
M1 (per unit)	6	4	24
M2 (per unit)	1	2	6
Profit (per unit)	5	4	

- Material M1: $6x_1 + 4x_2 \leq 24$
- Material M2: $x_1 + 2x_2 \leq 6$
- Production balance: $x_2 - x_1 \leq 1$
- Demand for B: $x_2 \leq 2$

Step 5: Define Non-negativity Constraints

Company ATLAS produces two products, A and B, using raw materials M1 and M2. The objective is to determine the production quantities of A and B to maximize profit. The production of B cannot exceed that of A by more than one unit, and the demand for product B is limited to at most two units. The resource requirements for each product and the availability of raw materials are summarized in the table below.

	Product A	**Product B**	**Availability (Units)**
M1 (per unit)	6	4	24
M2 (per unit)	1	2	6
Profit (per unit)	5	4	

$$x_1 \geq 0, \quad x_2 \geq 0$$

Example: Formulating an LP Model

Complete LP Model:

- Decision Variables:

$$x_1 = \text{units of Product A}, \quad x_2 = \text{units of Product B}$$

- Objective Function:

$$\max Z = 5x_1 + 4x_2$$

- Constraints:

$$\begin{aligned}
6x_1 + 4x_2 &\le 24 && \text{(Material M1)} \\
x_1 + 2x_2 &\le 6 && \text{(Material M2)} \\
x_2 - x_1 &\le 1 && \text{(Production balance)} \\
x_2 &\le 2 && \text{(Demand for B)} \\
x_1,\, x_2 &\ge 0 && \text{(Non-negativity)}
\end{aligned}$$

Example: Formulating an LP Model

Complete LP Model in Standard Form:

$$
\begin{aligned}
\text{maximize}: \quad & Z = 5x_1 + 4x_2 & \\
\text{subject to}: \quad & 6x_1 + 4x_2 \leq 24 & (1) \\
& x_1 + 2x_2 \leq 6 & (2) \\
& -x_1 + x_2 \leq 1 & (3) \\
& x_2 \leq 2 & (4) \\
& x_1, x_2 \geq 0 & (5)
\end{aligned}
$$

Four Assumptions in General LP Formulations

- General LP formulations apply to problems where the objective function and all constraints are linear algebraic expressions.
- There are four key assumptions in general LP formulations:
  - Proportionality
  - Additivity
  - Divisibility
  - Certainty

Four Assumptions in General LP Formulations

Proportionality:

- The contribution of each decision variable to the objective function and the constraints is **directly proportional** to the level of that decision variable.
- This means doubling the value of a decision variable will double its effect on the objective and constraints.

Examples Where Proportionality Holds

Maximize $Z = 5x_1 + 4x_2$

Minimize $Z = 0.5x_1 + 0.25x_2$

Inequality constraint $x_2 - x_1 \leq 1$

Equality constraint $2x_1 + 3x_2 = 50$

Examples Where Proportionality Does Not Hold

Maximize $Z = 5x_1 + 4x_2$ if $x_2 > 2$,

$5x_1 + x_2$ otherwise

Minimize $Z = 0.5x_1^{1.5} - x_1x_2$

Inequality constraint $x_2^2 - x_1 \leq 1$

Equality constraint $\log_2 x_1 + 2x_2 = 50$

Four Assumptions in General LP Formulations

Additivity:

- In LP formulations, the total effect of decision variables on the objective function and the constraints is the **sum** of their individual contributions.
- This means there are no cross-product terms, and each decision variable contributes independently.

Examples Where Additivity Holds	Examples Where Additivity Does Not Hold
Maximize $Z = 5x_1 + 4x_2$	Maximize $Z = 5x_1 + x_1x_2$
Minimize $Z = 0.5x_1 + 0.25x_2$	Minimize $Z = 0.5x_1 - x_1x_2$
Inequality constraint $x_2 - x_1 \leq 1$	Inequality constraint $x_1x_2^2 - x_1 \leq 1$
Equality constraint $2x_1 + 3x_2 = 50$	Equality constraint $2x_1x_2 = 50$

Four Assumptions in General LP Formulations

Divisibility:

- In LP formulations, decision variables can take any non-negative value, including both integer and non-integer values, as long as they satisfy the constraints.
- This means that decision variables are not restricted to whole numbers; they can be **fractional** as well.

Examples Where Divisibility Holds	
x_1	liters produced of product A
x_2	liters produced of product B
x_1	watts of electricity from A to B
x_2	watts of electricity from B to C

Examples Where Divisibility Does Not Hold	
x_1	salesmen assigned to branch A
x_2	salesmen assigned to branch B
x_1	number of trucks from A to B
x_2	number of trucks from B to C

Four Assumptions in General LP Formulations

Certainty:

- In LP formulations, all parameter values are assumed to be **known with certainty** and **remain constant**.
- This means that the values of the coefficients in the objective function and the constraints do not change during the optimization process.

Examples Where Certainty Holds	Examples Where Certainty Does not Hold
Maximize $Z = 5x_1 + 4x_2$ profits due to x_1, x_2 are constant	Maximize $Z = 5x_1 + 4x_2$ profits due to x_1, x_2 change a lot
Subject to $Ax_1 + Bx_2 = 50$ where $A = 2$ and $B = 3$	Subject to $Ax_1 + Bx_2 = 50$ where $A \geq 2$ and $B \leq 50$
The required quantities of x_1 and x_2 in this constraint are constant	The required quantities of x_1 and x_2 can change a lot during production

Graphical Method for Solving LP Models

Derive solutions to the problem based on the model.

- **Graphical Method**
  - For LP models with 2 decision variables
  - A two-dimensional graph is used to visualize decision variables, the objective function, constraints, the feasible region, the infeasible region, the search space, feasible solutions, and optimal solutions
  - Solutions are found by exploring key feasible solutions
- **Simplex Method and Its Variants**
  - For larger LP models with 3 or more decision variables
- **Exact Solver or Heuristic Approach**
  - Very large models may not be solvable to exact optimality within practical computation time
  - In such cases, we must resort to the heuristic approach

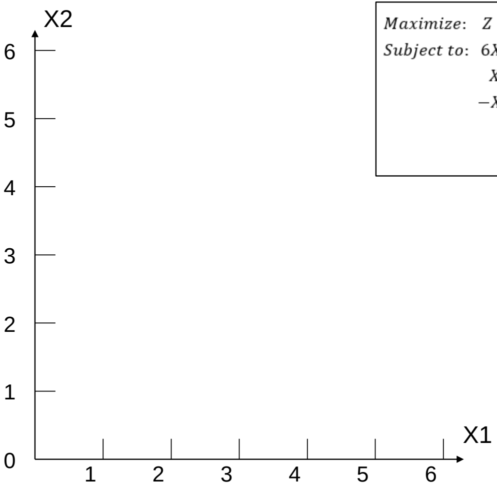
Graphical Method for Solving LP Models

Steps:

1. Draw a two-dimensional graph for $X_1, X_2 \geq 0$.
2. Draw the lines representing the constraints.
3. Identify the feasible and infeasible regions.
4. Identify the CPF(s) that contain the optimal solution(s).
5. Start with a solution in the feasible region and draw the line representing the objective function.
6. Identify the direction that improves the objective function by exploring better values for the objective function within the feasible region.
7. Verify the optimal solution(s) by solving the system of algebraic equations.
8. Identify the binding and non-binding constraints.

Graphical Method for Solving LP Models

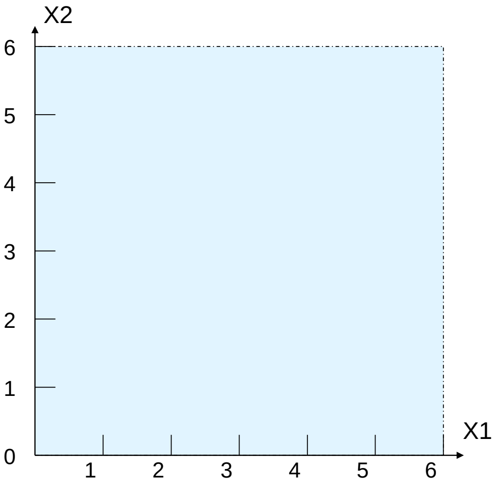
Example. Graphical method to solve the ATLAS LP model.



$$
\begin{aligned}
&\textit{Maximize:} && Z = 5X1 + 4X2 && \\
&\textit{Subject to:} && 6X1 + 4X2 \leq 24 && (1) \\
& && X1 + 2X2 \leq 6 && (2) \\
& && -X1 + X2 \leq 1 && (3) \\
& && X2 \leq 2 && (4) \\
& && X1, X2 \geq 0 && (5)
\end{aligned}
$$

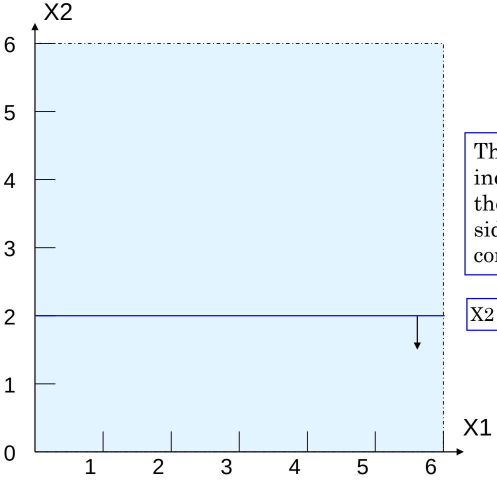
Step 1. Draw a two-dimensional graph for $X_1, X_2 \geq 0$.

$\boxed{X1 \geq 0 \text{ and } X2 \geq 0 \quad (5)}$



Step 2. Draw lines representing the constraints.

X1 ≥ 0 and X2 ≥ 0 (5)



The arrow indicates the feasible side of each constraint.

X2 ≤ 2 (4)

Step 2. Draw lines representing the constraints.

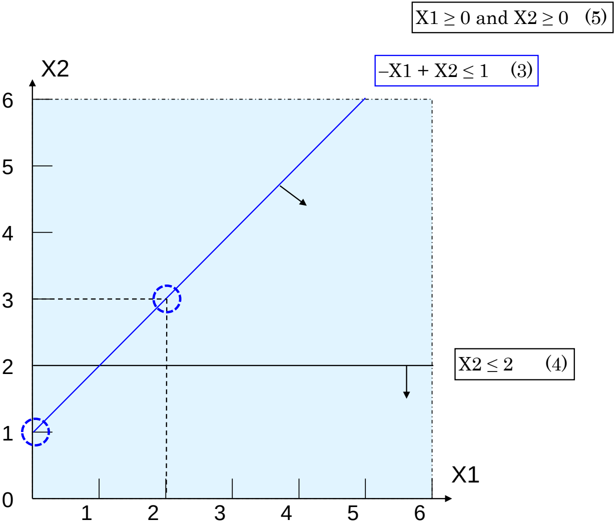
To plot a line, find any 2 points, e.g.

- If $X1 = 0$ Then $X2 \leq 1$

- If $X1 = 2$ Then $X2 \leq 3$

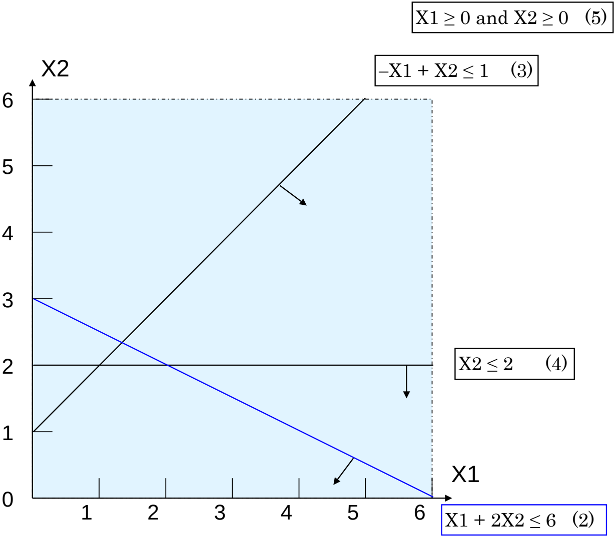
To identify feasible side, test a point out of the line, e.g.

$X1 = 0$
$X2 = 3$
violates constraint (3) hence not feasible.

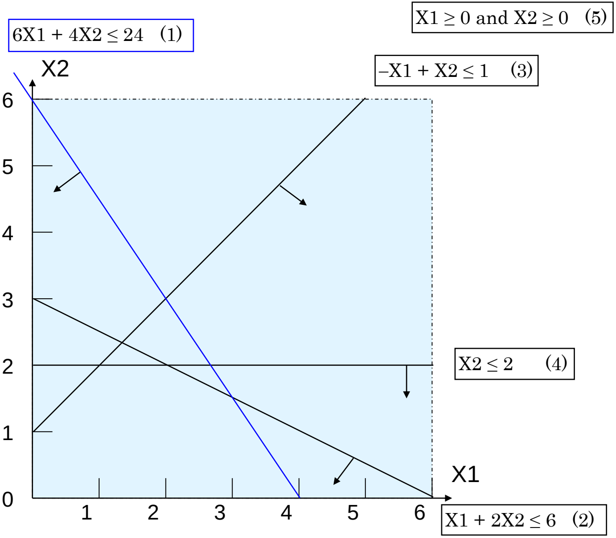


Step 2. Draw lines representing the constraints.

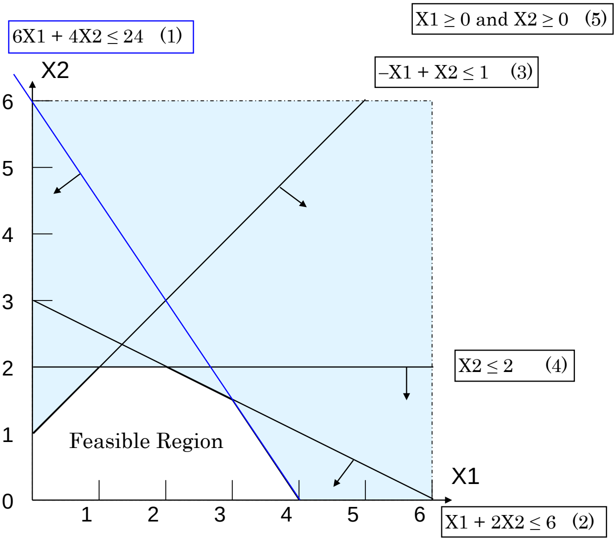
For example, the solution
X1 = 4
X2 = 2
satisfies constraints (3) and (4) but violates constraint (2) hence not feasible wrt these two constraints.



Step 2. Draw lines representing the constraints.



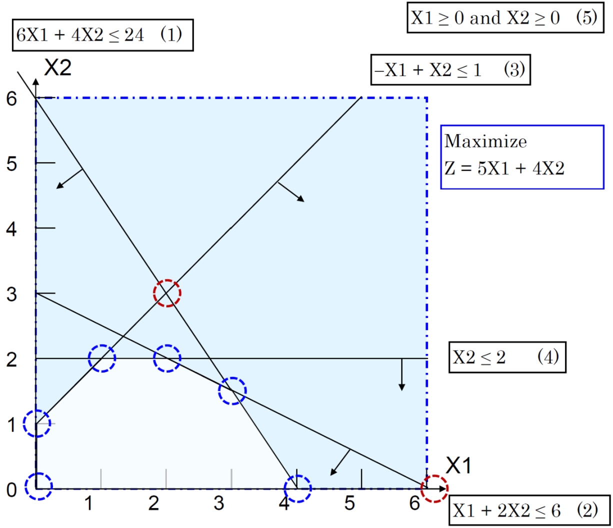
Step 3. Identify the feasible and infeasible regions.



Step 4. Identify the CPF(s) that contain the optimal solution(s).

An LP model with feasible solutions and a bounded feasible region has CPF (corner-point feasible) solutions and at least one optimal solution. There are also some infeasible corner-points.

The best CPF solution must be an optimal solution.



Step 5. Start with a solution in the feasible region and draw the line representing the objective function.

Plot line for some value of the objective function, e.g.

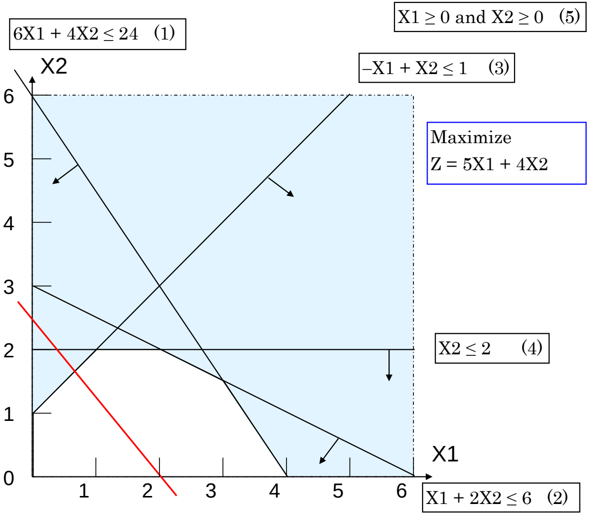
Assume
Z = 10

Then
10=5X1+4X2

Plot line for this objective function value.

- If X1 = 0
Then X2 = 2.5

- If X2 = 0
Then X1 = 2

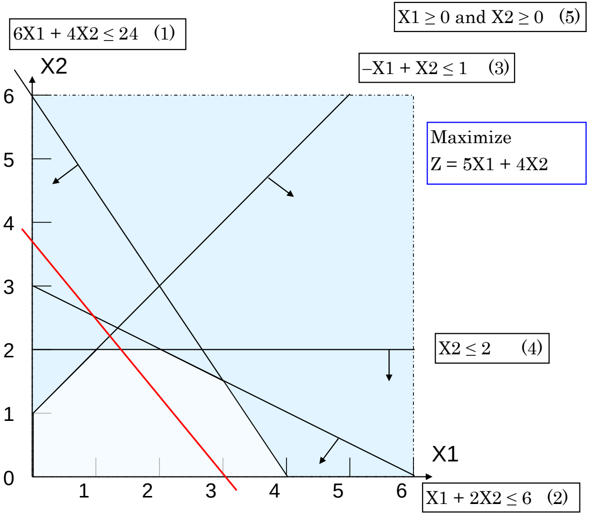


Step 6. Identify the direction that improves the objective function by finding better values for the objective function.

Identify the direction in which the objective function value improves.

Assume
Z = 15

Plot line for this objective function value.



Step 7. Verify the optimal solution(s) by solving the system of algebraic equations.

Intersection of (1) and (2) is uppermost point.

To solve in algebraic manner:

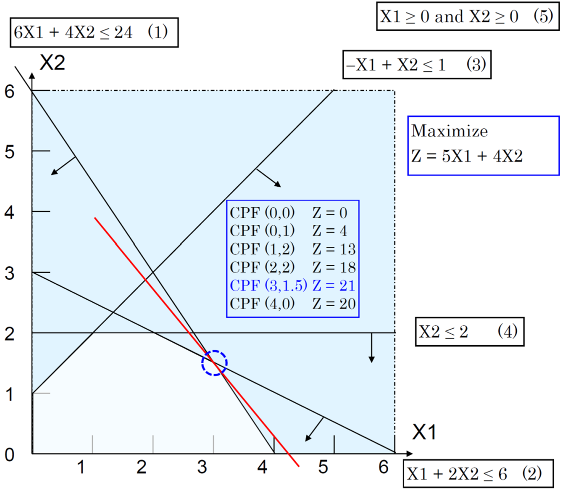
$6X1+4X2=24$
$X1+2X2=6$

$6X1+4X2=24$
$-2X1-4X2=-12$

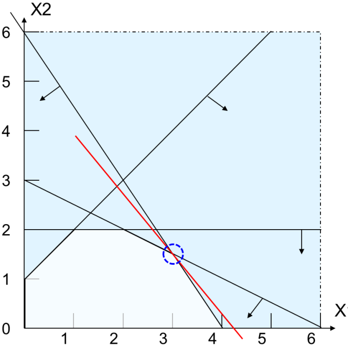
$4X1 = 12$
$X1 = 3$

$X1+2X2=6$
$3+2X2=6$
$X2 = 1.5$

$Z=5X1+4X2=21$



Step 8. Identify the binding and non-binding constraints.



Constraint (1) is binding on the optimal solution (X1 = 3 , X2 = 1.5) because:

$6X1 + 4X2 \leq 24$ (1)

$6(3) + 4(1.5) \leq 24$ then $24 \leq 24$

so, there is no 'slack' on the resource material M1.

Constraint (2) is also binding as it can be seen in the graph.

$$
\begin{aligned}
\textit{Maximize:} \quad & Z = 5X1 + 4X2 \\
\textit{Subject to:} \quad & 6X1 + 4X2 \leq 24 \quad (1) \\
& X1 + 2X2 \leq 6 \quad (2) \\
& -X1 + X2 \leq 1 \quad (3) \\
& X2 \leq 2 \quad (4) \\
& X1, X2 \geq 0 \quad (5)
\end{aligned}
$$

Constraint (3) is not-binding on the optimal solution (X1 = 3 , X2 = 1.5) because:

$-X1 + X2 \leq 1$ (3)

$-3 + 1.5 \leq 1$ then $-1.5 \leq 1$

so, there is 'slack' on the difference between the production of products A and B.

Constraint (4) is also not-binding as it can be seen in the graph.

Some Special Cases in LP Models

Some special cases can occur in an LP formulation:

- There is no feasible solution (and thus no optimal solution) because there is no feasible region.
- The feasible region is unbounded, which means there is no defined optimal solution.
- There are multiple optimal solutions to the model.

Some Special Cases in LP Models

There is **no feasible solution** **(and thus no optimal solution) because there is no feasible region.**

If we replace constraint:

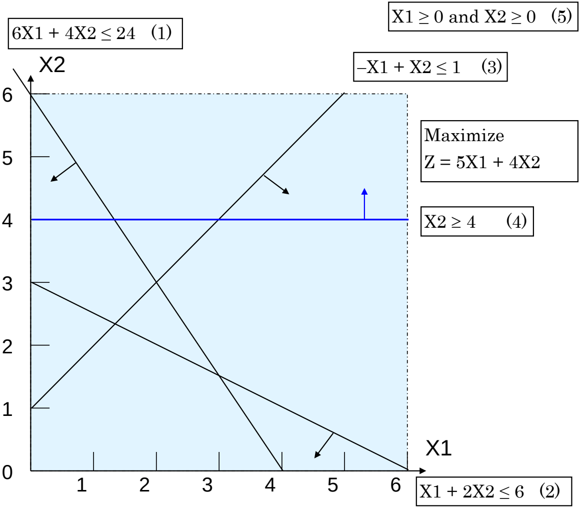
$X2 \leq 2$

with the constraint:

$X2 \geq 4$

[No Feasible Solution]

Please pay attention, optimization solvers report when a model is [infeasible].

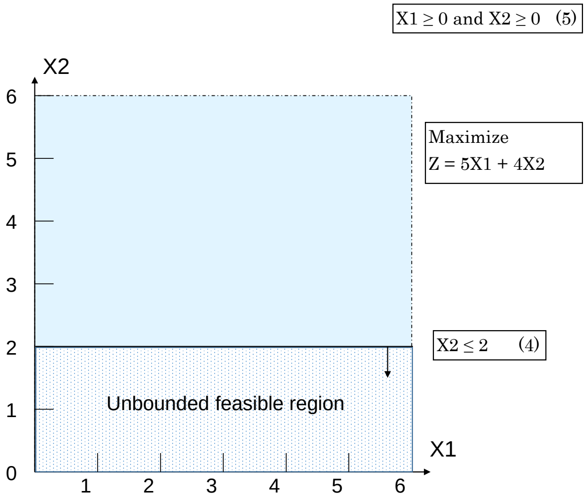


The feasible region is unbounded, which means there is no defined optimal solution.

If we remove all the functional constraints except (4)

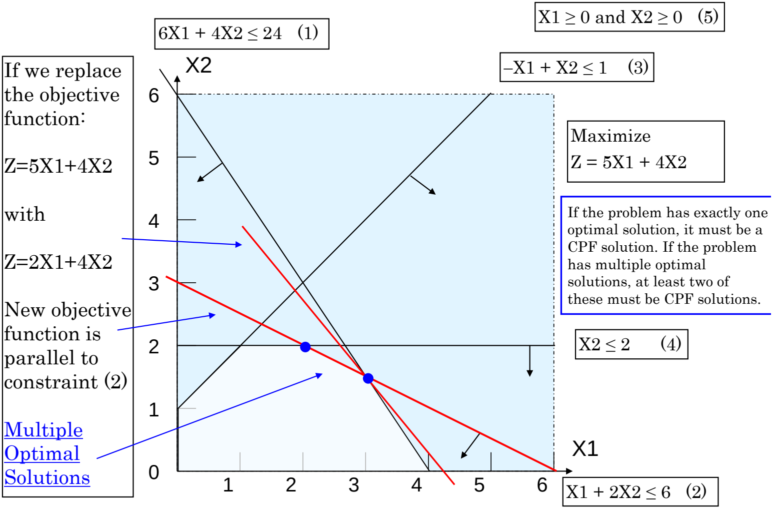
Unbounded Feasible Region

Please pay attention, optimization solvers report when a model is unbounded.



Some Special Cases in LP Models

There are multiple optimal solutions to the model.



Find Alternative Optimal Solutions

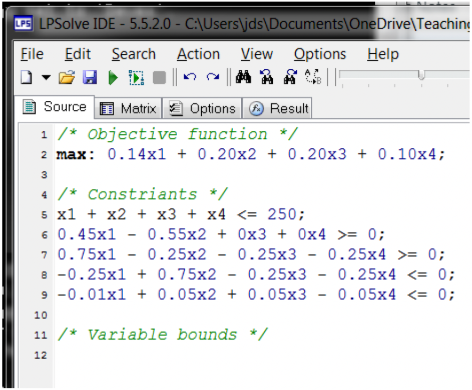
An optimization solver typically reports only one optimal solution—the first one it encounters.

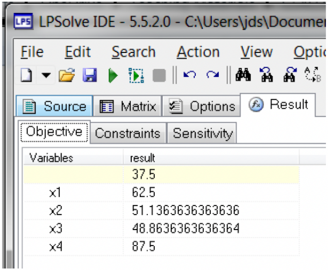
For a model with multiple optimal solutions, the solution a solver finds first can depend on the 'layout' of the model, as this influences the order in which the solver tackles the constraints and explores the search space.

If it is known that an LP model has multiple optimal solutions (e.g., the Bank ABC problem), then:

- How can alternative optimal solutions be found in Excel and LP-Solve?
- How many alternative optimal solutions can be found?

Finding Alternative Optimal Solutions

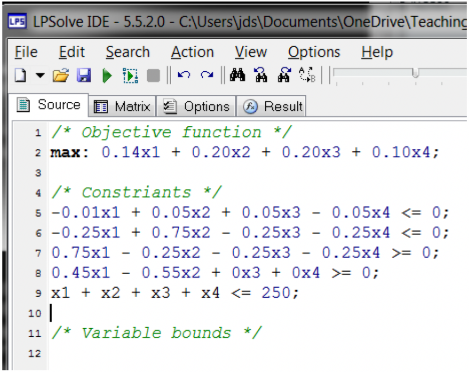


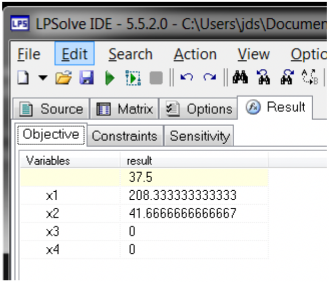


Adding constraints can help identify alternative optimal solutions by restricting the feasible region, for example:

- Add $x_1 \leq 62$, or
- Add $x_3 \geq 50$, etc.

Finding Alternative Optimal Solutions





Constraints can be added to identify other optimal solutions, for example:

- Add $x_3 + x_4 \geq 50$, or
- Add $x_1 - x_2 \leq 100$, etc.

Finding Alternative Optimal Solutions

- Adding constraints to the model modifies the feasible region, which can affect the optimal solution(s).
- When adding constraints, the feasible region may shrink or change shape.
- This can lead to:
  - Alternative optimal solutions that satisfy the new constraints.
  - No solution if the added constraints make the problem infeasible.
- It is important to ensure the added constraints are consistent with the original problem to avoid contradictions.