COMP4130 Linear and Discrete Optimization

Lecture 5: Network Flow Optimization

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1. **Transportation Problem**
2. **Minimum Cost Flow Problem**
3. **Maximum Flow Problem**

Additional Reading: *Introduction to Operations Research*:

- Chapter 9.1 *The Transportation Problem*
- Chapter 10.5 *The Maximum Flow Problem*
- Chapter 10.6 *The Minimum Cost Flow Problem*

Network Flow Optimization

Examples of Network Flow Optimization Problems:

- Design the optimal network structure or find the fastest data route in a communication network
- Find the shortest route in a transportation network or evacuation plan
- Minimize the distribution cost in supply chains
- Identify the critical path of activities in a project
- Design the best layout for an electronic circuit
- Optimize production scheduling and resource flows
- Determine the maximum cash flow in a financial network

Transportation Problem (TP)

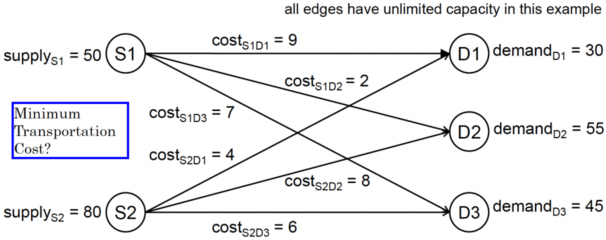
A transportation problem is defined on a directed network with:

- **Supply nodes (sources):** provide the total available commodity
- **Demand nodes (destinations):** require specified amounts

Each edge has:

- a **unit transportation cost** c_{ij} for sending one unit from source i to destination j
- **unlimited capacity** (no maximum flow constraint)

The objective is to distribute the entire supply to satisfy all demands while **minimizing the total transportation cost**.



Transportation Problem (TP)

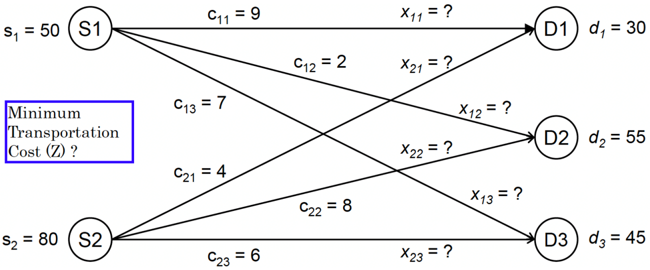
Key Properties:

- Flow moves directly from **supply nodes to demand nodes**
- Transportation cost is **proportional to flow** (LP proportionality condition)
- **Supply constraints:** all available supply must be distributed
- **Demand constraints:** all demand must be satisfied
- **Balancing:** dummy sources or destinations can be added if supply $\neq$ demand
- **Integrality:** if supplies and demands are integers, the optimal solution is guaranteed to be integer
- Applications arise mainly in **distribution and logistics networks**

Transportation Problem (TP)

Problem Setup:

- **Data:** node supplies, node demands, and cost for each edge.
- **Decision variables:** x_{ij} = amount transported from supply node i to demand node j.
- **Objective function:** minimize the total transportation cost across all edges.
- **Constraints:** all supply must be distributed and all demand must be satisfied (net flow balance).



Transportation Problem (TP)

Mathematical Formulation:

$$\text{Minimize:} \quad Z = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij}$$

Subject to:

$$\sum_{j=1}^{m} x_{ij} = S_i \quad \text{for } i = 1, \ldots, n \quad (1)$$

$$\sum_{i=1}^{n} x_{ij} = D_j \quad \text{for } j = 1, \ldots, m \quad (2)$$

$$x_{ij} \geq 0 \quad \text{for all } i \to j \quad (3)$$

Notation:

- n: number of supply nodes
- m: number of demand nodes
- c_{ij}: unit transportation cost from node i to node j
- x_{ij}: flow from node i to node j
- S_i: supply at source node i
- D_j: demand at destination node j

Remarks:

- Assumes the problem is balanced (total supply = total demand) and uncapacitated.
- If S_i and D_j are not integers, then the optimal x_{ij} values may also be non-integers.

Linear Program Formulation:

$$
\begin{aligned}
\text{Minimize:}\quad & Z = 9x_{11} + 2x_{12} + 7x_{13} + 4x_{21} + 8x_{22} + 6x_{23} && \\
\text{Subject to:}\quad & x_{11} + x_{12} + x_{13} = 50 && (1)\\
& x_{21} + x_{22} + x_{23} = 80 && (2)\\
& x_{11} + x_{21} = 30 && (3)\\
& x_{12} + x_{22} = 55 && (4)\\
& x_{13} + x_{23} = 45 && (5)\\
& x_{ij} \geq 0 \quad \forall i, j && (6)
\end{aligned}
$$

Notes:

- One decision variable per edge x_{ij}
- One equality constraint per node (net flow constraint)
- $i = 1, 2$ indexes supply nodes; $j = 1, 2, 3$ indexes demand nodes

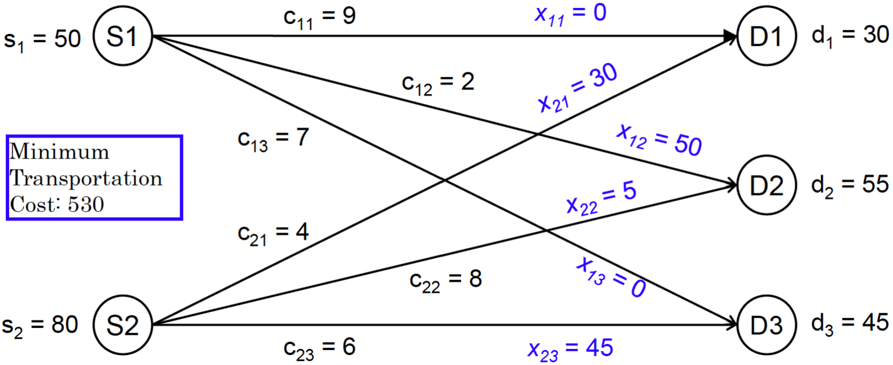
Transportation Problem (TP)

Two Alternative Layouts of the Spreadsheet Model:

	A	B	C	D	E	F	G	H	I	J
1	**TRANSPORTATION Problem**									
2										
3	**Source**	**Destination**	**Cost cij**		**Flow xij**		**Node**	**Net Flow bi**		**Supply si**
4	S1	D1	9		0		S1	50	=	50
5	S1	D2	2		50		S2	80	=	80
6	S1	D3	7		0		D1	30	=	30
7	S2	D1	4		30		D2	55	=	55
8	S2	D2	8		5		D3	45	=	45
9	S2	D3	6		45					
10										
11										
12		**Total Cost**	530							

	A	B	C	D	E	F	G	H
1	**TRANSPORTATION Problem**							
2								
3	**Cost cij**	**D1**	**D2**	**D3**				
4	**S1**	9	2	7				
5	**S2**	4	8	6				
6								
7	**Flow xij**	**D1**	**D2**	**D3**		**Net Flow bi**		**Supply si**
8	**S1**	0	50	0		50	=	50
9	**S2**	30	5	45		80	=	80
10								
11	**Net Flow bi**	30	55	45				
12		=	=	=				
13	**Supply si**	30	55	45				
14								
15	**Total Cost**	530						
16								

Optimal Solution:



Minimum Cost Flow Problem (MCFP)

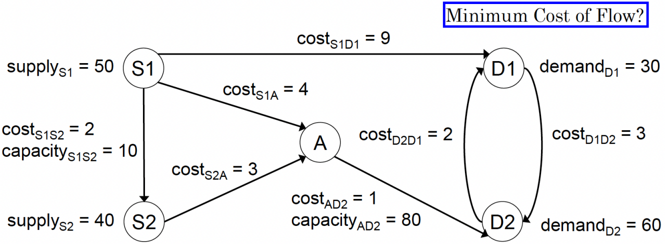
A minimum cost flow problem is defined on a directed network with:

- **Supply nodes:** provide a specified amount of flow
- **Demand nodes:** require a specified amount of flow
- **Transshipment nodes:** intermediate nodes with net flow $= 0$

Each edge has:

- a **unit cost** for transporting flow
- a non-negative **capacity** (maximum allowable flow)

The objective is to send the total supply through the network to meet all demands while **minimizing the total transportation cost**.



Minimum Cost Flow Problem (MCFP)

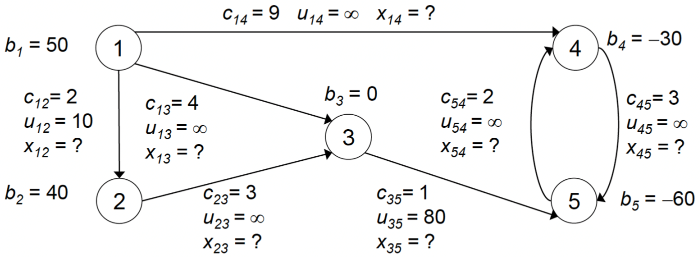
Key Properties:

- Flow may pass through transshipment nodes, which always have net flow $= 0$
- Transportation cost is **proportional to the flow amount** (LP proportionality condition)
- **Supply constraints:** all supply must leave the source nodes
- **Demand constraints:** all demand must be satisfied at the destination nodes
- Dummy sources and destinations can be added to balance supply and demand
- If supplies and demands are integers, then the optimal solution is guaranteed to be integer
- Applications include **distribution networks, logistics, communication routing, and production planning**

Minimum Cost Flow Problem (MCFP)

Problem Setup:

- **Data:** node supplies, node demands, and cost/capacity for each edge
- **Decision variables:** x_{ij} = flow transported along edge $(i \to j)$
- **Objective function:** minimize the total transportation cost across all edges
- **Constraints:**
  - Net flow for source, destination, and transshipment nodes must be satisfied
  - Edge capacities must not be exceeded



Minimum Cost Flow Problem (MCFP)

Mathematical Formulation:

$$\text{Minimize:} \quad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = b_i \quad \text{for } i = 1, \ldots, n \quad (1)$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for all edges } i \to j \quad (2)$$

Notation:

- n: number of nodes
- c_{ij}: unit cost on edge $i \to j$
- x_{ij}: flow on edge $i \to j$
- b_i: net flow at node i (positive = supply, negative = demand, zero = transshipment)
- u_{ij}: capacity of edge $i \to j$

Remarks:

- The MCFP is a generalization of the Transportation Problem.
- If b_i and u_{ij} are not integers, then the optimal x_{ij} may also be fractional.

Minimum Cost Flow Problem (MCFP)

Linear Program Formulation:

$$\text{Minimize:} \quad Z = 2x_{12} + 4x_{13} + 9x_{14} + 3x_{23} + x_{35} + 3x_{45} + 2x_{54}$$

$$\begin{aligned}
\text{Subject to:} \quad & x_{12} + x_{13} + x_{14} = 50 \quad (1) \\
& x_{23} - x_{12} = 40 \quad (2) \\
& x_{35} - x_{13} - x_{23} = 0 \quad (3) \\
& x_{45} - x_{14} - x_{54} = -30 \quad (4) \\
& x_{54} - x_{35} - x_{45} = -60 \quad (5) \\
& x_{12} \le 10, \; x_{35} \le 80 \quad (6) \\
& x_{ij} \ge 0 \quad \text{for all edges } i \to j \quad (7)
\end{aligned}$$

Notes:

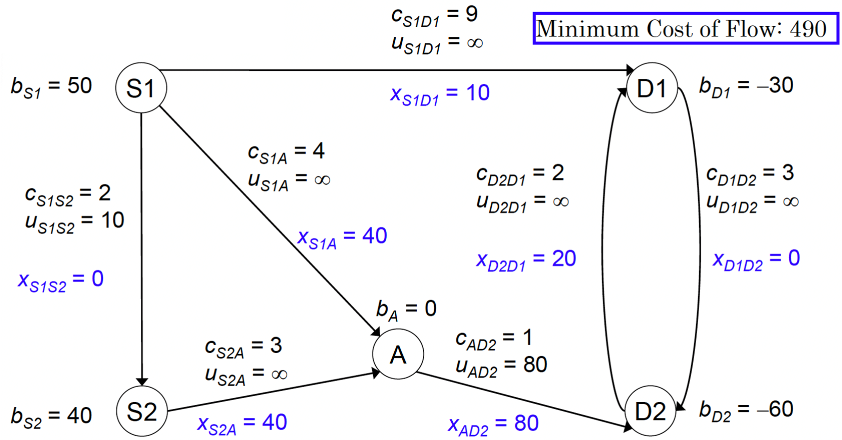
- One decision variable per edge
- One equality (net flow) constraint per node
- One constraint for each limited edge capacity
- $i, j = 1, \ldots, 5$ indicate nodes

Two Alternative Layouts of the Spreadsheet Model:

	A	B	C	D	E	F	G	H	I	J	K
1	MINIMUM COST FLOW Problem										
2											
3	From	To	Cost Cij	Capacity Uij		Flow Xij		Node	Net Flow Bi		Supply/Demand Bi
4	S1	D1	9			10		S1	50	=	50
5	S1	A	4			40		S2	40	=	40
6	S1	S2	2	10	>=	0		A	0	=	0
7	S2	A	3			40		D1	-30	=	-30
8	A	D2	1	80	>=	80		D2	-60	=	-60
9	D1	D2	3			0					
10	D2	D1	2			20					
11											
12		Total Cost	490								
13											

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	MINIMUM COST FLOW Problem												
2													
3	Cost Cij	S1	S2	A	D1	D2	Capacity Uij (<=)		S1	S2	A	D1	D2
4	S1		2	4	9			S1		10			
5	S2			3				S2					
6	A					1		A					80
7	D1					3		D1					
8	D2				2			D2					
9													
10	Flow Xij	S1	S2	A	D1	D2	OutFlow		NetFlow Bi		Supply/Demand Bi		
11	S1		0	40	10		50		50	=	50		
12	S2			40			40		40	=	40		
13	A					80	80		0	=	0		
14	D1					0	0		-30	=	-30		
15	D2				20		20		-60	=	-60		
16	InFlow	0	0	80	30	80							
17													
18	Total Cost	490											
19													

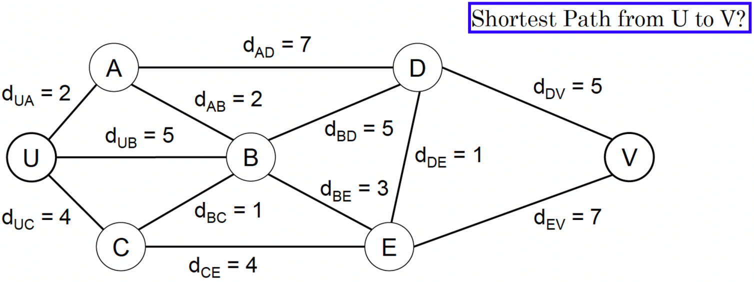
Optimal Solution:



Minimum Cost Flow Problem (MCFP)

Shortest Path as a Special Case of the MCFP:
While several algorithms exist for the shortest path problem (e.g., Dijkstra's, Floyd's), it can also be formulated and solved efficiently using linear programming.

Consider an undirected network with a non-negative distance assigned to each edge. The objective is to determine the shortest path between a specified origin node and destination node.



Minimum Cost Flow Problem (MCFP)

Modeling Shortest Path as an MCFP - Key Properties:

- Edge distance d_{ij} is treated as unit cost c_{ij}
- Edge capacities are set to $u_{ij} = \infty$
- Flow is binary: $x_{ij} = 1$ on edges in the path, $x_{ij} = 0$ otherwise
- Origin node becomes a supply node with $b_u = 1$
- Destination node becomes a demand node with $b_v = -1$
- All other nodes are transshipment nodes with $b_i = 0$
- Undirected edges are replaced by pairs of directed edges

Minimum Cost Flow Problem (MCFP)

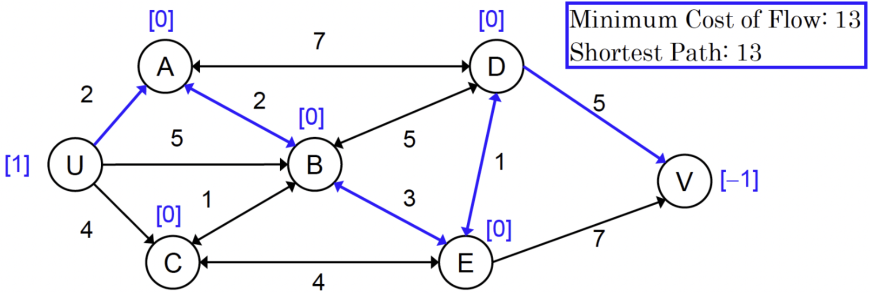
Modeling Shortest Path as an MCFP - Linear Program Formulation:

$$\begin{aligned}\text{Minimize:} \quad Z = {} & 2x_{UA} + 5x_{UB} + 4x_{UC} + 2(x_{AB} + x_{BA}) + (x_{BC} + x_{CB}) + 7(x_{AD} + x_{DA}) \\ & + 4(x_{CE} + x_{EC}) + 5(x_{BD} + x_{DB}) + 3(x_{BE} + x_{EB}) + (x_{DE} + x_{ED}) \\ & + 5x_{DV} + 7x_{EV}\end{aligned}$$

Subject to:

$$\begin{aligned}
x_{UA} + x_{UB} + x_{UC} &= 1 \quad (1) \\
x_{AB} + x_{AD} - x_{UA} - x_{BA} - x_{DA} &= 0 \quad (2) \\
x_{BA} + x_{BC} + x_{BD} + x_{BE} - x_{UB} - x_{AB} - x_{CB} - x_{DB} - x_{EB} &= 0 \quad (3) \\
x_{CB} + x_{CE} - x_{UC} - x_{BC} - x_{EC} &= 0 \quad (4) \\
x_{DA} + x_{DB} + x_{DE} + x_{DV} - x_{AD} - x_{BD} - x_{ED} &= 0 \quad (5) \\
x_{EB} + x_{EC} + x_{ED} + x_{EV} - x_{BE} - x_{CE} - x_{DE} &= 0 \quad (6) \\
-x_{DV} - x_{EV} &= -1 \quad (7) \\
x_{ij} \in \{0, 1\} \quad \text{for all edges } i \to j & \quad (8)
\end{aligned}$$

Modeling Shortest Path as an MCFP - Optimal Solution:



Maximum Flow Problem (MFP)

A maximum flow problem is defined on a **directed network** with:

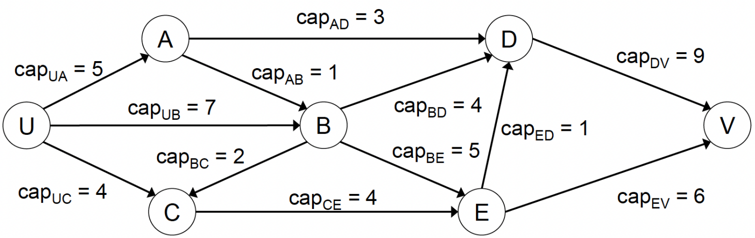
- **Source node(s):** where flow originates
- **Sink (destination) node(s):** where flow is received
- **Transshipment nodes:** intermediate nodes with net flow $= 0$

Each edge has:

- a non-negative **capacity**, representing the maximum allowable flow

The objective is to maximize the total flow that can be sent from the source(s) to the sink(s), possibly through transshipment nodes.

Maximum Flow from U to V?



Maximum Flow Problem (MFP)

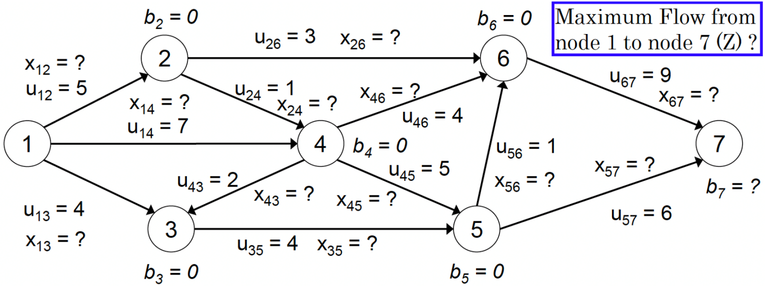
Key Properties:

- Flow may pass through transshipment nodes, which always have net flow $= 0$
- The objective is to maximize the total flow from the source(s) to the sink(s)
- Capacity constraints: flow on each edge cannot exceed its maximum capacity
- Applications include:
  - **Supply chains:** maximizing throughput of goods
  - **Communication networks:** maximizing data transmission
  - **Finance:** maximizing capital flow in financial systems

Maximum Flow Problem (MFP)

Problem Setup:

- **Data:** capacity u_{ij} for each edge $(i \to j)$
- **Decision variables:** x_{ij} = amount of flow sent along edge $(i \to j)$
- **Objective function:** maximize the total flow from the source(s) to the sink(s)
- **Constraints:**
  - Net flow at transshipment nodes must be zero (flow conservation)
  - Flow on each edge cannot exceed its capacity $(0 \leq x_{ij} \leq u_{ij})$



Maximum Flow Problem (MFP)

Mathematical Formulation:

$$\text{Maximize:} \quad Z = \sum_{j=2}^{n} x_{1j} \quad \text{or} \quad Z = \sum_{i=1}^{n-1} x_{in}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = b_i \quad \text{for } i = 2, \ldots, (n-1) \quad (1)$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for each edge } i \to j \quad (2)$$

Notation:

- n: number of nodes
- Node 1: source (supply node)
- Node n: sink (demand node)
- Nodes $2, \ldots, (n-1)$: transshipment nodes
- x_{ij}: flow on edge $i \to j$
- b_i: net flow at transshipment node i $(= 0)$
- u_{ij}: capacity of edge $i \to j$

Maximum Flow Problem (MFP)

Linear Program Formulation:

$$\text{Maximize:} \quad Z = x_{12} + x_{13} + x_{14} \quad \text{or} \quad Z = x_{57} + x_{67}$$

$$\begin{aligned}
\text{Subject to:} \quad -x_{12} + x_{24} + x_{26} &= 0 \quad (1) \\
-x_{13} - x_{43} + x_{35} &= 0 \quad (2) \\
-x_{14} - x_{24} + x_{43} + x_{45} + x_{46} &= 0 \quad (3) \\
-x_{35} - x_{45} + x_{56} + x_{57} &= 0 \quad (4) \\
-x_{26} - x_{46} - x_{56} + x_{67} &= 0 \quad (5) \\
0 \leq x_{ij} \leq u_{ij} \quad \text{for each edge } i &\to j \quad (6)
\end{aligned}$$

Notes:

- One decision variable per edge
- One equality (net flow) constraint per transshipment node
- One constraint per limited edge capacity
- $i, j = 1, \ldots, 7$ indicate nodes

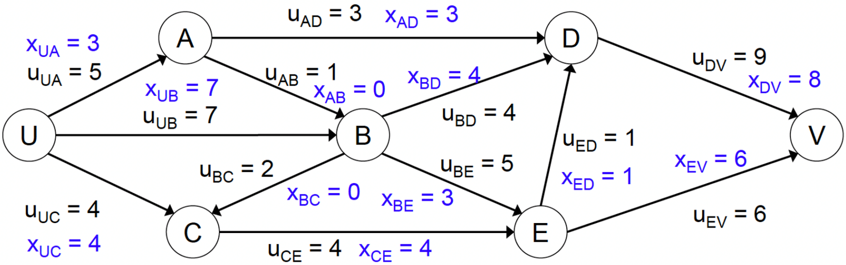
Maximum Flow Problem (MFP)

A Layout of the Spreadsheet Model

	A	B	C	D	E	F	G	H	I	J
1	**MAXIMUM FLOW Problem**									
2										
3			**From \ To**	**A**	**B**	**C**				
4		**Capacity Uij**		5	7	4		**Outflow from U**		**Inflow into V**
5			**U**	>=	>=	>=				
6		**Actual Flow Xij**		3	7	4		14		14
7										
8			**From \ To**	**D**	**B**					
9		**Capacity Uij**		3	1					
10			**A**	>=	>=					
11		**Actual Flow Xij**		3	0			3	**=**	3
12										
13			**From \ To**	**C**	**D**	**E**				
14		**Capacity Uij**		2	4	5				
15			**B**	>=	>=	>=				
16		**Actual Flow Xij**		0	4	3		7	**=**	7
17										
18			**From \ To**	**E**						
19		**Capacity Uij**		4						
20			**C**	>=						
21		**Actual Flow Xij**		4				4	**=**	4
22										
23			**From \ To**	**V**						
24		**Capacity Uij**		9						
25			**D**	>=						
26		**Actual Flow Xij**		8				8	**=**	8
27										
28			**From \ To**	**D**	**V**					
29		**Capacity Uij**		1	6					
30			**E**	>=	>=					
31		**Actual Flow Xij**		1	6			7	**=**	7
32										
33								**(Total Outflow from Node X)**	**=**	**(Total Inflow into Node X)**
34										

Optimal Solution:

$\boxed{\text{Maximum Flow from U to V : 14}}$



Exercise 1: MEDEQUIP

MEDEQUIP operates two production factories that supply three customers. The table shows the production capacity of each factory, the demand of each customer, and the unit shipping cost between each factory–customer pair. The objective is to determine the optimal shipping plan that meets all customer demands while minimizing the total transportation cost.

From / To	**Unit Shipping Cost (£)**			**Supply**
	Customer 1	**Customer 2**	**Customer 3**	
Factory 1	600	800	700	400
Factory 2	400	900	600	500
Demand	300	200	400	

- Identify the type of network flow problem (TP, MCFP, or MFP).
- Is the problem balanced (total supply $=$ total demand)?
- Formulate the mathematical model and its Linear Programming formulation.

Exercise 2: BUNEL

BUNEL is a company with three manufacturing plants and four warehouses. The table shows the production capacity of each plant, the demand of each warehouse, and the unit shipping cost for each plant-warehouse pair. The company must decide how much to ship from each plant to each warehouse in order to satisfy warehouse demand while minimizing the total shipping cost.

From / To	**Unit Shipping Cost (\$)**				**Capacity**
	Whse 1	**Whse 2**	**Whse 3**	**Whse 4**	
Plant 1	0.60	0.56	0.22	0.40	9,000
Plant 2	0.36	0.30	0.28	0.58	12,000
Plant 3	0.65	0.68	0.55	0.42	13,000
Requirements	7,500	8,500	9,500	8,000	

- Identify the type of network flow problem (TP, MCFP, or MFP).
- Is the problem balanced (total supply = total demand)?
- Formulate the mathematical model and its Linear Programming formulation.

Exercise 3: RIVERA NETWORK

RIVERA Transport Authority manages a pipeline network that delivers water from a source reservoir to several distribution nodes before reaching the main city. The table below shows the **maximum capacity** on each pipeline segment.
The objective is to determine the **maximum possible flow** that can be delivered from the reservoir (Source S) to the city (Sink T).

From / To	Pipeline Capacity (m^3/hour)			
	A	**B**	**C**	**T**
S	20	15	–	–
A	–	10	15	–
B	–	–	5	10
C	–	–	–	25

- Identify the type of network flow problem (TP, MCFP, or MFP).
- Formulate the mathematical model and its Linear Programming formulation.