REPORT

Executive Summary

In order to complete a computational project to reflect our learnings in this week's APS1022 course and also to implement and compare the Mean-variance optimization (MVO) model, robust mean-variance optimization, risk parity optimization with no short selling and market capitalization model, we prepared this final report to provide our analysis results and findings. This project aims to figure out how these four optimization models perform, and tests will be conducted on the out-of-sample data.

Monthly adjusted closing prices for 20 stocks from S&P 500 were used from Dec 30, 2004, to Sep 30, 2008, to generate the portfolio, and tests were conducted on October and November 2008's data. Return, variance, standard deviation, and Sharpe Ratio have all been calculated, and comparing October and November 2008's data shows that the risk parity optimization with no short selling and market capitalization model performed relatively not that good as compared to the MVO and robust MVO model. This is due to the significant drop in stock prices in October 2008, and with short selling not allowed, it would perform worse than that of MVO and robust MVO.

Efficient frontiers of the estimated MVO frontier, the true MVO frontier, the actual MVO frontier, the actual robust MVO frontier, and the estimated robust MVO frontier have been plotted. The true MVO frontier is the frontier on the top, followed by the actual MVO frontier and the estimated MVO frontier, which should also relate to the drop in price in October 2008, that caused the change in short selling position and affected the expected return. Among the robust MVO frontiers, the actual robust frontiers with the 90% and 95% confidence intervals are closer to the true MVO frontier, while the estimated robust frontiers with the 90% and 95% confidence intervals appear to be lower than the estimated MVO frontier, due to the box uncertainty constraint.

Comparison has been made on different portfolios on October and November 2008's data and their advantages and disadvantages have been discussed.

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1.0 Introduction

This project aims to implement and compare several different financial optimization models, including 1) Mean-variance optimization, 2) Robust mean-variance optimization, 3) Risk Parity optimization with no short selling, and 4) Market portfolio. The assets that consist of this project's investment portfolios are F, CAT, DIS, MCD, KO, PEP, WMT, C, WFC, JPM, AAPL, IBM, PFE, JNJ, XOM, MRO, ED, T, VZ, and NEM. They are all S&P 500 stocks.

To generate the four portfolios, the data used is the monthly adjusted closing prices for the stocks from 30-Dec-2004 to 30-Sep-2008 downloaded from yahoo finance. The sample mean, sample variance, and sample covariance are computed first based on the in-sample data.

Let r_{it} = return of asset i for month t, t = 1, …, T. The sample mean μ is computed arithmetically by the formula $\overline{r_i} = \frac{1}{T}\sum_{t=1}^{T} r_{it}$. The sample variance is computed by the square of the standard deviation function in MATLAB. The sample covariance Q is computed by the covariance function in MATLAB.

The risk-free rate r_f and lambda λ are computed next. For the risk-free rate, the average rate of 10-year Treasury Bill rates from 2005 to 2008 are used and converted into monthly rate, $r_f = \frac{(4.15+4.22+4.42+4.76+3.74)\%}{5*12} = 0.3548\%$ (10 year Treasury rate by year, n.d.). The lambda λ is the risk aversion coefficient in the Black-Litterman's model, and it is calculated by the formula $\lambda = \frac{E[r_{mkt}]-r_f}{\sigma_{mkt}^2}$ = 2.2228.

2.0 Optimization Models

2.1 Mean-variance Optimization

The Mean-variance optimization allows investors to obtain the highest level of returns at a given risk or the lowest level of risks at a given return. According to the project, the version of MVO used is to maximize the returns at the given level of risk while allowing short selling. The model maximizes the return while subtracting risk from the return.

The weights of the 20 assets sum up to 1 for the portfolio. In the model, x is the optimal weights of the stocks in the Mean-variance portfolio, μ is the expected return, Q is the covariance matrix, and λ is the risk aversion coefficient.

The model is formulated as below:

$$\max_{x} \mu^T x - \lambda x^T Q x$$

$$\text{Subject to } e^T x = 1$$

The result of the optimization is shown in the table below:

	Portfolio Return (%)	Portfolio Variance	Standard Deviation	Sharpe Ratio
October, 2008	0.4352	0.0316	0.1777	2.4286
November, 2008	0.4435	0.0328	0.1810	2.4304

Table 2.1 Mean-variance Optimization

2.2 Robust mean-variance optimization

The Robust mean-variance optimization is an alteration based on the nominal MVO assumptions. Under the nominal MVO assumptions, the investment decisions are based solely on the estimated parameters; therefore, the estimation errors are neglected. With the Robust mean-variance optimization, the uncertainty is considered. The two uncertainty sets introduced in the course are box uncertainty and ellipsoidal uncertainty. In this project, the box uncertainty is used.

The weights of the 20 assets sum up to 1 for the portfolio. In the model, x is the optimal weights of the stocks in the Robust mean-variance portfolio, μ is the expected return, Q is the covariance matrix, and λ is the risk aversion coefficient.

Since the box uncertainty set around the estimated expected returns, $\mu^{true} \in U(\mu) = \mu^{true} \in R^n: \left|\mu_i^{true} - \mu_i\right| \leq \delta_i, i = 1, \ldots, n,$ where μ^{true} is the vector of true expected returns and δ_i is the maximum 'distance' between μ_i - μ_i^{true}. To determine the distance δ_i, it can be set proportionally to the standard errors, so $\delta_i = \varepsilon(\frac{\sigma_i}{\sqrt{T}})$. The two-confidence interval considered in this project are 90% and 95%, so δ_i is calculated by multiplying 1.645 and 1.96 to the diagonal matrix of standard errors of the asset expected returns. y is set to 2, meaning that 200% short selling is allowed for this portfolio. R is set to 0.26, which is lower than the return of the Mean-variance portfolio.

The model is formulated as below:

$$\max_x \mu^T x - \lambda x^T Q x$$

$$\text{Subject to } \mu^T x - \delta^T y \geq R$$

$$1^T x = 1$$

$$y \geq x$$

$$y \geq -x$$

The result of the optimization is shown in the table below:

	Portfolio Return	Portfolio Variance	Standard Deviation	Sharpe Ratio
90% confidence October, 2008	0.2719	0.0176	0.1326	2.0240
90% confidence November, 2008	-0.1320	0.0184	0.1355	-1.0002
95% confidence October, 2008	0.3779	0.0263	0.1623	2.3061
95% confidence November, 2008	-0.2185	0.0282	0.1680	-1.3216

Table 2.2 Robust Mean-variance Optimization

2.3 Risk Parity optimization with no short selling

The risk parity portfolio diversifies risk by ensuring each asset in the investment universe contributes the same level of risk. According to the project, since there are 20 assets, each asset should contribute 1/20 of portfolio's total risk.

The weights of the 20 assets sum up to 1 for the portfolio. In the model, x is the optimal weights of of the stocks in the Risk Parity optimization portfolio with no short selling, Q is the covariance matrix, and $\theta \in R$ is an auxiliary unconstrained variable.

The model is formulated as below:

$$\min_{x,\theta} \sum_{i=1}^{n} (x_i(Qx)_i - \theta)^2$$

$$\text{Subject to} \quad 1^T x = 1$$

$$x \geq 0$$

The result of the optimization is shown in the table below:

	Portfolio Return	Portfolio Variance	Standard Deviation	Sharpe Ratio
October, 2008	-0.0484	1.0659e-04	0.0103	-5.0307
November, 2008	-0.0028	1.0458e-04	0.0102	-0.6193

Table 2.3 Risk Parity optimization with no short selling

2.4 Market portfolio

The market portfolio is the portfolio that includes all the assets in the investment universe, and the assets' weights are based on their market capitalizations. The market capitalization is the total value of a company's shares of stock, which is determined by the total shares of a specific stock multiplying its stock price.

To generate the market portfolio, the historical monthly adjusted close price of the 20 assets in September 2008 and their volumes are collected. The market capitalization is calculated by multiplying the adjusted close price by the corresponding stock's volumes. The optimal weight x is generated by dividing each stock's market capitalization by the total value of the investment universe.

The result of the portfolio is shown in the table below:

	Portfolio Return	Portfolio Variance	Standard Deviation	Sharpe Ratio
October, 2008	-0.1360	0.0023	0.0476	-2.9303
November, 2008	-0.1108	0.0025	0.0502	-2.2780

Table 2.4 Market Capitalization portfolio

2.5 Discussion of Results

When comparing the returns of the four different strategies, the Mean-variance portfolio performed significantly better than the other three portfolios. Based on the weights of the Mean-variance portfolio generated from the in-sample data, its portfolio returns in October and November 2008 are higher than 40%. The Robust Mean-variance portfolio has around 28% and 38% returns, corresponding to the 90% and 95% confidence levels in October 2008. However, in November 2008, its returns all dropped significantly to negative. It is less stable compared to the Mean-variance portfolio. The third portfolio is the risk parity portfolio, which has a slightly negative return in both the months of October and November. The fourth portfolio is the market portfolio; its returns are less than -10%. The Mean-variance portfolio outperformed the other three portfolios in terms of portfolio returns. The Robust Mean-variance portfolio's returns fluctuate a lot. The equal risk contribution and market portfolios are also stable but with negative returns because short selling is not allowed. The market portfolio also has a negative return because of the drop in the price in these months.

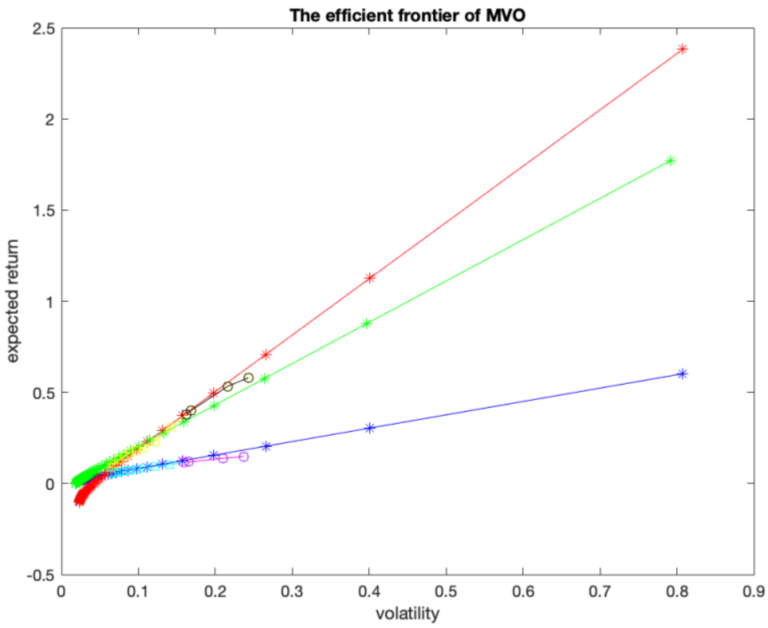
When comparing the variance and standard deviation of the four portfolios, the equal risk contribution portfolio has the lowest volatility. This is reasonable since this portfolio diversified the risks well by making each stock contribute the same amount of risk to the portfolio. The market portfolio has the second-lowest volatility because it should only suffer systematic risk, which is the risk that would affect the whole market instead of a particular stock. Both the Mean-variance portfolio and the Robust Mean-variance portfolio have a higher risk since these two models tradeoff between risk and return.

When comparing the Sharpe ratios of the four portfolios, only the Mean-variance portfolio remained positive in October and November 2008. Its Sharpe ratios for these two months are around 2.4, which is considered very good according to the definition of the Sharpe ratio. The Robust Mean-variance portfolio's Sharpe ratio varies significantly between these two months. Both the 90% and 95% confidence levels in October had Sharpe ratios over 2, but in November, they dropped to -1. For the equal risk contribution and market portfolios, their Sharpe ratios remain negative for these two months. Sharpe ratio is determined by the effect of return and variance. With higher returns and lower variance, the Sharpe ratio tends to be larger. Otherwise, the Sharpe ratio might also be negative, especially when the return is negative.

Therefore, based on the portfolio's return, variance, and Sharpe ratio, the Mean-variance portfolio performs the best. Its returns and Sharpe ratios are significantly higher than the other three portfolios. Its variance is higher, which is reasonable because it balances risks and rewards.

3.0 Efficient Frontiers

Below is the plot of the efficient frontier with the estimated MVO frontier, the true MVO frontier, the actual MVO frontier, the actual robust MVO frontier and the estimated robust MVO frontier:



Plot 1: Efficient Frontiers

From the plot above, the efficient frontiers are computed and expressed as follows:

1) Estimated MVO frontier: blue line, computed using the estimated expected return
2) True MVO frontier: red line, computed using the realized return for October 2008
3) Actual MVO frontier: green line, computed by applying the portfolio weights from the estimated frontier to the realized return for October 2008 and true covariance matrix
4) Actual robust frontier (90% confidence interval): yellow line, actual robust MVO frontier with 90% confidence interval computed using the realized return for October 2008 and true covariance matrix
5) Actual robust frontier (95% confidence interval): black line, actual robust MVO frontier with 95% confidence interval computed using the realized return for October 2008 and true covariance matrix
6) Estimated robust frontier (90% confidence interval): cyan line, estimated robust MVO frontier with 90% confidence interval computed using the estimated expected return
7) Estimated robust frontier (95% confidence interval): magenta line, estimated robust MVO frontier with 95% confidence interval computed using the estimated expected return

Among the MVO frontiers, from the plot above, it is visualized that the actual MVO frontier is in the middle, while the true MVO frontier is on the top and the estimated MVO frontier is on the bottom. This is very reasonable because the true MVO frontier is computed using the realized return for October 2008. From the adjusted close price of October 2008, it shows that the prices of all stocks experienced a significant drop, and since short selling is allowed in this case, it led to more short-selling happening during this time, which led to the growth of expected return. With the growth of expected return happening, the true MVO frontier becomes the frontier on the top, followed by the actual MVO frontier and the estimated MVO frontier.

Among the robust MVO frontiers, both the actual robust frontiers with the 90% and 95% confidence interval are closer to the true MVO frontier, while the estimated robust frontiers with the 90% and 95% confidence interval appear to be lower than the estimated MVO frontier. Since the robust frontiers are generated with the box uncertainty, although it provides a more stable portfolio, the uncertainty has been added as a penalty on the target return. This might cause the return to become lower than what it actually is, which has been reflected in the plot.

4.0 Conclusions

The portfolio return, standard deviation, variance, and Sharpe ratio of October 2008 shows that the return and Sharpe ratio are relatively bad when using risk parity optimization with no short selling and market capitalization. In contrast, the MVO and robust MVO are relatively better. In October 2008, the asset price dropped significantly, so if the risk parity optimization with no short selling portfolio is used, the estimate of the portfolio's weight would not be very accurate when applying to this month's data, since short selling is not allowed. As for the market capitalization portfolio, with the drop in the stock price, market capitalization has also experienced change, so the estimate of this when applying to October 2008's data would lead to a relatively poor and inaccurate portfolio. As a comparison, since the MVO model allows the existence of short selling, the MVO, and robust MVO model performed much better than risk parity optimization with no short selling and market capitalization under this scenario.

For November 2008, the change in the price became more stable after the significant drop in October 2008. Although there was still a stock that experienced a significant decline, other stocks' prices either rose or dropped a little bit, and the drop or rise was not that large compared to October 2008's adjusted close price. The return and Sharpe ratio for risk parity optimization with no short selling and market capitalization were still not good; however, they were already much better than the return and Sharpe ratio of October 2008's data.

Analyzing the result from October and November of 2008 shows that the robust MVO is relatively stable, either not the best or the worst, because the uncertainty set has been set for this model. With the constraint, the result would be relatively stable. Risk parity optimization with no short selling and market capitalization performs relatively not that good when there appears to be a significant drop in asset prices, but for the risk parity optimization, the use of leverage is an advantage of this model, which would help to allocate with respect to risk. Market capitalization cares more about the capitalization in the market, which is from a much broader aspect. In conclusion, these four portfolios have distinct advantages and disadvantages, and which portfolio performs the best would be based on the current situation in the market. In this case, MVO performs the best.

References

10 year Treasury rate by year. (n.d.). Retrieved June 10, 2022, from https://www.multpl.com/10-year-treasury-rate/table/by-year