REPORT

1.0 Summary

This project aims to compute the prices of seven options: Asian call, Asian put, lookback call, lookback put, floating lookback call, floating lookback put, and American put by using two methods.

These seven options can be classified into three different types of options:
a) Asian option is an option type where the payoff depends on the average price of the underlying asset over a certain period of time as opposed to standard options (American and European), where the payoff depends on the price of the underlying asset at a specific point in time(maturity) (Chen, 2022).
b) Lookback and floating lookback option is an option type where the payoff depends on the price of the optimal underlying asset occurring over the life of the option (Scott, 2022).
c) An American option is an option that can be exercised any time before the maturity date (Vaidya et al., 2022).

The first method is Monte Carlo. A Monte Carlo simulation is a model used to predict the probability of different outcomes when the intervention of random variables is present. 10000 sample sizes are used to generate random paths and calculate the option's payoff from the simulations.

The second method is Lattice (binomial options price model). Basically, it uses a binomial tree to calculate the option's payoff by using lattice paths. The Lattice divides the time interval to maturity into equal time steps. We can observe the price of an option at each time step from the Lattice. We calculate the probabilities and multiplication factors for prices going upward and downward to simulate the price process.

The results of the two methods are similar for each option. However, when calculating the price of an American put option, we find that the process using Monte Carlo simulation is much more complicated than the process using binomial Lattice.

2.0 Methods

2.1 Monte Carlo

Monte Carlo is a method used for option pricing by generating many random price paths by simulation, which relies on risk-neutral valuation. For each path, the payoff is calculated and then discounted back to the present value (Broadie & Glasserman, 1996). In this option valuation project, Monte Carlo Simulation is used as the first method to price Asian call, Asian put, lookback call, lookback put, floating lookback call, and floating lookback put.

The parameters and values used in the Monte Carlo method are listed below:

Parameter	Value
Current price of the underlying stock: S_0	$100
Strike price: K	$105
Volatility: σ	0.25
Risk-free rate: r	0.02
Maturity (years): T	1/6 year
Number of Steps: numSteps	8 weeks (2 months)
Number of Paths: numPaths	10000

For Monte Carlo simulation, the following formula is used:

$$S_{t,j+1} = S_{t,j}e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t}\varepsilon},$$

where:

r: risk-free rate

σ: volatility

Δt: a small period

ε: follows N (0,1)

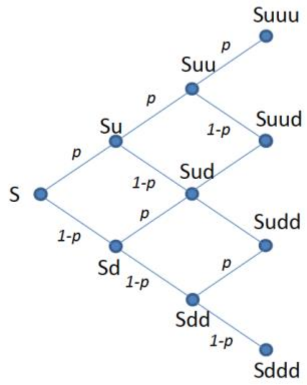
For the Asian put and call option, the payoff is determined by the average of the underlying prices over periods of time. For an Asian call option, the payoff is determined by the maximum between 0 and the average of a path minus the strike price. For an Asian put option, the payoff is determined by the maximum between 0 and the strike price minus the average of a path. For the lookback put and call option, the payoff is determined by the optimal price of the underlying asset over the life of the option. For a lookback call option, the payoff is determined by the maximum between 0 and the maximum of the path minus the strike price. For a lookback put option, the payoff is determined by the maximum between 0 and the strike price minus the maximum of the path. For a floating lookback call option, the payoff of each path is the maximum between 0 and the underlying asset's price at maturity minus the asset's minimum price along the path. For a floating lookback put option, the payoff of each path is the maximum among 0 and the asset's maximum price along the path minus the underlying asset's price at maturity. By simulating each path and then covert the payoff to the present value, the call and put prices of the Asian option, lookback option, and floating lookback option could be achieved by the Monte Carlo method.

For the American put option, the early exercise of the option is allowed, and when the price falls below the strike price, early exercise might exist. When calculating the price of an American put option using Monte Carlo, simulation occurs over the paths and steps, and stop time and remaining time are calculated. Then Black-Scholes formula is used to calculate the premium of American put when there is still time remaining along the path. The payoff is then computed and compared with the premium of American put to determine the exercise and the payoff. When the simulation along the paths and steps has been completed, the American put option price will be computed after calculating the present value of the payoff and calculating the average of the price.

The number of paths simulated over is set to be 10000 when conducting the Monte Carlo method, and the result will be shown in the following sections.

2.2 Lattice

In general, the lattice model is used to value derivatives. There is a binomial tree to compute the various paths the price of an underlying asset (options) takes over the derivative's life. The binomial option pricing model uses an iterative procedure, allowing for the specification of nodes, or points in time, during the time span between the valuation date and the option's expiration date. In this model, there are assumptions assuming that there are two possible outcomes; that is, the outcome is either move up or move down. The binomial tree graphically plots out the possible values that option prices can have over different time periods (Figure below).



The figure of the lattice model above illustrates 3 period lattice paths. We assume that the price (S) of the option can either go up (u) or down(d) for each period of time. There are 4 paths in total. In each period, we do the lattice multiplication to calculate the product. Lattice multiplication is easy to learn and allows us to quickly calculate the product even when the factors are large numbers. On the other hand, it has the disadvantage that we must draw the lattice before performing the multiplication.

In this lattice case, we compute the prices of the 7 options using the lattice paths. First of all, it is essential to draw the lattice paths before performing the multiplication. Since the unit time of lattice is one week, there are 8 periods in total and paths. After the lattice path is designed, we will use this in each option's algorithm according to the table below. For example, for the Asian and lookback options, the difference between the Monte Carlo method is that we use the generated prices in the lattice paths instead of simulating the price. The procedure is that first, we compute the option's payoff for the specified path. Repeat this procedure for the rest paths. The discount factor can be calculated given the conditions. Then, we use this discount factor to calculate the present payoff values. Finally, calculate the average of all present values, which will be the price. This procedure works for Asian call(put) options, lookback call(put)options, and floating lookback call(put) options.

Option type	Payoff
Asian call	$(\overline{S} - K)^+$
Asian put	$(K - \overline{S})^+$
Lookback call	$(S_{max} - K)^+$
Lookback put	$(K - S_{min})^+$
Floating lookback call	$(S_T - S_{min})^+$
Floating lookback put	$(S_{max} - S_T)^+$

The American put options that can be exercised at any time at the will of the holder of the option before the expiration date, therefore, we need to compare the payoff between exercising and holding at each node, and the principle is choosing the node with a higher value. And then calculate the previous node's payoff. Suppose still in 3-period lattice, the option price at maturity from top to bottom will be: [max $(u^3S - K, 0)$; max $(u^2dS - K, 0)$; max $(ud^2S - K, 0)$; max $(d^3S - K, 0)$]. Then calculate the payoff of the previous nodes by using the discount factor $e^{-r\triangle t}$. Basically, we use maximum functions to decide whether the option is exercised. The entire node will be calculated after the whole process.

3.0 Explanation of Major Functions in the Code

3.1 Monte Carlo

The two major functions used for the Monte Carlo simulation method in the code are the Geometric random walk and Black-Scholes models.

The formula to generate paths for a geometric random walk is:

$$S_{t,j+1} = S_{t,j}e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t}\varepsilon}$$ (Romanko, 2021),

where $S_{t,j}$ is the value of the underlying asset at step j on path t, and $S_{t,j+1}$ is the value at j+1 step on path t. The risk-free rate 2% is used. σ is the volatility and ε is the standard normal random variables $N(0,1)$.

After the generation of paths, the payoffs of different options are calculated. For call option, its payoff is max {S-K, 0}, and for put option, its payoff is max {K-S, 0}. The payoffs all discounted back with e^{-rT}assuming continuous compounding.

For the American put option pricing, the Black-Scholes model is used. By the Black-Scholes model, the American put option can be priced with the formula:

$$p = Ke^{-rT}N(d_2) - SN(d_1),$$

where:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

For the pricing model, the Geometric random walk function is used first, then the Black-Scholes formulas are called. There is a particular case when the option is exercised at maturity. If the American put option is exercised at maturity, its payoff will be zero if its price is higher than the strike price; otherwise, it will be the difference between its current price and its strike price.

3.2 Lattice

In each period, the option price can either move upward to Su with probability Pu or move downward to Sd with probability $P_d = 1 - P_u$. The u and d are computed as follows. The probability of going up is q, and the probability of going down would be 1-q.

$$u = e^{\sigma\sqrt{\triangle t}}$$

$$d = 1/u$$

$$q = e^{r\sqrt{\triangle t}} - d/_{u-d}$$

For Asian call and Asian put, we can use the probabilities of each node and the average prices of stocks in each path. The payoff depends on the average stock price in each path.

$$e^{-rf*dt} \times \sum(prob \times \max(\bar{S} - K, 0))$$

$$e^{-rf*dt} \times \sum(prob \times \max(K - \bar{S}, 0))$$

For the lookback call, we can use the probabilities of each node and the maximum prices of stocks in each path. The payoff depends on the maximum value of stock in each path.

$$e^{-rf*dt} \times \sum(prob \times \max(S_{max} - K, 0))$$

For the lookback put, we can use the probabilities of each node and the maximum prices of stocks in each path. The payoff depends on the minimum value of stock in each path.

$$e^{-rf*dt} \times \sum(prob \times \max(K - S_{min}, 0))$$

For the floating lookback call, we can use the list of weights and the minimum prices of stocks in each path. The payoff depends on the difference between a stock's final value and the minimum value.

$$e^{-rf*dt} \times \sum(prob \times \max(S_{final} - S_{min}, 0))$$

For the floating lookback put, we can use the list of weights and the maximum prices of stocks in each path. The payoff depends on the difference between a stock's final value and the maximum value.

$$e^{-rf*dt} \times \sum(prob \times \max(S_{max} - S_{final}, 0))$$

For the American put option, we have 8 periods. At each node, if moving upward, multiply the stock price at the present node with u and the probability of getting to the next node with q. In each node, we can decide whether to wait until the next period or exercise the option immediately. The price depends on the difference between strike and exercise price.

$$expected\ waiting\ profit = e^{-rf*dt} \times (q \times price_{up} + (1 - q) \times price_{down})$$
$$e^{-rf*dt} \times (q \times price_{up} + (1 - q) \times price_{down})$$

To make a decision between waiting or exercising, we need to compare the payoff to find the maximum. If exercising, the payoff is the difference between strike and exercise price. If waiting, the profit is the expected stock price for the next period.

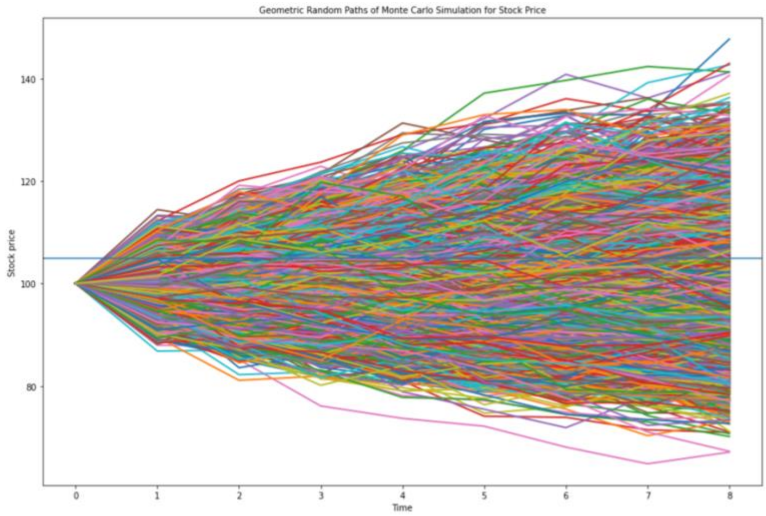
4.0 Visualization of the Solutions

4.1 Monte Carlo

The following is the table of the price of Asian call, Asian put, lookback call, lookback put, floating lookback call, floating lookback put and American put using the Monte Carlo method:

Option	Price
Asian call	0.6732
Asian put	5.4769
Lookback call	3.3715
Lookback put	11.0013
Floating lookback call	6.3294
Floating lookback put	6.3193
American put	6.9447

Below is the plot of the geometric random walks of the Monte Carlo simulations:



The y-axis of the plot corresponds to the price of stocks, while the x-axis corresponds to time measured in weeks. The x-axis ranges from 0 to 8, where 0 is the start time of the simulation. The end time of simulation is 8 since the time to maturity of the options is 2 months, which equals approximately 8 weeks. The y-axis ranges from 60 to 150, meaning that by the Monte Carlo simulation, the stock price may drop to 60 dollars or rise to 150 dollars. The starting point is 100, which is given in the project. The blue line is the strike price, which is 105 dollars.

4.2 Lattice

The following is the table of the price of Asian call, Asian put, lookback call, lookback put, floating lookback call, floating lookback put and American put using the Binomial Lattice method:

Option	Price
Asian call	0.6945
Asian put	5.5256
Lookback call	3.5854
Lookback put	11.2814

Floating lookback call	6.6172
Floating lookback put	6.5670
American put	7.0322

5.0 Comparison & Discussion

When using the Monte Carlo simulation to acquire the option's price, the price computed in the previous parts shows that the put price of Asian and lookback is higher than the call price of Asian and lookback. Whereas the floating lookback put price is a little bit lower than that of the call price. With the use of Monte Carlo simulation, it contains the approach of the random walk, which would help reduce variance and make the price relatively stable when the number of simulations is large enough. However, when calculating the price of American put using Monte Carlo, this approach seems rather challenging to use based on the characteristics of the American put, which allows exercising early. This brings many difficulties in the coding and computational process.

When Binomial Lattice is used, as discussed in the previous section, although the factor numbers are large, the calculation process is not complicated, indicating a significant advantage of the binomial lattice approach, which is mathematically simple. On the other hand, it has the disadvantage that lattice must be drawn before performing the multiplication, which requires a further step in addition to computation.

Based on the result in previous sections, it is shown that the prices of all seven types of options using both Monte Carlo and Binomial Lattice generate similar results. This could indicate that both methods used are appropriate and could help with pricing options.

6.0 Lessons Learned

From this project, the two approaches to option valuation are learned. Although both the Monte Carlo simulation and Lattice methods can calculate the call and put options prices with slight variations, they have advantages and disadvantages. They might be better for a specific option's valuation, so it is necessary to determine which method to use. Also, we learned that under the same current price, strike price, risk-free rate, and maturity settings, different options' call and put prices have a considerable difference. The floating lookback call option has the highest price in the call options, while the lookback put option has the highest price in the put options. This difference is caused by the various properties of the option types.

References

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