This is a function to estimate the distance from a CY with a defining polynomial f, to the discriminant locus.
The method is to find the discrete solutions x_i of grad f=0 in each patch.
Then find the minimal normalized distance between one of these points and the manifold f=0, by constrained minimization of d(x,y)^2.
The first routine uses the Euclidean distance in the patch,
and the second one uses the distance in P^n, which is cos^{-1} |z_1 \cdot \bar z_2|/|z_1|/|z_2|.
The second one is better motivated but somewhat slower.

Note that we have to run this once for each independent patch. Often symmetry will relate the results in different patches in which case we need not run it on all the patches.
For the function f, symmetry relates all patches.
For f2, symmetry relates (z1,z2,z3) and (z4,z5), so we need two runs.

Complex parameters might work but they are too slow.

```
EstimateDistanceInCPN[ f_, patch_, vars_] := Module[ {eqs, gradzero},
  eqs = Map[# == 0 &, Grad[f /. patch → 1 , vars ]];
  gradzero = DeleteDuplicates[Solve[eqs, vars]];
  dmin[sol_] := Norm[Map[(# /. sol) - # &, vars]];
  MinimalBy[
   Map[NMinimize[{dmin[#], 0 == f /. patch → 1 }, vars] &, gradzero], #[[1]] &]
 ]
```

In[156]:=
```
EstimateTrueDistanceInCPN[ f_, patch_, vars_] := Module[ {eqs, gradzero},
  eqs = Map[# == 0 &, Grad[f /. patch → 1 , vars ]];
  gradzero = DeleteDuplicates[Solve[eqs, vars]];
  dmin[sol_] := Abs[1 + Total[Map[# Conjugate[# /. sol] &, vars]]]^2 /
      (1 + Norm[vars]^2) / (1 + Norm[Map[(# /. sol) &, vars]]^2);
  res = MaximalBy[
    Map[NMaximize[{dmin[#], 0 == f /. patch → 1 }, vars] &, gradzero], #[[1]] &];
  Map[{ArcCos[#[[1]]] / Pi, #[[2]]} &, res]
 ]
```

Note the nonstandard sign of psi. The conifold is psi=5 in these conventions.

In[29]:=
```
f = z1^5 + z2^5 + z3^5 + z4^5 + z5^5 - psi z1 z2 z3 z4 z5
```

Out[29]=
$$z1^5 + z2^5 + z3^5 + z4^5 - \text{psi}\ z1\ z2\ z3\ z4\ z5 + z5^5$$

In[131]:=
```
EstimateDistanceInCPN[f /. psi → 0, z1, {z2, z3, z4, z5}]
```

Out[131]=
$$\left\{\left\{1., \left\{z2 \to -1.05395 \times 10^{-8}, z3 \to -8.23809 \times 10^{-9}, z4 \to -1.25487 \times 10^{-8}, z5 \to -1.\right\}\right\}\right\}$$

In[129]:=
```
EstimateDistanceInCPN[f /. psi → -1, z1, {z2, z3, z4, z5}]
```

Out[129]=
$$\{\{0.801359, \{z2 \to -0.195074, z3 \to -0.195074, z4 \to -0.195074, z5 \to -1.00131\}\}\}$$

In[130]:= **EstimateDistanceInCPN[f /. psi → -2, z1, {z2, z3, z4, z5}]**

Out[130]= {{0.61794, {z2 → -1.01674, z3 → -0.377812, z4 → -0.377812, z5 → -0.377812}}}

In[132]:= **EstimateDistanceInCPN[f /. psi → -3, z1, {z2, z3, z4, z5}]**

Out[132]= {{0.474414, {z2 → -0.559842, z3 → -0.559842, z4 → -0.559842, z5 → -1.06929}}}

In[133]:= **EstimateDistanceInCPN[f /. psi → -4, z1, {z2, z3, z4, z5}]**

Out[133]= {{0.388713, {z2 → -0.7491, z3 → -0.7491, z4 → -0.7491, z5 → -1.17858}}}

In[134]:= **EstimateDistanceInCPN[f /. psi → -4.5, z1, {z2, z3, z4, z5}]**

Out[134]= {{0.367729, {z2 → -1.25548, z3 → -0.845649, z4 → -0.845649, z5 → -0.845649}}}

In[136]:= **EstimateDistanceInCPN[f /. psi → -4.9, z1, {z2, z3, z4, z5}]**

Out[136]= {{0.359813, {z2 → -0.92316, z3 → -1.32608, z4 → -0.92316, z5 → -0.92316}}}

In[135]:= **EstimateDistanceInCPN[f /. psi → -6, z1, {z2, z3, z4, z5}]**

Out[135]= {{0.367686, {z2 → -1.13585, z3 → -1.13585, z4 → -1.55049, z5 → -1.13585}}}

In[180]:= **EstimateDistanceInCPN[f /. psi → 0.5, z1, {z2, z3, z4, z5}]**

Out[180]= {{0.922771 + 0. i, {z2 → 0.100663, z3 → -0.0818742, z4 → 0.100663, z5 → -0.99992}},
{0.922771 + 0. i, {z2 → 0.100663, z3 → -0.0818742, z4 → 0.100663, z5 → -0.99992}}}

In[169]:= **EstimateDistanceInCPN[f /. psi → 4, z1, {z2, z3, z4, z5}]**

Out[169]= {{0.670926, {z2 → 0.792333, z3 → 0.792333, z4 → -0.709644, z5 → -0.709644}}}

In[170]:= **EstimateDistanceInCPN[f /. psi → 6, z1, {z2, z3, z4, z5}]**

Out[170]= $\left\{\left\{1., \left\{z2 \to -1., z3 \to -1.09275 \times 10^{-8}, z4 \to -1.07643 \times 10^{-8}, z5 \to -1.00248 \times 10^{-8}\right\}\right\}\right\}$

In[139]:= **EstimateDistanceInCPN[f /. psi → 0.1 I, z1, {z2, z3, z4, z5}]**

Out[139]= $Aborted

In[157]:= **EstimateTrueDistanceInCPN[f /. psi → 0, z1, {z2, z3, z4, z5}]**

Out[157]= $\left\{\left\{0.333333, \left\{z2 \to -1., z3 \to 1.03689 \times 10^{-9}, z4 \to 1.09149 \times 10^{-9}, z5 \to 2.60861 \times 10^{-11}\right\}\right\}\right\}$

In[160]:= **EstimateTrueDistanceInCPN[f /. psi → -1, z1, {z2, z3, z4, z5}]**

Out[160]= {{0.243162, {z2 → -0.3269, z3 → -0.3269, z4 → -0.3269, z5 → -1.00473}}}

In[161]:= **EstimateTrueDistanceInCPN[f /. psi → -2, z1, {z2, z3, z4, z5}]**

Out[161]= {{0.157268, {z2 → -1.03769, z3 → -0.568556, z4 → -0.568556, z5 → -0.568556}}}

In[162]:= **EstimateTrueDistanceInCPN[f /. psi → -3, z1, {z2, z3, z4, z5}]**

Out[162]= {{0.0996586, {z2 → -1.11583, z3 → -0.796405, z4 → -0.796405, z5 → -0.796405}}}

In[163]:= **EstimateTrueDistanceInCPN[f /. psi → -4, z1, {z2, z3, z4, z5}]**

Out[163]= {{0.0659886, {z2 → -1.08758, z3 → -1.22171, z4 → -1.08758, z5 → -1.08758}}}

In[164]:=
```
EstimateTrueDistanceInCPN[f /. psi → -4.5, z1, {z2, z3, z4, z5}]
```
Out[164]=
```
{{0.0547978, {z2 → -1.23312, z3 → -1.23312, z4 → -1.23312, z5 → -1.23312}}}
```

In[165]:=
```
EstimateTrueDistanceInCPN[f /. psi → -4.9, z1, {z2, z3, z4, z5}]
```
Out[165]=
```
{{0.0481844, {z2 → -1.30991, z3 → -1.30991, z4 → -1.30991, z5 → -1.30991}}}
```

In[166]:=
```
EstimateTrueDistanceInCPN[f /. psi → -5.5, z1, {z2, z3, z4, z5}]
```
Out[166]=
```
{{0.0410062, {z2 → -1.4341, z3 → -1.4341, z4 → -1.4341, z5 → -1.4341}}}
```

In[167]:=
```
EstimateTrueDistanceInCPN[f /. psi → -6, z1, {z2, z3, z4, z5}]
```
Out[167]=
```
{{0.036718, {z2 → -1.54399, z3 → -1.54399, z4 → -1.54399, z5 → -1.54399}}}
```

In[171]:=
```
EstimateTrueDistanceInCPN[f /. psi → 4, z1, {z2, z3, z4, z5}]
```
Out[171]=
```
{{0.161229, {z2 → -0.715237, z3 → 0.827135, z4 → -0.715237, z5 → 0.827135}},
 {0.161229, {z2 → -0.715237, z3 → 0.827135, z4 → -0.715237, z5 → 0.827135}}}
```

In[172]:=
```
EstimateTrueDistanceInCPN[f /. psi → 6, z1, {z2, z3, z4, z5}]
```
Out[172]=
```
{{0.0274821, {z2 → 1.44221, z3 → 1.44221, z4 → 1.44221, z5 → 1.44221}}}
```

In[176]:=
```
f2 = f + phi (z3 z4^4 + z3^2 z4^3 + z3^3 z4^2 + z3^4 z4)
```
Out[176]=
$$z1^5 + z2^5 + z3^5 + z4^5 + \text{phi}\left(z3^4\, z4 + z3^3\, z4^2 + z3^2\, z4^3 + z3\, z4^4\right) - \text{psi}\, z1\, z2\, z3\, z4\, z5 + z5^5$$

In[168]:=
```
f
```
Out[168]=
$$z1^5 + z2^5 + z3^5 + z4^5 - \text{psi}\, z1\, z2\, z3\, z4\, z5 + z5^5$$

In[178]:=
```
EstimateDistanceInCPN[f2 /. {psi → 0.5, phi → 1}, z1, {z2, z3, z4, z5}]
```
Out[178]=
```
{{0.901534 + 0. i, {z2 → -0.999766, z3 → 0.149416, z4 → -0.125699, z5 → 0.131492}},
 {0.901534 + 0. i, {z2 → -0.999766, z3 → 0.149416, z4 → -0.125699, z5 → 0.131492}}}
```

In[181]:=
```
EstimateDistanceInCPN[f2 /. {psi → 0.5, phi → 1}, z4, {z1, z2, z3, z5}]
```
Out[181]=
$$\left\{\left\{0.584779 + 0.\,i, \left\{z1 \to -6.43016 \times 10^{-9},\right.\right.\right.$$
$$\left.\left.z2 \to -6.61161 \times 10^{-9}, z3 \to -1., z5 \to -9.1173 \times 10^{-9}\right\}\right\}, \left\{0.584779 + 0.\,i,$$
$$\left.\left.\left\{z1 \to -6.43016 \times 10^{-9}, z2 \to -6.61161 \times 10^{-9}, z3 \to -1., z5 \to -9.1173 \times 10^{-9}\right\}\right\}\right\}$$

In[179]:=
```
EstimateDistanceInCPN[f2 /. {psi → 0.5, phi → 2}, z1, {z2, z3, z4, z5}]
```
Out[179]=
```
{{0.835949 + 0. i, {z2 → -0.99723, z3 → 0.369146, z4 → -0.335788, z5 → 0.245961}},
 {0.835949 + 0. i, {z2 → -0.99723, z3 → 0.369146, z4 → -0.335788, z5 → 0.245961}}}
```

In[182]:=
```
EstimateDistanceInCPN[f2 /. {psi → 0.5, phi → 2}, z4, {z1, z2, z3, z5}]
```
Out[182]=
$$\left\{\left\{0.357282 + 0.\,i, \left\{z1 \to -7.27525 \times 10^{-9},\right.\right.\right.$$
$$\left.\left.z2 \to -7.18925 \times 10^{-9}, z3 \to -1., z5 \to -7.02754 \times 10^{-9}\right\}\right\}, \left\{0.357282 + 0.\,i,$$
$$\left.\left.\left\{z1 \to -7.27525 \times 10^{-9}, z2 \to -7.18925 \times 10^{-9}, z3 \to -1., z5 \to -7.02754 \times 10^{-9}\right\}\right\}\right\}$$

In[187]:=
```
EstimateTrueDistanceInCPN[f2 /. {psi → 0.5, phi → 0}, z1, {z2, z3, z4, z5}]
```
Out[187]=
```
{{0.299683, {z2 → -0.999563, z3 → 0.185917, z4 → -0.147868, z5 → 0.185917}},
 {0.299683, {z2 → -0.999563, z3 → 0.185917, z4 → -0.147868, z5 → 0.185917}}}
```

In[188]:=
```
EstimateTrueDistanceInCPN[f2 /. {psi → 0.5, phi → 0}, z4, {z1, z2, z3, z5}]
```
Out[188]=
```
{{0.299683, {z1 → -0.999563, z2 → 0.185917, z3 → -0.147868, z5 → 0.185917}},
 {0.299683, {z1 → -0.999563, z2 → 0.185917, z3 → -0.147868, z5 → 0.185917}}}
```

In[183]:=
```
EstimateTrueDistanceInCPN[f2 /. {psi → 0.5, phi → 1}, z1, {z2, z3, z4, z5}]
```
Out[183]=
```
{{0.282629, {z2 → 0.206801, z3 → -0.746158, z4 → -0.643832, z5 → 0.206801}},
 {0.282629, {z2 → 0.206801, z3 → -0.746158, z4 → -0.643832, z5 → 0.206801}}}
```

In[184]:=
```
EstimateTrueDistanceInCPN[f2 /. {psi → 0.5, phi → 1}, z4, {z1, z2, z3, z5}]
```
Out[184]=
```
{{0.157952, {z1 → 8.05528 × 10^-10, z2 → 7.79577 × 10^-10, z3 → -1., z5 → -8.82207 × 10^-11}},
 {0.157952, {z1 → 8.05528 × 10^-10, z2 → 7.79577 × 10^-10, z3 → -1., z5 → -8.82207 × 10^-11}}}
```

In[185]:=
```
EstimateTrueDistanceInCPN[f2 /. {psi → 0.5, phi → 2}, z1, {z2, z3, z4, z5}]
```
Out[185]=
```
{{0.230639, {z2 → 0.376849, z3 → 0.375025, z4 → -1.14105, z5 → -0.299264}},
 {0.230639, {z2 → 0.376849, z3 → 0.375025, z4 → -1.14105, z5 → -0.299264}}}
```

In[186]:=
```
EstimateTrueDistanceInCPN[f2 /. {psi → 0.5, phi → 2}, z4, {z1, z2, z3, z5}]
```
Out[186]=
```
{{0.0853865, {z1 → -4.37508 × 10^-9, z2 → -6.02107 × 10^-9, z3 → -1., z5 → -6.83388 × 10^-9}},
 {0.0853865, {z1 → -4.37508 × 10^-9, z2 → -6.02107 × 10^-9, z3 → -1., z5 → -6.83388 × 10^-9}}}
```