Iterative Improvement

Algorithm design technique for solving optimization problems

- **Start with a feasible solution**
- **Repeat the following step until no improvement can be found:**
  - **change the current feasible solution to a feasible solution with a better value of the objective function**
- **Return the last feasible solution as optimal**

Note: Typically, a change in a current solution is "small" (*local search*)

Major difficulty: Local optimum vs. global optimum

Important Examples

- simplex method
- Ford-Fulkerson algorithm for maximum flow problem
- maximum matching of graph vertices
- Gale-Shapley algorithm for the stable marriage problem

- local search heuristics

Linear Programming

Linear programming (LP) problem is to optimize a linear function of several variables subject to linear constraints:

maximize (or minimize) $c_1 x_1 + ... + c_n x_n$

subject to $a_{i\,1} x_1 + ... + a_{in} x_n \le (\text{or} \ge \text{or} =)\ b_i,\ i = 1,...,m$

$x_1 \ge 0, \ldots , x_n \ge 0$

The function $z = c_1 x_1 + ... + c_n x_n$ is called the *objective function*;

constraints $x_1 \ge 0, \ldots , x_n \ge 0$ are called *nonnegativity constraints*

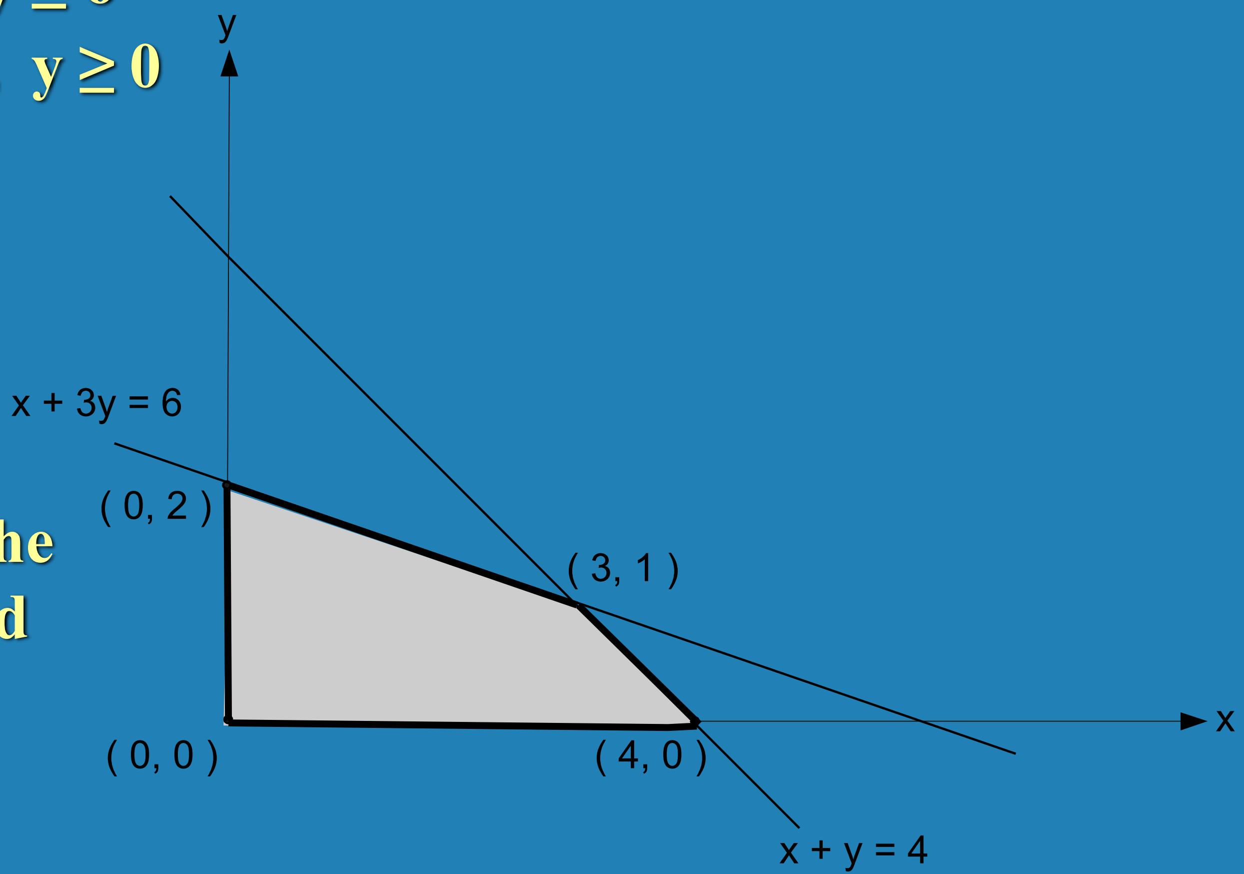
Example

maximize $3x + 5y$

subject to

$$x + y \le 4$$
$$x + 3y \le 6$$
$$x \ge 0, \; y \ge 0$$

Feasible region is the set of points defined by the constraints



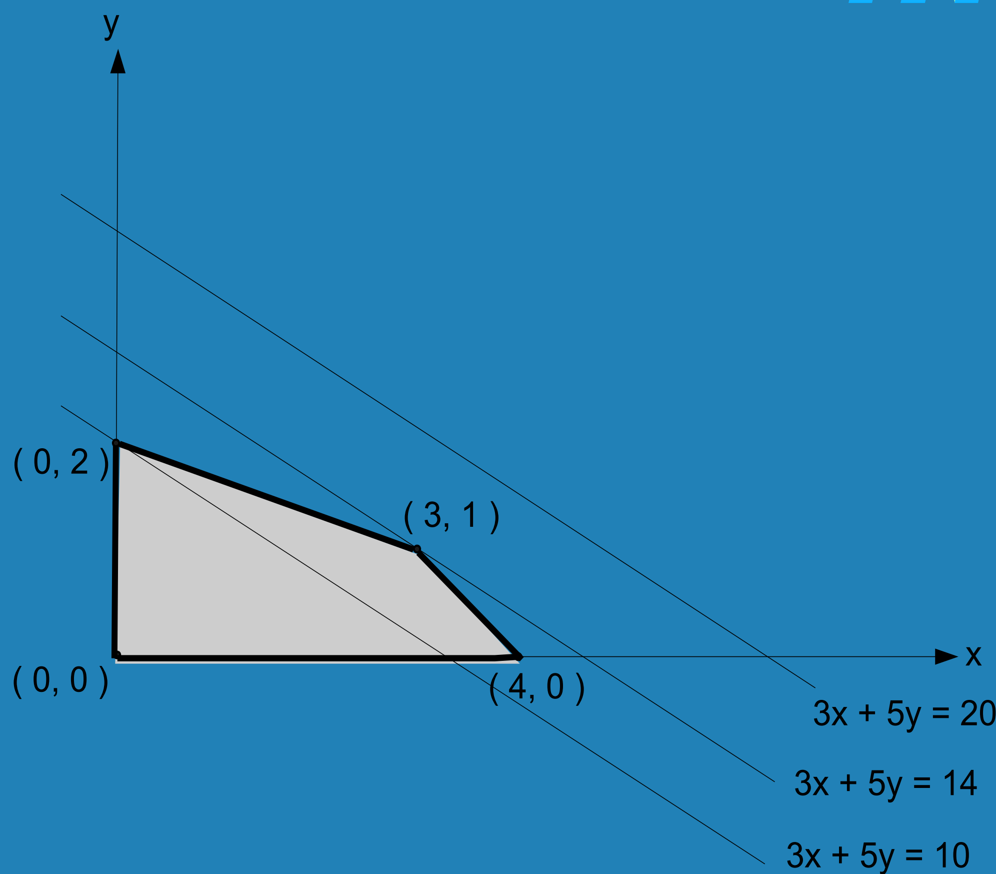
Geometric solution

maximize $\quad 3x + 5y$

subject to $\quad x + y \le 4$

$x + 3y \le 6$

$x \ge 0,\ y \ge 0$



Optimal solution: $x = 3$, $y = 1$

<u>Extreme Point Theorem</u> **Any LP problem with a nonempty bounded feasible region has an optimal solution; moreover, an optimal solution can always be found at an *extreme point* of the problem's feasible region.**

3 possible outcomes in solving an LP problem

- **has a finite optimal solution, which may no be unique**
- ***unbounded*: the objective function of maximization (minimization) LP problem is unbounded from above (below) on its feasible region**
- ***infeasible*: there are no points satisfying all the constraints, i.e. the constraints are contradictory**

The Simplex Method

- The classic method for solving LP problems;
  one of the most important algorithms ever invented

- Invented by George Dantzig in 1947

- Based on the iterative improvement idea:

  Generates a sequence of adjacent points of the problem's feasible region with improving values of the objective function until no further improvement is possible

Standard form of LP problem

- must be a maximization problem
- all constraints (except the nonnegativity constraints) must be in the form of linear equations
- all the variables must be required to be nonnegative

Thus, the general linear programming problem in standard form with m constraints and n unknowns ($n \geq m$) is

$$
\begin{aligned}
&\text{maximize} \quad c_1 x_1 + \ldots + c_n x_n \\
&\text{subject to} \quad a_{i\,1} x_1 + \ldots + a_{in} x_n = b_i, \quad i = 1,\ldots,m, \\
&\qquad\qquad\quad x_1 \geq 0, \ldots, x_n \geq 0
\end{aligned}
$$

Every LP problem can be represented in such form

Example

maximize $3x + 5y$

subject to

$$x + y \le 4$$
$$x + 3y \le 6$$
$$x \ge 0, \quad y \ge 0$$

$\Rightarrow$

maximize $3x + 5y + 0u + 0v$

subject to

$$x + y + u = 4$$
$$x + 3y + v = 6$$
$$x \ge 0, \quad y \ge 0, \quad u \ge 0, \quad v \ge 0$$

Variables *u* and *v*, transforming inequality constraints into equality constrains, are called *slack variables*

Basic feasible solutions

A *basic solution* to a system of m linear equations in n unknowns ($n \geq m$) is obtained by setting $n - m$ variables to 0 and solving the resulting system to get the values of the other m variables. The variables set to 0 are called *nonbasic*; the variables obtained by solving the system are called *basic*.

A basic solution is called *feasible* if all its (basic) variables are nonnegative.

Example

$$\begin{aligned} x + y + u \quad\quad &= 4 \\ x + 3y \quad\quad + v &= 6 \end{aligned}$$

(0, 0, 4, 6) is basic feasible solution

(x, y are nonbasic; u, v are basic)

There is a 1-1 correspondence between extreme points of LP's feasible region and its basic feasible solutions.

Simplex Tableau

maximize $z = 3x + 5y + 0u + 0v$

subject to

$$x + y + u = 4$$

$$x + 3y + v = 6$$

$$x \ge 0,\ y \ge 0,\ u \ge 0,\ v \ge 0$$

basic variables

objective row

	x	y	u	v	
u	1	1	1	0	4
v	1	3	0	1	6
	−3	−5	0	0	0

basic feasible solution

(0, 0, 4, 6)

value of z at (0, 0, 4, 6)

Outline of the Simplex Method

Step 0 [Initialization] Present a given LP problem in standard form and set up initial tableau.

Step 1 [Optimality test] If all entries in the objective row are nonnegative — stop: the tableau represents an optimal solution.

Step 2 [Find entering variable] Select (the most) negative entry in the objective row. Mark its column to indicate the entering variable and the pivot column.

Step 3 [Find departing variable] For each positive entry in the pivot column, calculate the θ-ratio by dividing that row's entry in the rightmost column by its entry in the pivot column. (If there are no positive entries in the pivot column — stop: the problem is unbounded.) Find the row with the smallest θ-ratio, mark this row to indicate the departing variable and the pivot row.

Step 4 [Form the next tableau] Divide all the entries in the pivot row by its entry in the pivot column. Subtract from each of the other rows, including the objective row, the new pivot row multiplied by the entry in the pivot column of the row in question. Replace the label of the pivot row by the variable's name of the pivot column and go back to Step 1.

Example of Simplex Method Application

maximize $z = 3x + 5y + 0u + 0v$

subject to

$$x + y + u = 4$$

$$x + 3y + v = 6$$

$$x \geq 0,\ y \geq 0,\ u \geq 0,\ v \geq 0$$

	x	y	u	v	
u	1	1	1	0	4
v	1	3	0	1	6
	−3	−5	0	0	0

	x	y	u	v	
u	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	2
y	$\frac{1}{3}$	1	0	$\frac{1}{3}$	2
	$-\frac{4}{3}$	0	0	$\frac{5}{3}$	10

	x	y	u	v	
x	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	3
y	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	1
	0	0	2	1	14

basic feasible sol. (0, 0, 4, 6)

$z = 0$

basic feasible sol. (0, 2, 2, 0)

$z = 10$

basic feasible sol. (3, 1, 0, 0)

$z = 14$

Notes on the Simplex Method

- Finding an initial basic feasible solution may pose a problem
- Theoretical possibility of cycling
- Typical number of iterations is between *m* and 3*m*, where *m* is the number of equality constraints in the standard form
- Worse-case efficiency is exponential
- More recent *interior-point algorithms* such as *Karmarkar's algorithm* (1984) have polynomial worst-case efficiency and have performed competitively with the simplex method in empirical tests

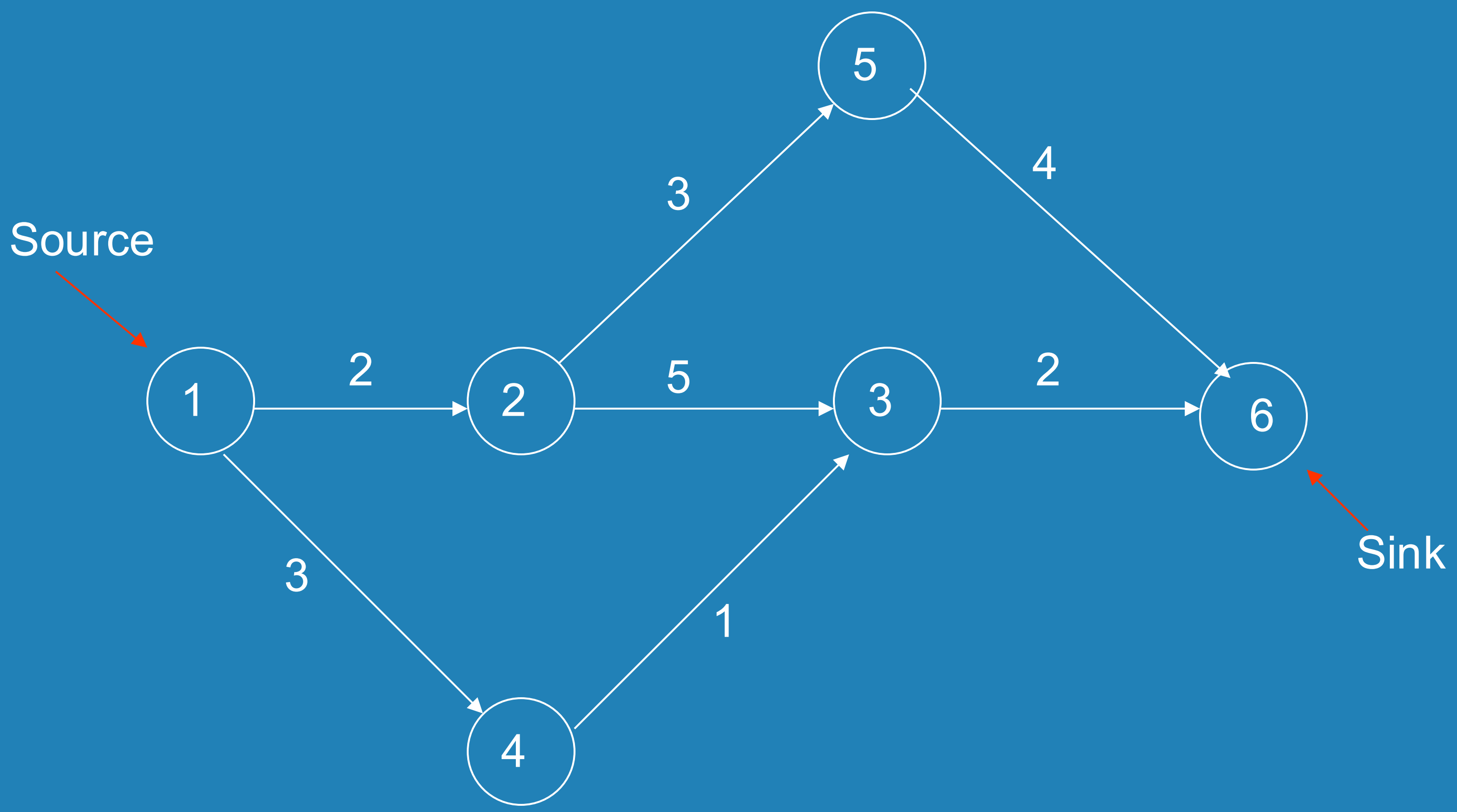
Maximum Flow Problem

Problem of maximizing the flow of a material through a transportation network (e.g., pipeline system, communications or transportation networks)

Formally represented by a connected weighted digraph with n vertices numbered from 1 to n with the following properties:

- contains exactly one vertex with no entering edges, called the *source* (numbered 1)
- contains exactly one vertex with no leaving edges, called the *sink* (numbered n)
- has positive integer weight u_{ij} on each directed edge $(i.j)$, called the *edge capacity*, indicating the upper bound on the amount of the material that can be sent from i to j through this edge

Example of Flow Network



Definition of a Flow

A *flow* is an assignment of real numbers x_{ij} to edges (i,j) of a given network that satisfy the following:

- ***flow-conservation requirements***

  The total amount of material entering an intermediate vertex must be equal to the total amount of the material leaving the vertex

$$\sum_{j:\,(j,i)\,\in\,E} x_{ji} = \sum_{j:\,(i,j)\,\in\,E} x_{ij} \quad \text{for } i = 2,3,\ldots, n\text{-}1$$

- ***capacity constraints***

$$0 \le x_{ij} \le u_{ij} \quad \text{for every edge } (i,j) \in E$$

Flow value and Maximum Flow Problem

Since no material can be lost or added to by going through intermediate vertices of the network, the total amount of the material leaving the source must end up at the sink:

$$\sum_{j:\,(1,j)\,\in\,E} x_{1j} = \sum_{j:\,(j,n)\,\in\,E} x_{jn}$$

The *value* of the flow is defined as the total outflow from the source (= the total inflow into the sink).

The *maximum flow problem* is to find a flow of the largest value (maximum flow) for a given network.

Maximum-Flow Problem as LP problem

Maximize $v = \sum_{j:\,(1,j) \in E} x_{1j}$

subject to

$$\sum_{j:\,(j,i) \in E} x_{ji} - \sum_{j:\,(i,j) \in E} x_{ij} = 0 \quad \text{for } i = 2, 3, \ldots, n-1$$

$$0 \le x_{ij} \le u_{ij} \quad \text{for every edge } (i,j) \in E$$