$SU(3)$ Introduction

1 Introduction

By definition, $SU(3) = \{U|U^\dagger U = \mathbb{1}, \det U = 1\}$, where U are 3x3 matrices. We can write the group elements U in the form $U = e^{-i\theta_j T_j}$, where T_j are called the generators of the group. Each of the above restriction on the group imposes a condition on these generators. The special property requires that the generators have a zero trace. The unitarity property requires that the generators are Hermitian. There are 8 linearly independent Hermitian, traceless matrices that satisfy these conditions. They are given as the Gell-Mann matrices.

$$ \tag{1} $$

These matrices are orthogonal because they are linearly independent, but we need a definition for what orthogonal means. For example, for normal vectors $\vec{v}$ and $\vec{u}$, we say that they are orthogonal if their dot product $\vec{v} \cdot \vec{u} = 0$. For these matrices, we will define the analog of the dot product as the trace of pairwise products $\mathrm{Tr}[\lambda_i \lambda_j] = 2\delta_{ij}$. The δ_{ij} shows that they are orthogonal, though the factor of 2 shows that they are not normalized in the normal sense that we expect elements to be normalized to 1. However, we have the freedom to normalize things to whatever we want, and having to carry around an extra factor of $\frac{1}{\sqrt{2}}$ is inconvienient enough that this is the preferred way.

However, with all that being said, there is a more common basis for the generators than the Gell-Mann matrices. We will define these as $T_i \equiv \frac{\lambda_i}{2}$. These new generators are normalized to a value of $\frac{1}{2}$, meaning $\mathrm{Tr}[T_i T_j] = \frac{1}{2}\delta_{ij}$. This is analogous to how the spin-$\frac{1}{2}$ operators are defined as $S_i = \frac{1}{2}\sigma_i$.

Before we move on, I want you to notice a few things about the Gell-Mann matrices. First, λ_1, λ_2, and λ_3 all have the Pauli matrices in the upper-left hand corner. Secondly, λ_4 and λ_5 look similar to σ_1 and σ_2 as if the matrices were spread between rows/columns 1 and 3. Similarly, λ_6 and λ_7 are similar with the σ_1 and σ_2 split between rows and columns 2 and 3. Lastly, the only two diagonal matrices are λ_3 and λ_8.

2 Eigenstates

As mentioned in the motivations for $SU(3)$, we are seeking to describe an isospin symmetry (and potentially others) between all the new particles found during the particle zoo era. Thus it makes sense to search for eigenstates describing this system. If our system exhibits $SU(3)$ symmetry, then its Hilbert space will be described by the vector space of irreducible representations of $SU(3)$. So we seek to find a set of basis states for the irreducible representations of $SU(3)$.

For now, we will focus on the fundamental representation of $SU(3)$, meaning the 3-dimensional representation. Like we said earlier, there are 2 diagonal Gell-Mann matrices: λ_3 and λ_8. It seems natural that we take the eigenstates of these to be the eigenstates of the system. However, we will make one small adjustment. We will define the hypercharge operator to be $Y \equiv \frac{2}{\sqrt{3}} T_8$.

$$Y = \frac{1}{3} \tag{2}$$

Thus, our eigenstates can be labelled by eigenvalues of these two matrices, which we shall call t_3 and y, making our states $|t_3, y\rangle$. However, we have 3 additional operators that form a complete set of commutating operators. The first 2 are 2 Casimir operators, which by definition commute with every generator. They are not important, but shown below just for interest. (Describe cubic Casimir)

$$C_1 = \sum_{i=1}^{8} T_i^2 \tag{3}$$

$$C_2 = \sum_{ijk} d_{ijk} F_i F_j F_k \tag{4}$$

The last commuting operator is $T_1^2 + T_2^2 + T_3^2$. It may be a bit surprising that this shows up in addition to the Casimir operator C_1. However, remember that λ_1, λ_2, and λ_3 form an $SU(2)$ subgroup. When you studied spin, you labelled eigenstates by eigenvalues of the angular momentum operators J^2 and J_z. It is the same concept here: we are using T_3 to label the eigenstates, so we also need the analog of J^2.

This gives us a form for our fulling classified and unique eigenstates. Here I have abused notation for the last 3 elements by labelling the eigenvalues the same as what I labelled the operators.

$$|t_3, y, I^2, C_1, C_2\rangle \tag{5}$$

Now that we have gone through the work of finding the fully qualified eigenstates, we are going to throw away most of it and just label our eigenstates by t_3 and y. The reason to ignore the Casimirs is because they commute with every generator and so will have the same value for all states in a given represnetation. Their eigenvalues will vary between representations, but we will be clear when talking about which specific representation we will be using.

We will ignore the I^2 eigenvalue for a similar reason. If we were looking at $SU(2)$, I^2 would be a Casimir and could be ignored for the reasons above. The different representations could be fully determined by the maximum value of t_3. We will continue to use the maximum weight analysis here, making I^2 unecessary.

I think perhaps make all the discussion above an aside and keep the happy path the most important stuff, i.e. talking about the (1,0,0), (0,1,0) etc quark states and their eigenvalues.

3 Fundamental Representation

4 Conjugate Representation