Particle Physics HW 7

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December 5, 2024

Question 21.1. Consider the decay $K \to 2\pi$.

(a) What is the orbital angular momentum of the pions?

Solution. We as always work in the rest frame of the decaying particle, meaning it has no orbital angular momentum. The K is a spin-0 particle, so it has total angular momentum 0. The pions are also spin-0, so they cannot have any orbital angular momentum if the total angular momentum must also be 0.

$$\boxed{L_\pi = 0}$$

□

(b) What are the allowed isospin states of the two pions?

Solution. As pions are bosons, their state must be symmetric. The symmetry for Clebsch Gordon coefficients is given by the equation below, where I_1 and I_1 are the isospin of the pions and I is the isospin of the coupled state.

$$\text{Symmetry: } (-1)^{I_1+I_2-I}$$

Now, pions form an isospin triplet, so $I_1 = I_2 = 1$. Thus,

$$\text{Symmetry: } (-1)^{2-I}$$

We see that we need I to be an even number, i.e. $I = 0, 2$ but not 1. We can also see this from the Clebsch-Gordon table for combining two spin-1 particles. The only states that are symmetric are $I = 0$ and $I = 2$.

$$\boxed{I = 0, 2}$$

□

(c) Is $K^+ \to \pi^+\pi^0$ allowed in weak decays, assuming only $|\Delta I| \leq 1$ is allowed?

Solution. The π^+ has $I_3 = +1$, while the π^0 has $I_3 = 0$. Looking at the Clebsch-Gordon table, we see that there are two possible states.

$$|2,0\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle) = \frac{1}{\sqrt{2}}(|\pi^+\pi^0\rangle + |\pi^0\pi^+\rangle)$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle) = \frac{1}{\sqrt{2}}(|\pi^+\pi^0\rangle - |\pi^0\pi^+\rangle)$$

However, we said that we need $I = 0$ or $I = 2$, so only $I = 2$ is valid here. Since $I = \frac{1}{2}$ for a K, we have $|\Delta I| = \frac{3}{2}$, so this is not allowed.

$$\boxed{|\Delta I| = \frac{3}{2}, \text{ so this process is forbidden}}$$

□

(d) Given the lifetimes and branching ratios for $K^+ \to \pi^+\pi^0$ and $K^0 \to \pi^+\pi^-$, calculate the ratio of the amplitudes for $|\Delta I| = \frac{3}{2}$ and $|\Delta I| = \frac{1}{2}$.

Solution. We seek a ratio of amplitudes $\langle f| S |i\rangle$ for these processes. The amplitudes are related to the partial decay rates. Here, ϕ is the phase space integral.

$$\Gamma(i \to f) = \phi| \langle f| S |i\rangle |^2$$

Rearranging, we find the amplitude in terms of the partial decay rate.

$$\langle f| S |i\rangle = \sqrt{\frac{\Gamma(i \to f)}{\phi}}$$

The partial decay rate is found from the branching ratio and total decay rate.

$$\Gamma(i \to f) = \mathrm{BR}(i \to f)\Gamma(i)$$

Lastly, because $\Gamma(i) = \tau_i^{-1}$, we find the amplitude in terms of what we are given: the branching ratio and the lifetime.

$$\langle f| S |i\rangle = \sqrt{\frac{\mathrm{BR}(i \to f)}{\tau_i \phi}}$$

We saw from part (c) that the process $K^+ \to \pi^+\pi^0$ has $|\Delta I| = \frac{3}{2}$. So now let's look at $K^0 \to \pi^+\pi^-$. The π^- has $I_3 = -1$. From the Clebsch-Gordon tables, we see that we have one possible state that is symmetric.

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle) = \\
&\qquad \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{6}}|2,0\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{3}}|0,0\rangle + \frac{1}{\sqrt{6}}|2,0\rangle - \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{3}}|0,0\rangle) \\
&\qquad = \frac{1}{\sqrt{3}}|2,0\rangle + \sqrt{\frac{2}{3}}|0,0\rangle
\end{aligned}$$

We see that it has a $|\Delta I| = \frac{3}{2}$ and $|\Delta I| = \frac{1}{2}$ component. We know that the $|\Delta I| = \frac{1}{2}$ will dominate, so we can suppress the other one.

Alright, we can put it all together now.

$$\left|\Delta I = \frac{3}{2}\right| \to \langle \pi^+\pi^0 | \, S \, | K^+ \rangle = \sqrt{\frac{\text{BR}(K^+ \to \pi^+\pi^0)}{\tau_{K^+}\phi}}$$

$$\left|\Delta I = \frac{1}{2}\right| \to \langle \pi^+\pi^- | \, S \, | K^0 \rangle = \sqrt{\frac{2}{3}} \langle 0,0 | \, S \, | K^0 \rangle$$
$$\langle 0,0 | \, S \, | K^0 \rangle = \sqrt{\frac{3\text{BR}(K^0 \to \pi^+\pi^-)}{2\tau_{K^0}\phi}}$$

Now we take the ratio of these.

$$R = \frac{\langle \pi^+\pi^0 | \, S \, | K^+ \rangle}{\langle 0,0 | \, S \, | K^0 \rangle} = \sqrt{\frac{\text{BR}(K^+ \to \pi^+\pi^0)}{\tau_{K^+}\phi}} \Big/ \sqrt{\frac{3\text{BR}(K^0 \to \pi^+\pi^-)}{2\tau_{K^0}\phi}}$$
$$= \sqrt{\frac{2\tau_{K^0}\text{BR}(K^+ \to \pi^+\pi^0)}{3\tau_{K^+}\text{BR}(K^0 \to \pi^+\pi^-)}}$$

And we plug in the numbers given.

$$\boxed{R = 0.038}$$

□

Question 21.2. Find the mass of the Δ^{++} and K^{*+} from the Dalitz plot given.

Solution. This is essentially a relativistic collision problem. We have an incoming particle at a stationary target in the lab frame, we will describe this collision by $12 \to 34$.

We are given a few things: $E_{1,\text{lab}}$, m_1, m_2, m_4, and $E_{4,\text{CM}}$, and we seek m_3. We will solve this by considering 4-momentum conservation in both frames.

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$$

In the CM frame, we have $\vec{p}_3 = -\vec{p}_4$. Thus,

$$\vec{p}_3^{\,2} = \vec{p}_4^{\,2} \to E_3^2 - m_3^2 = E_4^2 - m_4^2$$

We know everything except E_3. To get that, we'll consider conservation of energy.

$$E_3 + E_4 = E_{CM} = \sqrt{s}$$

Meanwhile,

$$s = (P^\mu_{1,\text{lab}} + P^\mu_{2,\text{lab}})^2 = P_1^2 + P_2^2 + 2P_1P_2 = m_1^2 + m_2^2 + 2E_{1,\text{lab}}m_2$$

Putting it all together,

$$(\sqrt{s} - E_4)^2 - m_3^2 = E_4^2 - m_4^2$$

Or,

$$m_3^2 = s + m_4^2 - 2E_4\sqrt{s}$$

Now we can plug in the numbers for the two processes. For $K^+p \to K^0\Delta^{++}$, we have $m_4 = 0.498$ GeV and $E_4 = 1.0549$ GeV. Our center of mass energy s is $s = m_p^2 + m_{K^+}^2 + 2E_{1,\text{lab}}m_p = 6.75$ GeV. This gives us $m_{\Delta^{++}} = 1.23$ GeV.

For the $K^+p \to K^{*+}p$, we have $m_4 = 0.938$ GeV and $E_4 = 1.3154$. This gives us $m_{K^{*+}} = 0.89$ GeV.

In summary,

$$\boxed{m_{\Delta^{++}} = 1.23 \text{ GeV} \qquad m_{K^{*+}} = 0.89 \text{ GeV}}$$

□

Question 22.1. Find the generators of $SU(2)$ in the $I = \frac{3}{2}$ representation. Show that these matrices satisfy the $SU(2)$ Lie algebra.

Solution. Because the states $|t_3\rangle$ are labelled by eigenvalues of T_3, finding that matrix is trivial.

$$\boxed{T_3 = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}}$$

Next, we can find $T_\pm$ easily since $(T_\pm)_{t_3 t_3'} = \sqrt{I(I+1) - t_3(t_3 \pm 1)}\delta_{t_3, t_3' \pm 1}$. We are in the $I = \frac{3}{2}$ representation.

$$T_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

We can use these to get T_1 and T_2 since $T_\pm = T_1 \pm iT_2$.

$$\begin{aligned} T_1 &= \frac{1}{2}(T_+ + T_-) \\ T_2 &= \frac{1}{2i}(T_+ - T_-) = \frac{i}{2}(T_- - T_+) \end{aligned}$$

Plugging in our matrices for T_+ and T_- gives us what we want.

$$\boxed{T_1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}}$$

$$\boxed{T_2 = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}}$$

Now we want to check the commutation relations of these matrices without explicitly computing them.

$$\begin{aligned} [T_i, T_j]_{t_3 t_3'} &= \sum_x \langle t_3| T_i |x\rangle \langle x| t_j |t_3'\rangle - \langle t_3| T_j |x\rangle \langle x| t_i |t_3'\rangle \\ &= \langle t_3| T_i \mathbb{1} T_j - T_j \mathbb{1} T_i |t_3'\rangle \\ &= \langle t_3| [T_i, T_j] |t_3'\rangle \end{aligned}$$

We know the generators of the group follow $[T_i, T_j] = i\epsilon_{ijk}T_k$, so we plug that in.

$$[T_i, T_j]_{t_3 t_3'} = \langle t_3| \, i\epsilon ijk T_k \, |t_3'\rangle = i\epsilon_{ijk}(T_k)_{t_3 t_3'}$$

Or,

$$\boxed{[T_i, T_j] = i\epsilon_{ijk}T_k}$$

$\square$

Question 23.1. Given the generators of $SU(3)$ satisfy the Lie algebra $[T_a, T_b] = if_{abc}T_c$, answer the following questions.

(a) Show the conjugate representation generators $\overline{T}_a \equiv -T_a^T$ satisfy the $SU(3)$ algebra.

Solution. This is straightforward from just computing the commutators.

$$\begin{aligned}[\overline{T}_a, \overline{T}_b] &= [-T_a^T, -T_b^T] = T_a^T T_b^T - T_b^T T_a^T = -[T_a, T_b]^T \\ &= -if_{abc}T_c^T = if_{abc}(-T_c^T) = if_{abc}\overline{T}_c\end{aligned}$$

We see it follows the same algebra.

$$\boxed{[\overline{T}_a, \overline{T}_b] = \overline{T}_c}$$

□

(b) Show the adjoint representation generators $(F_a)_{bc} \equiv -if_{abc}$ satisfy the $SU(3)$ algebra using the Jacobi identity.

Solution. We start by plugging in the commutation relations to the Jacobi identity. I'll do it term by term because it will get messy.

$$\begin{aligned}[[T_a, T_b], T_c] &= if_{abx}[T_x, T_c] = -f_{abx}f_{xcz} \\ [[T_b, T_c], T_a] &= if_{bcx}[T_x, T_a] = -f_{bcx}f_{xaz} \\ [[T_c, T_a], T_b] &= if_{cax}[T_x, T_b] = -f_{cax}f_{xbz}\end{aligned}$$

Putting it all together,

$$f_{abx}f_{xcz} + f_{bcx}f_{xaz} + f_{cax}f_{xbz} = 0$$

Now we do a bunch of index shuffling until we get the form we want. First we move the left term to the RHS of the equation. We also make the substitution $c \to x$ so that x and z are the two free indices on both sides.

$$f_{bxy}f_{yaz} + f_{xay}f_{ybz} = -f_{abc}f_{cxz}$$

The reason we do this is now we can write the RHS as $-f_{abc}f_{cxz}{=}{-}if_{abc}(F_c)_{xz}$. We can start to see the Lie algebra develop. Now on the LHS we want to swap around the first two indices for a few terms. Doing this gives us $f_{abc} = -fbac$.

$$-f_{bxy}f_{ayz} + f_{axy}f_{byz} = (-if_{bxy})(-if_{ayz}) - (-if_{axy})(-if_{byz}) = -[F_a, F_b]_{xz}$$

Putting it all together, the minus sign cancels and we get exactly what we want

$$\boxed{[F_a, F_b] = if_{abc}F_c}$$

□

(c) Find all states in the multiplet with the highest weight state of $t_3 = \frac{3}{2}$ and $y = 1$. Draw the weight diagram with the charge of each state.

Solution. Since this is the highest weight state, it is clear to see that this forms an isospin quadruplet. We also see there are three generators that will yield 0.

$$T_+ \left|\frac{3}{2}, 1\right\rangle = V_+ \left|\frac{3}{2}, 1\right\rangle = U_- \left|\frac{3}{2}, 1\right\rangle = 0$$

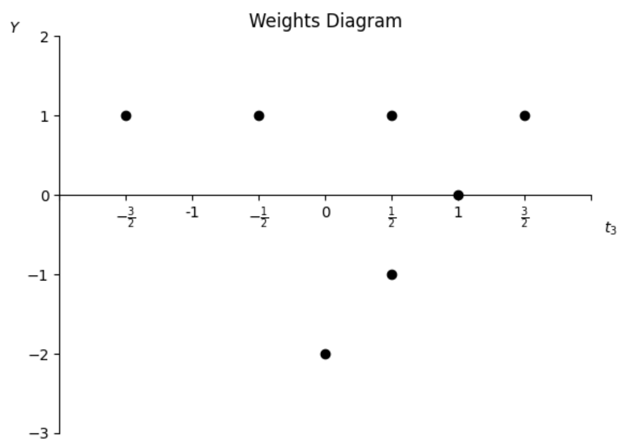
We now seek to see what sort of U-spin and V-spin multiplet this state is a part of. We will start with U-spin.

$$U_3 \left|\frac{3}{2}, 1\right\rangle = \left(\frac{3}{4}Y - \frac{1}{2}T_3\right) \left|\frac{3}{2}, 1\right\rangle = 0$$

Since this is the lowest U-spin weight state of this multiplet, we determine that it forms a singlet. Now we check V-spin.

$$V_3 \left|\frac{3}{2}, 1\right\rangle = \left(\frac{3}{4}Y + \frac{1}{2}T_3\right) \left|\frac{3}{2}, 1\right\rangle = \frac{3}{2}$$

This is the highest V-spin weight state, so this must be a quadruplet. Thus, we can start to fill out the weight diagram with what we know currently.



Now let's check the isospin of the state $|1, 0\rangle$, specifically if it is the highest weight state.

$$T_+ |1, 0\rangle = T_+V_+ \left|\frac{3}{2}, 1\right\rangle = [T_+, V_+] \left|\frac{3}{2}, 1\right\rangle = 0$$

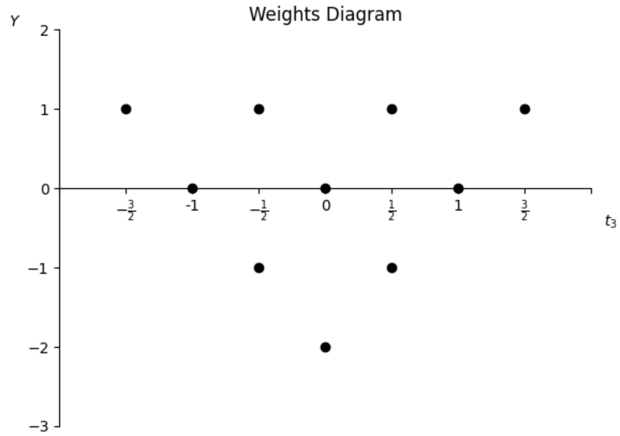
Thus, this is a highest weight state, and it forms an isospin triplet. Lastly, we will look at the U-spin of the state $|0, -2\rangle$.

$$U_- |0, -2\rangle = U_-V_+^3 \left|\frac{3}{2}, 1\right\rangle = [U_-, V_+^3] \left|\frac{3}{2}, 1\right\rangle = 0$$

Thus, $|0, -2\rangle$ is the lowest weight state of a U-spin multiplet. We can see that it is a quadruplet.

$$U_3 |0, -2\rangle = \left(\frac{3}{4}Y - \frac{1}{2}T_3\right) |0, -2\rangle = -\frac{3}{2}$$

So our weight diagram is (almost) fully filled out.



Our last task is to check the uniqueness of the middle state. Let's assume the middle state is a superposition of two states, $|a\rangle$ and $|b\rangle$. Then we will approach it from two directions.

$$V_- \left|\frac{1}{2}, 1\right\rangle = a\,|a\rangle + b\,|b\rangle$$

$$U_+ \left|\frac{1}{2}, -1\right\rangle = a'\,|a\rangle + b'\,|b\rangle$$

We can find the norm of these states.

$$\left\langle\frac{1}{2}, 1\right| V_+V_- \left|\frac{1}{2}, 1\right\rangle = \left\langle\frac{1}{2}, 1\right| [V_+, V_-] \left|\frac{1}{2}, 1\right\rangle = 2\left\langle\frac{1}{2}, 1\right| V_3 \left|\frac{1}{2}, 1\right\rangle = 2$$

$$\left\langle\frac{1}{2}, -1\right| U_-U_+ \left|\frac{1}{2}, -1\right\rangle = \left\langle\frac{1}{2}, -1\right| [U_-, U_+] \left|\frac{1}{2}, -1\right\rangle = 2\left\langle\frac{1}{2}, -1\right| U_3 \left|\frac{1}{2}, -1\right\rangle = 2$$

Thus, $a^2 + b^2 = a'^2 + b'^2 = 2$. Now we notice that $[U_+, V_-] = T_-$, so $[U_+, V_-] = T_- = 0$.

$$\begin{aligned}(U_+V_- - V_-U_+ - T_-)\,|1, 0\rangle &= 0\\ 2U_+ \left|\frac{1}{2}, -1\right\rangle - V_- \left|\frac{1}{2}, 1\right\rangle - \sqrt{2}\,|a\rangle &= 0\\ 2(a'\,|a\rangle + b'\,|b\rangle) - (a\,|a\rangle + b\,|b\rangle) - \sqrt{2}\,|a\rangle &= 0\\ (2a' - a - \sqrt{2})\,|a\rangle + (b' - b)\,|b\rangle &= 0\end{aligned}$$

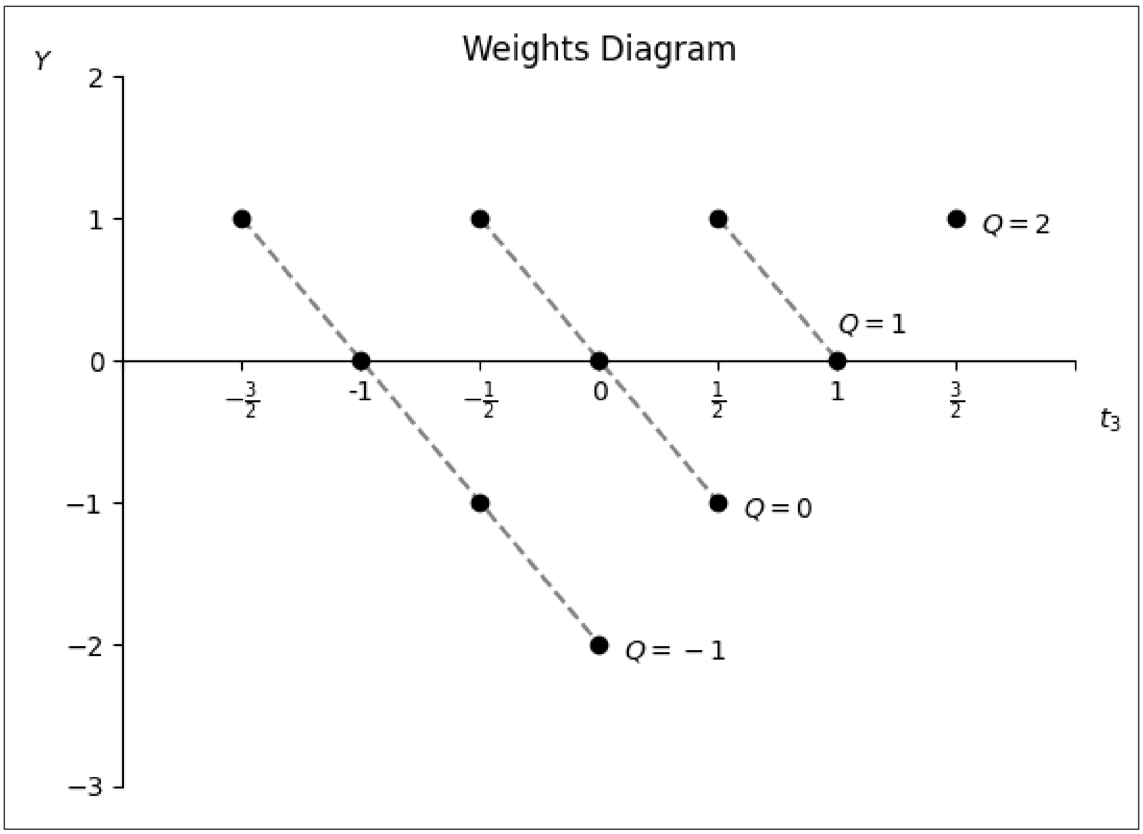
We take $|a\rangle$ and $|b\rangle$ to be orthogonal, so this requires $2a' - a - \sqrt{2} = 0$ and $b' - b = 0$. Solving these 4 equations yields us the coefficients.

$$b' = b = 0 \qquad a' = a = \sqrt{2}$$

Thus, we see that the central state is unique.

The last thing we need to do is write the charge for each state with $Q = T_3 + \frac{Y}{2}$. You can see that this equation defines a line $Y = 2(Q - T_3)$, i.e. lines with negative slope have a constant charge. So, we can label the charges of each state. See the diagram below.

□


Weights Diagram
Y
2
1
0
−1
−2
−3
$-\frac{3}{2}$
-1
$-\frac{1}{2}$
0
$\frac{1}{2}$
1
$\frac{3}{2}$
t_3
$Q=2$
$Q=1$
$Q=0$
$Q=-1$