University of São Paulo
Institute of Mathematics and Statistics
Bachelor of Applied Mathematics

Combinatorial methods in Banach space theory

Lucas Nunes Fernandes Teles

Undergraduate Thesis

MAP2080 - Graduation Project

Supervisor: Prof. Dr. Valentin Raphael Henri Ferenczi

São Paulo
2024

À memória das minhas avós, Marta e Nicinha.

Acknowledgements

To mention all those who inspired and helped get where I am today would lead to a reckless increase in the number of pages of this document, so I limit myself to only mentioning the people whose influence shaped the very words that came to be on this essay. I must begin by thanking professor Ferenczi, for his expert guidance and wisdom which motivate me greatly to pursue a career in the study of Banach spaces. I shall also extend my gratitude to my two previous supervisors, the first being professor Kawano, who first let me glimpse at the wonders of mathematical research and introduced me to functional analysis. The second being professor Goloshchapova, whose kind mentoring allowed me to explore the wonderful field of spectral theory while developing my writing and exposition skills. Without them I would have been hopeless in understanding any of this work.

I am also deeply thankful to Maurício, João, Dani, Bia, Arthur, André and Luigi, for their great insights, wonderful proof checking abilities, but most of all, for being great friends. And, for helping me remain with whatever is left of my sanity, I thank Melissa, who I'm proud to call my best friend.

Lastly, I dedicate my kindest words to those who are responsible for me being who I am in all positive matters: my family. How far I have come is directly a product of their efforts, and though the physical distance between us leaves much to, their support and love never fails to reach me. Especially, obrigado Mãe e Pai.

Resumo

Lucas Nunes Fernandes Teles. **Métodos combinatórios na teoria dos espaços de Banach**. Monografia (Bacharelado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2024.

Neste trabalho exploramos tópicos de análise funcional a fim de introduzir a aplicação de métodos de combinatória infinita na teoria dos espaços de Banach. Partimos de uma introdução às versões finita e infinita do teorema de Ramsey e apresentamos o teorema de Galvin-Prikry como um resultado de tipo Ramsey na teoria descritiva dos conjuntos, no processo estudando a topologia de Ellentuck e a propriedade de Baire.

Por fim, usamos do teorema de Ramsey e de conceitos da análise funcional, como finita representavidade, autovalores aproximados e ultrapotências para provar o teorema de Krivine como um resultado de caráter combinatório na teoria dos espaços de Banach.

Palavras-chave: Espaços de Banach. Teoria de Ramsey. Teorema de Krivine.

Abstract

Lucas Nunes Fernandes Teles. **Combinatorial methods in Banach space theory**. Undergraduate thesis (Bachelor). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2024.

In this essay we explore results in functional analysis in order to introduce the application of methods from infinite combinatorics in Banach space theory. We begin by presenting the finite and infinite versions of Ramsey's theorem, alongside Galvin-Prikry's theorem as an example of a Ramsey type result in descriptive set theory. In the process, we study the Ellentuck topology and the Baire property.

We proceed to use Ramsey's theorem and concepts from functional analysis, such as finite representability, approximate eigenvalues and ultrapowers to prove Krivine's theorem as an example of result in Banach space theory of combinatorial nature.

Keywords: Banach spaces. Ramsey theory. Krivine's theorem.

Contents

Acknowledgements **i**

Resumo **iii**

Abstract **v**

1 Introduction **1**

2 Infinite Ramsey theory **5**

2.1 Ramsey's theorems 5
2.2 To infinity and beyond 11

3 Ramsey methods in Banach space theory **23**

3.1 Refining sequences 24
3.2 Spectral theory and ultrapowers 52
3.3 Krivine's theorem 64

Bibliography **77**

Index **79**

Chapter 1

Introduction

While there were results of a similar nature before his time[1], Ramsey theory consolidated itself after the work of Frank P. Ramsey, and by the efforts of mathematicians such as Paul Erdős. In its center is the idea that "given a large enough structure one may find an adequately large regular substructure". To give precise meaning to this, the most appropriate is the language of colorings: given a set A, a k-coloring is a function

$$f : [A]^r \to k$$

from $[A]^r$, the set of all r-element subsets of A, into $k = \{0, 1, ..., k-1\}$. One could then view a coloring as a way of labeling all the different assortments of r elements of a given set A with colors, each represented by a number in k. Ramsey's theorems essentially assert that given n, if A is large enough, for any k coloring of $[A]^r$, A will contain a subset B of size n such that $[B]^r$ is monochromatic, i.e., where all its elements are mapped to the same color in k.

In spite of its simple statement this is an important result, one that has influenced much of modern mathematics. Although it is originally a result from logic [21], it has since permeated a wide range of different areas, functional analysis is no exception. Combinatorial methods have found their way into the toolkit of Banach space theorists and proven to be a rich addition to the theory. In this work we substantiate this notion through the careful study of Krivine's theorem.

We begin in Chapter 2 with a presentation of the finite and infinite versions of Ramsey's theorem. After that, we explore the question of how one

[1] See for instance Soifer's "Ramsey theory yesterday, today and tomorrow" [23].

could extend the infinite version of Ramsey's theorem. Though hopes of the statement above holding true for the first infinite cardinal $r = \aleph_0$ are ultimately futile, as shown by Erdős and Rado [8], we will find that under certain constraints a Ramsey type result is possible for this setting. In doing so, we journey briefly through descriptive set theory, exploring how Ellentuck's topology on $[A]^{\aleph_0}$ might relate to the combinatorial aspects of this set. Ultimately we arrive at Galvin-Prikry's theorem, which states that for any coloring of $[\mathbb{N}]^{\aleph_0}$ which partition this set into Borel sets, there is some infinite subset $N \subseteq \mathbb{N}$ such that $[N]^{\aleph_0}$ is monochromatic.

With a notion of what is Ramsey theory, in Chapter 3 we move towards a combinatorial result in Banach space theory from 1976 due to Krivine. This is a theorem concerning the presence of ℓ_p $(1 \leq p < \infty)$ or c_0 spaces in general Banach spaces. The ℓ_p and c_0 spaces are so pervasive in Banach space theory that it was only in 1972, decades into the study of the subject, that a Banach space that does not contain an isomorphic copy of either of these spaces was presented [25]. Krivine's theorem comes as a guarantee that, although we cannot always find an infinite-dimensional copy of these spaces in a general Banach space, we can always find finite, but arbitrarily large, copies of them. The meaning of this is made clearer in Section 3.1, with the concept of finite representability. As the length of the chapter suggests, much work is needed to arrive at Krivine's theorem. Our starting point is the infinite version of Ramsey's theorem, after that we explore many important techniques of Banach space theory. For instance, in Section 3.2 we present some results from spectral theory and develop the concept of ultrapowers of Banach spaces.

To develop Chapter 3 we made use of a variety of resources, the main two being the books of Milman and Schetchman [19] and Artstein-Avidan, Giannopoulos and Milman [2], which present the strategy[2] we follow to prove Krivine's theorem. The main difference between this work and books such as these in the presentation of Krivine's theorem, is the level of detail we present. At the cost of brevity, we give careful explanations to each step taken in our studies. We also try to highlight what we prove through lemmas and propositions as often as possible, in an attempt to both make these results useful for other purposes but also to give a clear view of what we are tackling

[2]We should mention that in our study, much time was spent trying to work around imprecisions in [19]. While both books use roughly the same strategy, we found [2] to be easier to follow.

at each point.

We expect the reader to be acquainted with the topics of a standard introductory course into functional analysis, with some notions of basic sequences[3] in Banach spaces. A reasonable familiarity with point-set topology is also expected (especially for Section 2.2). Lastly, we should mention that while we borrow from set theoretic notation at times, we assume $\mathbb{N} = \{1, 2, 3, ...\}$.

[3] As a reference in the subject we suggest Megginson's book [18].

Chapter 2

Infinite Ramsey theory

Frank P. Ramsey was a prominent mathematician, economist and philosopher of the early twentieth century and though he died at a young age, he managed to mark his place in history in each of these fields. For mathematicians his name is best known for the two theorems in combinatorics that were named after him. These results, first introduced in the context of mathematical logic [21] were brought to the forefront of mathematics by Erdős and Szekeres' seminal paper "A combinatorial problem in geometry" [9].

The influence of these ideas is perhaps better illustrated by the extent to which Ramsey-type theorems permeate modern mathematics[1]. In fact the main results we develop in this work are considered Ramsey-type theorems. It is worth noting that while this kind of theorem was already present before Ramsey's time (c.f. [23]), his name came to unify this study.

As this is primarily a text on Banach space theory, we limit ourselves to a brief exploration of infinite Ramsey theory. First we introduce Ramsey's theorems and then move to try and extend the infinite dimensional case, arriving at the Galvin-Prikry's theorem.

2.1 Ramsey's theorems

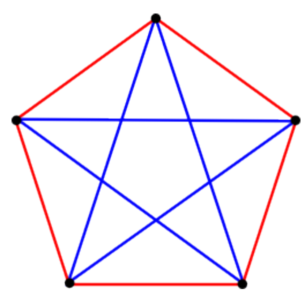
Ramsey's theorems have a very distinct flavor, perhaps the first experience a mathematician has with a result of this kind is the pigeonhole principle. Which in mathematical terms may be stated as follows:

[1]The interested reader could see the book of Graham and Rothschild [11].

Theorem 2.1.1 (Pigeonhole principle)**.** *If $n < m$ are natural numbers and $f : \{0, ..., m\} \to \{0, ..., n\}$ is a function, then there is some $k \in \{0, ..., n\}$ such that the pre-image $f^{-1}(k)$ has at least two elements.*

While to many this result is self-evident, the implications of this kind of thinking are far from that. An example of this is the quintessential example of Ramsey theory: The party problem. Suppose you have a party with n guests, and want to know if there are either three people that all know each other, or three people where any two of them don't know each other. One way of representing this is thinking of a (complete) graph where each vertex represents a person and the edge connecting them can be colored blue to represent that they know each other and red otherwise. The problem poses the question of how great n has to be so that one of these two possibilities is guaranteed to happen? Is there necessarily such an n?

The answer to the first question happens to be $n \geq 6$, for $n \leq 5$, there will be ways of coloring a graph to avoid there being a red or blue triangle (notice this is the graph equivalent to the conditions we posed above). An example of this follows.



However, for $n \geq 6$, this is guaranteed to happen [11]. While in this specific case we can present the precise value for n from which this works, we can answer the second question (concerning the existence of n) more broadly through Ramsey's theorem. To state it we need to introduce some notions that we borrow from the graph example:

Given a set A and $r \in \mathbb{N}$, we denote by $[A]^r$ the set of all r-element subsets of A (when $r = 2$ we may think of $[A]^r$ as the set of edges of a complete graph). To simplify our notation we borrow from the set-theoretic definition of the naturals and by $[n]^r$ we indicate $[\{0, 1, ..., n-1\}]^r$.

We call a function $f : [A]^r \to \{0, ..., k-1\}$, or simply $f : [A]^r \to k$ following the previous notation, a **k-coloring of** $[A]^r$. In the party problem this was the function that assigned the value 0 for an edge of the graph when two people didn't know each other (representing the color blue) and 1 when they did (representing the color red). What we sought was then a *monochromatic* subset of the graph. In general, given a specific k-coloring, we say that a subset $B \subseteq A$ is **monochromatic** if $f|_{[B]^r}$ is constant[2].

Theorem 2.1.2 (Ramsey 1930). *For any natural numbers $b, r, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for any set A with $|A| > n$, and any k-coloring of $[A]^r$, there exists a monochromatic subset B of A such that $|B| = b$.*

Often this theorem is stated directly in the context of graph-theory and in terms of Ramsey's number, $R(l_0, l_1)$, which is the smallest positive integer n for which any 2-coloring f of $[n]^2$ has a monochromatic subset B_0 of cardinality l_0 and color 0 or a monochromatic subset B_1 of cardinality l_1 and color 1 ($f[[B_1]^2] = \{1\}$). In this case the theorem asserts that for any given l_0 and l_1, $R(l_0, l_1)$ exists. As the only thing needed to prove this assertion is to show that for some n one of those subsets exists, we see that the graph-theoretic version follows easily from the theorem above by taking $r = 2$ and $b = \max\{l_0, l_1\}$.

There are many different versions of Ramsey's theorems and, as exemplified above, we often see one version implying another. In fact, we have not given a proof for Theorem 2.1.2 for this precise reason. We will prove a more general version of Ramsey's theorem and from it derive this one.

At first glance, this plurality of different theorems with the same name might be confusing. However these theorems are unified not only by the name of their originator but also by the fact that they all represent how, given a large enough structure, any assortment of different characteristics (which we represent by colors) given to the relationship of its elements (e.g. the edges of graphs) will always contain some uniform (monochromatic) subset.

In this section, our aim is to explore how far we may extend the limits of this fact. But first, we introduce some new notation: We'll write $a \to (b)^r_k$ to say that given any set A with cardinality $|A| = a$ and a k-coloring $f : [A]^r \to \{1, ..., k\}$, there always exists some monochromatic subset $B \subseteq A$ with cardinality $|B| = b$.

Remark. *The notation $a \to (b)^r_k$ abandons references to specific sets and focuses only on their cardinalities. The reason for this is simple: results like*

[2]It is common to say that $[B]^r$ is monochromatic in this case, instead of B. But it will serve us best to refer to B as monochromatic rather than $[B]^r$.

Ramsey's theorem (both in their finite and infinite versions), are completely transferable through bijections. That is, given a bijection $h : X \to A$ between sets A and X, and a coloring of A, then $f \circ \varphi$ is a coloring of X with monochromatic subsets of the same cardinalities as those of A. Hence, if A has no k-colorings without monochromatic subsets of a certain size, the same goes for X.

One natural next step in generalizing Theorem 2.1.2, especially given our preference for the use of cardinalities, is to consider the possibility of a and b to be infinite. It is easy to see that for any given $b, k, r \in \mathbb{N}$, $\aleph_0 \to (b)^r_k$ is a consequence of Theorem 2.1.2. Now, for which (if any) $r, k \in \mathbb{N}$ would we have $\aleph_0 \to (\aleph_0)^r_k$?

Theorem 2.1.3 (Ramsey 1930)**.** *For any countable set A, positive integer n and finite k-coloring of the family $[A]^n$, there is an infinite monochromatic subset B of A. That is, $\aleph_0 \to (\aleph_0)^n_k$.*

Proof. Clearly, $\aleph_0 \to (\aleph_0)^1_k$ for any $k \in \mathbb{N} \setminus \{0\}$, since it is the pre-image of a function with finite image and infinite domain. Assuming by induction that $\aleph_0 \to (\aleph_0)^r_k$, for a given $r \in \mathbb{N}$, we'll consider an arbitrary k-coloring $f : [A]^{r+1} \to k$ for the subsets of A with $r + 1$ elements.

Taking an arbitrary element $a_0 \in A$, we can define a function $g : [A \setminus \{a_0\}]^r \to k$ by setting

$$g(\{x_0, ..., x_{r-1}\}) = f(\{x_0, ..., x_{r-1}, a_0\}).$$

By the assumption ($\aleph_0 \to (\aleph_0)^r_k$) we know that there is an infinite monochromatic subset $B_0 \subseteq A \setminus \{a_0\}$ with respect to g. In terms of the coloring given by f we have that all $(r + 1)$-element subsets of $B_0 \cup \{a_0\}$ which include a_0 have the same color.

As the set B_0 we get is also infinite, we may repeat this process recursively and get a sequence of distinct points $a_0, a_1, a_2, ...$ and infinite subsets $B_0, B_1, B_2, ...$ of A such that $a_{j+1} \in B_j$ for all $j \geq 0$ and

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq B_3 \supseteq ...$$

and for which, given an a_n, all $(r + 1)$-element subsets of $B_n \cup \{a_n\}$ which include a_n have the same color. While the color is uniform among these subsets, it might not be the same color when we change n. We deal with this fact by creating sets $S_0, ..., S_{k-1}$ which hold all indexes $n \in \mathbb{N}$ for which the

$(r+1)$-element sets in $B_n \cup \{a_n\}$ with a_n have the colors $0, ..., k-1$. Since we have a finite number of sets whose union gives $\mathbb{N}$, one of them has infinite elements. Let S_p, for $p \in \{0, ..., k-1\}$, be that set. Then,

$$B = \{a_n : n \in S_p\}$$

is monochromatic with respect to f. To verify this, take any set $\{a_{n_0}, ..., a_{n_r}\}$ in $[B]^{r+1}$. Without loss of generality we may assume that $n_0 < n_1 < ... < n_r$, and by our construction of the sets $B_n \cup \{a_n\}$ we have that $a_{n_j} \in B_{n_0}$ for $1 \leq j \leq r$, and so $f(\{a_{n_0}, ..., a_{n_r}\}) = p$. Since our choice was arbitrary we have that B is monochromatic. $\square$

Now we can prove the finite version of Ramsey's theorem using an argument that is sometimes called the Compactness Principle.

Proof of Theorem 2.1.2. Suppose, by contradiction, that there are b, k, r natural numbers but no $n \in \mathbb{N}$ such that $n \to (b)^r_k$. That is, for every $n \in \mathbb{N}$, $[n]^r$ admits some k-coloring with no monochromatic subset of cardinality b. For every $n \in \mathbb{N}$, let C_n be the set of these colorings. Clearly C_n is non-empty.

Given any k-coloring f of $[n+p]^r$ (for $n, p \in \mathbb{N}$) in C_{n+p}, it is easy to see that we may restrict f to be a k-coloring of $[n]^r$ by ignoring all the sets in $[n+p]^r$ with elements $n, n+1, ..., p-1$. We also note that this k-coloring of $[n]^r$ must not contain a monochromatic subset of cardinality b (since this is a restriction of f, such a set would also be monochromatic in $[n+p]^r$).

Denoting by C^p_n the set of all restrictions of functions in C_{n+p} to $[n]^r$, we have found that $C^p_n \subseteq C_n$ and, given that C_{n+p} is non-empty, $C^p_n \neq \emptyset$.

Now, take $f \in C^{p+1}_n$. There is some $g \in C_{n+p+1}$ such that $f = g|_{[n]^r}$. If we let $h := g|_{[n+p]^r}$, then $h \in C_{n+p}$ and further restricting, $f = h|_{[n]^r} \in C^p_n$. Also, $g|_{[n+1]^r} \in C^p_{n+1}$ since it is a restriction of $g \in C_{(n+1)+p}$. We summarize this below:

1. $f \in C^{p+1}_n \implies f \in C^p_n$. That is, $C^p_n \supseteq C^{p+1}_n$, for all $n, p \in \mathbb{N}$.

2. Every $f \in C^{p+1}_n$ has an extension in C^p_{n+1}, for all $n, p \in \mathbb{N}$.

From 1 it follows that, for any $m \in \mathbb{N}$,

$$C_m \supseteq C^1_m \supseteq C^2_m \supseteq C^3_m \supseteq ...$$

If we think of these colorings without monochromatic subsets of cardinality b as "problematic", our strategy hereafter is to take some problematic k-coloring on $[m]^r$, extend it to some problematic k-coloring of $[m+1]^r$, then extend that one to a larger problematic k-coloring of $[m+2]^r$, and so on. To do this we have to prove that there is some k-coloring of $[m]^r$ that has extensions to $[m+p]^r$ for any $p \in \mathbb{N}$, which we do as follows: From the fact that C_m has finitely many, say q, elements (there are finite k-colorings of $[m]^r$), we have that the intersection $C_m^\infty := \cap_{p=0}^\infty C_m^p$ is non-empty (if it were, there would be $p_1, ..., p_q$ indices for which $C_m^{p_j}$ would not contain the j-th element of C_m, and by taking p_0 to be the maximum of $p_1, ..., p_q$, we would have $C_m^{p_0}$ empty, which cannot happen).

Now, take $f \in C_m^\infty$. Then, $f \in C_m^{p+1}$ and, by fact 2, f has an extension in C_{m+1}^p, for all $p \in \mathbb{N}$. Let A_0 be the set of extensions of f in C_{m+1} and A_p be the set of extensions of f in C_{m+1}^p, for any $p \in \mathbb{N}$. Then,

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$$

and, just as before, $\bigcap_{n=0}^\infty A_n \neq \emptyset$. So, given any $f \in C_m^\infty$ we know there is some extension of f in $\bigcap_{n=0}^\infty A_n \subseteq C_{m+1}^\infty$.

Finally, we take some $f_1 \in C_1^\infty$, let $f_2 \in C_2^\infty$ be an extension of f_1, and continue this recursively where $f_{n+1} \in C_{n+1}^\infty$ is an extension of f_n. Having this sequence of colorings we can define a coloring on $[\mathbb{N}]^r$ as

$$f_\infty(\{n_1, ..., n_r\}) := f_{n_r}(\{n_1, ..., n_r\}),$$

assuming that $n_1 < n_2 < ... < n_r$.

Then f_∞ is a k-coloring of $\mathbb{N}$ which must not have a monochromatic subset $B = \{m_1, m_2, ..., m_b\}$ (where we assume $m_1 < ... < m_b$). Because if it did, one would only need to take $t = m_b$ and $\{m_1, ..., m_b\}$ would be monochromatic for $f_\infty|_{[m_b]^r} = f_t$. Which is a contradiction to Theorem 2.1.3. □

Some versions of Theorem 2.1.3 deal only with 2-colorings. While this may seem a weaker result, it is actually sufficient to consider this scenario. We verify this in the following proposition.

Proposition 2.1.4. *For any $r \in \mathbb{N}$, $\aleph_0 \to (\aleph_0)_2^r \implies \aleph_0 \to (\aleph_0)_k^r$ for all $k \geq 2$.*

Proof. If we have $a \to (a)_2^r$ for any given $r \in \mathbb{N}$, clearly

$$a \to (a)_2^r \implies a \to (a)_2^r.$$

Now suppose that $a \to (a)^r_2$ and $a \to (a)^r_k$. Given a set A with $|A| = a$ and any $k+1$-coloring $\chi : [A]^r \to \{0, ..., k\}$ we can define $\phi : \{0, ..., k\} \to \{0, ..., k-1\}$ such that:

$$\phi(x) = \begin{cases} x, & \text{if } x \in \{0, ..., k-2\}, \\ k-1, & \text{if } x \in \{k-1, k\}. \end{cases}$$

It is easy to see then that $\phi \circ \chi$ is a k-coloring of $[A]^r$ and since $a \to (a)^r_k$ we have that there exists $B \subseteq A$ such that $|B| = a$ and B is monochromatic with respect to $\phi \circ \chi$. If $\phi \circ \chi[[B]^r] \in \{0, ..., k-2\}$ then B is also monochromatic with respect to χ. Otherwise we have that $\chi|_{[B]^r}$ is a 2-coloring and by $a \to (a)^r_2$ we have that there is $C \subseteq B$ with $|C| = a$ and such that C is monochromatic with respect to $\chi|_{[B]^r}$, and therefore, χ. That is, $a \to (a)^r_{k+1}$.

By induction, $a \to (a)^r_k$ for any $k \in \mathbb{N}$ and since the choice for r was arbitrary, we have proven the statement. $\square$

2.2 To infinity and beyond

To go beyond even the infinite version of Ramsey's theorem we might consider the possibility of a Ramsey theorem for "infinite colorings" of $[\mathbb{N}]^r$, however this doesn't work in any meaningful capacity, since we can always enumerate the elements of $[\mathbb{N}]^r$ and give each a different color. What we will actually find to be be productive is the case where r is infinite, that is, when we have the set $[A]^{\aleph_0}$ of all infinite subsets of a countable set A.

To get to this we might try to adapt the proof of Theorem 2.1.3, but that won't work since our result depends on induction on r. Actually, we will see that we cannot achieve something as general in nature as the statement of Theorem 2.1.3 for $[\mathbb{N}]^{\aleph_0}$. This was shown by Erdős and Rado in 1952 [8].

In their counterexample they show, for any given countable set A, a 2-coloring f of $[A]^{\aleph_0}$ defined as such: by the well ordering theorem, we know there exists a well-order $<$ of $[A]^{\aleph_0}$. Now take any $B \in [A]^{\aleph_0}$, we will define $f(B)$ to be 0 if B is the $<$-smallest element of $[B]^{\aleph_0}$ and $f(B) = 1$ otherwise. Clearly this is a well-defined 2-coloring. Then, for any countable $B \subseteq A$, $[B]^{\aleph_0}$ has a $<$-smallest element B_0 and, as such, it is also the $<$-smallest element of $[B_0]^{\aleph_0} \subseteq [A]^{\aleph_0}$ and therefore has color 0. Since $|[B]^{\aleph_0}| > 2$, there is some other element with color 1. So there can be no monochromatic subset C of cardinality $\aleph_0$.

To see that there is a counterexample for any k-coloring is only a matter of noticing that any 2-coloring might be restated as a k-coloring by simply changing its codomain.

With this in mind we look for conditions we may impose to advance beyond Theorem 2.1.3. To do this, we turn to topological structures.

The space $[\mathbb{N}]^{\aleph_0}$ might be seen as a subset of the set 2^ω, the set of all functions from $\mathbb{N}$ to $\{0,1\}$, by the injection $F : [\mathbb{N}]^{\aleph_0} \to 2^\omega$ which maps sets to their characteristic functions. That is, for any $B \in [\mathbb{N}]^{\aleph_0}$, $F(B) := \chi_B$ where

$$\chi_B(n) = \begin{cases} 1, & \text{if } n \in B, \\ 0, & \text{otherwise.} \end{cases}$$

This is useful insofar as 2^ω has a canonical topology τ_{2^ω} given by the product of the discrete topology on $\{0,1\}$, $\tau_2 = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. By the definition of the product topology we have that τ_{2^ω} is generated by the basis

$$\left\{ \prod_{n\in\mathbb{N}} U_n : U_n \in \tau_2 \ \forall n \in \mathbb{N}, |\{U_n \in \tau_2 : U_n \neq \{0,1\}\}| < \aleph_0 \right\}.$$

This topology then induces a subspace topology on $F([\mathbb{N}]^{\aleph_0})$, τ', which is easily transferred back to $[\mathbb{N}]^{\aleph_0}$ by the pre-images of F,

$$\tau = \{F^{-1}[U] : U \in \tau'\}.$$

One might view this as the topology[3] where any non-empty basic open set U consists of all sets in $[\mathbb{N}]^{\aleph_0}$ which contain $\{n_1, ..., n_k\}$ but don't contain $\{m_1, ..., m_p\}$ for some specific finite selection of numbers $n_1, ..., n_k, m_1, ..., m_p \in \mathbb{N}$. In other words, $U \in \tau$ is a non-empty basic open set if, and only if, there is $\{n_1, ..., n_k, m_1, ..., m_p\} \subseteq \mathbb{N}$ such that

$$U = \{A \in [\mathbb{N}]^{\aleph_0} : \{n_1, ..., n_k\} \subseteq A \text{ and } \{m_1, ..., m_p\} \subseteq \mathbb{N} \setminus A\}. \tag{2.1}$$

We will take this to be the "usual" topology on $[\mathbb{N}]^{\aleph_0}$, we say this as we'll explore yet another topology on $[\mathbb{N}]^{\aleph_0}$:

Definition 2.2.1 (Ellentuck topology)**.** Let a, A be finite and infinite subsets of $\mathbb{N}$, respectively. We will write $a < A$ when $\max(a) < \min(A)$ and when this is the case we will denote by $[a, A]$ the following set:

$$[a, A] = \{S \in [\mathbb{N}]^{\aleph_0} : a \subseteq S \subseteq A \cup a\}.$$

[3] It is easy to verify that τ is in fact a topology by using that $F : [\mathbb{N}]^{\aleph_0} \to F([\mathbb{N}]^{\aleph_0})$ is a bijection. We leave this to the reader.

The topology generated by the basis[4] $\{[a, A] : a \in [\mathbb{N}]^{<\aleph_0}, A \in [\mathbb{N}]^{\aleph_0}, a < A\}$ is called the **Ellentuck topology** and we will denote it by τ_E.

We may then think of $[a, A]$ as the set of all subsets of $\mathbb{N}$ that begin with a (when following the ordering of natural numbers) and continue with some infinite selection of elements in A. If $a = \{n_1, ..., n_k\}$ and $A = \{n_{k+1}, n_{k+2}, ...\}$ we have:

$$B \in [a, A] \iff B = \{\underbrace{n_1, n_2, n_3, ..., n_k}_{\text{all the elements of } a}, \underbrace{n_{k+j_1}, n_{k+j_2}, n_{k+j_3}, ...}_{\text{infinite elements of } A}\}.$$

Remark. *Henceforth whenever we mention a set $[a, A]$ we take it to mean that a is a finite subset of $\mathbb{N}$, A an infinite subset of $\mathbb{N}$ with $a < A$ and satisfying the above mentioned definition. We may then take a set $B \subseteq A$ and construct $[a, B]$ under the implicit assumption that B is infinite.*

With this we can easily see that the Ellentuck topology is a refinement of the usual topology: Let $U \in \tau$ be a non-empty basic open set and $n_1, ..., n_k, m_1, ..., m_s$ the numbers which determine this set in the notation of Equation (2.1). Let $m = \max\{n_1, ..., n_k, m_1, ..., m_p\}$ and denote by $\mathcal{A}$ the set of all subsets of $\{1, ..., m\}$ which contain $\{n_1, ..., n_k\}$ but which don't contain any m_j for $1 \leq j \leq p$. Then we may write

$$U = \bigcup_{a \in \mathcal{A}} [a, \{m+1, m+2, ...\}]$$

Since the basic open sets of τ_E generate all non-empty basic open sets of τ, $\tau \subseteq \tau_E$.

Definition 2.2.2. Let X be a set. We call a collection $\mathcal{I}$ of subsets of X an **ideal** on X if

1. $\emptyset \in \mathcal{I}$.

2. If $B \in \mathcal{I}$ and $A \subseteq B$, then $A \in \mathcal{I}$.

3. If $B_1, ..., B_k \in \mathcal{I}$, then $\bigcup_{n=1}^k B_n \in \mathcal{I}$.

[4]To see that this is indeed a basis for a topology, we need to verify that the intersection of sets $[a, A]$ and $[b, B]$ is always either empty (if either $a \neq b$ or $A \cap B$ is finite), or it is of the form $[a, A \cap B]$.

We call $\mathcal{I}$ a **σ-ideal** if aside from this we have:

4. If $\{B_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ then $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{I}$.

Ideals define a concept of "smallness" of sets (the elements of the ideal being the sets considered small). With it, we are able to consider a notion of equivalence dependent on the (symmetric) difference of two sets being considered small.

Definition 2.2.3. Let (E, τ) be a topological space and $A \subseteq X$. We say that A has the **Baire property** if $A =^* U$ for some $U \in \tau$. That is, if, for some $U \in \tau$,

$$A\Delta U = (A \setminus U) \cup (U \setminus A) \in \mathcal{I}$$

where $\mathcal{I}$ is the σ-ideal[5] of meager sets[6].

Similarly to how our concept of smallness is made to obey the rules of an ideal, we will find that the family of sets with the Baire property has a nice structure, one we call a *σ-algebra*.

Definition 2.2.4. Let X be a set. We call a collection $\mathcal{A}$ of subsets of X a **σ-algebra** if

1. $X \in \mathcal{A}$.

2. If $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$.

3. If $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

In this context, we say that a σ-algebra on E is generated by a subset X of $\mathcal{P}(E)$ if it is the intersection of all σ-algebras on E containing X. Hereafter we use the notation $\text{Int}(S)$ to refer to the interior of a set S in a given topological space (E, τ).

Given the fact that the set operations which define a σ-algebra are frequently used in topology, it might not be surprising that there is a deep and important connection between these concepts. Indeed, it is natural to consider, and those interested in measure theory would have a hard time avoiding doing so, the σ-algebra generated by a topology.

[5]The verification of this fact is straightforward and therefore omitted.

[6]A set B in a topological space X is called *meager* if it may be written as $B = \bigcup_{n=1}^{\infty} B_n$ where each B_n is nowhere dense ($\text{Int}(\overline{B_n}) = \emptyset$ for all $n \in \mathbb{N}$).

Definition 2.2.5. Let (E, τ) be a topological space, we call the σ-algebra generated by τ to be the class of Borel sets and any set in this class a **Borel set**.

While the family of Borel sets is already vast, the σ-algebra we are after, the one that will characterize the Baire property, contains it.

Proposition 2.2.6. *Let (E, τ) be a topological space. Then the set of all subsets with the Baire property is the σ-algebra generated by the set of all open sets and meager sets.*

Proof. Let $S \subseteq \mathcal{P}(E)$ be the set of all subsets of E with the Baire Property. First, we note that all open sets $U \in \tau$ trivially satisfy $U =^* U$ and so are in S. Similarly, any meager set M satisfies $M =^* \emptyset$ since $M \Delta \emptyset = M$ and thus $M \in S$.

Before we verify that S is a σ-algebra, note that for any open set U, $\overline{U} \setminus U$ is meager (if V is open and $V \subseteq \overline{U} \setminus U$, then $\overline{U} \setminus V$ is a closed subset of $\overline{U}$ containing U and so it must be $\overline{U}$, which implies $V = \emptyset$). Similarly, if F is closed, $F \setminus \operatorname{Int}(F)$ is meager.

Now, if $A \in S$ then $A =^* U$ for some open set U. Then

$$(E \setminus A) \Delta (E \setminus U) = A \Delta U \implies E \setminus A =^* E \setminus U,$$

and

$$E \setminus A =^* (\operatorname{Int}(E \setminus U)) \cup ((\overline{E \setminus U}) \setminus \operatorname{Int}(E \setminus U)) =^* \operatorname{Int}(E \setminus U).$$

So $A \in S \implies E \setminus A \in S$.

Taking $A_1, A_2, ... \in S$, we have $U_1, U_2, ... \in \tau$ such that $A_n \Delta U_n = M_n \in \mathcal{I}$ for all $n \in \mathbb{N}$.

$$\begin{aligned} \bigcup_{n \in \mathbb{N}} A_n \Delta \bigcup_{n \in \mathbb{N}} U_n &= \left(\bigcup_{n \in \mathbb{N}} A_n \setminus \bigcup_{n \in \mathbb{N}} U_n \right) \cup \left(\bigcup_{n \in \mathbb{N}} U_n \setminus \bigcup_{n \in \mathbb{N}} A_n \right) \\ &= \bigcup_{n \in \mathbb{N}} \left(\left(A_n \setminus \bigcup_{m \in \mathbb{N}} U_m \right) \cup \left(U_n \setminus \bigcup_{m \in \mathbb{N}} A_m \right) \right) \\ &\subseteq \bigcup_{n \in \mathbb{N}} A_n \Delta U_n = \bigcup_{n \in \mathbb{N}} M_n. \end{aligned}$$

Since $\mathcal{I}$ is a σ-ideal, we find that $\bigcup_{n \in \mathbb{N}} M_n \in \mathcal{I}$ and so $\bigcup_{n \in \mathbb{N}} A_n \in S$.

Finally, to see that S is the smallest of such σ-algebras, take $A \in S$. Then, there is some $U \in \tau$ satisfying $A \Delta U = M$ where M is meager. We may then verify that

$$M \setminus U = ((A \setminus U) \cup (U \setminus A)) \setminus U = (A \setminus U),$$

and

$$U \setminus M = U \setminus ((A \setminus U) \cup (U \setminus A)) = U \setminus (U \setminus A) = U \cap A.$$

So, $A = M \Delta U$ and, given any σ-algebra Q containing the open and meager sets, $A \in Q$. So $S \subseteq Q$, and since Q was chosen arbitrarily we conclude that S is in fact the smallest σ-algebra containing the open and meager sets. $\square$

With these new notions in mind we go back to the combinatorial aspect of our studies, and introduce two definitions. In these, it will be convenient to simplify the notation for the complement $B \setminus A$ as $\sim A$, when $A \subseteq B$ and it is clear that B is the set in question.

Definition 2.2.7. Let $X \subseteq [\mathbb{N}]^{\aleph_0}$. We say that X is a **Ramsey set** if there is some set $[\emptyset, A]$ such that either

1. $[\emptyset, A] \subseteq X$, or
2. $[\emptyset, A] \subseteq \sim X$.

If we think of 2-colorings as partitions, we see that a set X is Ramsey if it partitions $[\mathbb{N}]^{\aleph_0}$ in a way that leaves some subset $[A]^{\aleph_0} = [\emptyset, A]$ fully contained in, or outside of, X. While this notion already characterizes the types of 2-colorings we are after, a stronger condition will be convenient.

Definition 2.2.8. Let $X \subseteq [\mathbb{N}]^{\aleph_0}$. We say that X is a **completely Ramsey set** if, for any set $[a, A]$ there is some subset $B \subseteq A$ such that either

1. $[a, B] \subseteq X$, or
2. $[a, B] \subseteq \sim X$.

Clearly, by taking $[a, A] := [0, \mathbb{N}] = [\mathbb{N}]^{\aleph_0}$ we see that any completely Ramsey set is Ramsey set.

With this, we may begin to work our way towards some connection between the Ellentuck topology and combinatorics. As somewhat foreshadowed by the previous definitions, this will come by means of a connection between a set having the Baire property and being completely Ramsey. To this end, we begin by seeing that the Ellentuck topology is made up entirely of completely Ramsey sets:

Lemma 2.2.9. *Every open set U in the Ellentuck topology is completely Ramsey.*

Proof. Let U be an arbitrary non-empty open set on the Ellentuck topology ($\emptyset$ is trivially completely Ramsey). We wish to show that for any set $[a, A]$ there is an infinite subset $B \subseteq A$ for which we fall into one of two situations:

1. $[a, B] \subseteq U$, or

2. $[a, B] \subseteq \sim U$.

Take any $[a, A]$, if there is some $B \subseteq A$ for which $[a, B] \subseteq U$, we have what we wished for and we call $[a, A]$ *good*, otherwise we call it *bad*.

It is clear that if $[a, A]$ is bad, and $B \subseteq A$, then $[a, B]$ is bad. We will verify the following property which we denote by $(\star)$:

> If $[a, A]$ is bad, there is some $B \subseteq A$, with the property that we may take any $n \in B$ and we will still have that $[a \cup \{n\}, B/n]$ is bad (where $B/n = B \setminus \{1, ..., n\}$), in which case we say $[a, B]$ is *very bad*.

Note as well that if $[a, B]$ is very bad, then so is $[a, C]$ for any $C \subseteq B$, since $[a \cup \{n\}, C/n] \subseteq [a \cup \{n\}, B/n]$ for all $n \in C \subseteq B$.

To prove that $[a, A]$ bad implies the existence of $B \subseteq A$ with $[a, B]$ very bad, suppose, on the contrary, that for any infinite subset $B \subseteq A$ there is some $n \in B$ and some $C \subseteq B/n$ with $[a \cup \{n\}, C] \subseteq U$. Since $A \subseteq A$, there is $n_0 \in A$ and $B_0 \subseteq A/n_0$ for which $[a \cup \{n_0\}, B_0] \subseteq U$. Given that $[a, A]$ is bad, there is no $C \subseteq B_0 \subseteq A$ with $[a, C] \subseteq U$, and again we can find $n_1 \in B_0$ and some $B_1 \subseteq B_0/n_1$ such that $[a \cup \{n_1\}, B_1] \subseteq U$. Repeating this process recursively we'll have

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq ...$$

and a set of natural numbers $B_\infty := \{n_0, n_1, n_2, ...\}$.

Since, for any $w \in [a, B_\infty]$ we have $w = a \cup C$ with $C \subseteq B_\infty$, by taking $n_k := \min(C)$ we find that $w \in [a \cup \{n_k\}, B_k] \subseteq U$, and $[a, B_\infty] \subseteq U$, contradicting the fact that $[a, A]$ was bad.

We will now use a similar argument as above, and the fact it was proving, to show that there is $B \subseteq A$ with $[a, B] \subseteq \sim U$.

To construct this set B we will first take $B_0 \subseteq A$ for which $[a, B_0]$ is bad and let $n_0 := \min B_0$, then $[a \cup \{n_0\}, B_0/n_0]$ is bad, and so is $[a, B_0/n_0]$ (by the simple fact that $B_0/n_0 \subseteq B$). Using ($\star$) again, we know there is some $B_1 \subseteq B_0/n_0$ such that $[a \cup \{n_0\}, B_1]$ is very bad. In which case we define $n_1 := \min B_1$ and we have that $[a, B_1/n_1]$, $[a \cup \{n_0\}, B_1/n_1]$ and $[a \cup \{n_0, n_1\}, B_1/n_1]$ are all bad. If we proceed recursively, we will have sets

$$A \supseteq B_1 \supseteq B_2 \supseteq ...$$

and a sequence $n_1 < n_2 < ...$ satisfying that $[a \cup b, B_k/n_k]$ is bad for any $b \subseteq \{n_1, ..., n_k\}$ and every $k \geq 1$. Let $B := \{n_0, n_1, n_2, ...\}$ and it remains only to verify that $[a, B] \subseteq \sim U$:

Suppose it is false (i.e. $[a, B] \cap U \neq \emptyset$). Since U and $[a, B]$ are both open, $[a, B] \cap U$ is a non-empty open set and there is some basic open set $[a', B'] \subseteq ([a, B] \cap U)$. This means that a' is of the form $a' = a \cup b$ for some $b \subseteq B$. Let $n_t := \max b$, and by what we just proved, $[a \cup b, B/n_t]$ is bad. However, $[a', B'] \subseteq [a \cup b, B/n_t]$ and $[a \cup b, B/n_t] \subseteq U$, which would mean $[a \cup b, B/n_t]$ is good, a contradiction. $\square$

It is easy to see that this result extends to closed sets: Let X be a non-empty closed set, since $\sim X$ is an open set it is completely Ramsey, and for any set $[a, A]$ there is $B \subseteq A$ where either $[a, B] \subseteq \sim X$ or $[a, B] \subseteq \sim (\sim X) = X$.

Now with the open and closed sets dealt with, we aim towards the nowhere dense sets in the Ellentuck topology.

Lemma 2.2.10. *Every nowhere dense set X in the Ellentuck topology is completely Ramsey. In fact, for any set $[a, A]$ there is $B \subseteq A$ with $[a, B] \subseteq \sim X$.*

Proof. $\overline{X}$ is closed and so is completely Ramsey, which means that for any set $[a, A]$ there is $B \subseteq A$ where either $[a, B] \subseteq \overline{X}$ or $[a, B] \subseteq \sim \overline{X}$. Since X is nowhere dense, it contains no non-empty open sets and so the first option is impossible. That leaves us with

$$[a, B] \subseteq \sim \overline{X} \subseteq \sim X,$$

which means X is completely Ramsey. $\square$

A natural next step is to "extend" this to meager sets. In doing so we inadvertently prove a remarkable property of the Ellentuck topology: A set is meager if, and only if, it is nowhere dense.

Proposition 2.2.11. *Every meager set X in the Ellentuck topology is nowhere dense and completely Ramsey. In fact, for any set $[a, A]$ there is $B \subseteq A$ with $[a, B] \subseteq \sim X$*

Proof. Let $X = \bigcup_{n \in \mathbb{N}} X_n$ where each X_n os nowhere dense. To show $\mathrm{Int}(\overline{X}) = \emptyset$, we verify that given any non-empty basic open set $[a, A]$, there is some $B \subseteq A$ such that $[a, B] \subseteq \sim X$, thus showing that $[a, A] \not\subseteq \overline{X}$.

Let $[a, A]$ be any non-empty basic open set. By Lemma 2.2.10, there is $B_1 \subseteq A$ such that $[a, B_1] \subseteq \sim X_1$. If we let $n_1 := \min B_1$ we have

- $[a \cup \{n_1\}, B_1/n_1] \subseteq [a, B_1] \subseteq \sim X_1$.
- $[a, B_1/n_1] \subseteq \sim X_1$.

Since X_2 is also nowhere dense, there is $B_2 \subseteq B_1/n_1$ where $[a \cup \{n_1\}, B_2]$ is contained in $\sim X_2$. Defining $n_2 := \min B_2$, we note that

- $[a, B_2/n_2] \subseteq [a, B_1/n_1] \subseteq \sim X_2$.
- $[a \cup \{n_1, n_2\}, B_2/n_2] \subseteq [a \cup \{n_1\}, B_2] \subseteq \sim X_2$.

From this we get that

$$[a \cup b, B_2/n_2] \subseteq \bigcap_{i=1}^{2} (\sim X_i)$$

for every $b \subseteq \{n_1, n_2\}$. If we proceed recursively, as we did in Lemma 2.2.9, we'll find

$$A \supseteq B_0 \supseteq B_1 \supseteq B_2 \supseteq ...$$

and a sequence $n_0 < n_1 < n_2 < ...$ satisfying

$$[a \cup b, B_k/n_k] \subseteq \bigcap_{i=1}^{k} (\sim X_i)$$

for every $k \geq 0$ and any $b \subseteq \{n_0, ..., n_k\}$. If we let $B = \{n_1, n_2, ...\}$ we'll have that any element of $[a, B]$ will be in $\bigcap_{i=1}^{k} (\sim X_i)$ for all $k > 0$. That is,

$$[a, B] \subseteq \bigcap_{i=1}^{\infty} (\sim X_i) = \sim \bigcup_{i=1}^{\infty} X_i = \sim X.$$

$\square$

So far we have seen that open, closed and meager sets are all completely Ramsey. In light of the notion of a σ-algebra, one might wonder if the class of completely Ramsey sets satisfy some similar properties. Ellentuck's theorem tells us that not only does it satisfy the notion of a σ-algebra, but it coincides exactly with the σ-algebra of sets with the Baire property!

Theorem 2.2.12 (Ellentuck)**.** *For any $X \subseteq [\mathbb{N}]^{\aleph_0}$, X is completely Ramsey if, and only if, X has the Baire property in the Ellentuck topology.*

Proof. ($\Rightarrow$) Let X be a completely Ramsey set and $[a, A]$ an arbitrary non-empty basic open set. Then there is some $B \subseteq A$ such that either

1. $[a, B] \subseteq X \implies [a, B] \subseteq \operatorname{Int}(X) \implies X \setminus \operatorname{Int}(X) \subseteq \sim [a, B]$, or
2. $[a, B] \subseteq \sim X \implies X \subseteq \sim [a, B] \implies X \setminus \operatorname{Int}(X) \subseteq \sim [a, B]$.

Either way since $\sim [a, B]$ is a closed set, $\overline{X \setminus \operatorname{Int}(X)} \subseteq \sim [a, B]$ and

$$[a, B] \nsubseteq \overline{X \setminus \operatorname{Int}(X)} \implies [a, A] \nsubseteq \overline{X \setminus \operatorname{Int}(X)}$$

Hence the set $\overline{X \setminus \operatorname{Int}(X)}$ can contain no basic open sets and so it is nowhere dense. With this, we have that $X \Delta \operatorname{Int}(X) = X \setminus \operatorname{Int}(X)$ is meager and so X has the Baire property.

($\Leftarrow$) Suppose that X has the Baire property, then there is some open set U such that $X \Delta U$ is meager. Let $[a, A]$ be any non-empty basic open set. Since $X \Delta U$ is nowhere dense there is some $B \subseteq A$ such that $[a, B] \subseteq \sim X \Delta U$. By Lemma 2.2.9, U is completely Ramsey and there is some $C \subseteq B$ such that $[a, C] \subseteq [a, B] \subseteq \sim X \Delta U$ and

1. $[a, C] \subseteq U \implies [a, C] \subseteq X$, or
2. $[a, C] \subseteq \sim U \implies [a, C] \subseteq \sim X$.

So we have found $C \subseteq A$ for which $[a, C] \subseteq X$ or $[a, C] \subseteq \sim X$ for an arbitrary choice of $[a, A]$, which means that X is completely Ramsey. □

With this we have tangible progress in our quest for a Ramsey-like results for the set $[\mathbb{N}]^{\aleph_0}$: We have added some practical structure to $[\mathbb{N}]^{\aleph_0}$ (the Ellentuck topology) that is directly related to sets we find "combinatorially nice" (completely Ramsey sets). It remains only to make use of this structure and get the combinatorial result we were after.

Theorem 2.2.13 (Galvin-Prikry)**.** *Let $[\mathbb{N}]^{\aleph_0} = P_0 \cup P_1 \cup ... \cup P_{k-1}$ be a partition where each set P_i is Borel (in the usual topology) for $0 \leq i \leq k-1$. Then there is some infinite subset $A \subseteq \mathbb{N}$ and an integer $i \in \{0, ..., k-1\}$ such that $[A]^{\aleph_0} \subseteq P_i$.*

Proof. First, we note that since the Ellentuck topology is finer than the usual topology for $[\mathbb{N}]^{\aleph_0}$, we have that $P_0, ..., P_{k-1}$ are also Borel sets in the Ellentuck topology. In what follows we use the

We prove this by induction. For $k = 1$ it is immediate. For $k = 2$ we have $[\mathbb{N}]^{\aleph_0} = P_0 \cup P_1$ then $P_1 = \sim P_0$. Since P_0 is a Borel set, it is in the σ-algebra of open sets which is contained in the family of sets with the Baire property (Theorem 2.2.6). So P_0 has the Baire property and, by Theorem 2.2.12, is completely Ramsey. Then, for $[\emptyset, \mathbb{N}]$ is a non-empty basic open set and there is $A \subseteq \mathbb{N}$, infinite, such that $[A]^{\aleph_0} = [\emptyset, A] \subseteq P_0$ or $[A]^{\aleph_0} \subseteq \sim P_0 = P_1$.

Now, suppose the theorem is valid for $k = s$, we show that it is also true for $s + 1$. Let $[\mathbb{N}]^{\aleph_0} = P_0 \cup ... \cup P_s$, we may then define $Q_0 := P_0 \cup ... \cup P_{s-1}$ and $Q_1 := P_s$, we'll have $[\mathbb{N}]^{\aleph_0} = Q_0 \cup Q_1$. Since the statement is true for $s = 2$, there is some $A \subseteq \mathbb{N}$ such that $[A]^{\aleph_0} \subseteq Q_0$ or $[A]^{\aleph_0} \subseteq Q_1 = P_s$. In the latter we case we are done. In the former, we would find that

$$[A]^{\aleph_0} = (P_0 \cap [A]^{\aleph_0}) \cup (P_1 \cap [A]^{\aleph_0}) \cup ... \cup (P_{s-1} \cap [A]^{\aleph_0}).$$

Since A is countable, this is the same as the statement of the theorem for s, which we are assuming to be true. Then there will be $B \subseteq A \subseteq \mathbb{N}$ such that $B \subseteq P_i \cap [A]^{\aleph_0} \subseteq P_i$ for some $i \in \{0, ..., s-1\}$. In any case we have that the theorem would also be valid for $k = s + 1$, which concludes our proof. □

Lastly we should mention that another important class of sets that is related to Ramsey sets, and studied in Ellentuck's paper [7], is that of analytic sets.

Definition 2.2.14. Let X be a polish space (a separable and completely metrizable topological space). A set $A \subseteq X$ is said to be **analytic**[7] if there is a polish space Y and a continuous function $f : Y \to A$ such that $f(Y) = A$.

In fact, the main purpose of Ellentuck's paper was to give another proof for the following theorem due to Silver in 1970 [22].

Theorem 2.2.15 (Silver)**.** *Every analytic set $A \subseteq [\mathbb{N}]^{\aleph_0}$ (in the usual topology) is completely Ramsey.*

[7]Other equivalent definitions can be found in Kechris' book [14].

Chapter 3

Ramsey methods in Banach space theory

The spaces ℓ_p ($1 \leq p < \infty$) and c_0 are commonplace in any course on functional analysis. Their initial appearance as examples of Banach spaces is not surprising when one considers how these spaces arise naturally from generalizations of finite-dimensional Euclidean spaces and in studies of sequence convergence. What is in fact remarkable is how they maintain their relevance all throughout Banach space theory.

Aleksander Pełczyński published in 1960 what came to be a famous result of Banach space theory concerning these spaces [20]. In his paper he proved that the spaces ℓ_p ($1 \leq p < \infty$) and c_0 are *prime*. We say that a Banach space X is **prime** if any infinite-dimensional complemented subspace of X is isomorphic to X.

This language is evocative of prime numbers in number theory, and illustrates how these spaces may be thought of as building blocks for Banach spaces. And, while these are not the only prime spaces (other separable prime spaces where constructed by Gowers and Maurey in 1997 [10]), they are the simplest representatives of this class of Banach spaces.

As a confirmation of the prominence of ℓ_p and c_0 spaces in the construction of infinite-dimensional Banach spaces, the first example of an infinite-dimensional Banach space which did not contain a subspace isomorphic to some ℓ_p or c_0 was built by Tsirelson only in 1974 [25]. While we know from this that we cannot expect an infinite-dimensional Banach space to contain an isomorphic copy of ℓ_p or c_0 in general, we may hope for the next best thing, that this is true *locally*[1].

It is with this in mind that we journey in this chapter through some applications of Ramsey theory to Banach space theory. Our main goal is proving Krivine's theorem for real Banach spaces, which, as the length of this chapter suggests, requires some work. Of course, on the way various techniques and ideas of a more general nature are explored; such as basic sequences, spectral theory and some notions of ultrapowers of Banach spaces.

In the first section, we begin our work directly towards Krivine's theorem as we explore the main tool we use to investigate this local presence of the spaces ℓ_p and c_0: finite representability. We do so using the work of Brunel and Sucheston (c.f [3] and [4]). In the second section we develop some results concerning spectral theory and ultrapowers, and relate them to what we did in the previous section. Lastly, we state and the prove Krivine's theorem in the final section, following Lemberg's approach [17].

The combinatorial aspects of our studies are concentrated in the first section, wherein we use the infinite version of Ramsey's theorem in our quest to get stronger properties for the sequences we have at hand. All in all, while Krivine's theorem is combinatorial in nature, the arguments towards it are mostly analytic, well within the domain of a functional analyst.

3.1 Refining sequences

One may consider that the essence of Ramsey theory is the pursuit of "regular substructures" given a large enough structure. In standard functional analysis fashion, the structures that we shall investigate are sequences of vectors in normed spaces. Specifically, we will use the infinite version of Ramsey's theorem (Theorem 2.1.3) to hone some properties of given sequences. One such property that will be recurring in this section is the following:

Definition 3.1.1. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space X is **invariant under spreading**, or **1-spreading**, if for any $n \in \mathbb{N}$ and scalars $a_1, ..., a_n$,

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| = \left\|\sum_{i=1}^{n} a_i x_{k_i}\right\|,$$

for any natural numbers $k_1 < k_2 < ... < k_n$.

[1] In Banach space theory the word "local" is used to refer to finite-dimensional subspaces.

Example 3.1.2. *The standard Schauder bases for c_0 and ℓ_p, with $1 \leq p < \infty$, are* 1*-spreading.*

This, however, is quite a strong condition for a sequence to satisfy, one we hope to work our way towards. As our discussion of sequences is in many ways motivated by basic sequences, we investigate briefly what is reasonable to expect of these sequences in terms of invariance under spreading.

It is easy to construct a non-normalized basic sequence which won't be 1-spreading. But given that normalizing a basic sequence is a routine practice, it is still useful to present an example of a normalized Schauder basis which isn't 1-spreading:

Example 3.1.3. *Let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis for c_0. The sequence $(g_n)_{n\in\mathbb{N}}$ defined as $g_1 := e_1$, $g_2 = e_2$ and $g_n := e_2 + ... + e_n$ for all $n > 2$ is a normalized basic sequence which isn't* 1*-spreading.*

Proof. First we verify that $(g_n)_{n\in\mathbb{N}}$ is a Schauder basis for c_0. For this, define $g_1^*(x) = e_1^*(x)$ and $g_n^*(x) = e_n^*(x) - e_{n+1}^*(x)$ for $n > 1$ (where e_n^* are the coefficient functionals of e_n for all $n \in \mathbb{N}$) and let $x = (x_n)_{n\in\mathbb{N}} \in c_0$ be some arbitrary vector. Then, given $m \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=1}^{m} g_n^*(x) g_n &= e_1^*(x)e_1 + \sum_{n=2}^{m} (e_n^*(x) - e_{n+1}^*(x)) g_n \\
&= x_n e_1 + \sum_{n=2}^{m} x_n g_n - \sum_{n=3}^{m+1} x_n g_{n-1} \\
&= x_1 e_1 + x_2 e_2 + \sum_{n=3}^{m} x_n (g_n - g_{n-1}) + x_{m+1} g_m \\
&= \sum_{n=1}^{m} x_n e_n + x_{m+1} g_m
\end{aligned}$$

and

$$\begin{aligned}
\left\| x - \sum_{n=1}^{m} g_n^*(x) g_n \right\| &\leq \left\| x - \sum_{n=1}^{m} x_n e_n \right\| + \left\| \sum_{n=1}^{m} x_n e_n - \sum_{n=1}^{m} g_n^*(x) g_n \right\| \\
&= \left\| x - \sum_{n=1}^{m} x_n e_n \right\| + \| x_{m+1} g_m \|.
\end{aligned}$$

From the fact that $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis, that $(g_n)_{n\in\mathbb{N}}$ is normalized and $(x_n)_{n\in\mathbb{N}} \in c_0$, we have

$$\left\| x - \sum_{n=1}^{m} g_n^*(x) g_n \right\| \xrightarrow{m\to\infty} 0.$$

To see that it is, in fact, a Schauder basis, it is enough to show that if $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of scalars for which $\|\sum_{n=1}^{m} \alpha_n g_n\| \xrightarrow{m\to\infty} 0$, then $\alpha_n = 0$ for all $n \in \mathbb{N}$.

Suppose $(\alpha_n)_{n\in\mathbb{N}}$ is such. Then, for any $\varepsilon > 0$, there is some $m_0 \in \mathbb{N}$ such that $\| \sum_{n=1}^{m} \alpha_n g_n\| < \varepsilon$ for all $m \geq m_0$. So,

$$\left\| \sum_{n=1}^{m} \alpha_n f_n \right\| = \sup\{|a_1|, |a_2|\} \cup \left\{ \left| \sum_{n=2}^{k} \alpha_n \right| : 1 \leq k \leq m \right\} < \varepsilon, \qquad \forall m \geq m_1.$$

Therefore, $|a_1|, |a_2| < \varepsilon$ and $| \sum_{n=2}^{m} \alpha_n| < \varepsilon$ so

$$|\alpha_m| = \left| \sum_{n=2}^{m} \alpha_n - \sum_{n=2}^{m-1} \alpha_n \right| \leq 2\varepsilon,$$

for all $m \geq 3$. Since the choice of ε was arbitrary, we have that $\alpha_m = 0$, for all $m \in \mathbb{N}$. Finally, to see that $(g_n)_{n\in\mathbb{N}}$ isn't 1-spreading we need only consider the following

$$\|g_1 + g_2\| = \|e_1 + e_2\| \neq \|2e_2 + e_3\| = \|g_2 + g_3\|.$$

□

One way to weaken this condition is to think of some form of *crude* invariance, that is, where there is some $K > 0$ such that

$$\frac{1}{K} \leq \left\| \sum_{i=1}^{n} a_i x_{k_i} \right\| \leq K \tag{3.1}$$

for all choices of scalars $a_1, ..., a_n$ with $\sum_{i=1}^{n} |a_n| = 1$ and natural numbers $k_1 < ... < k_n$. The reason we specify that $\sum_{i=1}^{n} |a_n| = 1$ is to deal with scaling: Suppose Equation (3.1) were valid for some choice of scalars (not all being zero), then we could always scale theses scalars by some α big enough as to make $\| \sum_{i=1}^{n} \alpha a_i x_{k_i}\| = |\alpha| \| \sum_{i=1}^{n} a_i x_{k_i}\| > K$.

Now, thinking of a basic sequence $(x_n)_{n\in\mathbb{N}}$, if $(\|x_n\|)_{n\in\mathbb{N}}$ isn't semi-normalized, clearly (3.1) will not be satisfied (take $n = 1$ and $a_1 = 1$). So we move forward with the assumption that there exists $0 < c < C$ such that

$$c \leq \|x_n\| \leq C \qquad \text{for all } n \in \mathbb{N}.$$

To achieve an inequality of the sort we are after, we look for bounds on the variation of $\|\sum_{i=1}^n a_i x_i\|$ independent of the choice of scalars $a_1, ..., a_n$. By the triangle inequality,

$$\left\|\sum_{i=1}^n a_i x_i\right\| \leq n \sup_{1\leq i\leq n} |a_i|\|x_i\| \leq n \sup_{1\leq i\leq n} |a_i| C. \tag{3.2}$$

For a lower bound, the basic sequence criterion is a promising start point: Let M be the basis constant of $(x_n)_{n\in\mathbb{N}}$, then

$$\left\|\sum_{i=1}^s a_i x_i\right\| \leq M \left\|\sum_{i=1}^n a_i x_i\right\|$$

for any $s < n$ and $a_1, ..., a_n$ scalars. Considering $s = 1$, we verify that $\|a_1 x_1\| \leq M \|\sum_{i=1}^n a_i x_i\|$. By re-ordering the linear combinations we have $\|a_j x_j\| \leq M\|\sum_{i=1}^n a_i x_i\|$ for any $1 \leq j \leq n$. Then,

$$\frac{c}{M} \sup_{1\leq i\leq n} |a_i| \leq \left\|\sum_{i=1}^n a_i x_i\right\|. \tag{3.3}$$

Since $\sum_{i=1}^n |a_n| = 1$, we know that $\frac{1}{n} \leq \sup_{1\leq i\leq n} |a_i| \leq 1$ and, putting together Equations (3.2) and (3.3),

$$\frac{c}{nM} \leq \left\|\sum_{i=1}^n a_i x_i\right\| \leq nC.$$

Notice that these bounds are independent of our starting choice of indices for the vectors $x_1, ..., x_n$. However, one "issue" remains: our desired bounds K are dependent of n, the size of the linear combinations we are considering. This turns out to be unavoidable, which leads us to use as a starting point a *looser* notion of invariance.

Definition 3.1.4. We define[2] a sequence to be **loosely spreading** if for any $n \in \mathbb{N}$ there exists constants $0 < c(n) < C(n)$, depending on n, such that

$$c(n) \leq \left\| \sum_{i=1}^{n} a_i x_{k_i} \right\| \leq C(n)$$

for any choices of natural numbers $k_1 < k_2 < ... < k_n$ and scalars $a_1, ..., a_n$ where $\sum_{i=1}^{n} |a_n| = 1$.

We may then summarize what we verified leading up to this definition as follows:

Example 3.1.5. *Any semi-normalized basic sequence is loosely spreading.*

The next result is our first dive into a combinatorial result in Banach space theory. In its essence, it says that a loosely spreading sequence $(x_n)_{n\in\mathbb{N}}$ must have an *almost invariant under spreading* subsequence, that is, a subsequence $(x_{m_n})_{n\in\mathbb{N}}$ that satisfies Equation (3.4). In fact, it is stronger than this as it is able to handle any pre-determined sequence of ε_n's as opposed to a single fixed one. In precise terms:

Proposition 3.1.6. *Let X be a normed space and $(x_n)_{n\in\mathbb{N}}$ a loosely spreading sequence. Then given any decreasing sequence of positive non-zero real numbers $\varepsilon_1 \geq \varepsilon_2 \geq ...$, there exists an increasing sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that for all $n \geq 1$ and scalars $a_1, ..., a_n$ we have*

$$\left\| \sum_{i=1}^{n} a_i x_{j_i} \right\| \leq (1 + \varepsilon_n) \left\| \sum_{i=1}^{n} a_i x_{k_i} \right\| \tag{3.4}$$

when $j_1, ..., j_n, k_1, ..., k_n$ are elements of $(m_n)_{n\in\mathbb{N}}$ satisfying $m_n < j_1 < ... < j_n$ and $m_n < k_1 < ... < k_n$.

Proof. First we will see that, given any sequence $(A_n)_{n\in\mathbb{N}}$ of non-empty finite sets A_n of scalar n-tuples such that $\sum_{i=1}^{n} |a_i| = 1$ for all $(a_1, ..., a_n) \in A_n$, we can construct $(m_n)_{n\in\mathbb{N}}$ if we reduce the choice of scalars $a_1, ..., a_n$ to the tuples of the set A_n for all $n \in \mathbb{N}$. Later we will see that this is enough.

Let $\varepsilon_1 > \varepsilon_2 > ... > 0$ and take $(A_n)_{n\in\mathbb{N}}$ to be a sequence as described above. We now define sets $N_{k,|A_k|} \subseteq \mathbb{N}$, for all $k \in \mathbb{N}$, recursively, focusing on the linear combinations of m vectors at a time:

[2]Please note this definition was not found by the author in the literature, we merely associated this terminology to simplify our notation in the following also already well established results.

- For $m = 1$, the only possible element of A_1 is $a_1 = 1$. So, let $N_{1,0} := \mathbb{N}$, since $(x_n)_{n\in\mathbb{N}}$ is loosely spreading, there are $0 < c(1) < C(1)$ such that

$$c(1) \leq \|a_1 x_k\| \leq C(1), \qquad \forall k \in \mathbb{N}.$$

Now, we may partition $[c(1), C(1)]$ into a finite number intervals $P_0, ..., P_{r-1}$ of size at most δ_1 (for now taken to be an arbitrary positive real number).

With this partition we define an r-coloring of $f : [N_{1,0}]^1 \to r$ as

$$f(\{j\}) = q \iff \|a_1 x_j\| \in P_q.$$

By the pigeonhole principle, there is some $q \in \{0, ..., r-1\}$ where $f^{-1}(P_q) = \{n_{1,1}, n_{1,2}, ...\}$ is an infinite subset of $N_{1,0}$. Let $N_1 := f^{-1}(P_q)$ and we are done.

- Suppose that for all $s \leq m$ we have N_s such that

$$\left\|\sum_{i=1}^{s} a_i x_{j_i}\right\| \leq \left(1 + \frac{\varepsilon_s}{2}\right) \left\|\sum_{i=1}^{s} a_i x_{k_i}\right\|$$

for any $(a_1, ..., a_s) \in A_s$ and any $j_1 < ... < j_r$ and $k_1 < ... < k_s$ indices in N_s. We will now define N_{m+1}:

Let $N_{m+1,0} := N_m$ and take $(a_{1,1}, ..., a_{1,m+1}) \in A_{m+1}$. Then there are $0 < c(m+1) < C(m+1)$ satisfying

$$c(m+1) \leq \left\|\sum_{i=1}^{m+1} a_{i,1} x_{j_i}\right\| \leq C(m+1)$$

for all $j_1 < ... < j_{m+1}$ taken from $N_{m+1,0}$. We may then partition $[c(m+1), C(m+1)]$ into intervals $P_0, ..., P_{r-1}$ of size at most δ_{m+1} and define a coloring $f : [N_{m+1,0}]^{m+1} \to r$ where

$$f(\{j_1, ..., j_{m+1}\}) = q \iff \left\|\sum_{i=1}^{m+1} a_{i,1} x_{j_i}\right\| \in P_q.$$

By Ramsey's theorem (2.1.3), there is an infinite monochromatic subset of $N_{m+1,0}$ which we'll denote by $N_{m+1,1}$. Now, repeat this process with any $(a_{2,1}, ..., a_{2,m+1}) \in A_{m+1} \setminus \{(a_{1,1}, ..., a_{m+1,1})\}$ in place of

$(a_{1,1}, ..., a_{1,m+1})$ and $N_{m+1,1}$ instead of $N_{m+1,0}$. By iterating this we reach the set $N_{m+1,|A_{m+1}|}$, which we'll call N_{m+1}. The sets

$$N_{m+1,|A_{m+1}|} \subseteq N_{m+1,2} \subseteq ... \subseteq N_{m+1,1}$$

share the property that, for any $1 \leq s \leq |A_{m+1}|$,

$$\left| \left\| \sum_{i=1}^{m+1} a_{s,i} x_{j_i} \right\| - \left\| \sum_{i=1}^{m+1} a_{s,i} x_{k_i} \right\| \right| < \delta_{m+1}$$

for any $j_1 < ... < j_{m+1}$ and $k_1 < ... < k_{m+1}$ in $N_{m+1,s}$. Since each one of the sets $N_{m+1,s}$ is a subset of the previous, if one chooses the indices in N_{m+1} the inequality above holds for any choice of scalars in A_{m+1}.

Once again if we let $\delta_{m+1} := \varepsilon_{m+1} c(m+1)/2$, we are able to verify that, as before,

$$\left\| \sum_{i=1}^{m+1} a_i x_{j_i} \right\| \leq \left(1 + \frac{\varepsilon_{m+1}}{2}\right) \left\| \sum_{i=1}^{m+1} a_i x_{k_i} \right\|$$

holds for any $(a_1, ..., a_{m+1}) \in A_{m+1}$ and all choices of indices $j_1 < ... < j_{m+1}$ and $k_1 < ... < k_{m+1}$ in N_{m+1}.

By the sufficiency of considering the sets A_n (yet to be proven) what we have achieved now is a sequence of sets $N_1 \supseteq N_2 \supseteq N_3 \supseteq ...$ satisfying that, for any $n \geq 1$ and *any* choice of scalars $a_1, ..., a_n$,

$$\left\| \sum_{i=1}^{n} a_i x_{j_i} \right\| \leq \left(1 + \frac{\varepsilon_n}{2}\right) \left\| \sum_{i=1}^{n} a_i x_{k_i} \right\| \tag{3.5}$$

whenever $j_1 < ... < j_n$ and $k_1 < ... < k_n$ are taken in N_n. Since we want one definitive sequence $(m_n)_{n \in \mathbb{N}}$ that handles all choices of n, we simply define it as the diagonal sequence of $(N_n)_{n \in \mathbb{N}}$ (for all $n \in \mathbb{N}$ we take m_n to be the n-th element of N_n) and so we have the property we sought.

Finally, we need only verify that there is indeed a sequence $(A_n)_{n \in \mathbb{N}}$ of finite subsets of $\mathbb{N}$ that allows us to make the jump to the assertion of Equation (3.5) from what preceded:

Fix any $n \geq 1$ and let S^n be the set of all scalar n-tuples such that $\sum_{i=1}^{n} |a_i| = 1$. That is, the unit sphere of ℓ_1^n. Given that $(x_n)_{n \in \mathbb{N}}$ is loosely spreading, let $C(1) > 0$ be such that $\|x_i\| \leq C(1)$ for all $i \in \mathbb{N}$. And since

ℓ_1^n is finite-dimensional, S^n is compact. By covering it with balls of radius $\delta/C(1)$ (where δ may depend only on n) around each point, we may take a finite subcover. Let A_n be the set of vectors at the center of the balls of this subcover.

To see that this construction of A_n suffices, suppose Equation (3.5) is satisfied for all scalars in A_n and take some $(a_1, ..., a_n) \in S^n$. Then, there is $(a_1', ..., a_n')$ in A_n such that

$$\|(a_1, ..., a_n) - (a_1', ..., a_n')\|_Z \leq \delta/C(1)$$

and for any $j_1 < ... < j_n$,

$$\begin{aligned}\left\|\sum_{i=1}^n a_i x_{j_i} - \sum_{i=1}^n a_i' x_{j_i}\right\| &\leq \sum_{i=1}^n |a_i - a_i'| \|x_{j_i}\| \\ &\leq \|(a_1, ..., a_n) - (a_1', ..., a_n')\|_Z C(1) \leq \delta.\end{aligned}$$

Thus, by the inverse triangle inequality

$$\left| \left\|\sum_{i=1}^n a_i x_i\right\| - \left\|\sum_{i=1}^n a_i' x_i\right\| \right| \leq \delta.$$

Finally, choose $\delta = (\varepsilon_n c(n))/(4 + \varepsilon_n)$, then

$$\begin{aligned}\left\|\sum_{i=1}^n a_i x_{j_i}\right\| &\leq \left\|\sum_{i=1}^n a_i' x_{j_i}\right\| + \delta \leq \left(1 + \frac{\varepsilon_n}{2}\right) \left\|\sum_{i=1}^n a_i' x_{k_i}\right\| + \delta \\ &\leq \left(1 + \frac{\varepsilon_n}{2}\right) \left(\left\|\sum_{i=1}^n a_i x_{k_i}\right\| + \delta \right) + \delta \\ &= \left(1 + \frac{\varepsilon_n}{2}\right) \left\|\sum_{i=1}^n a_i x_{k_i}\right\| + \frac{4 + \varepsilon_n}{2}\delta \\ &= \left(1 + \frac{\varepsilon_n}{2}\right) \left\|\sum_{i=1}^n a_i x_{k_i}\right\| + \frac{\varepsilon_n}{2}c(n) \\ &= (1 + \varepsilon_n) \left\|\sum_{i=1}^n a_i x_{k_i}\right\|\end{aligned}$$

for any $j_1 < ... < j_n$ and $k_1 < ... < k_n$ in N_n. □

Remark. *Notice that in Equation (3.4) the choice of* ε_n *to be used in the inequality depends solely on the fact that* $j_1, k_1 > m_n$. *Although the sums also depend on the value of* n, *one may simply choose to sum over* $a_1, ..., a_t$ *for any* $t < n$ *and let* $a_{t+1}, ..., a_n$ *be* 0.

One condition for this property of "almost invariance under spreading", is that on both sides of the inequality (3.4) one must have linear combinations with the same number of terms. In the following corollary we use our almost invariant under spreading sequence to create another sequence that can handle a similar, but "unbalanced" (in the number of terms), inequality.

Corollary 3.1.7. *Let* $(m_n)_{n\in\mathbb{N}}$ *be the sequence given in Proposition 3.1.6. Then the sequence*

$$y_n := x_{m_{2n}} - x_{m_{2n-1}}, \qquad \textit{for all } n \in \mathbb{N}$$

satisfies

$$\left\|\sum_{i\in A} a_i y_i\right\| \leq (1+\varepsilon_r)^2 \left\|\sum_{i\in B} a_i y_i\right\|$$

given any $r \in \mathbb{N}$ *and finite sets* $A \subseteq B \subseteq \mathbb{N}$ *such that* $\min B \geq |B| \geq r$.

Proof. Fix a finite set of natural numbers $B = \{i_1, ..., i_r\}$ where $i_1 < ... < i_r$ for some $r \in \mathbb{N}$ and scalars $\{a_i : i \in B\}$. Now, for every $q \in \mathbb{N}$ we will select indices of $\{m_n : n \in \mathbb{N}\}$ according to a particular function

$$j_q : (A \times \{1,2\}) \cup ((B \setminus A) \times \{1, ..., q+1\}) \to \{m_n : n \in \mathbb{N}\}$$

defined as follows: Let i_1 be the smallest element of B and let $t_1 := |B|(q+1)$. We then define

$$j_q(i_1, k) := m_{t_1+k} \tag{3.6}$$

for every $k \in \{1,2\}$ if $i_1 \in A$ or $k \in \{1, ..., q+1\}$ if $i_1 \in B \setminus A$. Then, let t_2 be the index of the greatest number of the form "$j_q(i_1, k)$". That is,

$$t_2 := \begin{cases} i_1 + 2, & \text{if } i_1 \in A, \\ i_1 + q + 1, & \text{if } i_1 \notin A. \end{cases}$$

We proceed defining

$$j_q(i_2, k) := m_{t_2+k}$$

for every $k \in \{1,2\}$ if $i_2 \in A$ or $k \in \{1,...,q+1\}$ if $i_2 \in B \setminus A$. We iterate this process from Equation (3.6) with t_3 instead of t_2 and so on until j_q is defined for all of its domain.

With these functions in hand, see that for every $k \in \{1,...,q\}$, by Proposition 3.1.6,

$$\left\| \sum_{i\in A} a_i(x_{j_q(i,2)} - x_{j_q(i,1)}) + \sum_{i\in B\setminus A} a_i(x_{j_q(i,k+1)} - x_{j_q(i,k)}) \right\| \leq (1+\varepsilon_r) \left\| \sum_{i\in B} a_i y_i \right\|.$$

Then let $q \in \mathbb{N}$ and $n \in \mathbb{N}$ be arbitrary with $n \leq q$. We will use this to expand the expression

$$\left\| n\sum_{i\in A} a_i(x_{j_q(i,2)} - x_{j_q(i,1)}) + \sum_{k=1}^{n}\sum_{i\in B\setminus A} a_i(x_{j_q(i,k+1)} - x_{j_q(i,k)}) \right\| \tag{3.7}$$

with the triangle inequality as

$$\begin{aligned}(3.7) &\leq \sum_{k=1}^{n} \left\| \sum_{i\in A} a_i(x_{j_q(i,2)} - x_{j_q(i,1)}) + \sum_{i\in B\setminus A} a_i(x_{j_q(i,k+1)} - x_{j_q(i,k)}) \right\| \\ &\leq \sum_{k=1}^{n}(1+\varepsilon_r) \left\| \sum_{i\in B} a_i y_i \right\| = n(1+\varepsilon_r) \left\| \sum_{i\in B} a_i y_i \right\|.\end{aligned}$$

By dividing both sides by n, we have

$$\left\| \sum_{i\in A} a_i(x_{j_q(i,2)} - x_{j_q(i,1)}) + \frac{1}{n} \sum_{i\in B\setminus A} a_i(x_{j_q(i,q+1)} - x_{j_q(i,1)}) \right\| \leq (1+\varepsilon_r) \left\| \sum_{i\in B} a_i y_i \right\|.$$

By the fact that $(x_n)_{n\in\mathbb{N}}$ is loosely spreading, there is some $C > 0$ such that $\| \sum_{i\in B\setminus A} a_i(x_{j_q(i,q+1)} - x_{j_q(i,1)})\| < C$ for any $q \in \mathbb{N}$. By the reverse triangle inequality,

$$\left\| \sum_{i\in A} a_i(x_{j_q(i,2)} - x_{j_q(i,1)}) \right\| - \frac{1}{n} \left\| \sum_{i\in B\setminus A} a_i(x_{j_q(i,q+1)} - x_{j_q(i,1)}) \right\| \leq (1+\varepsilon_r) \left\| \sum_{i\in B} a_i y_i \right\|$$

and

$$\left\| \sum_{i\in A} a_i(x_{j_q(i,2)} - x_{j_q(i,1)}) \right\| - \frac{1}{n} C \leq (1+\varepsilon_r) \left\| \sum_{i\in B} a_i y_i \right\|.$$

Now, to get the inequality we desire, we need only deal with the specific indices of x on the left-hand side, which is no problem when we have almost invariance under spreading. That is, by Proposition 3.1.6, for all $q \in \mathbb{N}$ we have $r < j_q(i_1, 1) < \ldots < j_q(i_r, k)$ (for either $k = 2$ if $i_r \in A$ or $k = q + 1$ if $i_r \notin A$) and

$$\left\|\sum_{i\in A} a_i y_i\right\| - \frac{1}{n}C \leq (1+\varepsilon_r)\left\|\sum_{i\in A} a_i(x_{j_q(i,2)} - x_{j_q(i,1)})\right\| - \frac{1+\varepsilon_r}{n}C$$
$$\leq (1+\varepsilon_r)^2\left\|\sum_{i\in B} a_i y_i\right\|.$$

Since this holds for q and n arbitrarily large, we are left with

$$\left\|\sum_{i\in A} a_i y_i\right\| \leq (1+\varepsilon_r)^2\left\|\sum_{i\in B} a_i y_i\right\|.$$

□

To move forward in "refining" the properties of a loosely spreading sequence, we will need to rely on a looser connection between the sequence we will achieve and our original sequence than simply taking subsequences. This "connection" we aim to maintain as we strive for sequences with better properties is defined in what follows.

Definition 3.1.8. Let X and Y be infinite-dimensional Banach spaces. We say that Y is K-**finitely representable** in X if for every finite dimensional subspace E of Y there exists a subspace F of X and an isomorphism $T : E \to F$ with distortion no greater than K, i.e. $\|T\| \cdot \|T^{-1}\| \leq K$. Y is said to be **crudely finitely representable** on X if it is K-finitely representable in X for some $K > 1$. Lastly, Y is said **finitely representable** in X if it is K-finitely representable for all $K > 1$.

The concept of finite representability was introduced by James in 1967 [13] and serves as the local counterpart to a Banach space X containing an isomorphic copy of an (infinite-dimensional) Banach space Y. As our approach makes great use of sequence, we translate these notions to this context.

Definition 3.1.9. Let X and Y be infinite-dimensional Banach spaces. We say that a sequence $(y_n)_{n\in\mathbb{N}}$ in Y is **K-block finitely representable** on a sequence $(x_n)_{n\in\mathbb{N}}$ in X if for each $n \in \mathbb{N}$ there exists $u_1, ..., u_n$ vectors in X of the form $u_j = \sum_{i=m_{j-1}}^{m_j - 1} \alpha_i x_i$ for all $j \in \{1, ..., n\}$ (where $m_0 < m_2 < ... < m_n$), such that

$$\left\| \sum_{i=1}^{n} a_i u_i \right\| \leq \left\| \sum_{i=1}^{n} a_i y_i \right\| \leq K \left\| \sum_{i=1}^{n} a_i u_i \right\| \tag{3.8}$$

holds for all scalars $a_1, ..., a_n$. Similarly, if $(y_n)_{n\in\mathbb{N}}$ is K-block finitely representable in $(x_n)_{n\in\mathbb{N}}$ for some K, we say that $(y_n)_{n\in\mathbb{N}}$ is **crudely block finitely representable** on $(x_n)_{n\in\mathbb{N}}$. If it is so for all $K > 1$, we say that $(y_n)_{n\in\mathbb{N}}$ is **block finitely representable** on $(x_n)_{n\in\mathbb{N}}$.

The fact that for block finite representability we want to prove the existence of blocks, gives us some flexibility in how we may state the above definition. That is, there are blocks $z_1, ..., z_n$ for which Equation (3.8) holds for a given arbitrary $K > 1$ if, and only if, there are blocks $\tilde{u}_1, ..., \tilde{u}_n$ ($\tilde{u}_i = \sqrt{K} z_i$, for $1 \leq i \leq n$) where

$$\frac{1}{\sqrt{K}} \left\| \sum_{i=1}^{n} a_i \tilde{u}_i \right\| \leq \left\| \sum_{i=1}^{n} a_i y_i \right\| \leq \sqrt{K} \left\| \sum_{i=1}^{n} a_i \tilde{u}_i \right\|.$$

holds for all scalars $a_1, ..., a_n$. Which in turn means that, for block finite representability, Equation (3.8) is equivalent to saying that for any $\varepsilon > 0$ there are blocks $u_1, ..., u_n$ such that

$$\frac{1}{1+\varepsilon} \left\| \sum_{i=1}^{n} a_i u_i \right\| \leq \left\| \sum_{i=1}^{n} a_i y_i \right\| \leq (1+\varepsilon) \left\| \sum_{i=1}^{n} a_i u_i \right\|$$

holds for all scalars $a_1, ..., a_n$.

It is easy to see that the relation of block finite representability is transitive. To ease our work in what follows, we frame it as a Lemma:

Lemma 3.1.10. *Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be sequences in Banach spaces X, Y and Z, respectively. Then, if $(y_n)_{n\in\mathbb{N}}$ is block finitely representable in $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ is block finitely representable in $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ is block finitely representable in $(x_n)_{n\in\mathbb{N}}$.*

Proof. Suppose $(y_n)_{n\in\mathbb{N}}$ is block finitely representable in $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ is block finitely representable in $(y_n)_{n\in\mathbb{N}}$. Fix $n \in \mathbb{N}$ and some $\varepsilon > 0$, then there are blocks $u_1, ..., u_n$ and $v_1, ..., v_n$ of $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, respectively, such that

$$\frac{1}{1+\varepsilon}\left\|\sum_{i=1}^{n} a_i u_i\right\| \leq \left\|\sum_{i=1}^{n} a_i y_i\right\| \leq (1+\varepsilon)\left\|\sum_{i=1}^{n} a_i u_i\right\| \tag{3.9}$$

and

$$\frac{1}{1+\varepsilon}\left\|\sum_{i=1}^{n} a_i v_i\right\| \leq \left\|\sum_{i=1}^{n} a_i z_i\right\| \leq (1+\varepsilon)\left\|\sum_{i=1}^{n} a_i v_i\right\| \tag{3.10}$$

for any $n \in \mathbb{N}$ and scalars $a_1, ..., a_n$. For all $1 \leq j \leq n$ we may write $v_j = \sum_{i=m_{j-1}}^{m_j-1} \alpha_j y_i$ where $m_0 < ... < m_n$ are natural numbers and α_i scalars for all $i \in \{1, ..., m_n - 1\}$. And, clearly, $\tilde{u}_j := \sum_{i=m_{j-1}}^{m_j-1} \alpha_j u_i$, for all $j \in \mathbb{N}$, defines a sequence of disjoint successive blocks of $(x_n)_{n\in\mathbb{N}}$. So, if we put together (3.9) and (3.10) we have that for any $\varepsilon > 0$

$$\frac{1}{(1+\varepsilon)^2}\left\|\sum_{j=1}^{n} a_j \sum_{i=m_{j-1}}^{m_j-1} \alpha_j u_i\right\| \leq \left\|\sum_{i=1}^{n} a_i z_i\right\| \leq (1+\varepsilon)^2\left\|\sum_{j=1}^{n} a_j \sum_{i=m_{j-1}}^{m_j-1} \alpha_j u_i\right\|$$

holds for all scalars $a_1, ..., a_n$. Since the choices of $\varepsilon > 0$ and $n \in \mathbb{N}$ were arbitrary, we have that $(z_n)_{n\in\mathbb{N}}$ is block finitely representable in $(x_n)_{n\in\mathbb{N}}$. □

Proposition 3.1.11. *Let $(x_n)_{n\in\mathbb{N}}$ be a loosely spreading sequence. Then there exists a sequence $(\overline{y}_n)_{n\in\mathbb{N}}$ (of nonzero vectors) in some Banach space Y which is block finitely representable in $(x_n)_{n\in\mathbb{N}}$ and*

i. Is invariant under spreading.

ii. Satisfies

$$\left\|\sum_{i\in A} a_i \overline{y}_i\right\| \leq \left\|\sum_{i\in B} a_i \overline{y}_i\right\|$$

for all $A \subseteq B \subseteq \mathbb{N}$ with B finite and any scalars $\{a_i : i \in B\}$.

Proof. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive non-zero real numbers converging to 0 and $(y_n)_{n\in\mathbb{N}}$ be the sequence given by Corollary 3.1.7 for $(x_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$. We define a norm $\|\cdot\|_0$ on c_{00}, the space of finitely supported scalar sequences, as follows:

$$\left\|\sum_{i=1}^{m} a_i e_i\right\|_0 := \lim_{n\to\infty}\left\|\sum_{i=1}^{m} a_i y_{n+i}\right\| \tag{3.11}$$

where $(e_n)_{n\in\mathbb{N}}$ is the unit vector basis on c_{00}. To see that the limit does in fact exist for an arbitrary $\sum_{i=1}^m a_ie_i$, we first note that any linear combination of $(y_n)_{n\in\mathbb{N}}$ is a linear combination of $(x_n)_{n\in\mathbb{N}}$ and thus it is also loosely spreading and satisfies property (3.4) for some sequence $(m_n)_{n\in\mathbb{N}}$. Then, for $m_s \geq m$

$$\left\|\sum_{i=1}^m a_iy_{j_i}\right\| - \left\|\sum_{i=1}^m a_iy_{k_i}\right\| \leq \varepsilon_s \left\|\sum_{i=1}^m a_iy_{k_i}\right\| \leq \varepsilon_s \sum_{i=1}^m |a_i|\|y_i\| \leq \varepsilon_s \sum_{i=1}^m |a_i|C(2) \tag{3.12}$$

whenever $m_s < j_1 < ... < j_m$ and $m_s < k_1 < ... < k_m$ are elements of $(m_n)_{n\in\mathbb{N}}$. As this would work analogously if we switched the places of $j_1, ..., j_m$ and $k_1, ..., k_m$, we have

$$\left|\left\|\sum_{i=1}^m a_iy_{j_i}\right\| - \left\|\sum_{i=1}^m a_iy_{k_i}\right\|\right| \leq \varepsilon_s \sum_{i=1}^m |a_i|C(2).$$

Given that $\sum_{i=1}^m |a_i|C(2)$ is fixed we can choose s to make ε_s arbitrarily small and then for all $r, t \geq s$ we'll have that

$$\left|\left\|\sum_{i=1}^m a_iy_{r+i}\right\| - \left\|\sum_{i=1}^m a_iy_{t+i}\right\|\right|$$

is as small as we wished, which means that $(\|\sum_{i=1}^m a_iy_{n+i}\|)_{n\in\mathbb{N}}$ is a Cauchy sequence and the limit in Equation (3.11) does exist.

Let Y be the completion of $(c_{00}, \|\cdot\|_0)$ and represent by $(\overline{y}_n)_{n\in\mathbb{N}}$ the sequence $(e_{n+1} - e_n)_{n\in\mathbb{N}}$. Then, whenever we have $j_1 < ... < j_m$,

$$\left\|\sum_{i=1}^m a_ie_i\right\|_0 = \lim_{n\to\infty}\left\|\sum_{i=1}^m a_iy_{n+i}\right\| = \lim_{n\to\infty}\left\|\sum_{i=1}^m a_iy_{n+j_i}\right\| = \left\|\sum_{i=1}^m a_ie_{j_i}\right\|_0.$$

So $(e_n)_{n\in\mathbb{N}}$ is in fact invariant under spreading. And from this it follows that $(\overline{y}_n)_{n\in\mathbb{N}}$ is also 1-spreading:

$$\left\|\sum_{i=1}^m a_i\overline{y}_i\right\|_0 = \left\|\sum_{i=1}^m a_i(e_{i+1} - e_i)\right\|_0 = \left\|\sum_{i=1}^m a_i(e_{j_i+1} - e_{j_i})\right\|_0 = \left\|\sum_{i=1}^m a_i\overline{y}_{j_i}\right\|_0.$$

To prove the second property, first consider $A \subseteq B \subseteq \mathbb{N}$ finite subsets and fix some choice of scalars $\{a_i : i \in B\}$. Now, for each n we construct a family $\{\sigma_k : k \in B\}$ of subsets of $\mathbb{N}$ satisfying the following:

- If $k \in A$ we let $\sigma_k = \{r_{k,1}, r_{k,2}\}$ for two natural numbers $r_{k,1} < r_{k,2}$.
- If $k \in B \setminus A$, we let $\sigma_k = \{r_{k,1}, ..., r_{k,n+1}\}$ where $r_{k,1} < ... < r_{k,n+1}$ are natural numbers.
- If $k_1, k_2 \in B$ and $k_1 \leq k_2$ then $\max \sigma_{k_1} < \min \sigma_{k_2}$.

To construct a family like this we can write $B = \{t_1, ..., t_r\}$ for some $r \in \mathbb{N}$ and satisfying $t_1 < ... < t_r$, and then define the sets σ_k along the order of the natural numbers, always choosing σ_{k+1} as to satisfy the last point above. Since we always take finite natural numbers we can always proceed to the next step up until we've constructed σ_{t_r} and we're done.

With these sets and the fact that $(e_n)_{n\in\mathbb{N}}$ is 1-spreading, we have that for all $1 \leq s \leq n$

$$\begin{aligned}\left\|\sum_{i\in B} a_i \overline{y}_i\right\|_0 &= \left\|\sum_{i\in B} a_i(e_{i+1} - e_i)\right\|_0 \\ &= \left\|\sum_{i\in A} a_i(e_{r_{i,2}} - e_{r_{i,1}}) + \sum_{i\in B\setminus A} a_i(e_{r_{i,s+1}} - e_{r_{i,s}})\right\|_0\end{aligned}$$

If we sum these equalities we get

$$\begin{aligned}n\left\|\sum_{i\in B} a_i \overline{y}_i\right\|_0 &= \sum_{s=1}^{n}\left\|\sum_{i\in A} a_i(e_{r_{i,2}} - e_{r_{i,1}}) + \sum_{i\in B\setminus A} a_i(e_{r_{i,s+1}} - e_{r_{i,s}})\right\|_0 \\ &\geq \left\|n\sum_{i\in A} a_i(e_{r_{i,2}} - e_{r_{i,1}}) + \sum_{s=1}^{n}\sum_{i\in B\setminus A} a_i(e_{r_{i,s+1}} - e_{r_{i,s}})\right\|_0 \\ &= \left\|n\sum_{i\in A} a_i(e_{r_{i,2}} - e_{r_{i,1}}) + \sum_{i\in B\setminus A} a_i(e_{r_{i,n+1}} - e_{r_{i,1}})\right\|_0 \\ &\geq n\left\|\sum_{i\in A} a_i(e_{r_{i,2}} - e_{r_{i,1}})\right\|_0 - \left\|\sum_{i\in B\setminus A} a_i(e_{r_{i,n+1}} - e_{r_{i,1}})\right\|_0\end{aligned}$$

and so, using again the invariance under spreading of $(e_n)_{n\in\mathbb{N}}$,

$$n\left\|\sum_{i\in B} a_i\overline{y}_i\right\|_0 \geq n\left\|\sum_{i\in A} a_i(e_{r_{i,2}}-e_{r_{i,1}})\right\|_0 - \sum_{i\in B\setminus A}|a_i|\left\|(e_{r_{i,n+1}}-e_{r_{i,1}})\right\|_0$$
$$\geq n\left\|\sum_{i\in A} a_i\overline{y}_i\right\|_0 - 2\sum_{i\in B\setminus A}|a_i|\left\|e_1\right\|_0.$$

From the fact that $2\sum_{i\in B\setminus A}|a_i|\left\|e_1\right\|_0$ has a fixed value, and

$$\left\|\sum_{i\in A} a_i\overline{y}_i\right\|_0 - \frac{2}{n}\sum_{i\in B\setminus A}|a_i|\left\|e_1\right\|_0 \leq \left\|\sum_{i\in B} a_i\overline{y}_i\right\|_0$$

holds for all $n\in\mathbb{N}$, we have what we wished to prove.

Lastly, we verify that $(\overline{y}_n)_{n\in\mathbb{N}}$ is block finitely representable in $(x_n)_{n\in\mathbb{N}}$. Since y_n is equal to $x_{m_{2n}}-x_{m_{2n-1}}$, for all $n\in\mathbb{N}$, by Lemma 3.1.10, it is clearly enough to show that $(\overline{y}_n)_{n\in\mathbb{N}}$ is block finitely representable in $(y_n)_{n\in\mathbb{N}}$. Similarly, given that $(\overline{y}_n)_{n\in\mathbb{N}}=(e_{n+1}-e_n)_{n\in\mathbb{N}}$, we need only show that $(e_n)_{n\in\mathbb{N}}$ is block finitely representable in $(y_n)_{n\in\mathbb{N}}$.

Fix $\varepsilon>0$ and $n\in\mathbb{N}$ and let D be an arbitrary but finite set of scalar n-tuples satisfying $\sum_{i=1}^n|a_n|=1$. From what we showed in Proposition 3.1.6, we know that it is enough to prove that there are blocks $u_1,...,u_n$ of $(y_n)_{n\in\mathbb{N}}$ such that for all scalar n-tuples $(a_1,...,a_n)\in D$ we have

$$\frac{1}{1+\varepsilon}\left\|\sum_{i=1}^n a_iu_i\right\| \leq \left\|\sum_{i=1}^n a_ie_i\right\|_0 \leq (1+\varepsilon)\left\|\sum_{i=1}^n a_iu_i\right\|.$$

To see this, consider some $\delta>0$. From the definition of $\|\cdot\|_0$ we know that for each $(a_1,...,a_n)\in D$, there is $m(a_1,...,a_n)$ such that for all $s\geq m(a_1,...,a_n)$,

$$\left|\left\|\sum_{i=1}^n a_iy_{s+i}\right\| - \left\|\sum_{i=1}^n a_ie_i\right\|_0\right| < \delta.$$

Let $m:=\max\{m(a_1,...,a_n):(a_1,...,a_n)\in D\}$, $u_i=y_{m+i}$ for $1\leq i\leq n$ and

$$c_1=\min\left\{\left\|\sum_{i=1}^n a_ie_i\right\|_0 : (a_1,...,a_n)\in D\right\}$$

$$c_2 = \min\left\{\left\|\sum_{i=1}^{n} a_i u_i\right\| : (a_1, ..., a_n) \in D\right\}.$$

It is easily seen from the definition of $(y_n)_{n\in\mathbb{N}}$, and from the definition of the norm $\|\cdot\|_0$ that $(u_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ are both loosely spreading, from which it follows that c_1 and c_2 are both non-zero. So, let $\delta = \varepsilon \min\{c_1, c_2\}$. Then,

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i y_{m+i}\right\| &\le \left\|\sum_{i=1}^{n} a_i e_i\right\|_0 + \delta \le \left\|\sum_{i=1}^{n} a_i e_i\right\|_0 + \varepsilon c_1 \\ &\le (1+\varepsilon)\left\|\sum_{i=1}^{n} a_i e_i\right\|_0\end{aligned}$$

So,

$$\frac{1}{1+\varepsilon}\left\|\sum_{i=1}^{n} a_i u_i\right\| \le \left\|\sum_{i=1}^{n} a_i e_i\right\|_0.$$

Similarly,

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i e_i\right\|_0 &\le \left\|\sum_{i=1}^{n} a_i y_{m+i}\right\| + \delta \le \left\|\sum_{i=1}^{n} a_i u_i\right\| + \varepsilon c_2 \\ &\le (1+\varepsilon)\left\|\sum_{i=1}^{n} a_i u_i\right\|.\end{aligned}$$

□

One feature of this sequence $(\overline{y}_n)_{n\in\mathbb{N}}$ is that the choice of signs in the linear combinations $\sum_{i=1}^{n} a_i \overline{y}_i$ has a limited influence on the norm of the resulting linear combination. That is,

Corollary 3.1.12. *Let $(x_n)_{n\in\mathbb{N}}$ be a loosely spreading sequence. Then, $(\overline{y}_n)_{n\in\mathbb{N}}$ defined in Proposition 3.1.11 satisfies*

$$\left\|\sum_{i\in A} a_i \overline{y}_i\right\| \le 2\left\|\sum_{i\in B} \epsilon_i a_i \overline{y}_i\right\|.$$

for any given finite sets $A \subseteq B \subseteq \mathbb{N}$, all scalars a_i and all signs $\epsilon_i \in \{-1, 1\}$, where $i \in B$.

Proof. Let $A = \{j_1, ..., j_n\} \subseteq B$ be finite subsets of $\mathbb{N}$, $\{a_i : i \in B\}$ a set of scalars and $\{\epsilon_i : i \in B\}$ a set of signs. We define $D = \{1, ..., 2n\}$ and let $d_{2i-1} := a_{j_i}$ and $d_{2i} = -a_{j_i}$ for all $i \in \{1, ..., n\}$. Now, by using both properties given by Proposition 3.1.11, we verify:

$$\left\| \sum_{i\in A} a_i \overline{y}_i \right\| = \left\| \sum_{i=1}^{n} a_{j_i} \overline{y}_{j_i} \right\| \leq \left\| \sum_{i=1}^{2n} d_i \overline{y}_i \right\| = \left\| \sum_{i\in D} d_i \overline{y}_i \right\| .$$

We may then separate from $\sum_{i\in D} d_i \overline{y}_i$ the sum of n vectors by choosing, for every $i \in \{1, .., n\}$, d_{2i-1} when $\epsilon_{j_i} = 1$ and d_{2i} whenever $\epsilon_{j_i} = -1$, as to produce a sum with the chosen signs for the scalars:

$$\begin{aligned} \left\| \sum_{i\in A} a_i \overline{y}_i \right\| &\leq \left\| \sum_{i=1}^{n} \epsilon_{j_i} a_{j_i} \overline{y}_{p_i} - \sum_{i=1}^{n} \epsilon_{j_i} a_{j_i} \overline{y}_{q_i} \right\| \\ &\leq \left\| \sum_{i=1}^{n} \epsilon_{j_i} a_{j_i} \overline{y}_{p_i} \right\| + \left\| \sum_{i=1}^{n} \epsilon_{j_i} a_{j_i} \overline{y}_{q_i} \right\| \end{aligned}$$

where $p_1 < ... < p_n$, $q_1 < ... < q_n$ and $\{p_1, ..., p_n\} \cup \{q_1, ..., q_n\} = B$. Since $(\overline{y}_n)_{n\in\mathbb{N}}$ is invariant under spreading,

$$\left\| \sum_{i\in A} a_i \overline{y}_i \right\| \leq \left\| \sum_{i=1}^{n} \epsilon_{j_i} a_{j_i} \overline{y}_{j_i} \right\| + \left\| \sum_{i=1}^{n} \epsilon_{j_i} a_{j_i} \overline{y}_{j_i} \right\| = 2 \left\| \sum_{i\in A} \epsilon_i a_i \overline{y}_i \right\| .$$

If we use the second property of Proposition 3.1.11 we get

$$\left\| \sum_{i\in A} a_i \overline{y}_i \right\| \leq 2 \left\| \sum_{i\in A} \epsilon_i a_i \overline{y}_i \right\| \leq 2 \left\| \sum_{i\in B} \epsilon_i a_i \overline{y}_i \right\| .$$

□

With the next theorem we will study, we aim to relate a loosely spreading sequence $(x_n)_{n\in\mathbb{N}}$ to a sequence with properties stronger than the ones seen so far. We will look for a sequence that, while still being invariant under spreading, is "invariant under different choices of signs" for the scalars in their linear combinations, a condition which is known as 1-unconditionality.

Definition 3.1.13. Let X be a normed space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ is **1-unconditional** if

$$\left\|\sum_{i=1}^{n} \epsilon_i a_i z_i\right\| = \left\|\sum_{i=1}^{n} a_i z_i\right\|$$

for all $n \in \mathbb{N}$, scalars $a_1, ..., a_n$ and $\epsilon_1, ..., \epsilon_n \in \{-1, 1\}$.

As per usual, much of our interest lies in basic sequences and, similarly to invariance under spreading, there are many familiar examples of such sequences which are 1-unconditional:

Example 3.1.14. *The standard Schauder bases for c_0 and ℓ_p for $1 \leq p < \infty$, are 1-unconditional.*

For a basic sequence which is not 1-unconditional, we make use of Example 3.1.3.

Example 3.1.15. *The basic sequence $(g_n)_{n\in\mathbb{N}}$ of Example 3.1.3 is not 1-unconditional.*

Proof.

$$\|g_2 + g_3\| = \|2e_2 + e_3\| \neq \|e_3\| = \| - g_2 + g_3\|.$$

and so $(g_n)_{n\in\mathbb{N}}$ is not 1-unconditional. □

It turns out 1-unconditionality has a number of interesting consequences. One of these, for instance, addresses the apparent advantage real normed spaces have over complex normed spaces, given that we can use these signs to turn any $a \in \mathbb{R}$ into an $|a|$, while the same doesn't work when a has a non-zero imaginary component.

Lemma 3.1.16. *Let X be a normed space and $(x_n)_{n\in\mathbb{N}}$ a K-unconditional sequence on X. Then, for any $n \in \mathbb{N}$ and $a_1, ..., a_n$ scalars we have*

$$\left\|\sum_{i=1}^{n} \alpha_i a_i x_i\right\| \leq \left\|\sum_{i=1}^{n} a_i x_i\right\|,$$

for any $\alpha_1, ..., \alpha_n \in [-1, 1]$. Furthermore, if X is complex,

$$\left\|\sum_{i=1}^{n} \theta_i a_i x_i\right\| \leq 2\left\|\sum_{i=1}^{n} a_i x_i\right\|,$$

for any $\theta_1, ..., \theta_n$ in the complex unit circle.

Proof. Let $n \in \mathbb{N}$ be fixed and take $a_1, ..., a_n$ to be arbitrary scalars.

We begin by proving that the inequality holds if we add α_j to a single $j \in \{1, ..., n\}$ (i.e. $\alpha_i = 1$ for $i \neq j$). Without loss of generality we assume $j = n$. To prove the inequality we are after, we will make use of a specific convex combination:

Fix $\alpha \in [-1, 1]$ and $\beta = (1 - \alpha)/2 \geq 0$. Then, one may check that

$$\begin{aligned}\left\|\sum_{i=1}^{n-1} a_i x_i + \alpha a_n x_n\right\| &= \left\|(1-\beta)\sum_{i=1}^{n} a_i x_i + \beta\left(\sum_{i=1}^{n-1} a_i x_i - a_n x_n\right)\right\| \\ &\leq (1-\beta)\left\|\sum_{i=1}^{n} a_i x_i\right\| + \beta\left\|\sum_{i=1}^{n-1} a_i x_i - a_n x_n\right\|.\end{aligned}$$

By the K-unconditionality of $(x_n)_{n\in\mathbb{N}}$,

$$\left\|\sum_{i=1}^{n-1} a_i x_i + \alpha a_n x_n\right\| \leq (1-\beta)\left\|\sum_{i=1}^{n} a_i x_i\right\| + \beta\left\|\sum_{i=1}^{n} a_i x_i\right\| \leq \left\|\sum_{i=1}^{n} a_i x_i\right\|.$$

Note that this holds for all scalars $a_1, .., a_n$ and $\alpha \in [-1, 1]$ and that it is still true when α is multiplying any a_j with $1 \leq j \leq n$. To get to the general inequality, it is a simple matter of recursively applying this while adding $\alpha_1, \alpha_2, ..., \alpha_n$.

For the second assertion, suppose X is complex, fix $n \in \mathbb{N}$ again and let $\theta_1, ..., \theta_n$ be in the complex unit circle. We may then represent $\theta_j = \alpha_j + i\beta_j$ where $\alpha_j, \beta_j \in [-1, 1]$ for $1 \leq j \leq n$. Then, by the triangle inequality and what we have just proven,

$$\left\|\sum_{i=1}^{n} \theta_i a_i x_i\right\| \leq \left\|\sum_{i=1}^{n} \alpha_i a_i x_i\right\| + \left\|i\sum_{i=1}^{n} \beta_i a_i x_i\right\| \leq 2\left\|\sum_{i=1}^{n} a_i x_i\right\|.$$

□

Lastly, we verify that unconditional sequences behave monotonously.

Lemma 3.1.17. *Let X be a Banach space and $(x_n)_{n\in\mathbb{N}}$ a 1-unconditional sequence in X. Then,*

$$\left\|\sum_{i\in A} a_i x_i\right\| \leq \left\|\sum_{i\in B} a_i x_i\right\|$$

for any finite subsets $A \subseteq B \subseteq \mathbb{N}$ and $\{a_i : i \in B\}$ scalars. In particular, if $x_n \neq 0$ for all $n \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}}$ is a basic sequence.

Proof. Suppose that there is some 1-unconditional sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$, $m \in \mathbb{N}$, $A \subseteq B \subseteq \{1, ..., m\}$ and scalars $\{a_i : i \in B\}$ such that

$$\left\|\sum_{i\in A} a_i x_i\right\| > \left\|\sum_{i\in B} a_i x_i\right\|.$$

Then,

$$\begin{aligned}
2\left\|\sum_{i\in A} a_i x_i\right\| &\leq \left\|\sum_{i\in A} a_i x_i + \sum_{i\in B\setminus A} a_i x_i\right\| + \left\|\sum_{i\in A} a_i x_i - \sum_{i\in B\setminus A} a_i x_i\right\| \\
&= \left\|\sum_{i\in A} a_i x_i + \sum_{i\in B\setminus A} a_i x_i\right\| + \left\|\sum_{i\in A} a_i x_i + \sum_{i\in B\setminus A} a_i x_i\right\| \\
&< 2\left\|\sum_{i\in A} a_i x_i\right\|
\end{aligned}$$

and we arrive at a contradiction.

If $x_n \neq 0$ for all $n \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}}$ satisfies the basic sequence criterion. □

Finally, we consolidate our efforts so far in the following theorem:

Theorem 3.1.18. *Let $(x_n)_{n\in\mathbb{N}}$ be a loosely spreading sequence in a Banach space X. Then there exists a normalized basic sequence $(z_n)_{n\in\mathbb{N}}$ in some Banach space Y which is block finitely representable in $(x_n)_{n\in\mathbb{N}}$ and*

i. Is invariant under spreading.

ii. Is 1*-unconditional.*

Proof. From Proposition 3.1.11 we have a 1-spreading sequence $(\overline{y}_n)_{n\in\mathbb{N}}$, in a Banach space Y, which is block finitely representable in $(x_n)_{n\in\mathbb{N}}$ and satisfies Corollary 3.1.12. With Lemma 3.1.10 we reduce our problem to proving that there is some $(z_n)_{n\in\mathbb{N}}$ which is block finitely representable in $(\overline{y}_n)_{n\in\mathbb{N}}$ (and satisfies the remaining conditions of the statement). It is easy to see that by normalizing $(\overline{y}_n)_{n\in\mathbb{N}}$ we don't lose the block finite representability on $(x_n)_{n\in\mathbb{N}}$, so as a further simplification, we shall assume that it is the case that $\|\overline{y}_n\| = 1$ for all $n \in \mathbb{N}$.

Now we will deal separately with the only two possibilities we have. The first being that

$$\left\|\sum_{i=1}^{n}\overline{y}_i\right\| \xrightarrow{n\to\infty} \infty.$$

In this case, let $k, n \in \mathbb{N}$ be fixed and consider $A_{k,1}, ..., A_{k,n}$ to be ordered tuples of $2k$ consecutive integers, that is,

$$A_{k,i} = (j_i, j_i + 1, ..., j_i + 2k - 1),$$

with $j_i + 2k < j_{i+1}$ for all $i \in \{1, ..., n-1\}$. We will use these integers to define blocks of the form

$$u_{k,i} := \overline{y}_{j_i} - \overline{y}_{j_i+1} + \overline{y}_{j_i+2} - \overline{y}_{j_i+3} + ... - \overline{y}_{j_i+2k-1}.$$

By Corollary 3.1.12, we see that

$$\|u_{k,i}\| \geq \frac{1}{2}\left\|\sum_{i=1}^{2k}\overline{y}_i\right\| \xrightarrow{k\to\infty} \infty. \tag{3.13}$$

To simplify our calculations, we shall work with the normalized blocks $v_{k,i}$, defined as $v_{k,i} := u_{k,i}/\|u_{k,i}\|$.

Given $a_1, ..., a_n$ scalars and $\epsilon_1, ..., \epsilon_n$ in $\{-1, 1\}$, we define for every $A_{k,i}$ the linear isometry $T_i : \text{span}\{\overline{y}_j : j \in A_{k,i}\} \to \text{span}\{\overline{y}_j : j \in A_{k,i}\}$ given by the linear extension of $T_i(\overline{y}_j) = -\overline{y}_{j+1}$, if $\epsilon_i = -1$, and by $T_i = I$, if $\epsilon_i = 1$. Whenever $\epsilon_i = -1$, we have

$$\begin{aligned}\|T_i v_{k,i} - v_{k,i}\| &= \|u_{k,i}\|^{-1}\| - (\overline{y}_{j_i+1} - ... - \overline{y}_{j_i+2k}) - (\overline{y}_{j_i} - ... - \overline{y}_{j_i+2k-1})\| \\ &= \|u_{k,i}\|^{-1}\|\overline{y}_{j_i+2k} - \overline{y}_{j_i}\|\end{aligned}$$

which we may, by the invariance under spreading, rewrite as

$$\|T_i v_{k,i} - v_{k,i}\| = \|u_{k,1}\|^{-1}\|\overline{y}_2 - \overline{y}_1\|.$$

And, when $\epsilon_i = 1$ the difference $\|T_i v_{k,i} - v_{k,i}\|$ is clearly 0. Then

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i T_i(v_{k,i}) - \sum_{i=1}^{n} a_i v_{k,i}\right\| &\leq \|u_{k,1}\|^{-1}\sum_{i=1}^{n}|a_i|\|\overline{y}_2 - \overline{y}_1\| \\ &\leq \|u_{k,1}\|^{-1}\sum_{i=1}^{n}|a_i|(\|\overline{y}_2\| + \|\overline{y}_1\|) \\ &= 2\|u_{k,1}\|^{-1}\sum_{i=1}^{n}|a_i| \\ &\leq 2\|u_{k,1}\|^{-1} n \max_{1\leq i\leq n}|a_i|.\end{aligned}$$

Since $|a_i| \leq \|a_i v_{k,i}\| \leq \|\sum_{j=1}^{n} a_j v_{k,j}\|$ for all $i \in \{1, ..., n\}$ (by Proposition 3.1.11), it holds, for all $k \in \mathbb{N}$, that

$$\left\|\sum_{i=1}^{n} a_i T_i(v_{k,i}) - \sum_{i=1}^{n} a_i v_{k,i}\right\| \leq 2n\|u_{k,1}\|^{-1}\left\|\sum_{i=1}^{n} a_i v_{k,i}\right\|.$$

From Equation (3.13) we know that for any given $m \in \mathbb{N}$ there is some large enough $k(m) \in \mathbb{N}$ such that

$$\left\|\sum_{i=1}^{n} a_i T_i(v_{k(m),i}) - \sum_{i=1}^{n} a_i v_{k(m),i}\right\| \leq \frac{1}{m}\left\|\sum_{i=1}^{n} a_i v_{k(m),i}\right\|$$

and

$$\left(1 - \frac{1}{m}\right)\left\|\sum_{i=1}^{n} a_i v_{k(m),i}\right\| \leq \left\|\sum_{i=1}^{n} a_i T_i(v_{k(m),i})\right\| \leq \left(1 + \frac{1}{m}\right)\left\|\sum_{i=1}^{n} a_i v_{k(m),i}\right\|.$$

Remark. *Note that while we omit the indices k and n, T_i also depends on them. By taking different values of k we are also changing the functions T_i.*

The reason we defined the functions T_i as such was precisely to introduce a given choice of signs $\epsilon_1, ..., \epsilon_n$ to $v_{k,1}, ..., v_{k,n}$. While there was a shift in the indices of $\overline{y}_n$, to make the above calculations possible, it is inconsequential in face of invariance under spreading (keeping in mind that we purposefully left gaps between the sets A_i):

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i T_i(v_{k,i})\right\| &= \left\|\sum_{\substack{1\le i\le n\\ \epsilon_i=1}} a_i\epsilon_i v_{k,i} + \sum_{\substack{1\le i\le n\\ \epsilon_i=-1}} a_i\epsilon_i\|u_{k,i}\|^{-1}(\overline{y}_{j_i+1} - \ldots + \overline{y}_{j_i+2k})\right\| \\ &= \left\|\sum_{\substack{1\le i\le n\\ \epsilon_i=1}} a_i\epsilon_i v_{k,i} + \sum_{\substack{1\le i\le n\\ \epsilon_i=-1}} a_i\epsilon_i\|u_{k,i}\|^{-1}(\overline{y}_{j_i} - \ldots + \overline{y}_{j_i+2k-1})\right\| \\ &= \left\|\sum_{\substack{1\le i\le n\\ \epsilon_i=1}} a_i\epsilon_i v_{k,i} + \sum_{\substack{1\le i\le n\\ \epsilon_i=-1}} a_i\epsilon_i v_{k,i}\right\| = \left\|\sum_{i=1}^{n} a_i\epsilon_i v_{k,i}\right\|.\end{aligned}$$

So,

$$\left(1-\frac{1}{m}\right)\left\|\sum_{i=1}^{n} a_i v_{k(m),i}\right\| \le \left\|\sum_{i=1}^{n} a_i\epsilon_i v_{k(m),i}\right\| \le \left(1+\frac{1}{m}\right)\left\|\sum_{i=1}^{n} a_i v_{k(m),i}\right\|. \tag{3.14}$$

With this we have a sense of “crude uncondionality”. To get a sequence that is actually 1-unconditional, we will use the same argument as in the end of Proposition 3.1.6: For every $k \in \mathbb{N}$,

$$\left\|\sum_{i=1}^{n} a_i v_{k,i}\right\| \le \sum_{i=1}^{n} |a_i|\|v_{k,i}\| = \sum_{i=1}^{n} |a_i|,$$

so it follows that $(\|\sum_{i=1}^{n} a_i v_{k,i}\|)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$ and has a converging subsequence $((\|\sum_{i=1}^{n} a_i v_{k_s,i}\|)_{s\in\mathbb{N}}$. As the subsequence $(k_n)_{n\in\mathbb{N}}$ is dependent on the choice of $(a_1, \ldots, a_n, 0, \ldots) \in c_{00}$ and since c_{00} is separable, just as we did before, we recursively take subsequences $(k_{r+1,n})_{n\in\mathbb{N}}$ of the previous index subsequence $(k_{r,n})_{n\in\mathbb{N}}$ for all vectors in the dense countable subset of c_{00} (we'll assume to be the set of sequences on c_{00} with elements on $\mathbb{Q}$ or $\mathbb{Q}+i\mathbb{Q}$), guaranteeing that for the p-th vector of this set, say $(b_1, \ldots, b_q, 0, \ldots)$, the sequence $(\|\sum_{i=1}^{q} b_i v_{k_{r,n},i}\|)_{n\in\mathbb{N}}$ converges for any choice of $r \ge p$.

We denote by $(k_n)_{n\in\mathbb{N}}$ the diagonal sequence $(k_{n,n})_{n\in\mathbb{N}}$ from the subsequences we got on the previous step. As this sequence still holds the property that for any vector $(b_1, \ldots, b_q, 0, \ldots)$ of the chosen dense subset of c_{00}, $(\|\sum_{i=1}^{q} b_i v_{k_n,i}\|)_{n\in\mathbb{N}}$ converges, we have that this will also work for any vector $(a_1, ..., a_m, 0, ...)$ in c_{00}, since given any $\varepsilon > 0$ we need only take a rational sequence $(b_1, ..., b_m, 0, ...)$ satisfying $\sup_{1\leq i\leq m} |a_i - b_i| < \varepsilon/(4m)$ and n_0 to be such that $\| \sum_{i=1}^m b_i v_{k_n,i} - \sum_{i=1}^m b_i v_{k_{n_0},i}\| < \varepsilon/2$ for all $n \geq n_0$. Then,

$$\begin{aligned}
\left\|\sum_{i=1}^m a_i v_{k_n,i} - \sum_{i=1}^m a_i v_{k_{n_0},i}\right\| \leq& \left\|\sum_{i=1}^m (a_i - b_i)(v_{k_n,i} - v_{k_{n_0},i})\right\| \\
&+ \left\|\sum_{i=1}^m b_i(v_{k_n,i} - v_{k_{n_0},i})\right\| \\
\leq& \sum_{i=1}^m |a_i - b_i| \|v_{k_n,i} - v_{k_{n_0},i}\| + \frac{\varepsilon}{2} \\
\leq& m \sup_{1\leq i\leq m} |a_i - b_i| (\|v_{k_n,i}\| + \|v_{k_{n_0},i}\|) + \frac{\varepsilon}{2} \\
\leq& 2m\frac{\varepsilon}{4m} + \frac{\varepsilon}{2} = \varepsilon, \qquad \forall n \geq n_0.
\end{aligned}$$

With this, we have a function $\| \cdot \|_0 : c_{00} \to [0, +\infty)$ such that

$$\|(a_1, ..., a_m, 0, ...)\|_0 := \lim_{n\to\infty} \left\|\sum_{i=1}^m a_i v_{k_n,i}\right\|.$$

As the notation suggests, this turns out to be a norm. For the most part, the verification of this fact follows easily from the fact that $\| \cdot \|$ is a norm and the limit is linear. And $\|x\|_0 = 0 \iff x = 0$ follows from the properties of $(v_{k_n,i})_{n\in\mathbb{N}}$.

Finally, we take $(e_n)_{n\in\mathbb{N}}$ to be the unit vector basis in the completion Z of $(c_{00}, \| \cdot \|_0)$ and verify the following:

- It is block finitely representable in $(x_n)_{n\in\mathbb{N}}$.
  Let $m \in \mathbb{N}$ and $\varepsilon > 0$, by the same argument using the definition of the norm $\| \cdot \|_0$ as in Proposition 3.1.11.

- It is 1-unconditional.
  Given any choice of signs $\epsilon_1, ..., \epsilon_m \in \{-1, 1\}$ by the inverse triangle

inequality and Equation (3.14)

$$\left| \left\| \sum_{i=1}^{m} a_i e_i \right\|_0 - \left\| \sum_{i=1}^{m} a_i \epsilon_i e_i \right\|_0 \right| = \left| \lim_{n\to\infty} \left\| \sum_{i=1}^{m} a_i v_{k_n,i} \right\| - \lim_{n\to\infty} \left\| \sum_{i=1}^{m} a_i \epsilon_i v_{k_n,i} \right\| \right|$$
$$= \lim_{n\to\infty} \left| \left\| \sum_{i=1}^{m} a_i v_{k_n,i} \right\| - \left\| \sum_{i=1}^{m} a_i \epsilon_i v_{k_n,i} \right\| \right| = 0.$$

- It is 1-spreading.
  It follows from the fact that $(v_n)_{n\in\mathbb{N}}$ is 1-spreading. Take $j_1 < ... < j_n$, then

$$\left\| \sum_{i=1}^{m} a_i e_i \right\|_0 = \lim_{n\to\infty} \left\| \sum_{i=1}^{m} a_i v_{k_n,i} \right\| = \lim_{n\to\infty} \left\| \sum_{i=1}^{m} a_i v_{k_n,j_i} \right\| = \left\| \sum_{i=1}^{m} a_i e_{j_i} \right\|_0 .$$

We now turn to the second case[3], in which there is some $M > 0$ such that

$$\left\| \sum_{i=1}^{n} \overline{y}_i \right\| \le M, \qquad \forall n \in \mathbb{N}.$$

Let $(e_n)_{n\in\mathbb{N}}$ be the unit vector basis in $(c_0, \|\cdot\|_\infty)$, the (Banach) space of scalar sequences converging to zero, with the norm given by

$$\|(a_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}} |a_n|.$$

We will proceed to show that $(e_n)_{n\in\mathbb{N}}$ is the desired $(z_n)_{n\in\mathbb{N}}$ sequence. From Examples 3.1.2 and 3.1.14 we have that $(e_n)_{n\in\mathbb{N}}$ is invariant under spreading and 1-unconditional, which leaves us to prove that it is block finitely representable in $(y_n)_{n\in\mathbb{N}}$:

Let $m \in \mathbb{N}$ and $a_1, ..., a_m$ be scalars. First note that from the fact that $(e_n)_{n\in\mathbb{N}}$ is 1-unconditional and normalized we have

$$\left\| \sum_{i=1}^{m} a_i e_i \right\|_\infty = \sup_{1\le i\le m} |a_i| \le \left\| \sum_{i=1}^{m} a_i \overline{y}_i \right\| .$$

[3]Since $(\|\sum_{i=1}^{n} \overline{y}_i\|)_{n\in\mathbb{N}}$ is non-decreasing, this is the only remaining possibility

By Hahn-Banach's theorem we assert the existence of a norm one bounded linear functional $\varphi : Y \to \mathbb{R}$ such that $\varphi(\sum_{i=1}^m a_i \overline{y}_i) = \| \sum_{i=1}^m a_i \overline{y}_i \|$. Furthermore, we define $\epsilon_i = \varphi(\overline{y_i})/|\varphi(\overline{y_i})|$ when $\varphi(\overline{y_i}) \neq 0$, and $\epsilon_i = 0$ otherwise, for all $i \in \{1, ..., m\}$. By Corollary 3.1.12,

$$\begin{aligned}\left\| \sum_{i=1}^m a_i \overline{y}_i \right\| &= \varphi\left(\sum_{i=1}^m a_i \overline{y}_i \right) \leq \sum_{i=1}^m |a_i| |\varphi(\overline{y}_i)| \\ &\leq \sum_{i=1}^m \max_{1\leq i\leq m} |a_i| \epsilon_i \varphi(\overline{y}_i) = \varphi\left(\sum_{i=1}^m \epsilon_i \overline{y}_i \right) \max_{1\leq i\leq m} |a_i| \\ &\leq \left\| \sum_{i=1}^m \epsilon_i \overline{y}_i \right\| \max_{1\leq i\leq m} |a_i| \leq 2M \left\| \sum_{i=1}^m a_i e_i \right\|_\infty\end{aligned}$$

holds for any $m \in \mathbb{N}$ and any real scalars $a_1, ..., a_m$. If Y is complex and $a_1, ..., a_m$ are arbitrary complex scalars of the form $a_j = \alpha_j + i\beta_j$ for all $j \in \{1, ..., m\}$, then

$$\begin{aligned}\left\| \sum_{j=1}^m (\alpha_j + i\beta_j) \overline{y}_j \right\| &\leq \left\| \sum_{j=1}^m \alpha_j \overline{y}_j \right\| + \left\| i \sum_{j=1}^m \beta_j \overline{y}_j \right\| \\ &\leq 2M \left\| \sum_{j=1}^m \alpha_j e_j \right\|_\infty + 2M \left\| \sum_{j=1}^m \beta_j e_j \right\|_\infty \\ &\leq 4M \left\| \sum_{j=1}^m (\alpha_j + i\beta_j) e_j \right\|_\infty .\end{aligned}$$

Altogether we have

$$\left\| \sum_{i=1}^m a_i e_i \right\|_\infty \leq \left\| \sum_{i=1}^m a_i \overline{y}_i \right\| \leq 4M \left\| \sum_{i=1}^m a_i e_i \right\|_\infty$$

for all $m \in \mathbb{N}$ and scalars $a_1, ..., a_m$. To get to block finite representability, we need only construct the appropriate blocks. To do this, first we define β_n, for all $n \in \mathbb{N}$, to be the least positive real numbers such that

$$\left\| \sum_{i=n}^m a_i \overline{y}_i \right\| \leq \beta_n \left\| \sum_{i=n}^m a_i e_i \right\|_\infty$$

for all $m > n$ and all $a_n, ..., a_m$ scalars. Then $(\beta_n)_{n\in\mathbb{N}}$ is a monotonously decreasing sequence contained in $[1, 4M]$ and converges to some $\beta \geq 1$. Given any fixed $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}$ such that $\beta_n < \sqrt{1+\varepsilon}\beta$ for all $n \geq n_0$.

From the definition of β_{n_0} we know that there is some $n_1 \in \mathbb{N}$ and some choice of scalars $b_{n_0}, ..., b_{n_1-1}$ such that

$$\beta_{n_0} \left\| \sum_{i=n_0}^{n_1-1} b_i e_i \right\|_\infty < \sqrt{1+\varepsilon} \left\| \sum_{i=n_0}^{n_1-1} b_i \overline{y}_i \right\|,$$

We then fix $u_1 := \sum_{i=n_0}^{n_1-1} b_i \overline{y}_i$ which we can, and do, assume to be normalized. Suppose, that $u_k = \sum_{i=n_{k-1}}^{n_k-1} b_i \overline{y}_i$ has norm 1 and choose $n_{k+1} > n_k$ and $b_{n_k}, ..., b_{n_{k+1}-1}$ to be such that $u_{k+1} := \sum_{i=n_k}^{n_{k+1}-1} b_i \overline{y_i}$ has norm 1 and satisfies

$$\beta_{n_k} \left\| \sum_{i=n_k}^{n_{k+1}-1} b_i e_i \right\|_\infty < \sqrt{1+\varepsilon} \left\| \sum_{i=n_k}^{n_{k+1}-1} b_i \overline{y}_i \right\|.$$

By induction we have a normalized sequence of disjoint and successive blocks $u_1, u_2, ...$ of $(\overline{y}_n)_{n\in\mathbb{N}}$. And, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left\| \sum_{i=1}^{n} a_i e_i \right\|_\infty &\geq \max\left\{|a_j| : 1 \leq j \leq n\right\} = \max\left\{ |a_j| \left\| \sum_{i=n_{j-1}}^{n_j-1} b_i \overline{y}_i \right\| : 1 \leq j \leq n \right\} \\
&\geq \max\left\{ |a_j| \frac{\beta_{n_{j-1}}}{\sqrt{1+\varepsilon}} \left\| \sum_{i=n_{j-1}}^{n_j-1} b_i e_i \right\|_\infty : 1 \leq j \leq n \right\} \\
&\geq \frac{\beta}{\beta_{n_0}\sqrt{1+\varepsilon}} \max\left\{ \beta_{n_0} \left\| a_j \sum_{i=n_{j-1}}^{n_j-1} b_i e_i \right\|_\infty : 1 \leq j \leq n \right\} \\
&\geq \frac{1}{1+\varepsilon} \beta_{n_0} \left\| \sum_{j=1}^{n} a_j \sum_{i=n_{j-1}}^{n_j-1} b_i e_i \right\|_\infty \geq \frac{1}{1+\varepsilon} \left\| \sum_{j=1}^{n} a_j \sum_{i=n_{j-1}}^{n_j-1} b_i \overline{y}_i \right\| \\
&\geq \frac{1}{1+\varepsilon} \left\| \sum_{j=1}^{n} a_j u_j \right\|,
\end{aligned}$$

so, by the triangle inequality,

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_\infty \leq \left\| \sum_{i=1}^{n} a_i u_i \right\| \leq (1+\varepsilon) \left\| \sum_{i=1}^{n} a_i e_i \right\|_\infty$$

holds for all scalars $a_1, ..., a_n$. Which means that $(e_n)_{n\in\mathbb{N}}$ is block finitely representable in $(\overline{y}_n)_{n\in\mathbb{N}}$, as required. $\square$

3.2 Spectral theory and ultrapowers

Aside from what we have proved so far, some concepts are still needed. These concepts represent important tools in Banach space theory by themselves. Specifically, we make use of notions from spectral theory and from the study of ultraproducts of Banach spaces. While a proper discussion of the origins of spectral theory requires more paragraphs than we can afford here, and is already neatly done by Steen in [24], we note that the notion of the spectrum we adopt came from the work of Hilbert in the the turn of the twentieth century which largely predates functional analysis itself.

Let us begin with the notion of the spectrum of a bounded linear operator. As it heavily relies on complex analysis, there is a frequent assumption that the Banach spaces discussed in what follows are complex. To better serve the reader, we still mention this fact, when relevant, in the statement of definitions and results.

Definition 3.2.1. Let X be a complex Banach space and $T : X \to X$ a bounded linear operator. We define the set $\rho(T)$, called the **resolvent of** T to be the set of all scalars λ, called **regular values**, for which $T - \lambda I$ is a bijection. We call the set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ the **spectrum of** T and the scalars in $\sigma(T)$ **spectral values**.

By the open mapping theorem, we see that the scalar $\lambda \in \mathbb{C}$ is a regular value of T if, and only if, $T - \lambda I$ is an isomorphism. Intuitively, we think of spectral values as $-\lambda I$ perturbations to operators which disturb them from being isomorphisms.

Clearly, if λ is an eigenvalue of T, that is, $\ker(T - \lambda I) \neq \{0\}$, then $\lambda \in \sigma(T)$. In fact, if X is a finite-dimensional space, the spectrum of T is precisely the set of eigenvalues of T. As we usually assume that a given Banach space is infinite-dimensional, we have more possibilities for spectral values. One way in which these perturbations λI take advantage of the structure of infinite-dimensional spaces, is by "approximating the behavior of an eigenvector through a sequence". By this we mean the following: we call a scalar λ an **approximate eigenvalue** if there is a sequence $(x_n)_{n\in\mathbb{N}}$

in X of norm one vectors, called an **approximate eigenvector**, such that

$$\|Tx_n - \lambda x_n\| \to 0.$$

For finite dimensional complex vector spaces, it is well-known that any linear transformation has eigenvalues. While there are bounded linear operators without eigenvalues, one may question if there can be a bounded linear operator on a complex Banach space with empty spectrum. The answer to this is given by the following theorem.

Theorem 3.2.2. *Let $X \neq \{0\}$ be a complex Banach space and $T : X \to X$ a bounded linear operator. Then the spectrum of T is non-empty and compact in $\mathbb{C}$.*

A reader so inclined as to search for the proof of this fact could check Kreyszig's book [15]. It turns out, we can not only guarantee that the spectrum of a bounded linear operator is non-empty, but also that it contains an approximate eigenvalue.

Lemma 3.2.3. *Let $X \neq \{0\}$ be a complex Banach space and $T : X \to X$ a bounded linear operator. Then $\partial\sigma(T) \neq \emptyset$ and every λ in $\partial\sigma(T)$ (i.e. $\sigma(T) \setminus \mathrm{Int}(\overline{\sigma(T)})$) is an approximate eigenvalue of T.*

Proof. Since $\mathbb{C}$ is connected and $\sigma(T)$ is compact and non-empty, $\sigma(T)$ can't be an open set, so $\partial\sigma(T) \neq \emptyset$.

Take $\lambda \in \partial\sigma(T)$. Then there are $(\lambda_n)_{n\in\mathbb{N}} \subseteq \rho(T)$ such that $\lambda_n \to \lambda$. Suppose, by contradiction, that $\|(T - \lambda_n I)^{-1}\| \leq M$ for some $M > 0$. Then, there is some $n > 1$ such that

$$\begin{aligned}\|I - (T - \lambda I)(T - \lambda_n I)^{-1}\| &\leq \|(T - \lambda_n I)^{-1}\|\|(T - \lambda_n I) - (T - \lambda I)\| \\ &\leq M|\lambda_n - \lambda| < 1.\end{aligned}$$

Let $S = I - (T - \lambda I)(T - \lambda_n I)^{-1}$. It is then a standard verification that since $\|S\| < 1$, $\sum_{i=0}^{\infty} S^i$ is the inverse of $I - S = (T - \lambda I)(T - \lambda_n I)^{-1}$. From this we get that $(T - \lambda I)$ is invertible, which is a contradiction because $\sigma(T)$ is closed. So we may take $(\lambda_{n_j})_{j\in\mathbb{N}}$ such that

$$\|(T - \lambda_{n_j} I)^{-1}\| \xrightarrow{j\to\infty} \infty.$$

Choose a subsequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_j\| = 1$ for all $j \in \mathbb{N}$ satisfying $\|(T-\lambda_{n_j} I)^{-1}x_j\| \xrightarrow{j\to\infty} \infty$. Then, let $v_j := \frac{(T-\lambda_{n_j}I)^{-1}x_j}{\|(T-\lambda_{n_j}I)^{-1}x_j\|}$. So,

$$\|(T-\lambda I)v_j\| \leq \|(T-\lambda_{n_j}I)v_j\| + |\lambda_{n_j} - \lambda|\|v_j\|,$$

and $(v_n)_{n\in\mathbb{N}}$ is an approximate eigenvector, since $\|(T-\lambda I)v_n\| \xrightarrow{n\to\infty} 0$. □

With an approximate eigenvalue at hand we move to the last ingredient needed for the result we seek in spectral theory (Proposition 3.2.11); some notions of ultrapowers of Banach spaces are needed. As a historical note, Krivine himself, alongside Dacunha-Castelle, was responsible for the introduction of ultraproducts into Banach space theory in 1972 [6]. Ever since, the concept has kept deep ties with finite-representability and the local study of Banach spaces [12].

To build an ultrapower, we first need to present the notion of a filter, the dual concept to the ideal, presented in Definition 2.2.2.

Definition 3.2.4. Let X be a set. We call a collection $\mathcal{F}$ of subsets of X a **filter** on X if

1. $\emptyset \notin \mathcal{F}$
2. If $A \subseteq B$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$.
3. If $A_1, ..., A_k \in \mathcal{F}$, then $\bigcap_{n=1}^k A_n \in \mathcal{F}$.

Additionally, if $\mathcal{F}$ is maximal with respect to inclusion (i.e. if there is no filter $\mathcal{G}$ on X where $\mathcal{F} \subsetneq \mathcal{G}$), we call it an **ultrafilter**.

As a notion dual to an ideal, filters represent a notion of largeness, and with it we may, for instance, define different concepts of convergence.

Let $\mathcal{F}$ be a filter on a set I. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ of real values **converges to a through $\mathcal{F}$** if, for any $\varepsilon > 0$, we have

$$\{n \in \mathbb{N} : a_n \in (a-\varepsilon, a+\varepsilon)\} \in \mathcal{F}.$$

In which case we denote it by $\lim_{\mathcal{F}} a_n = a$. As it is standard for limits in analysis, $\lim_{\mathcal{F}} a_n$ need not exist for a given sequence $(a_n)_{n\in\mathbb{N}}$, but when it does, it is unique (to see this, suppose the opposite and since $\mathbb{R}$ is Hausdorff, we would be able to find $A, B \in \mathcal{F}$ disjoint, which is a contradiction to the definition of a filter). In this context, we present two examples of filters on $\mathbb{N}$:

Example 3.2.5. *Let $m \in \mathbb{N}$, then $\mathcal{F}_m := \{A \in \mathcal{P}(\mathbb{N}) : m \in A\}$ is an ultrafilter on $\mathbb{N}$, and $\lim_{\mathcal{F}_m} a_n = a$ if, and only if, $a_m = a$.*

Example 3.2.6. *The set $\mathcal{F}_\infty := \{A \in \mathcal{P}(\mathbb{N}) : \exists n(n \in \mathbb{N} \wedge [n, +\infty) \subseteq A)\}$ is a filter on $\mathbb{N}$ and $\lim_{\mathcal{F}_\infty} a_n = a$ if, and only if, $\lim_{n\to\infty} a_n = a$.*

Since these are straightforward verifications, we leave them to the reader. Among the ultrafilters of a given set, in this case $\mathbb{N}$, those of the form $\mathcal{F}_n$ are known as **principal ultrafilters**, whereas filters which are not of the form $\mathcal{F}_n$, for some $n \in \mathbb{N}$, are called **non-principal ultrafilters**. These play an important role in studies relating to ultrafilters, to see why this distinction is relevant, we make use of a fundamental property of ultrafilters:

Lemma 3.2.7. *Let X be a set and $\mathcal{U}$ an ultrafilter on X. Then, for any $A \subseteq X$, we either have $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

While we already have a characterization for convergence on principal ultrafilters, for non-principal ultrafilters we'll find quite a different scenario.

Lemma 3.2.8. *Every non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ contains $\mathcal{F}_\infty$.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and take some $m \in \mathbb{N}$. By Lemma 3.2.7, either $\{1, ..., m\} \in \mathcal{U}$ or $[m, +\infty) \in \mathcal{U}$. However, since $\mathcal{U}$ is non-principal, there are $A_1, ..., A_m \in \mathcal{U}$ where $n \notin A_n$ for $1 \leq n \leq m$. Which means $\{1, ..., m\} \notin \mathcal{U}$, because we would have

$$\emptyset = \{1, ..., m\} \cap \bigcap_{n=1}^{m} A_n \in \mathcal{U}.$$

Given that $m \in \mathbb{N}$ was arbitrary, we have found that $\mathcal{F}_\infty \subseteq \mathcal{U}$. □

More broadly, a filter of the form $\mathcal{F}_\infty$ on an arbitrary set X is known as the *cofinite filter* on X, and the property above works exactly in the same way.

With this, we may now view a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ as a way to generalize the concept of convergence of a sequence of real numbers (since $\mathcal{F}_\infty \subseteq \mathcal{U}$, we have that $\lim_{n\to\infty} a_n = a \implies \lim_{\mathcal{U}} a_n = a$), which is naturally very promising for analysis. But before diving into their use in our studies, it is important to address the existence of non-principal ultrafilters on $\mathbb{N}$. A easy way of seeing that these do in fact exist, is by using another fundamental lemma which states that any filter has an unique extension to an ultrafilter. With this, we may construct an ultrafilter from $\mathcal{F}_\infty$, which clearly won't be principal. The proof this follows a standard argument using Zorn's lemma.

Lemma 3.2.9. *Let X be a set and $\mathcal{F}$ an filter on X. Then, there exists an ultrafilter on X with $\mathcal{F} \subseteq \mathcal{U}$.*

Finally, we explore some other properties of limits on non-principal ultrafilters on $\mathbb{N}$.

Lemma 3.2.10. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and $(a_n)_{n\in\mathbb{N}}$ a sequence of real numbers. Then, if $(a_n)_{n\in\mathbb{N}} \in \ell_\infty$, $\lim_{\mathcal{U}} a_n$ exists. Furthermore, if $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\lim_{\mathcal{U}} a_n \geq 0$.*

Proof. Suppose, by contradiction, that there is a sequence of real numbers $(a_n)_{n\in\mathbb{N}} \in \ell_\infty$ which doesn't converge through $\mathcal{U}$. There is some $K > 0$ such that $(a_n)_{n\in\mathbb{N}} \subseteq [-K, K] =: X$ and then, for every $x \in X$ there are $\varepsilon(x) > 0$ such that

$$\{n \in \mathbb{N} : a_n \in (x - \varepsilon(x), x + \varepsilon(x)\} \notin \mathcal{U}.$$

Since X is compact, $\{B_{\varepsilon(x)}(x) : x \in X\}$ is an open covering of X and admits a finite subcover $\{B_{\varepsilon(x_1)}(x_1), ..., B_{\varepsilon(x_n)}(x_m)\}$. From the fact that $\mathcal{U}$ is an ultrafilter, we have $\{n \in \mathbb{N} : a_n \notin (x_i - \varepsilon(x_i), x_i + \varepsilon(x_i)\} \in \mathcal{U}$ for all $1 \leq i \leq m$ and so

$$\emptyset = \bigcap_{i=1}^{m} \{n \in \mathbb{N} : a_n \notin (x_i - \varepsilon(x_i), x_i + \varepsilon(x_i)\} \in \mathcal{U}$$

which contradicts the definition of a filter.

Now, let $a_0 := \lim_{\mathcal{U}} a_n$ for $(a_n)_{n\in\mathbb{N}} \in \ell_\infty$ such that $a_n \geq 0$ for all $n \in \mathbb{N}$, and suppose that $a_0 < 0$. Then, for $\varepsilon = |a_0|/2$, we would have $\{n \in \mathbb{N} : a_n \in B(a_0, \varepsilon)\} = \emptyset \in \mathcal{U}$, a contradiction. So $a_0 \geq 0$. $\square$

Now, to make use of these structures, we begin by constructing a vector space: Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{U}$ a nonprincipal ultrafilter on $\mathbb{N}$, we define the ℓ_∞-product of X as the normed space of bounded sequences in X

$$\ell_\infty(X) := \left\{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : \sup_{n\in\mathbb{N}} \|x_n\| < \infty\right\}$$

with norm $\|(x_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}} \|x_n\|$. From the fact that X is complete, we have that $\ell_\infty(X)$ is Banach. We then use a nonprincipal ultrafilter to define a seminorm on $\ell_\infty(X)$

$$\|(x_n)_{n\in\mathbb{N}}\|_{\mathcal{U}} := \lim_{\mathcal{U}} \|x_n\|.$$

We denote the kernel of $\|\cdot\|_{\mathcal{U}}$ by $c_{0,\mathcal{U}}(X)$. We may then consider the quotient space $\ell_\infty(X)/c_{0,\mathcal{U}}(X)$ together with the quotient norm[4]

$$\|[(x_n)_{n\in\mathbb{N}}]_{\mathcal{U}}\| := \inf\{\|(y_n)_{n\in\mathbb{N}}\|_{\mathcal{U}} : (y_n)_{n\in\mathbb{N}} \in [(x_n)_{n\in\mathbb{N}}]_{\mathcal{U}}\}.$$

Since $c_{0,\mathcal{U}}(X)$ is closed in $\ell_\infty(X)$ this is a Banach space called an **ultrapower** of X and denoted by $X_{\mathcal{U}}$. Naturally, the quotient norm coincides with

$$\|[(x_n)_{n\in\mathbb{N}}]_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|(x_n)_{n\in\mathbb{N}}\|, \qquad \text{for all } (x_n)_{n\in\mathbb{N}} \in \ell_\infty(X).$$

While we keep our discussion of this concept brief[5], there are several natural questions one might ask after the construction of an ultrapower of a Banach space. For instance, how could we relate a bounded linear operator $T : X \to Y$ to an operator between ultrapowers of X and Y? A natural way to do this is defining $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ by

$$T_{\mathcal{U}}([(x_n)_{n\in\mathbb{N}}]_{\mathcal{U}}) := [(Tx_n)_{n\in\mathbb{N}}]_{\mathcal{U}}.$$

To see that this map is consistent with the equivalence classes of $X_{\mathcal{U}}$ and $Y_{\mathcal{U}}$, we need only verify that

$$\lim_{\mathcal{U}} \|x_n - y_n\| = 0 \implies \lim_{\mathcal{U}} \|Tx_n - Ty_n\| = 0 \tag{3.15}$$

for any $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell_\infty(X)$. If $T = 0$ this is trivial, and given that for each $\varepsilon > 0$ we have that $\{n \in \mathbb{N} : \|x_n - y_n\| < \varepsilon/\|T\|\} \in \mathcal{U}$ we know that

$$\{n \in \mathbb{N} : \|Tx_n - Ty_n\| < \varepsilon\} \supseteq \{n \in \mathbb{N} : \|T\|\|x_n - y_n\| < \varepsilon\} \in \mathcal{U}$$

which means that for all $\varepsilon > 0$, $\{n \in \mathbb{N} : \|Tx_n - Ty_n\| < \varepsilon\} \in \mathcal{U}$, the verification of Equation (3.15). $T_{\mathcal{U}}$ is then called an **ultrapower of** T, and is a bounded linear operator. Linearity follows trivially so we focus only on the boundedness, which is a simple consequence of property 2 of Lemma 3.2.10 when we consider $(a_n)_{n\in\mathbb{N}} = (\|T\| - \|Tx_n\|)_{n\in\mathbb{N}}$ for any $(x_n)_{n\in\mathbb{N}}$ in the sphere of $\ell_\infty(X)$.

[4] The fact that it is a norm is an easy verification using that $\|\cdot\|_{\mathcal{U}}$ is a seminorm and $c_{0,\mathcal{U}}$ its kernel.

[5] The interested reader could see Heinrich's book [12] for a more in-depth discussion of the topic

Remark. *To simplify our notation going forward, we may choose to avoid using the subscript* $_\mathcal{U}$*, but we recommend the reader to keep in mind the details of the constructions we are using. A specific example of this is when denoting vectors in ultraproducts, it is often important that these are equivalence classes of sequences of vectors in* X*. In light of this, and to illustrate our omission of the subscript* $_\mathcal{U}$*, we denote the equivalence class* $[(x_n)_{n\in\mathbb{N}}]_\mathcal{U}$*, of a given Banach space ultrapower* $X_\mathcal{U}$*, simply as* $[(x_n)_{n\in\mathbb{N}}]$*.*

Now we can phrase the final proposition we wish to explore in this section, which essentially asserts that given any two commuting bounded linear operators on a complex Banach space one can always find a sequence that is an approximate eigenvector to both operators simultaneously (where their associated approximates eigenvalues need not be the same).

Proposition 3.2.11. *Let* $X \neq \{0\}$ *be a complex Banach space and let* $T : X \to X$ *and* $S : X \to X$ *be bounded linear operators that commute, i.e. where* $ST = TS$*. Let* λ *be any approximate eigenvalue of* T*. Then there exists* $\mu \in \sigma(S)$ *and a sequence* $(u_n)_{n\in\mathbb{N}}$ *in the unit sphere of* X *such that*

$$\|Tu_n - \lambda u_n\| \xrightarrow{n\to\infty} 0 \quad \textit{and} \quad \|Su_n - \mu u_n\| \xrightarrow{n\to\infty} 0.$$

Proof. Let $X \neq \{0\}$ be a complex Banach space, $T : X \to X$ and $S : X \to X$ be bounded linear operators that commute and let λ be an approximate eigenvalue of T with an approximate eigenvector $(x_n)_{n\in\mathbb{N}}$. If $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, we may consider the ultrapowers $T_\mathcal{U} : X_\mathcal{U} \to X_\mathcal{U}$ and $S_\mathcal{U} : X_\mathcal{U} \to X_\mathcal{U}$ we find that

1. $T_\mathcal{U}$ and $S_\mathcal{U}$ commute.
   For any $[(x_n)_{n\in\mathbb{N}}] \in X_\mathcal{U}$,

$$\begin{aligned} T_\mathcal{U} \circ S_\mathcal{U}[(x_n)_{n\in\mathbb{N}}] &= [(TSx_n)_{n\in\mathbb{N}}] = [(STx_n)_{n\in\mathbb{N}}] \\ &= S_\mathcal{U} \circ T_\mathcal{U}[(x_n)_{n\in\mathbb{N}}]. \end{aligned}$$

2. λ is an eigenvalue of $T_\mathcal{U}$ and $[(x_n)_{n\in\mathbb{N}}]$ is an eigenvector of $T_\mathcal{U}$ corresponding to λ.

$$(T_\mathcal{U} - \lambda I)[(x_n)_{n\in\mathbb{N}}] = [(Tx_n - \lambda x_n)_{n\in\mathbb{N}}] = 0,$$

   since $\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0 \implies \lim_\mathcal{U} \|Tx_n - \lambda x_n\| = 0.$

If we then set E to be the eigenspace of $T_{\mathcal{U}}$ corresponding to λ, we have that E is a closed non-trivial subspace of $X_{\mathcal{U}}$ (as it is the kernel of $T_{\mathcal{U}} - \lambda I$) and thus a non-trivial complex Banach space on its own. To see why this is interesting to us, notice first that given any $\overline{x} \in E$:

$$T_{\mathcal{U}}(S_{\mathcal{U}}\overline{x}) = S_{\mathcal{U}}(T_{\mathcal{U}}\overline{x}) = S_{\mathcal{U}}(\lambda\overline{x}) = \lambda(S_{\mathcal{U}}\overline{x}).$$

which means that $S_{\mathcal{U}}\overline{x}$ lies in E for every $\overline{x} \in E$, which means that E is an $S_{\mathcal{U}}$-invariant subspace.

With this, we may denote the restriction of $S_{\mathcal{U}}$ to E as $S_{\mathcal{U}}|_E : E \to E$ and apply Lemma 3.2.3 to find an approximate eigenvalue μ with corresponding approximate eigenvalue $(\overline{y}_n)_{n\in\mathbb{N}} \subseteq E$. Naturally, λ is also an approximate eigenvalue of $T_{\mathcal{U}}$ with approximate eigenvector $(\overline{y}_n)_{n\in\mathbb{N}}$. Then,

$$\lim_{n\to\infty} \|T_{\mathcal{U}}\overline{y}_n - \lambda\overline{y}_n\|_{\mathcal{U}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|S_{\mathcal{U}}\overline{y}_n - \mu\overline{y}_n\|_{\mathcal{U}} = 0. \tag{3.16}$$

We'll focus now on transferring this property to X. To do this, first see that since $(\overline{y}_n)_{n\in\mathbb{N}}$ is an approximate eigenvalue, $\|\overline{y}_m\|_{\mathcal{U}} = 1$ for all $m \in \mathbb{N}$. We prove then that, for any fixed $m \in \mathbb{N}$, there is some $(z_{m,n})_{n\in\mathbb{N}} \in \ell_\infty(X)$ such that $(z_{m,n})_{n\in\mathbb{N}} \in \overline{y}_m$ and $\|z_{m,n}\| = 1$ for all $n, m \in \mathbb{N}$:

Let $(y_{m,n})_{n\in\mathbb{N}}$ be some element of $\overline{y}_m$ and $(z_{m,n})_{n\in\mathbb{N}}$ its normalization (whenever $y_{m,n} = 0$, simply let $z_{m,n}$ be an arbitrary norm 1 vector). Since $\|\overline{y_m}\| = \lim_{\mathcal{U}} \|y_{m,n}\| = 1$, for any $0 < \varepsilon < 1$,

$$A := \{n \in \mathbb{N} : \|y_{m,n}\| \in (1-\varepsilon, 1+\varepsilon)\} \in \mathcal{U}.$$

Then $1 - \varepsilon \leq \|y_{m,n}\| \leq 1 + \varepsilon$ for all $n \in A$ and

$$\|z_{m,n} - y_{m,n}\| \leq \left|1 - \frac{1}{\|y_{m,n}\|}\right| \|y_{m,n}\| \leq \frac{\varepsilon}{\|y_{m,n}\|}\|y_{m,n}\| = \varepsilon$$

for all $n \in A$. So,

$$A \subseteq \{n \in \mathbb{N} : \|y_{m,n} - z_{m,n}\| < \varepsilon\} \in \mathcal{U}.$$

and since this is valid for all $0 < \varepsilon < 1$, it is also true for any $\varepsilon > 0$, and $\lim_{\mathcal{U}} \|y_{m,n} - z_{m,n}\| = 0$, which confirms that $(z_{m,n})_{n\in\mathbb{N}} \in \overline{y}_m$.

With this in mind, let $(z_{m,n})_{n\in\mathbb{N}}$ be a normalized representative of $\overline{y}_m$, for all $m \in \mathbb{N}$. We'll use this and Equation (3.16) to get our desired sequence in X. First, fix $k \in \mathbb{N}$ and take $m_k \in \mathbb{N}$ such that

$$\|T_{\mathcal{U}}\overline{y}_{m_k} - \lambda\overline{y}_{m_k}\|_{\mathcal{U}} < \frac{1}{2^k} \quad \text{and} \quad \|S_{\mathcal{U}}\overline{y}_{m_k} - \overline{y}_{m_k}\|_{\mathcal{U}} < \frac{1}{2^k}.$$

which we rewrite as

$$\lim_{\mathcal{U}} \|Tz_{m_k,n} - \lambda z_{m_k,n}\| < \frac{1}{2^k} \quad \text{and} \quad \lim_{\mathcal{U}} \|Sz_{m_k,n} - \mu z_{m_k,n}\| < \frac{1}{2^k}.$$

From this there is $A, B \in \mathcal{U}$ such that

$$\|Tz_{m_k,n} - \lambda z_{m_k,n}\| < \frac{1}{2^{k-1}} \quad \text{and} \quad \|Sz_{m_k,s} - \mu z_{m_k,s}\| < \frac{1}{2^{k-1}}$$

for all $n \in A$ and $s \in B$. Given that $A \cap B \in \mathcal{U}$ and $A \cap B$ cannot be empty, we may take some $n_k \in A \cap B$. Finally, if we define $u_k := z_{m_k,n_k}$ for all $k \in \mathbb{N}$, we will have a sequence $(u_n)_{n\in\mathbb{N}}$ in the unit sphere of X satisfying

$$\|Tu_n - \lambda u_n\| \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \|Su_n - \mu u_n\| \xrightarrow{n\to\infty} 0,$$

as claimed. □

One application of this proposition, that will come to be useful in the next section, consists of finding approximate eigenvalues and a shared approximate eigenvector for specific "spreading operators", operators tailored to some 1-spreading sequence that will alow us to relate their behavior to certain spectral values. To "spread" the vectors of the given sequence we take advantage of the ordering of the rationals (actually, of $\mathbb{Q}^+ \cap (0,1)$), so it is worth noting that changing the indexing set of a sequence changes what it means for that sequence to be 1-spreading, as $\mathbb{N}$ and $\mathbb{Q}$ have fundamentally different linear orders (there is no order isomorphism between them).

Proposition 3.2.12. *Let $Y \neq \{0\}$ be a complex Banach space and $(v_r)_{r\in\mathbb{Q}^+\cap(0,1)}$ a non-null sequence in Y that*

i. is 1-spreading. That is, $\|\sum_{i=1}^n a_i v_{r_i}\| = \|\sum_{i=1}^n a_i v_{s_i}\|$ for all $n \in \mathbb{N}$, scalars $a_1, ..., a_n$ and positive rationals $0 < r_1 < ... < r_n < 1$ and $0 < s_1 < ... < s_n < 1$.

ii. satisfies $\|\sum_{i=1}^n a_i v_{r_i}\| = \|\sum_{i=1}^n \theta_i a_i v_{r_i}\|$ for all $n \in \mathbb{N}$, scalars $a_1, ..., a_n$, $r_1, ..., r_n \in \mathbb{Q}^+ \cap (0,1)$ and $\theta_1, ..., \theta_n$ scalars in the unit sphere of $\mathbb{C}$.

Let $W_0 := \operatorname{span}\{v_r : r \in \mathbb{Q}^+ \cap (0,1)\}$ be a subspace of Y and W its closure. Then, the operators $T, S : W \to W$ given by

$$Tv_r = v_{\frac{r}{2}} + v_{\frac{r+1}{2}} \quad \textit{and} \quad Sv_r = v_{\frac{r}{3}} + v_{\frac{r+1}{3}} + v_{\frac{r+2}{3}}$$

for all $r \in \mathbb{Q}^+ \cap (0,1)$*, have approximate eigenvalues* λ *and* μ*, respectively, and an approximate eigenvector* $(u_n)_{n\in\mathbb{N}}$ *such that*

$$\|Tu_n - \lambda u_n\|_Y \xrightarrow{n\to\infty} 0 \quad \textit{and} \quad \|Su_n - \mu u_n\|_Y \xrightarrow{n\to\infty} 0.$$

Furthermore, we can assume that $1 \le \lambda \le 2$*,* $1 \le \mu \le 3$*, and that* $(u_n)_{n\in\mathbb{N}}$ *is a sequence of vectors in* W_0 *with non-negative real coordinates.*

Proof. Let Y be a Banach space, $(v_r)_{r\in\mathbb{Q}^+\cap(0,1)}$ a sequence in Y and W_0, all as described. We begin by constructing linear operators $T_0, S_0 : W_0 \to W_0$ from the linear extension of

$$T_0 v_r = v_{\frac{r}{2}} + v_{\frac{r+1}{2}} \quad \text{and} \quad S_0 v_r = v_{\frac{r}{3}} + v_{\frac{r+1}{3}} + v_{\frac{r+2}{3}}$$

for all $r \in \mathbb{Q}^+ \cap (0,1)$. This way, T_0 maps any given vector x from W_0 to a sum of two vectors with same distribution as x and coordinates in $\mathbb{Q}^+\cap(0,\frac{1}{2})$ and $\mathbb{Q}^+ \cap (\frac{1}{2},1)$, respectively. Similarly, S_0 maps any vector from W_0 to the sum of three vectors, each with the same distribution as x but coordinates in $\mathbb{Q}^+ \cap (0,\frac{1}{3})$, $\mathbb{Q}^+ \cap (\frac{1}{3},\frac{2}{3})$ and $\mathbb{Q}^+ \cap (\frac{2}{3},1)$. This simple construction allows us to easily check that

$$\begin{aligned} T_0 S_0 \left(\sum_{i=1}^n a_i v_{r_i}\right) &= T_0 \left(\sum_{i=1}^n a_i v_{\frac{r_i}{3}} + \sum_{i=1}^n a_i v_{\frac{r_i+1}{3}} + \sum_{i=1}^n a_i v_{\frac{r_i+2}{3}}\right) \\ &= \sum_{j=0}^5 \sum_{i=1}^n a_i v_{\frac{r_i+j}{6}} = S_0 \left(\sum_{i=1}^n a_i v_{\frac{r_i}{2}} + \sum_{i=1}^n a_i v_{\frac{r_i+1}{2}}\right) \\ &= S_0 T_0 \left(\sum_{i=1}^n a_i v_{r_i}\right), \end{aligned}$$

for all $\sum_{i=1}^n a_i v_{r_i} \in W_0$, so T_0 and S_0 commute. And, using that $(v_r)_{r\in\mathbb{Q}^+\cap(0,1)}$ is 1-spreading, we see that

$$\begin{aligned} \left\| T_0 \left(\sum_{i=1}^n a_i v_{r_i}\right)\right\|_Y &= \left\| \sum_{i=1}^n a_i v_{\frac{r_i}{2}} + \sum_{i=1}^n a_i v_{\frac{r_i+1}{2}} \right\|_Y \\ &\le \left\| \sum_{i=1}^n a_i v_{\frac{r_i}{2}} \right\|_Y + \left\| \sum_{i=1}^n a_i v_{\frac{r_i+1}{2}} \right\|_Y \\ &= 2 \left\| \sum_{i=1}^n a_i v_{r_i} \right\|_Y \end{aligned}$$

and

$$\begin{aligned}
\left\|S_0\left(\sum_{i=1}^n a_i v_{r_i}\right)\right\|_Y &= \left\|\sum_{i=1}^n a_i v_{\frac{r_i}{3}} + \sum_{i=1}^n a_i v_{\frac{r_i+1}{3}} \sum_{i=1}^n a_i v_{\frac{r_i+2}{3}}\right\|_Y \\
&\leq \left\|\sum_{i=1}^n a_i v_{\frac{r_i}{3}}\right\|_Y + \left\|\sum_{i=1}^n a_i v_{\frac{r_i+1}{3}}\right\|_Y + \left\|\sum_{i=1}^n a_i v_{\frac{r_i+2}{3}}\right\|_Y \\
&= 3\left\|\sum_{i=1}^n a_i v_{r_i}\right\|_Y ,
\end{aligned}$$

for an arbitrary $\sum_{i=1}^n a_i v_{r_i} \in W_0$, so both operators are bounded. Using Lemma 3.1.16 and the fact that $(v_r)_{r\in\mathbb{Q}^+\cap(0,1)}$ is 1-unconditional we can go further and prove that

$$\begin{aligned}
\left\|T_0\left(\sum_{i=1}^n a_i v_{r_i}\right)\right\|_Y &= \left\|\sum_{i=1}^n a_i v_{\frac{r_i}{2}} + \sum_{i=1}^n a_i v_{\frac{r_i+1}{2}}\right\|_Y \\
&\geq \left\|\sum_{i=1}^n a_i v_{\frac{r_i}{2}}\right\|_Y = \left\|\sum_{i=1}^n a_i v_{r_i}\right\|_Y
\end{aligned}$$

and

$$\begin{aligned}
\left\|S_0\left(\sum_{i=1}^n a_i v_{r_i}\right)\right\|_Y &= \left\|\sum_{i=1}^n a_i v_{\frac{r_i}{3}} + \sum_{i=1}^n a_i v_{\frac{r_i+1}{3}} \sum_{i=1}^n a_i v_{\frac{r_i+2}{3}}\right\|_Y \\
&\geq \left\|\sum_{i=1}^n a_i v_{\frac{r_i}{3}}\right\|_Y = \left\|\sum_{i=1}^n a_i v_{r_i}\right\|_Y ,
\end{aligned}$$

for all $\sum_{i=1}^n a_i v_{r_i} \in W_0$.

Then, we extend T_0 and S_0 to operators $T : W \to W$ and $S : W \to W$, respectively. From what we just proved we immediately get that, for any $x \in W$,

$$\|x\|_Y \leq \|Tx\|_Y \leq 2\|x\|_Y \quad \text{and} \quad \|x\|_Y \leq \|Sx\|_Y \leq 3\|x\|_Y. \tag{3.17}$$

And, given that T and S commute on W_0, $TS - ST$ is a bounded linear operator which is null on a dense subset of its domain so, by continuity, $TS = ST$. Then, by Proposition 3.2.11, there are $\lambda, \mu \in \mathbb{C}$ and $(\tilde{w}_n)_{n\in\mathbb{N}}$ in the sphere of W such that

$$\|T\tilde{w}_n - \lambda\tilde{w}_n\|_Y \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \|S\tilde{w}_n - \mu\tilde{w}_n\|_Y \xrightarrow{n\to\infty} 0.$$

Since W_0 is dense in W, we may choose a sequence $(w'_n)_{n\in\mathbb{N}} \subseteq W_0$ such that $\|w'_n - \tilde{w}_n\|_Y \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. It follows then, that $1 - \frac{1}{2^n} \leq \|w'_n\|_Y \leq 1 + \frac{1}{2^n}$ and if we define $w_n := \frac{w'_n}{\|w'_n\|_Y}$ we have that

$$\|w_n - \tilde{w}_n\|_Y \leq \|w_n - w'_n\|_Y + \|w'_n - \tilde{w}_n\|_Y \leq \frac{1/2^n}{\|w'_n\|_Y}\|w'_n\|_Y + \frac{1}{2^n} = \frac{1}{2^{n-1}}$$

for all $n \in \mathbb{N}$, so $\|w_n - \tilde{w}_n\| \to 0$. So there is a sequence $(w_n)_{n\in\mathbb{N}}$ in the sphere of W_0 such that

$$\|Tw_n - \lambda w_n\|_Y \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \|Sw_n - \mu w_n\|_Y \xrightarrow{n\to\infty} 0.$$

Now, since each w_k is an element of W_0 we may write $w_k = \sum_{i=1}^{n_k} b_{k,i} v_{r_i}$ (where $r_1 < ... < r_{n_k}$) and define $u_k := \sum_{i=1}^{n_k} |b_{k,i}| v_{r_i}$ for all $k \in \mathbb{N}$. So that

$$\begin{aligned}
\|Tu_k - |\lambda| u_k\|_Y &= \left\| \sum_{i=1}^{n_k} |b_{k,i}| v_{\frac{r_i}{2}} + \sum_{i=1}^{n_k} |b_{k,i}| v_{\frac{r_i+1}{2}} - \sum_{i=1}^{n_k} |\lambda| |b_{k,i}| v_{r_i} \right\|_Y \\
&\leq \left\| \sum_{i=1}^{n_k} b_{k,i} v_{\frac{r_i}{2}} + \sum_{i=1}^{n_k} b_{k,i} v_{\frac{r_i+1}{2}} - \sum_{i=1}^{n_k} \lambda b_{k,i} v_{r_i} \right\|_Y \\
&= \|Tw_k - \lambda w_k\|_Y
\end{aligned}$$

by ii. and Lemma 3.1.16 with an appropriate choice of θ's and α's. Which means that $\|Tu_n - |\lambda| u_n\| \to 0$. Analogously, we have $\|Su_n - |\mu| u_n\| \to 0$. Lastly, from property ii. we check that $\|u_k\|_Y = \|w_k\|_Y = 1$ for all $k \in \mathbb{N}$, so $(u_n)_{n\in\mathbb{N}}$ is an approximate eigenvector of T and S with respect to $|\lambda|$ and $|\mu|$, respectively. By this, Equation (3.17) and the inverse triangle inequality, we have that

$$1 - |\lambda| \leq \|Tu_n\|_Y - \|\lambda u_n\|_Y \leq |\|Tu_n\|_Y - \|\lambda u_n\|_Y| \leq \|Tu_n - \lambda u_n\|_Y$$

for all $n \in \mathbb{N}$, so $1 - |\lambda| \leq 0$ and $|\lambda| \geq 1$. On the other hand, suppose that $|\lambda| > 2$. Then

$$\lambda - \|Tu_k\|_Y \leq |\lambda - \|Tu_k\|_Y| = |\|Tu_k\|_Y - \|\lambda u_k\|_Y| \leq \|Tu_k - \lambda u_k\|_Y$$

for all $k \in \mathbb{N}$. If we choose $s \in \mathbb{N}$ such that $\|Tu_s - \lambda u_s\|_Y \leq \frac{|\lambda|-2}{2}$ which implies that $\|Tu_s\|_Y > 2$ for a unit vector u_s, contradicting Equation (3.17). Doing the same for S, we verify that

$$1 \leq |\lambda| \leq 2 \quad \text{and} \quad 1 \leq |\mu| \leq 3,$$

as required. □

3.3 Krivine's theorem

In the first section we based ourselves in the work of Brunel and Sucheston (c.f [3] and [4]), while in the second section we introduced the tools needed to, and began to present, Lemberg's proof of Krivine's theorem [17]. Now, we tie these ideas together.

To do this it will be convenient to introduce some notation: We define by the **support** of a sequence of scalars $(a_i)_{i\in D}$, for some linearly ordered set D (for our purposes D will be $\mathbb{N}$, $\mathbb{Q}^+$ or $\mathbb{Q}^+ \cap (0,1)$), the set of indices $i \in D$ such that $a_i \neq 0$. It is precisely in this sense that c_{00} is the space of finitely supported scalar sequences. An interesting special case for when two sequences $(a_i)_{i\in D}$ and $(b_i)_{i\in D}$ have disjoint support is when

$$\sup\{i \in D : a_i \neq 0\} < \inf\{i \in D : b_i \neq 0\}.$$

When this is the case, or either sequence is null, we write $(a_i)_{i\in D} \prec (b_i)_{i\in D}$. Notice this is precisely what we were constructing in the proof of Proposition 3.1.11. Lastly, we say that two sequences $\sum_{i=1}^{\infty} a_i e_i$ and $\sum_{i=1}^{\infty} b_i e_i$ have the same **distribution** if the non-zero coefficients of $(a_1, ..., a_n, ...)$ are the same, and disposed in the same order, as those of $(b_1, ..., b_n, ...)$.

Proposition 3.3.1. *Let X be a real Banach space, $(x_n)_{n\in\mathbb{N}}$ a loosely spreading sequence in X. Then there are two real scalars $1 \leq \lambda \leq 2$ and $1 \leq \mu \leq 3$, a real Banach space Z and a normalized basic sequence $(e'_n)_{n\in\mathbb{N}}$ in Z that is block finitely representable in $(x_n)_{n\in\mathbb{N}}$,*

i. is 1-spreading;

ii. is 1-unconditional; and

iii. satisfies

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z = \left\| z_1 + \lambda^k \mu^s \sum_{i=1}^{m} e'_{n_1+i} + z_2 \right\|_Z,$$

for all $m, k, s, n_1 \in \mathbb{N}$ and $z_1 \prec \sum_{i=1}^{m2^k3^s} e'_{n_1+i} \prec z_2$ with z_1, z_2 are in $\mathrm{span}\{e'_n : n \in \mathbb{N}\}$.

Proof. Let X be a Banach space and $(x_n)_{n\in\mathbb{N}}$ a loosely spreading sequence in X. From Theorem 3.1.18, we may assume that $(x_n)_{n\in\mathbb{N}}$ is 1-spreading and 1-unconditional. Now, let c_{00} be the space of finitely supported complex sequences and $(v_r)_{r\in\mathbb{Q}^+}$ it's standard basis indexed by $\mathbb{Q}^+$ (that is, $v_r(j) = \delta_{r,j}$ for all $r, j \in \mathbb{Q}^+$). Using $(x_n)_{n\in\mathbb{N}}$ we define $\|\cdot\|_Y : c_{00} \to \mathbb{R}$ as

$$\left\|\sum_{i=1}^{n} a_i v_{r_i}\right\|_Y := \left\|\sum_{i=1}^{n} |a_i| x_i\right\|$$

for any $n \in \mathbb{N}$, $r_1 < ... < r_n$ indices (where we use the order of the rationals) and $a_1, ..., a_n$ scalars. To see that this is well defined, we need only check that adding terms with $a_j = 0$ doesn't change the value of the function, which is immediate from the fact that $(x_n)_{n\in\mathbb{N}}$ is 1-spreading. It is easy then to see that $\|\cdot\|_Y$ is a norm.

Furthermore, from the properties of $(x_n)_{n\in\mathbb{N}}$ we see that for all $r_1 < ... < r_n$, $s_1 < ... < s_n$ indices and $\theta_1, ..., \theta_n$ in the unit sphere of $\mathbb{C}$,

$$\left\|\sum_{i=1}^{n} a_i v_{r_i}\right\|_Y = \left\|\sum_{i=1}^{n} |a_i| x_i\right\| = \left\|\sum_{i=1}^{n} a_i v_{s_i}\right\|_Y$$

and

$$\left\|\sum_{i=1}^{n} a_i v_{r_i}\right\|_Y = \left\|\sum_{i=1}^{n} |a_i| x_i\right\| = \left\|\sum_{i=1}^{n} \theta_i a_i v_{r_i}\right\|_Y, \tag{3.18}$$

meaning that $(v_r)_{r\in\mathbb{Q}^+}$ is 1-spreading and 1-unconditional. Now, let Y be the completion of $(c_{00}, \|\cdot\|_Y)$ and W the closure of

$$W_0 := \operatorname{span}\{v_r : r \in \mathbb{Q}^+ \cap (0,1)\}.$$

Let $T, S : W \to W$ be the linear operators defined by

$$Tv_r = v_{\frac{r}{2}} + v_{\frac{r+1}{2}} \quad \text{and} \quad Sv_r = v_{\frac{r}{3}} + v_{\frac{r+1}{3}} + v_{\frac{r+2}{3}}$$

for all $r \in \mathbb{Q}^+ \cap (0,1)$. By Proposition 3.2.12, T and S are bounded and there are $(r_n)_{n\in\mathbb{N}}$ indices in $\mathbb{Q}^+ \cap (0,1)$ and $(m_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $r_1 < r_2 < ...$ and there is a normalized sequence $(u_n)_{n\in\mathbb{N}} \subseteq W_0$, of vectors $u_k = \sum_{i=1}^{m_k} b_{k,i} v_{r_i}$ (where $b_{k,i} \geq 0$ for all $k \in \mathbb{N}$ and $1 \leq i \leq m_k$), such that

$$\|Tu_n - \lambda u_n\|_Y \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \|Su_n - \mu u_n\|_Y \xrightarrow{n\to\infty} 0.$$

for $1 \leq \lambda \leq 2$ and $1 \leq \mu \leq 3$. Similarly to what we did in Proposition 3.2.11, we will use a non-principal ultrafilter $\mathcal{U}$ to construct a new Banach space which turns our approximate eigenvector $(u_n)_{n\in\mathbb{N}}$ into a proper eigenvector for both $T_{\mathcal{U}}$ and $S_{\mathcal{U}}$. First we define for all $j, k \in \mathbb{N}$,

$$u_{k,j} := \sum_{i=1}^{m_k} b_{k,i} v_{j+r_i}$$

as the vector with the same distribution as u_k but with support on the interval $(j, j+1)$. It is clear then that any two distinct vectors in the sequence $(u_{k,n})_{n\in\mathbb{N}}$ have disjoint support, and it follows from the properties of $(v_r)_{r\in\mathbb{Q}^+}$ that $(u_{k,n})_{n\in\mathbb{N}}$ is 1-unconditional and 1-spreading.

Now, let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, then we denote by Z_0 the space c_{00} of finitely supported sequences of *real* numbers together with the norm

$$\left\|\sum_{i=1}^{m} a_i e'_i\right\|_Z := \lim_{\mathcal{U}} \left\|\sum_{i=1}^{m} a_i u_{n,i}\right\|_Y$$

and unit vector basis $(e'_n)_{n\in\mathbb{N}}$.From the fact that $(u_n)_{n\in\mathbb{N}}$ and thus $(u_{n,j})_{n\in\mathbb{N}}$, for any $j \in \mathbb{N}$, are normalized, we have that

$$\left\|\sum_{i=1}^{m} a_i u_{n,i}\right\|_Y \leq \sum_{i=1}^{m} |a_i| \|u_{n,i}\|_Y = \sum_{i=1}^{m} |a_i|$$

and so $(\|\sum_{i=1}^{m} a_i u_{n,i}\|)_{n\in\mathbb{N}}$ is bounded and the limit truly exists for all $m \in \mathbb{N}$ and scalars $a_1, ..., a_m$. We then denote by Z the completion of $(Z_0, \|\cdot\|_Z)$. By Lemma 3.1.17 and the definition of the norm $\|\cdot\|_Z$, it is easy to see that $(e'_n)_{n\in\mathbb{N}}$ is a normalized, 1-unconditional and 1-spreading basic sequence in Z. From the definition of the norm $\|\cdot\|_Z$, we can also conclude that for any $\varepsilon > 0$, $m \in \mathbb{N}$ and $a_1, ..., a_m$ real scalars with $\sum_{i=1}^{m} |a_i| = 1$, there is some n_0 such that

$$\left| \left\|\sum_{i=1}^{m} a_i e'_i\right\|_Z - \left\|\sum_{i=1}^{m} a_i u_{n_0,i}\right\|_Y \right| < \varepsilon. \tag{3.19}$$

Each n_0 depends on the choice of scalars $a_1, ..., a_m$, but since we take them from the sets

$$V_\varepsilon := \left\{ n \in \mathbb{N} : \left| \left\|\sum_{i=1}^{m} a_i e'_i\right\|_Z - \left\|\sum_{i=1}^{m} a_i u_{n,i}\right\|_Y \right| < \varepsilon \right\}$$

in the ultrafilter, and given that the intersection of any arrangement of finitely many of these sets has non-empty intersection, we may take n_0 to be an element of this intersection so that (3.19) is true for any choice of $a_1, ..., a_m$ among this arrangement. By what we have seen in the proof of Proposition 3.1.6, this is enough to guarantee the existence of some n_0 which satisfies (3.19) while not depending on the choice of $a_1, ..., a_m$ (given that $\sum_{i=1}^m |a_i|$). That is, for all $\varepsilon > 0$ and $m \in \mathbb{N}$, there is some $n_0 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^m a_i u_{n_0,i} \right\|_Y - \varepsilon \le \left\| \sum_{i=1}^m a_i e'_i \right\|_Z \le \left\| \sum_{i=1}^m a_i u_{n_0,i} \right\|_Y + \varepsilon$$

for any real scalars $a_1, ..., a_m$ with $\sum_{i=1}^m |a_i|$. Since $(e'_i)_{n\in\mathbb{N}}$ is a normalized basic sequence, by Example 3.1.5, it is loosely spreading and

$$\left(1 - \frac{\varepsilon}{C(m)}\right) \left\| \sum_{i=1}^m a_i u_{n_0,i} \right\|_Y \le \left\| \sum_{i=1}^m a_i e'_i \right\|_Z \le \left(1 + \frac{\varepsilon}{c(m)}\right) \left\| \sum_{i=1}^m a_i u_{n_0,i} \right\|_Y .$$

By scaling we easily extend the above inequalities to any choice of real scalars $a_1, ..., a_m$, ditching the requirement that $\sum_{i=1}^m |a_i| = 1$. Furthermore, from the fact that the choice of ε was arbitrary, and that $c(m)$ and $C(m)$ are both fixed, we have that

$$(1 - \varepsilon) \left\| \sum_{i=1}^m a_i u_{n_0,i} \right\|_Y \le \left\| \sum_{i=1}^m a_i e'_i \right\|_Z \le (1 + \varepsilon) \left\| \sum_{i=1}^m a_i u_{n_0,i} \right\|_Y \tag{3.20}$$

holds for all real scalars $a_1, ..., a_m$. To prove that $(e'_n)_{n\in\mathbb{N}}$ is block finitely representable in $(x_n)_{n\in\mathbb{N}}$, we simply need to adapt the block sequence $(u_n)_{n\in\mathbb{N}}$ (notice that the blocks u_k are disjoint and successive in the order of the rationals) into a block sequence of $(x_n)_{n\in\mathbb{N}}$: Let $s_0 = 0$, $s_k := \sum_{i=1}^{k-1} m_i$ and $q_k := \sum_{j=1}^{m_k} b_{n_0,j} x_{s_k+j}$ for all $k \in \mathbb{N}$. It is easy to see that $(q_n)_{n\in\mathbb{N}}$ is a block sequence of $(x_n)_{n\in\mathbb{N}}$. Then, by the definition of $\|\cdot\|_Y$,

$$\begin{aligned} \left\| \sum_{i=1}^m a_i u_{n_0,i} \right\|_Y &= \left\| \sum_{i=1}^m a_i \left(\sum_{j=1}^{m_i} b_{n_0,j} v_{i+r_j} \right) \right\|_Y \\ &= \left\| \sum_{i=1}^m a_i \left(\sum_{j=1}^{m_i} b_{n_0,j} x_{s_i+j} \right) \right\| = \left\| \sum_{i=1}^m a_i q_i \right\| \end{aligned}$$

holds for all real scalars $a_1, ..., a_n$. Substituting by the above Equation in (3.20) we conclude that $(e'_n)_{n\in\mathbb{N}}$ is block finitely representable in $(x_n)_{n\in\mathbb{N}}$.

We continue by fixing two vectors u and v in $\operatorname{span}\{v_r : r \in \mathbb{Q}^+ \cap (0,1)\}$ and $\operatorname{span}\{v_r : r \in \mathbb{Q}^+ \cap (3,4)\}$ respectively. Recall that $u \prec v$ if, and only if, $\max \operatorname{supp}_{\mathbb{Q}} u < \min \operatorname{supp}_{\mathbb{Q}} v$. Then,

$$u \prec u_{n,1} \prec u_{n,2} \prec v$$

for every $n \in \mathbb{N}$. Here we have made use of intervals in the positive rationals to keep these vectors disjoint. But the usefulness of this construction doesn't end there, the fact that the interval $\mathbb{Q}^+ \cap (0,1)$ doesn't have a maximal or minimal elements, means that there are infinite vectors v_r to the left and right of the support of any given finitely supported vector. Consider $Tu_n \in W_0$, since this is finitely supported on $\mathbb{Q}^+ \cap (0,1)$, we can find $u^n, v^n \in W_0$ with the same distribution as u and v, respectively, and such that $u^n \prec Tu_n \prec v^n$. By the fact that $(v_r)_{r\in\mathbb{Q}^+}$ is 1-unconditional and 1-spreading, it follows that

$$\begin{aligned}
&|\|u + u_{n,1} + u_{n,2} + v\|_Y - \|u + \lambda u_{n,1} + v\|_Y| \\
&= |\|u^n + Tu_n + v^n\|_Y - \|u^n + \lambda u_n + v^n\|_Y| \\
&\le \|(u^n + Tu_n + v^n) - (u^n + \lambda u_n + u_{n,2} + v^n)\|_Y \\
&\le \|Tu_n - \lambda u_n\|_Y,
\end{aligned}$$

and so

$$|\|u + u_{n,1} + u_{n,2} + v\|_Y - \|u + \lambda u_{n,1} + v\|_Y| \xrightarrow{n\to\infty} 0. \tag{3.21}$$

Now we'll transfer this result into our new space Z. For this, fix some $j \in \mathbb{N}$ and take any $z_1, z_2 \in Z_0$ such that $z_1 \prec e'_j \prec e'_{j+1} \prec z_2$. We may write $z_1 = \sum_{i=1}^{j-1} c_i e'_i$ and $z_2 = \sum_{i=j+2}^{m} c_i e'_i$ for some choice of $m \in \mathbb{N}$ and $c_1, ..., c_{j-1}, c_{j+2}, ..., c_m$ scalars.

Since our choice of u, v above was arbitrary, we may consider them to have the same distribution as z_1 and z_2. Then, by Equation (3.21),

$$\begin{aligned}
\|z_1 + e'_j + e'_{j+1} + z_2\|_Z &= \lim_{\mathcal{U}} \left\| \sum_{i=1}^{j-1} c_i u_{n,i} + u_{n,j} + u_{n,j+1} + \sum_{i=j+2}^{m} c_i u_{n,i} \right\|_Y \\
&= \lim_{\mathcal{U}} \left\| \sum_{i=1}^{j-1} c_i u_{n,i} + \lambda u_{n,j} + \sum_{i=j+2}^{m} c_i u_{n,i} \right\|_Y \\
&= \|z_1 + \lambda e'_j + z_2\|_Z.
\end{aligned}$$

If we do the same for S instead of T and μ instead of λ, we find the following identities:

$$\|z_1 + e'_j + e'_{j+1} + z_2\|_Z = \|z_1 + \lambda e'_j + z_2\|_Z \tag{3.22}$$

for any $z_1, z_2 \in Z_0$ with $z_1 \prec e'_j \prec e'_{j+1} \prec z_2$, and,

$$\|z_1 + e'_j + e'_{j+1} + e_{j+2} + z_2\|_Z = \|z_1 + \mu e'_j + z_2\|_Z \tag{3.23}$$

for any $z_1, z_2 \in Z_0$ with $z_1 \prec e'_j \prec e'_{j+2} \prec z_2$. Now, take $m, k, s, n_1 \in \mathbb{N}$ and $z_1, z_2 \in Z_0$ such that $z_1 \prec \sum_{i=1}^{m2^k3^s} e'_{n_1+i} \prec z_2$. Then,

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\| = \left\| z_1 + \sum_{i=1}^{2} e'_{n_1+i} + \sum_{i=3}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|,$$

by Equation (3.22) with $\tilde{z}_1 = z_1$ and $\tilde{z}_2 = \sum_{i=3}^{m2^k3^s} e'_i + z_2$, we have

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z = \left\| z_1 + \lambda e'_{n_1+1} + \sum_{i=3}^{4} e'_{n_1+i} + \sum_{i=5}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z.$$

Using Equation (3.23) with $\tilde{z}_1 = z_1 + \lambda e'_1$ and $\tilde{z}_2 = \sum_{i=5}^{m2^k3^s} e'_{n_1+i} + z_2$, we have

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z = \left\| z_1 + \lambda \sum_{i=1}^{2} e'_{n_1+i} + \sum_{i=5}^{6} e'_{n_1+i} + \sum_{i=7}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z.$$

Iterating

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_i + z_2 \right\|_Z = \left\| z_1 + \lambda \sum_{i=1}^{2j} e'_{n_1+i} + \sum_{i=2j+1}^{2(j+1)} e'_{n_1+i} + \sum_{i=2(j+1)+1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z$$

for every $1 \le j \le m2^{k-2}3^s$, we find

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z = \left\| z_1 + \lambda \sum_{i=1}^{m2^{k-1}3^s} e'_{n_1+i} + z_2 \right\|_Z.$$

Since we have shown this with arbitrary $m, k, s \in \mathbb{N}$ and $z \in Z_0$, it is clear that we can use the above equation k times to get

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z = \left\| z_1 + \lambda^k \sum_{i=1}^{m3^s} e'_{n_1+i} + z_2 \right\|_Z . \tag{3.24}$$

Repeating all this for μ, with

$$(3.24) = \left\| z_1 + \lambda^k \left(\mu^t \sum_{i=1}^{3j} e'_{n_1+i} + \sum_{i=3j+1}^{3(j+1)} e'_{n_1+i} + \sum_{i=3(j+1)+1}^{m3^{s-t}} e'_{n_1+i} \right) + z_2 \right\|_Z$$

for all $1 \leq t \leq s$ and $1 \leq j \leq m3^{s-t-1}$ and, we get

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z = \left\| z + \sum_{i=1}^{m} \lambda^k \mu^s e'_{n_1+i} + z_2 \right\|_Z .$$

Letting $m = 1$ and $z_1, z_2 = 0$ we see that

$$\left\| \sum_{i=1}^{2^k3^s} e'_i \right\|_Z = \lambda^k \mu^s$$

for any k and s non-negative integers. □

So far we have sought to improve the properties of sequences we find to be block finitely representable in $(x_n)_{n\in\mathbb{N}}$, for $(x_n)_{n\in\mathbb{N}}$ an arbitrary loosely spreading sequence in a real Banach space X. With the previous proposition, we have all we need, we must only verify that indeed the above sequence is related to the basis of c_0 or some ℓ_p $(1 \leq p < \infty)$ as we originally hoped for. In fact, we will see that $(e'_n)_{n\in\mathbb{N}}$ is isometrically equivalent[6] to $(e_n)_{n\in\mathbb{N}}$ in either c_0 or some ℓ_p. To verify this we make use of the following fact:

Lemma 3.3.2. *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a function that is monotonously increasing and multiplicative (i.e. $f(xy) = f(x)f(y)$ for all x, y in $\mathbb{R}^+$) where $f(1) > 0$, there is some $\alpha \geq 0$ such that $f(x) = x^\alpha$.*

[6]We say that two basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in Banach spaces X and Y are **isometrically equivalent** if $\|\sum_{i=1}^n a_i x_i\|_X = \|\sum_{i=1}^n a_i y_i\|_Y$ for any $n \in \mathbb{N}$ and scalars $a_1, ..., a_n$.

Proof. We define $g : \mathbb{R} \to \mathbb{R}$ as $g(x) := \ln(f(e^x))$. Then,

$$g(x+y) = \ln(f(e^{x+y})) = \ln(f(e^x e^y)) = \ln(f(e^x)) + \ln(f(e^y)) = g(x) + g(y)$$

for any $x, y \in \mathbb{R}$, and so $g(0) = 0$ (since $g(x) = g(0) + g(x)$ for all $x \in \mathbb{R}$). From there, take any rational $\frac{n}{m}$, then

$$ng(1) = g(n) = g\left(m\frac{n}{m}\right) = mg\left(\frac{n}{m}\right)$$

meaning that $g(\frac{n}{m}) = g(1)\frac{n}{m}$. Now let $\alpha = g(1)$ we check that $g(x) = \alpha x$ for all $x \in \mathbb{R}^+$. By the fact that f is monotonously increasing, so is g and so, choosing $y \in \mathbb{Q}$ with $y \leq x$

$$g(x) = g(y) + g(x-y) = \alpha y + g(x-y) \geq \alpha y$$

since $x - y \geq 0$ means that $g(x-y) \geq g(0) = 0$. Analogously, if we choose $z \in \mathbb{Q}$ with $x \leq z$ we check that

$$\alpha z = g(z) = g(x) + g(z-x) \geq g(z).$$

By the fact that we may choose y, z as above arbitrarily close to x, we have that $g(x) = \alpha x$. Then, $\ln(f(e^x)) = \alpha x$ for all $x \in \mathbb{R}^+$ and $f(e^x) = e^{\alpha x}$, so $f(x) = e^{\alpha \ln x} = x^\alpha$ for all $x \in \mathbb{R}^+$. □

As we shall see, the value of λ (or equivalently, of μ) will determine wether $(e'_n)_{n\in\mathbb{N}}$ is isometrically equivalent to the basis of c_0 or ℓ_p ($1 \leq p < \infty$). If $\lambda = 1$ or $\mu = 1$ we have c_0. If $\lambda > 1$ and $\mu > 1$, we have

$$\lambda = 2^{1/p} \quad \text{and} \quad \mu = 3^{1/p}.$$

This will constrain the behavior of the function $h(d) = \| \sum_{i=1}^{d} e'_i \|_Z$, which in turn will imply the respective isometric equivalence and will allow us to conclude the following:

Theorem 3.3.3 (Krivine 1976)**.** *Let X be a real Banach space and $(x_n)_{n\in\mathbb{N}}$ a loosely spreading sequence on X. Then either the unit vector basis of ℓ_p (for some $1 \leq p < \infty$) is block finitely representable in $(x_n)_{n\in\mathbb{N}}$ or the unit vector basis of c_0 is block finitely representable in $(x_n)_{n\in\mathbb{N}}$.*

Proof. Let X be a Banach space and $(x_n)_{n\in\mathbb{N}}$ a loosely spreading sequence in X. From Proposition 3.3.1 we know there is a real Banach space Z and a normalized basic sequence $(e'_n)_{n\in\mathbb{N}}$ in Z that is block finitely representable in $(x_n)_{n\in\mathbb{N}}$. Furthermore, we have the existence of $1 \leq \lambda \leq 2$ and $1 \leq \mu \leq 3$ such that

$$\left\| z_1 + \sum_{i=1}^{m2^k3^s} e'_{n_1+i} + z_2 \right\|_Z = \left\| z_1 + \lambda^k \mu^s \sum_{i=1}^{m} e'_{n_1+i} + z_2 \right\|_Z , \tag{3.25}$$

for all $m, k, s, n_1 \in \mathbb{N}$, $z_1, z_2 \in \operatorname{span}\{e'_n : n \in \mathbb{N}\}$ and $z_1 \prec \sum_{i=1}^{m2^k3^s} e'_{n_1+i} \prec z_2$. In particular,

$$\left\| \sum_{i=1}^{2^k3^s} e'_i \right\|_Z = \lambda^k \mu^s, \qquad \text{for all } k, s \in \mathbb{N}. \tag{3.26}$$

First, consider the possibility that $\lambda = 1$ or $\mu = 1$. Then, $\| \sum_{i=1}^{2^k} e'_i \|_Z = 1$ for all $k \in \mathbb{N}$ or $\| \sum_{i=1}^{3^s} e'_i \|_Z = 1$ for all $s \in \mathbb{N}$. In either case, from Lemma 3.1.17 we see that for any $n \in \mathbb{N}$ there is some $k \in \mathbb{N}$ or $s \in \mathbb{N}$ such that $n \leq 2^k$ or $n \leq 3^s$ so $\|\sum_{i=1}^{n} e'_i\|_Z \leq 1$. From Lemma 3.1.17, we also have $1 = \|e'_1\| \leq \| \sum_{i=1}^{n} e'_i\|$, so

$$\left\| \sum_{i=1}^{n} e'_i \right\|_Z = 1$$

for all $n \in \mathbb{N}$. Now, take some $m \in \mathbb{N}$ and $a_1, ..., a_m$ scalars (not all being zero). By Lemma 3.1.17, we have

$$|a_i| \|a_i e'_i\|_Z \leq \left\| \sum_{i=1}^{m} a_i e'_i \right\|_Z$$

for any $1 \leq i \leq m$. Then,

$$\sup_{1\leq i\leq m} |a_i| \leq \left\| \sum_{i=1}^{m} a_i e'_i \right\|_Z .$$

By the 1-unconditionality of $(e'_n)_{n\in\mathbb{N}}$ and Lemma 3.1.16,

$$\begin{aligned}\left\|\sum_{i=1}^{m} a_i e'_i\right\|_Z &= \left\|\sum_{i=1}^{m} |a_i| e'_i\right\|_Z = \left\|\sup_{1\le i\le m} |a_i| \sum_{j=1}^{m} \frac{|a_j|}{\sup_{1\le t\le m} |a_t|} e'_j\right\|_Z \\ &\le \sup_{1\le i\le m} |a_i| \left\|\sum_{i=1}^{m} a_i e'_i\right\|_Z = \sup_{1\le i\le m} |a_i|.\end{aligned}$$

Considering $(e_n)_{n\in\mathbb{N}}$ to be the unit vector basis of c_0, by the two previous calculations we have

$$\left\|\sum_{i=1}^{m} a_i e'_i\right\|_Z = \left\|\sum_{i=1}^{m} a_i e_i\right\|_\infty .$$

Since this works for any linear combination of $(e_n)_{n\in\mathbb{N}}$, we have that the unit vector basis $(e_n)_{n\in\mathbb{N}}$ of c_0 is is isometrically equivalent to, and thus block finitely representable in, $(e'_n)_{n\in\mathbb{N}}$ and we are done. This is however, only the case when λ or μ are 1.

If $\lambda, \mu > 1$, we define a function $f : \{2^k/3^s : k, s \in \mathbb{N}\} \to \mathbb{R}^+$ by $f(2^k/3^s) := \lambda^k/\mu^s$, for all $k, s \in \mathbb{N}$. It is clear then that f is multiplicative. And from the fact that $(e'_n)_{n\in\mathbb{N}}$ is 1-unconditional we get

$$\lambda^k \mu^s = \left\|\sum_{i=1}^{2^k 3^s} e'_i\right\|_Z \le \left\|\sum_{i=1}^{2^l 3^r} e'_i\right\|_Z = \lambda^l \mu^r$$

whenever $2^k 3^s \le 2^l 3^r$, i.e. f is monotonously increasing.

To prove that $\{2^k/3^s : k, s \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$, it is enough to show that its image through a homeomorphism, say $\ln(x)$, is dense. To verify that the image

$$\{\ln(2^k/3^s) : k, s \in \mathbb{N}\} = \{\ln 2(k - s\log_2^3) : k, s \in \mathbb{N}\},$$

which we may see as an additive group, is dense in $\mathbb{R}$, it suffices to notice that it isn't a cyclic group, which follows from the fact that $\log_3 2$ is irrational.

By defining $f(a) := \sup\{f(2^k/3^s) : k, s \in \mathbb{N}, 2^k/3^s \le a\}$ for any $a \in \mathbb{R}^+$, we then have an extension of f into the non-negative real numbers $\mathbb{R}^+$ (which we still denote by f), that is easily seen to still be monotonously increasing and multiplicative. From Lemma 3.3.2 we have $f(x) = x^\alpha$ for some $\alpha \in \mathbb{R}^+$. Since f isn't constant, we know that $\alpha > 0$ and there is some $p \ge 1$ for which

$f(x) = x^{1/p}$. From this we have $\lambda = f(2) = 2^{1/p}$ and $\mu = f(3) = 3^{1/p}$. In what follows let $(e_n)_{n\in\mathbb{N}}$ be the standard basis of ℓ_p.

If we define $h(d) = \left\|\sum_{i=1}^d e'_i\right\|_Z$ for all $d \in \mathbb{N}$, we easily see that h is monotonously increasing (from the 1-unconditionality of $(e'_n)_{n\in\mathbb{N}}$) and subadditive. By Equation (3.25) we have

$$h(2^k 3^s) = \left\|\lambda^k \mu^s e'_i\right\|_Z = (2^k 3^s)^{1/p}$$

for all $k, s \in \mathbb{N}$. We'll now check that $h(d) = d^{1/p}$ holds for all $d \in \mathbb{N}$: First, fix $d \in \mathbb{N}$. From the fact that $\{\frac{2^k}{3^s} : k, s \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$, for any $n \in \mathbb{N}$, we choose $k_n, s_n, t_n \in \mathbb{N}$ such that $s_n \geq n$ and

$$\frac{2^{k_n}}{3^{s_n}} \leq d \leq \frac{2^{k_n}}{3^{s_n}} + \frac{1}{3^n}$$

by multiplying everything by 3^s we get

$$2^k \leq 3^{s_n} d \leq 2^{k_n} + 3^{s_n - n}$$

and from the fact that h is increasing and subadditive we have

$$\begin{aligned} & h\left(2^{k_n}\right) \leq h(3^{s_n} d) \leq h\left(2^{k_n}\right) + h\left(3^{s_n-n}\right) \\ \implies & (2^{k_n})^{1/p} \leq h(d)(3^{s_n})^{1/p} \leq (2^{k_n})^{1/p} + (3^{s_n-n})^{1/p} \\ \implies & \left(\frac{2^{k_n}}{3^{s_n}}\right)^{1/p} \leq h(d) \leq \left(\frac{2^{k_n}}{3^{s_n}}\right)^{1/p} + \left(\frac{1}{3^n}\right)^{1/p}. \end{aligned}$$

Since $\lim_{n\to\infty}(1/3^n)^{1/p} = 0$, we have $\lim_{n\to\infty}(2^{k_n}/3^{s_n})^{1/p} = h(d)$, but from the continuity of $x \mapsto x^{1/p}$ and $2^{k_n}/3^{s_n} \xrightarrow{n\to\infty} d$ we find that

$$h(d) = \lim_{n\to\infty} \left(\frac{2^{k_n}}{3^{s_n}}\right)^{\frac{1}{p}} = d^{\frac{1}{p}}.$$

Fix any choice of $n, k_1, ..., k_n, s_1, ..., s_n \in \mathbb{N}$ and let $c : \{1, ..., n\} \to \mathbb{N}$ be given by $c(t) := \sum_{i=1}^t 2^{k_i} 3^{s_i}$. From the 1-unconditionality of $(e'_n)_{n\in\mathbb{N}}$ and the fact that $\lambda = 2^{1/p}$ and $\mu = 3^{1/p}$ we get

$$\begin{aligned} \left\|\sum_{i=1}^n (2^{k_i} 3^{s_i})^{\frac{1}{p}} e'_i\right\|_Z &= \left\|\sum_{i=1}^n \lambda^{k_i} \mu^{s_i} e'_i\right\|_Z = \left\|\lambda^{k_1} \mu^{s_1} e'_1 + \sum_{i=2}^n \lambda^{k_i} \mu^{s_i} e'_i\right\|_Z \\ &= \left\|\lambda^{k_1} \mu^{s_1} e'_1 + \sum_{i=2}^n \lambda^{k_i} \mu^{s_i} e'_{c(1)+i}\right\|_Z \end{aligned}$$

Using that $(e'_n)_{n\in\mathbb{N}}$ is 1-spreading and Equation (3.25), we get

$$\begin{aligned}\left\|\sum_{i=1}^{n}(2^{k_i}3^{s_i})^{\frac{1}{p}}e'_i\right\|_Z &= \left\|\sum_{i=1}^{2^{k_1}3^{s_1}} e'_i + \sum_{i=2}^{n}\lambda^{k_i}\mu^{s_i}e'_{c(1)+i}\right\|_Z \\ &= \left\|\sum_{i=1}^{c(1)} e'_i + \lambda^{k_2}\mu^{s_2}e'_{c(1)+2} + \sum_{i=3}^{n}\lambda^{k_i}\mu^{s_i}e'_{c(2)+i}\right\|_Z \\ &= \left\|\sum_{i=1}^{c(1)} e'_i + \sum_{i=c(1)+2}^{c(2)+1} e'_i + \sum_{i=3}^{n}\lambda^{k_i}\mu^{s_i}e'_{c(2)+i}\right\|_Z \\ &= \left\|\sum_{i=1}^{c(2)} e'_i + \sum_{i=3}^{n}\lambda^{k_i}\mu^{s_i}e'_{c(2)+i}\right\|_Z .\end{aligned}$$

By repeating this with

$$\left\|\sum_{i=1}^{n}(2^{k_i}3^{s_i})^{\frac{1}{p}}e'_i\right\|_Z = \left\|\sum_{i=1}^{c(t-1)} e'_i + 2^{k_t}3^{s_t}e'_{c(t-1)+t} + \sum_{i=t+1}^{n}\lambda^{k_i}\mu^{s_i}e'_{c(t)+i}\right\|_Z ,$$

for every $t \in \{2, ..., n\}$, we get

$$\left\|\sum_{i=1}^{n}(2^{k_i}3^{s_i})^{\frac{1}{p}}e'_i\right\|_Z = \left\|\sum_{i=1}^{c(n)} e'_i\right\|_Z = \left(\sum_{i=1}^{n}2^{k_i}3^{s_i}\right)^{\frac{1}{p}} = \left\|\sum_{i=1}^{n}(2^{k_i}3^{s_i})^{\frac{1}{p}}e_i\right\|_p .$$

It is clear that if we have this for any scalar coefficients, we would have that $(e'_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ are isometrically equivalent. From what we have seen in our treatment of Proposition 3.1.6, it would be enough to show that this holds for some set of scalar n-tuples dense in the unit sphere of ℓ_p^n.

For this final step, we use yet again the density of $\{2^k/3^s : k, s \in \mathbb{N}\}$ in $\mathbb{R}^+$. Fix $n \in \mathbb{N}$ and let $(a_1, ..., a_n)$ be an arbitrary n-tuple on ℓ_p^n. Without loss of generality (since we have 1-unconditionality), we may assume these are non negative scalars. Take $\varepsilon > 0$ and choose $k_1, ..., k_n$ and $s_1, ..., s_n$ such that $|2^{k_i}/3^{s_i} - a_i| < ((\varepsilon)^p/n)^{1/p}$ for $1 \leq i \leq n$. Then,

$$\|(a_1, ..., a_n) - ((2^{k_1}/3^{s_1}), ..., (2^{k_n}/3^{s_n}))\|_p \leq \varepsilon$$

and the set of n-tuples of this form is dense in ℓ_p^n. From this and the fact that $x \mapsto x^{1/p}$ is continuous, we get that the set of n-tuples of the form $((2^{k_1}/3^{s_1})^{1/p}, ..., (2^{k_n}/3^{s_n})^{1/p})$ is dense in ℓ_p^n. It is not hard to see that the set of the normalized sequences

$$\frac{1}{(\sum_{i=1}^n 2^{k_i}/3^{s_i})^{1/p}} \left(\epsilon_1 \left(2^{k_1}/3^{s_1}\right)^{1/p}, ..., \epsilon_n \left(2^{k_n}/3^{s_n}\right)^{1/p}\right),$$

for $\epsilon_1, ..., \epsilon_n \in \{-1, 1\}$ and $k_1, ..., k_n, s_1, ..., s_n \in \mathbb{N}$, is dense in the sphere of ℓ_p^n. Lastly, since we can scale all coordinates by an arbitrarily large power of 3 while dividing the n-tuple as a whole by said power without changing the vector they represent, we have thus found what we wished for: the set of n-tuples

$$\frac{1}{(\sum_{i=1}^n 2^{k_i}3^{s_i})^{1/p}}(\epsilon_1(2^{k_1}3^{s_1})^{1/p}, ..., \epsilon_n(2^{k_n}3^{s_n})^{1/p})$$

for $\epsilon_1, ..., \epsilon_n \in \{-1, 1\}$ and $k_1, ..., k_n, s_1, ..., s_n \in \mathbb{N}$, is dense in the unit sphere of ℓ_p^n. So,

$$\left\|\sum_{i=1}^n a_i e_i'\right\|_Z = \left\|\sum_{i=1}^n a_i e_i\right\|_p$$

holds for all real scalars $a_1, ..., a_n$. As the choice of $n \in \mathbb{N}$ was arbitrary, we have that $(e_n)_{n\in\mathbb{N}}$ is block finitely representable in $(e'_n)_{n\in\mathbb{N}}$, and so in $(x_n)_{n\in\mathbb{N}}$. $\square$

Bibliography

[1] F. Albiac and N. J. Kalton. *Topics in Banach Space Theory.* Springer, 2006.

[2] S. Artstein-Avidan, V. D. Milman, and A. Giannopoulos. *Asymptotic Geometric Analysis, part II.* Mathematical surveys and monographs, vol. 261, 2021.

[3] A Brunel and L. Sucheston. On B-convex Banach spaces. *Math. Systems Theory*, 7(4):294–299, 1974.

[4] A Brunel and L. Sucheston. On J-convexity and some ergodic super-properties of Banach spaces. *Trans. Amer. Math. Soc.*, 204:79–90, 1975.

[5] T. J. Carlson and S. G. Simpson. *Topological Ramsey Theory.* In: Nešetřil, J., Rödl, V. (eds.) Mathematics of Ramsey Theory, pp. 172 – 183. John Wiley & Sons, 1990.

[6] D. Dacunha-Castelle and J. L. Krivine. Application des ultraproduits à l'étude des espaces et des algèbres de Banach. *Studia Mathematica*, 41(3):315–334, 1972.

[7] E. Ellentuck. A new proof that analytic sets are ramsey. *Journal of Symbolic Logic*, 39(1):163 – 165, 1974.

[8] P. Erdös and R. Rado. Combinatorial theorems on classifications of subsets of a given set. *Proc. London Math. Soc.*, 3(1):417–439, 1952.

[9] P. Erdös and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[10] W. T. Gowers and B. Maurey. Banach spaces with small spaces of operators. *Mathematische Annalen*, 307:543–568, 1997.

[11] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey Theory, second edition*. John Willey & Sons, 1990.

[12] S. Heinrich. Ultraproducts in Banach space theory. *Journal für die reine und angewandte Mathematik*, 313:72–104, 1980.

[13] R. C. James. Some self-dual properties of normed linear spaces. In R. D. Anderson, editor, *Symposium on Infinite Dimensional Topology*, pages 159–175. Princeton University Press, 1972.

[14] A. S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, 1995.

[15] E. Kreyszig. *Functional Analysis*. John Wiley & Sons, 1978.

[16] J. L. Krivine. Sous-espaces de dimension finie des espaces de Banach réticulés. *Ann. of Math.*, 104(1):1–29, 1976.

[17] H. Lemberg. Nouvelle démonstration d'un théorème de J.L. Krivine sur la finie représentation de lp dans un espace de Banach. *Israel J. Math.*, 39(4):341–348, 1981.

[18] R. E. Megginson. *An introduction to Banach Space Theory*. Springer, 1998.

[19] V. D. Milman and G. Schechtman. *Asymptotic Theory of Finite Dimensional Normed Spaces Corr. 2. printing*. Springer-verlag, 2001.

[20] A. Pełczyński. Projections in certain Banach spaces. *Studia Mathematica*, 19:209–228, 1960.

[21] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, s2-30(1):264–286, 1930.

[22] J. Silver. Every analytic set is Ramsey. *Journal of Symbolic Logic*, 35(1):60 – 64, 1970.

[23] A. Soifer, editor. *Ramsey Theory Yesterday, Today and Tomorrow*. Birkhäuser Boston, 2010.

[24] L. A. Steen. Highlights in the history of spectral theory. *The American Mathematical Monthly*, 80(4), 1973.

[25] B. S. Tsirelson. Not every Banach space contains an imbedding of ℓ_p or c_0. *Functional Analysis and Its Applications*, 8(2):138–141, 1974.

Index

2^ω, 12
$=^*$, 14
B/n, 17
$X_{\mathcal{U}}$, 57
$[A]^r$, 6
$[a, A]$, 12
$\mathrm{Int}(\cdot)$, 14
χ_B, 12
$\ell_\infty(X)$, 56
$\partial\sigma(T)$, 53
$\rho(T)$, 52
σ-algebra, 14
σ-ideal, 14
$\sigma(T)$, 52
$\sim A$, 16
$\sum_{i=1}^\infty a_i e_i \prec \sum_{i=1}^\infty b_i e_i$, 64
$a \to (b)^r_k$, 7
$a < A$, 12
c_0, 49
$c_{0,\mathcal{U}}$, 57
c_{00}, 36
k-coloring, 6

analytic set, 21
approximate
  eigenvalue, 52
  eigenvector, 53

Baire property, 14
Borel set, 15

convergence through filters, 54

Ellentuck topology, 12
Ellentuck's theorem, 20

filter, 54

Galvin-Prikry theorem, 21

ideal, 13
Invariance under spreading, 24

Krivine's theorem, 71

monochromatic set, 7
multiplicative function, 70

pigeonhole principle, 5
polish space, 21
prime Banach space, 23

Ramsey set, 16
  completely, 16
Ramsey's theorem, 7, 8
regular value, 52
representable
  K-block finitely, 35
  K-finitely, 34

block finitely, 35
crudely block finitely, 35
crudely finitely, 34
finitely, 34
resolvent
set, 52

sequence
1-spreading, 24
1-unconditional, 42
basic, 28
distribution of, 64
isometrically equivalent, 70
loosely spreading, 28
support of, 64
Silver's theorem, 21
spectral value, 52
spectrum of T, 52

ultrafilter, 54
nonprincipal, 55
principal, 55
ultrapower
of a Banach space, 57
of an operator, 57