Greek Geometry: Regular Polyhedra

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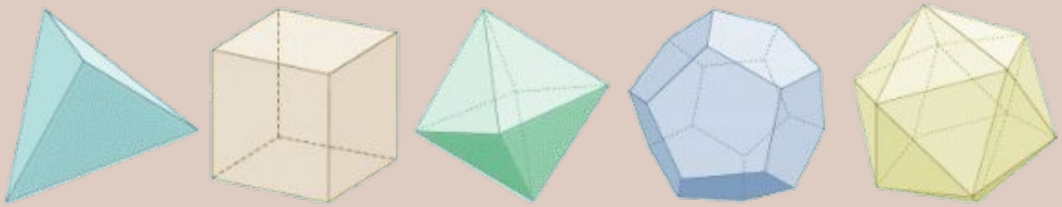
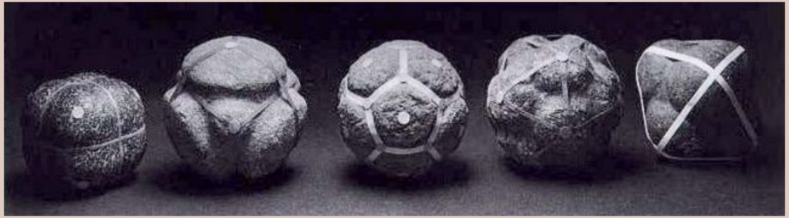
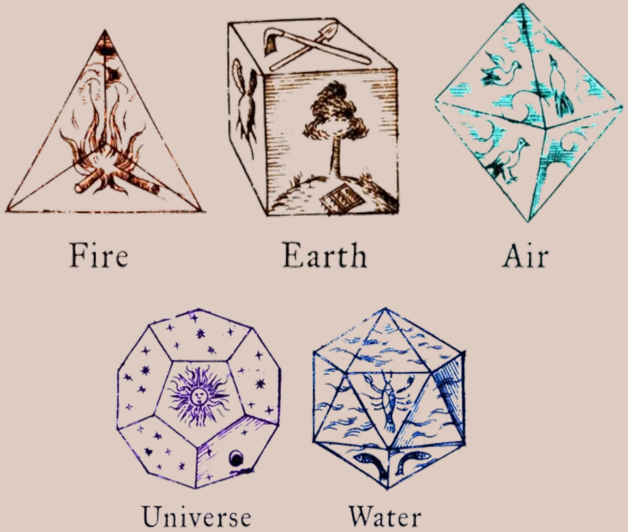


Background





Historical

Stones carved by Neolithic people in shapes resembling spheres found in Scotland; around 4000 years ago

Pythagoras of Samos: inventor of the regular polyhedron

Theaetetus: discovered the regular octahedron and icosahedron around 360 BC

Euclid (325-265 BC): studied regular polyhedra in *The Elements*; limit to five solids

Athenian philosopher Plato: associated regular polyhedra with the four basic elements and the heavens

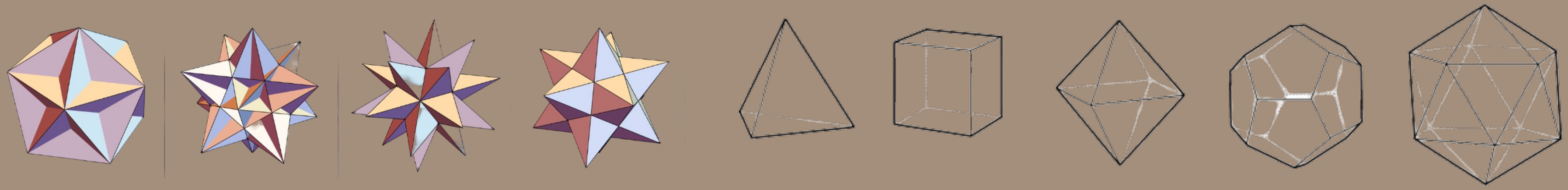
- Plato's theory development of universe based on the five regular polyhedra → Platonic solids

Artists of the Renaissance: drawing semi-regular polyhedra and other non-convex polyhedra

Johannes Kepler (1571-1630): revived study of polyhedra and discovers two-star polyhedra

Leonhard Euler (1707-1787): proved formula which links the number of vertices, edges, and faces for convex polyhedra

Introduction



Formal Definition:
- A three-dimensional solid with congruent regular polygon faces, straight edges of equal length, and identical vertices where the same number of faces meet.

9 regular polyhedra in total:
- Kepler–Poinsot polyhedra: discovered independently by mathematicians Johannes Kepler and Louis Poinsot
  - Unique given that they contain intersecting facial planes and have pentagrams as faces/vertex figures
  - Small-stellated dodecahedron: 12 intersecting pentagrammic faces, with five pentagrams at each vertex
  - Great dodecahedron: 12 intersecting pentagonal faces (six pairs of parallel pentagons), with five pentagons meeting at each vertex
  - Great stellated dodecahedron: 12 intersecting pentagrammic faces, with three pentagrams at each vertex
  - Great icosahedron (Johnson solid): 20 intersecting triangular faces, with five triangles meeting at each vertex in a pentagrammic sequence
- Convex regular polyhedra (Platonic solids): tetrahedron, cube, octahedron, dodecahedron, icosahedron

Introduction

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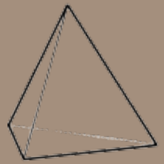


Tetrahedron

4 equilateral triangle faces

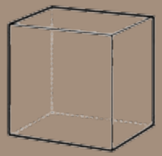
6 straight edges

4 vertex corners, each equidistant from each other

6 planes of symmetry

no parallel face

(b)



Cube

6 identical square faces

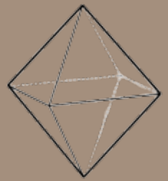
12 straight edges

8 vertex corners, each equidistant from each other

90° angles between any two faces or surfaces

opposite faces and edges parallel

(c)



Octahedron

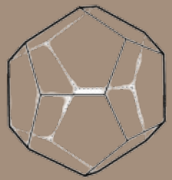
8 equilateral triangle faces

12 straight edges

6 vertex corners, each equidistant from each other

4 edges meet at each vertex

(d)



Dodecahedron

12 pentagonal faces

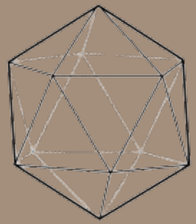
30 straight edges

20 vertex corners, each equidistant from each other

160 diagonals

3 edges meet at each vertex

(e)



Icosahedron

20 equilateral triangle faces

30 straight edges

12 vertex corners, each equidistant from each other

greatest volume and number of faces of any Platonic solid

Properties and Principles

Properties:

- Regular Faces:
  - Faces consist of regular polygons with congruent sides/angles of each face
- Uniform Symmetry:
  - Rotational Symmetry - polyhedron looks the same after rotation around its center
  - Reflection Symmetry - polyhedron looks the same after reflection across a plane
- Angles/Lengths:
  - Uniform and equal angles between faces and edge lengths
- Constant Vertex Configuration:
  - Same # of faces meet at each vertex
  - All the vertex figures of the polyhedron are regular polygons
  - The vertices of a convex regular polyhedron all lie on a sphere

Principles:

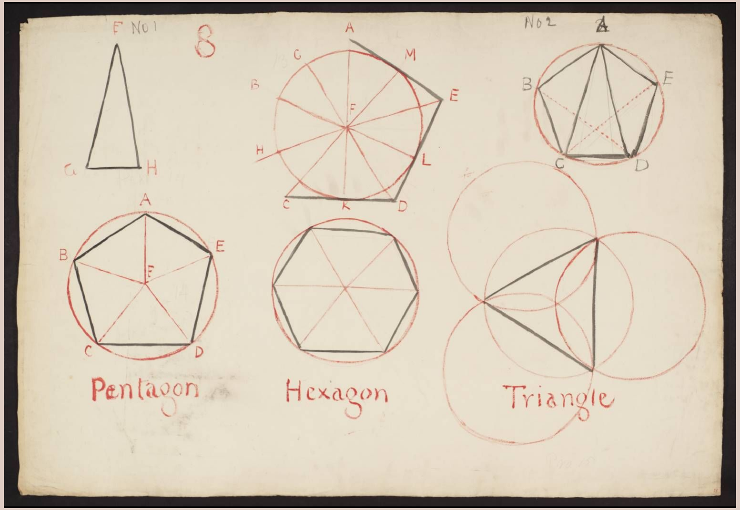
- Euler's Formula:
  - Any convex polyhedron are related by V(vertices) - E(edges) + F(faces) = 2
  - Fundamental relationship between polyhedra topological properties
- Duality Principles:
  - Polyhedra's certain properties are translated to those of their dual polyhedra
- Convexity
  - Regular polyhedra have no indentations to ensure straight line segments joining any two inner points
- Symmetry Groups
  - Describes all possible symmetries including rotations, reflections, etc. that leave the polyhedron unchanged

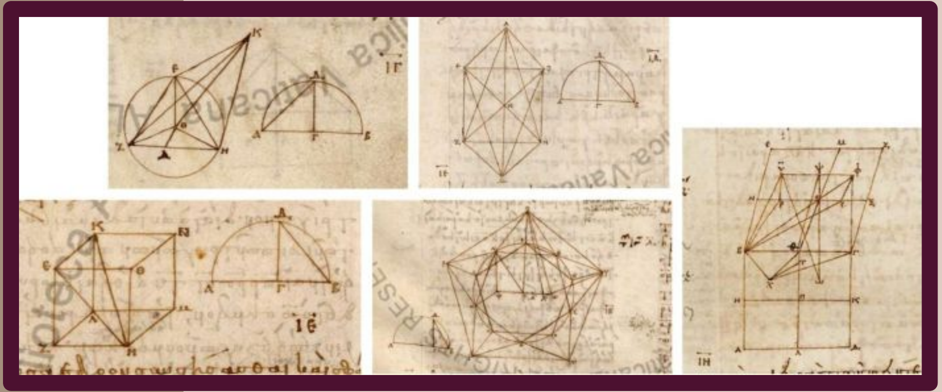
Motivations

Goal: to understand and classify geometric shapes in 3D-space
- Interest in symmetrical properties of polyhedra; uniformity of faces, edges, and angles
- Develop a systematic way to classify other geometrical shapes and expand mathematical knowledge

Goal: to understand and construct the five Platonic solids
- Use of ruler and compass for constructions
- Important in ancient geometry
- Practical applications in architecture, art, astronomy

Developments
1. Concept of symmetry groups
   a. Better understand geometric object symmetries and polyhedra symmetries
2. Group theory
   a. Understand relationship between different symmetries and classifying polyhedra

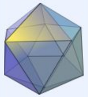
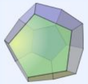




Proof of Existence

Why are there only five Platonic solids?

- For all five shapes, at least three faces meet at each vertex
- The sum of the internal angles that meet at a vertex must be less than 360°
  - Shapes flatten out at 360°
- Ex: a regular triangle has internal angles of 60° → maximum of five triangles can meet at each vertex
  - Tetrahedron, octahedron, icosahedron
- Ex: a square has internal angles of 90° so a maximum of three squares can meet
  - Cube
- Ex: a regular pentagon has internal angles of 108° → maximum of three pentagons can meet at each vertex
  - Dodecahedron

At each vertex:	Angles at Vertex (Less than 360°)	Solid	
3 triangles meet	180°	tetrahedron	
4 triangles meet	240°	octahedron	
5 triangles meet	300°	icosahedron	
3 squares meet	270°	cube	
3 pentagons meet	324°	dodecahedron	

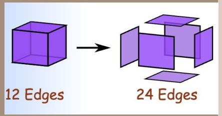
Proof of Existence

Why are there only five Platonic solids? (Using Topology)

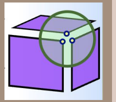
- Euler's Formula: V(vertices) - E(edges) + F(faces) = 2
  - Any convex polyhedron will meet this criteria

Let s = number of sides each face of a Platonic solid can have
Let m = number of faces that meet at a corner

When pulling a solid apart (cutting each face free):



- There are twice as many edges as compared to the original number → 2E
  - Equivalent to the number of sides/face (s) times F (number of faces) → sF = 2E
- One corner will become several corners, depending on the number of faces that meet at a corner (m) times the number of vertices of the original solid (V) → mV
  - Same number of corners as edges → mV = 2E



Plug into F + V - E = 2 to get 2(E/s) + 2(E/m) - E = 2

- The number of edges cannot be < 0
- Rearrange to get the final expression 1/s + 1/m > ½
- Only five possible combinations that work!

s	m	1/s+1/m	> 0.5 ?
3	3	0.666...	✓
3	4	0.583...	✓
4	3	0.583...	✓
4	4	0.5	✗
5	3	0.533...	✓
3	5	0.533...	✓
5	4	0.45	✗
4	5	0.45	✗
5	5	0.4	✗
etc...	...	...	✗

Proof of Constructability of Polyhedra

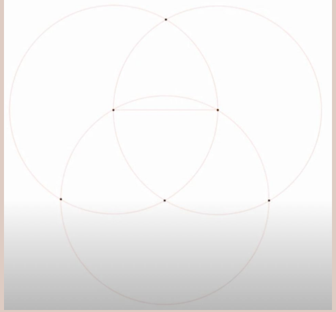
Objective: Explore the step-by-step geometric construction of Platonic solids using classical tools.

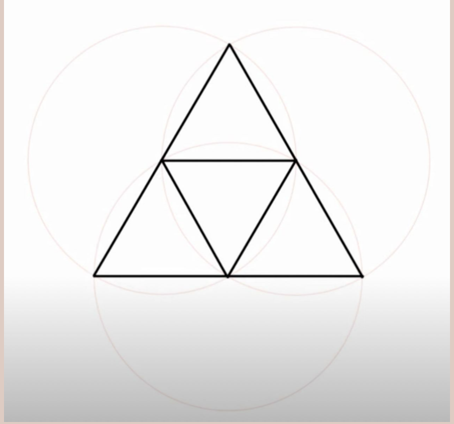
Tools Required:

1. Compass: Essential for drawing arcs and perfect circles.
2. Straightedge: Crucial for drawing straight lines without measurements.

Constructing a Tetrahedron:

Step 1: Draw an equilateral triangle using the compass and straightedge.
Step 2: Construct a circle around the triangle by setting the compass to the length of one side and drawing an arc from each vertex.
Step 3: Keeping the compass at the same radius, draw an arc from each vertex of the triangle upwards to locate the fourth vertex.
Step 4: Join this new vertex with straight lines to each of the three vertices of the base triangle to complete the tetrahedron.



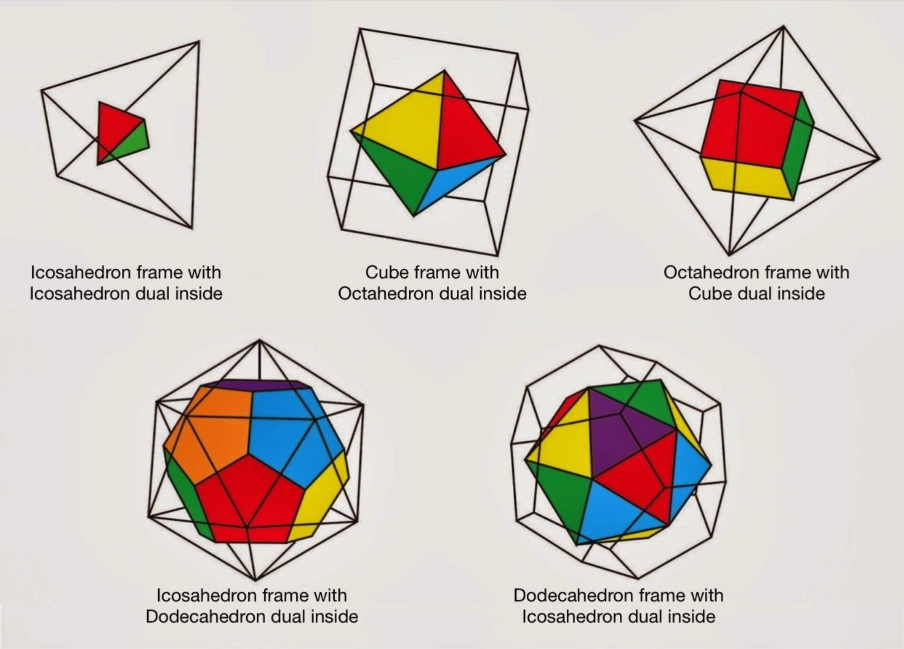


Advancements in Mathematics: The Role of Regular Polyhedra

Key Understanding:
Regular Polyhedra are fundamental in understanding three-dimensional symmetries (rotations, reflections, inversions)

Leading Contributions:
Early development of group theory (closure, associativity, identity elements, inverse elements)

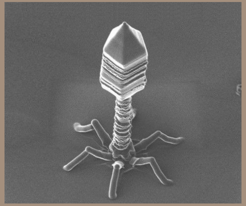
Key Principle:
Dual polyhedra share identical symmetry groups, which illustrates deep geometric relationships. It shows that the inherent symmetries of a shape are not merely a feature such as the length of its edges or the angles between its faces but are instead a more fundamental property related to how vertices, edges, and faces are arranged and connected in space.



Significance & Applications

Significance

Occurrences in Nature:
- Tetrahedron, cube, and octahedron found in crystal shapes
- New carbon molecules discovered are hypothesized to have the shape of a slightly rounded icosahedron
- Biological organisms and cells (such as the myovirus) can take the shape of dodecahedrons or icosahedrons



In Philosophy:
- Plato philosophized a theory of matter which connected the five elements as being comprised of tiny copies of one of the five regular solids

Applications

Technology:
- Tetrahedrons used in electronics, icosahedrons used in geophysical modelling, polyhedral speakers used for sound energy

Architectural designs
- Great Pyramids of Giza, Mayan Ruins

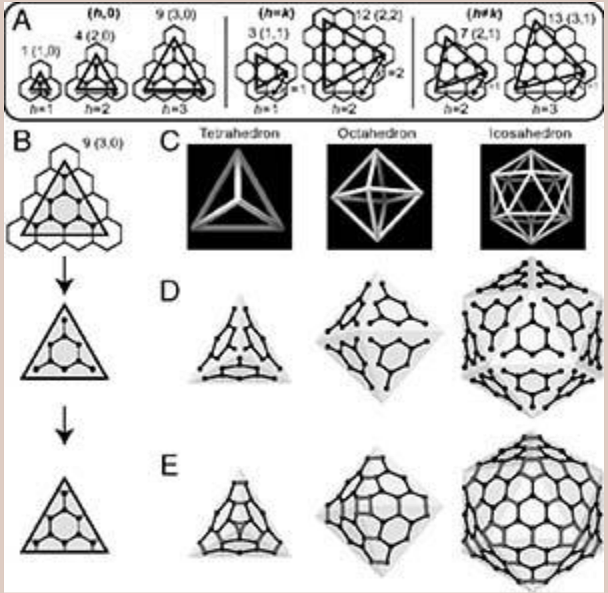
Future Development

Regular Polyhedra in non-Euclidean Space

- Studies of hyperbolic and elliptic space reveals complex polyhedra which only take regular form in this space

New Class of Polyhedra Discovered

- In 2014, researchers discovered a fourth class of convex equilateral polyhedra with polyhedral symmetry
- Start with a tetrahedron, an octahedron, or an icosahedron (facets of these polyhedra are equilateral triangles)
- Draws a triangle on a mesh or tiling of hexagons, creating Goldberg triangles.
- Next, place the triangles on each of the
- polyhedron's facets.
- Add edges to join the vertices of the Goldberg triangles



Conclusion

New Developments building on Old Concepts

- The study of three dimensional symmetries leading to the field of modern algebra and analysis

- Allow mathematicians to develop a more systematic way to classify other geometric shapes and expand mathematical knowledge into various fields

- Further applications to biology and the natural sciences: viruses are examples of naturally occurring polyhedra, understanding their shape can aid the study of their effects

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