ML Homework week 2 _ DSEB 62

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Exercise 1

Gaussian distribution:

$$p(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

a) To prove that Gaussian distribution is normalized, we need to prove:

$$\int_{-\infty}^{\infty} p(x|\mu,\sigma^2) = 1$$

or $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1 \quad (1)$$

We assume that $\mu = 0$

$$\Rightarrow (1) \Leftrightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{-x^2}{2\sigma^2} dx = 1$$

$$\Leftrightarrow \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}$$

let $$I = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}} dx$$

$$\rightarrow I^2 = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}} dx \int_{-\infty}^{\infty} e^{\frac{-y^2}{2\sigma^2}} dy$$

$$\Leftrightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2\sigma^2}} dx\,dy$$

let $x = r\cos\theta$ $\rightarrow x^2+y^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2$

$y = r\sin\theta$

The Jacobian of the change of variables is given by:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta = r$$

$$\Leftrightarrow \partial(x,y) = r\,\partial(r,\theta)$$

$\Rightarrow$ We have:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{\frac{-r^2}{2\sigma^2}}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} e^{\frac{-r^2}{2\sigma^2}}\, r\,dr$$

Then, let $u = r^2 \Leftrightarrow du = 2dr$

$$\Rightarrow I^2 = \pi \int_0^{\infty} e^{\frac{-u}{2\sigma^2}}\, du$$

$$= \pi \left[e^{\frac{-u}{2\sigma^2}} \cdot (-2\sigma^2)\right]\Big|_0^{\infty}$$

$$= -2\pi\sigma^2 \cdot (0-1)$$

$$= 2\pi\sigma^2$$

$$\Rightarrow \boxed{I = \sqrt{I^2} = \sqrt{2\pi\sigma^2} \quad \text{(proof)}}$$

b) We have $E(X) = \int_{-\infty}^{\infty} x\, p(x|\mu,\sigma^2)\, dx$

$$\Rightarrow E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

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Let $t = \dfrac{x-\mu}{\sigma\sqrt{2}} \Rightarrow \begin{cases} \dfrac{dt}{dx} = \dfrac{1}{\sigma\sqrt{2}} \Leftrightarrow dx = \sigma\sqrt{2}\,dt \\ x = \sigma\sqrt{2}t + \mu \end{cases}$

$$\Rightarrow E(X) = \int_{-\infty}^{\infty} (\sigma\sqrt{2}t + \mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2} \cdot \sigma\sqrt{2}\,dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} (\sigma\sqrt{2} + \mu) e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}} \cdot \left(\int_{-\infty}^{\infty} \sigma\sqrt{2} t e^{-t^2} dt + \int_{-\infty}^{\infty} \mu e^{-t^2} dt \right)$$

$$= \frac{1}{\sqrt{\pi}} \cdot \left[\sigma\sqrt{2} \cdot \left(-\frac{1}{2} e^{-t^2} \right) \Big|_{-\infty}^{\infty} + \mu\sqrt{\pi} \right]$$

$$= \frac{1}{\sqrt{\pi}} \cdot (0 + \mu\sqrt{\pi})$$

$$\boxed{\Leftrightarrow E(X) = \mu}$$

c) We have

$$\text{var}(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) dx - \mu^2$$

Let $t = \dfrac{x-\mu}{\sigma\sqrt{2}} \Rightarrow \begin{cases} dx = \sigma\sqrt{2}\,dt \\ x = \sigma\sqrt{2}t + \mu \end{cases}$

$$\Rightarrow \text{var}(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} (\sigma\sqrt{2}t + \mu)^2 \cdot e^{-t^2} \sigma\sqrt{2}\,dt - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\sigma\sqrt{2}t + \mu)^2 e^{-t^2} dt - \mu^2$$

$$\Leftrightarrow \text{var}(X) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (2\sigma^2 t^2 + 2\sqrt{2}\sigma\mu t + \mu^2) e^{-t^2} dt - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} \left(2\sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt + 2\sqrt{2}\sigma\mu \left(\frac{-1}{2} e^{-t^2} \right) \Big|_{-\infty}^{\infty} + \mu^2\sqrt{\pi} \right) - \mu^2$$