General Information

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Class: Applied Mathematics - 33/2023

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In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

from math import comb
from collections import Counter
```

Note for doing exercise

Poisson distribution = discrete probability distribution of a number of events in fixed interval of time with two main conditions:

- Events occur with some constant mean rate.
- Events are independent of each other and independent of time.

The PMF (probability mass function) of a Poisson distribution is given by:

$$p(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

where:

- λ: real number, $\lambda = E(X) = \mu$
- k: the number of occurrences

The CDF (cumulative distribution function) of a Poisson distribution is given by:

$$F(k, \lambda) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!}$$

In [ ]:
```python
# from scipy import special

def poisson_pmf(_k: int, _lambda: float):
    """Python implementation of the PMF (probability mass function) of a Poisson distribution.

    Args:
        _k (int): the number of occurrences
        _lambda (float): $\lambda$: real number, $\lambda = E(X) = \mu$
    """
    return (np.power(_lambda, _k) * np.exp(-_lambda)) / np.math.factorial(_k)
```

In [ ]:
```python
def poisson_cdf(_k: int, _lambda: float):
    """Python implementation of the CDF (cumulative distribution function) of a Poisson distribution.

    Args:
        _k (int): the number of occurrences
        _lambda (float): $\lambda$: real number, $\lambda = E(X) = \mu$
    """
    cdf = 0
    for i in range(_k+1):
        cdf += poisson_pmf(i, _lambda)

    return cdf
```

In [ ]:
```python
# Poisson distribution example in Python
k = np.arange(0, 17)
k
```

Out[ ]:
```
array([ 0,  1,  2,  3,  4,  5,  6,  7,  8,  9, 10, 11, 12, 13, 14, 15, 16])
```

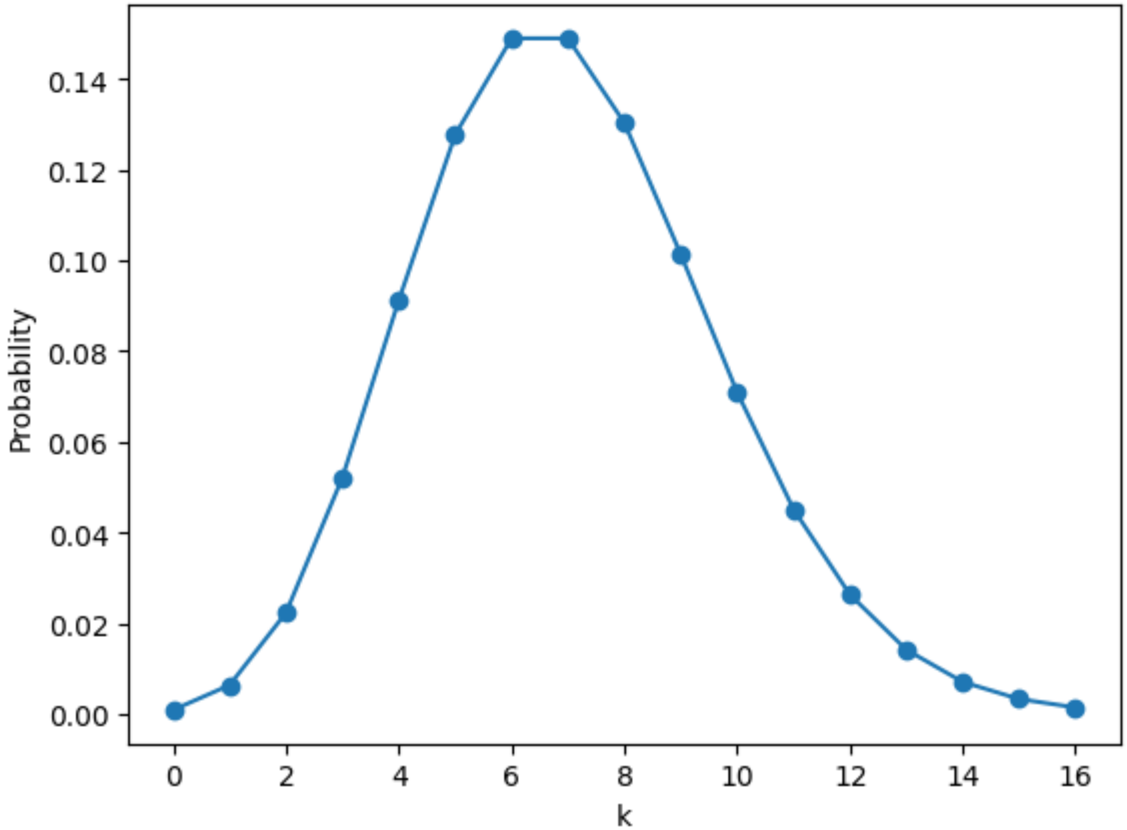
In [ ]:
```python
pmf = []

for ik in k:
    pmf.append(poisson_pmf(ik, _lambda=7))

pmf = np.round(pmf, 5)
```

In [ ]:
```python
for val, prob in zip(k, pmf):
    print(f"k-value {val} has probability = {prob}")
```

```
k-value 0 has probability = 0.00091
k-value 1 has probability = 0.00638
k-value 2 has probability = 0.02234
k-value 3 has probability = 0.05213
k-value 4 has probability = 0.09123
k-value 5 has probability = 0.12772
k-value 6 has probability = 0.149
k-value 7 has probability = 0.149
k-value 8 has probability = 0.13038
k-value 9 has probability = 0.1014
k-value 10 has probability = 0.07098
k-value 11 has probability = 0.04517
k-value 12 has probability = 0.02635
k-value 13 has probability = 0.01419
k-value 14 has probability = 0.00709
k-value 15 has probability = 0.00331
k-value 16 has probability = 0.00145
```

In [ ]:

```
plt.plot(k, pmf, marker='o')
plt.xlabel('k')
plt.ylabel('Probability')

plt.show()
```



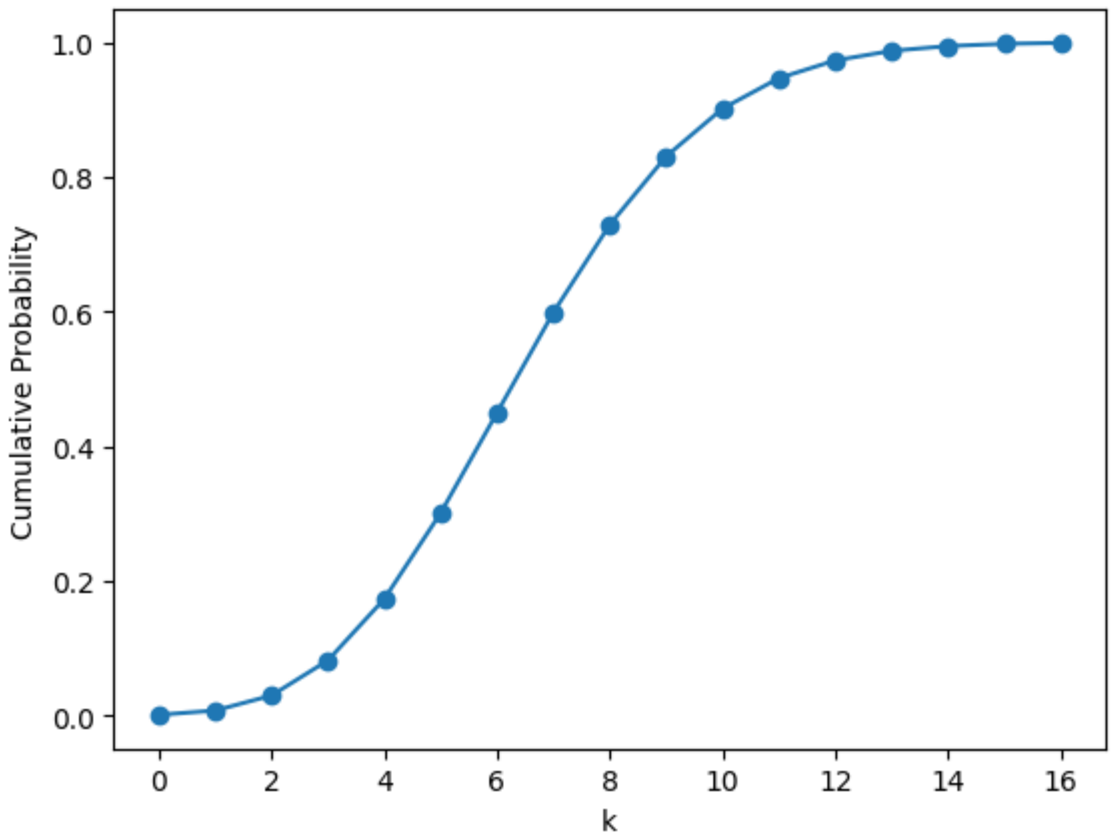
In [ ]:
```
cdf = []

for ik in k:
    cdf.append(poisson_cdf(ik, _lambda=7))

cdf = np.round(cdf, 5)
```

In [ ]:
```
for val, prob in zip(k, cdf):
    print(f"k-value {val} has probability = {prob}")
```

```
k-value 0 has probability = 0.00091
k-value 1 has probability = 0.0073
k-value 2 has probability = 0.02964
k-value 3 has probability = 0.08177
k-value 4 has probability = 0.17299
k-value 5 has probability = 0.30071
k-value 6 has probability = 0.44971
k-value 7 has probability = 0.59871
k-value 8 has probability = 0.72909
k-value 9 has probability = 0.8305
k-value 10 has probability = 0.90148
k-value 11 has probability = 0.94665
k-value 12 has probability = 0.973
k-value 13 has probability = 0.98719
k-value 14 has probability = 0.99428
k-value 15 has probability = 0.99759
k-value 16 has probability = 0.99904
```

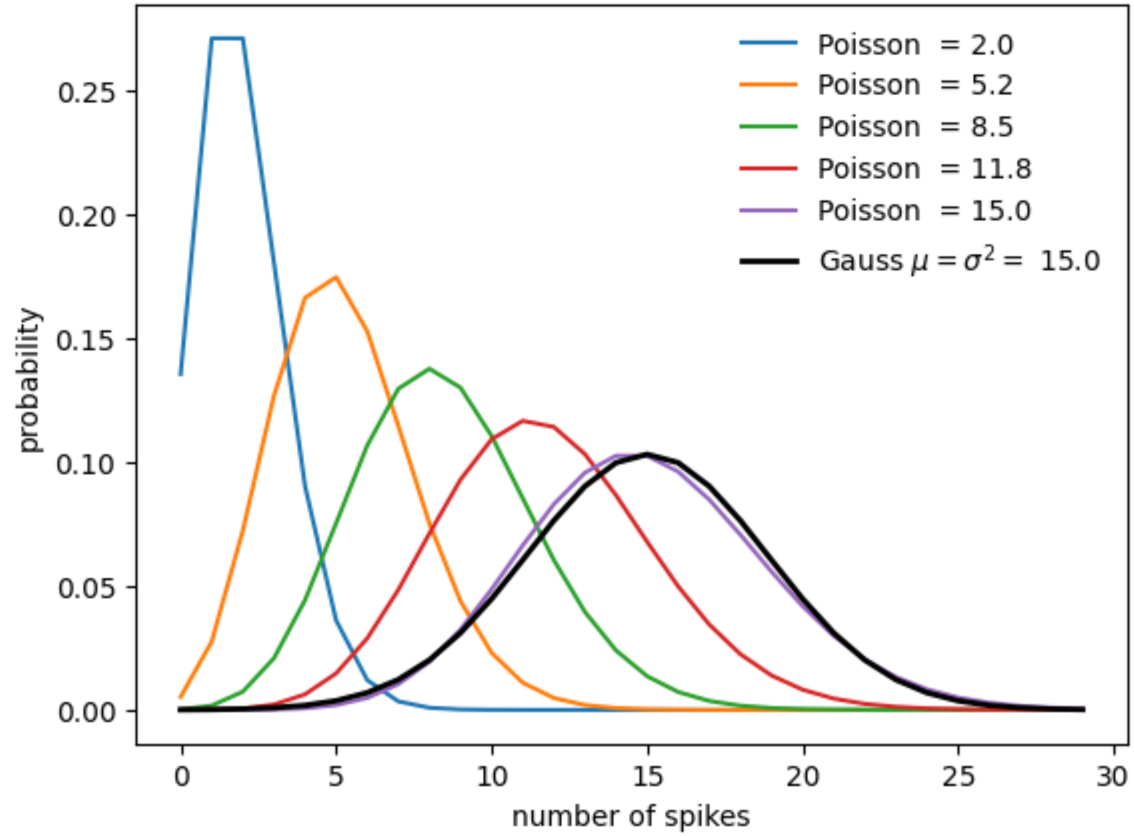
In [ ]:

```
plt.plot(k, cdf, marker='o')
plt.xlabel('k')
plt.ylabel('Cumulative Probability')

plt.show()
```


Cumulative Probability
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
k

In [ ]:
```
x = np.arange(30)

for lam in np.linspace(2, 15, 5):
    pmf = []
    for ix in x:
        pmf.append(poisson_pmf(ix, lam))

    plt.plot(x, pmf, label='Poisson  = %.1f' % lam)

def gaussian_pdf(x, mu, sigma_sq):
    return 1/(np.sqrt(2*np.pi*sigma_sq))*np.exp(-(x-mu)**2/sigma_sq/2)

plt.plot(gaussian_pdf(x, lam, lam), 'k', label=f"Gauss $\mu = \sigma^2 =$ {lam}", lw=2)
plt.legend(frameon=False)
plt.xlabel('number of spikes')
plt.ylabel('probability')
```

Out[ ]:
```
Text(0, 0.5, 'probability')
```



Stochastic Process - Homework Assignment 02

Exercise 01

Defects occur along the length of a filament according to a Poisson distribution of rate of $\lambda = 2$ per foot

(a) Calculate the probability that there are no defects in the first foot of the filament.

For $s > 0, t > 0$, the random variable has the Poisson distribution

$$P[N(t+s) - N(s) = n] \sim \text{Poisson}(\lambda t)$$

Due this Poisson process so that $N(0) = 0$.

We need to calculate the probability that there are no defects in the first foot of the filament:

$$\begin{aligned} P[N(1) - N(0) = 0] &= P[N(1) - 0 = 0] \qquad \text{since} \quad N(0) = 0 \\ &= P[N(1) = 0] \\ &= (2 \cdot 1)^0 \cdot \frac{e^{(-2 \cdot 1)}}{0!} \\ &= 0.13534 \end{aligned} \tag{1}$$

(b) Calculate the conditional probability that there are no defects in the second foot of the filament, given that the first foot contained a single defect.

Due the property of Poisson process, $N(t)$ has independent increments. So for any time points $t < t_1 < t_2 < \cdots < t_n$, the increment $N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$ are independent random variables.

We need to calculate the conditional probability that there are no defects in the second foot of the filament, given that the first foot contained a single defect. That means

$$\begin{aligned} P[N(2) - N(1) = 0 \mid N(1) - N(0) = 1] &= P[N(2) - N(1) = 0] \\ &= (2 \cdot 1)^0 \cdot \frac{e^{(-2 \cdot 1)}}{0!} \\ &= 0.13534 \end{aligned} \tag{2}$$

Exercise 02

Customers arrive at a service facility according to a Poisson process of rate λ customer/hour. Let $N(t)$ be the number of customers that have arrived up to time t.

(a) What is $P[N(t) = k]$ for $k = 0, 1, \ldots$?

$$P[N(t) = k] = e^{(-\lambda t)} \frac{(\lambda t)^k}{k!}, \text{ for } k = 0, 1, \ldots$$

(b) Consider fixed times $0 < s < t$. Determine the conditional probability $P[N(t) = n + k | N(s) = n]$ and the expected value $\mathbb{E}[N(t)N(s)]$.

Determine the conditional probability $P[N(t) = n + k | N(s) = n]$. We have

$$\begin{aligned} P[N(t) = n + k | N(s) = n] &= P[N(t - s) = k | N(s) = n] \\ &= \frac{P[N(t - s) = k \cap N(s) = n]}{P[N(s) = n]} \\ &= \frac{P[N(t - s) = k] P[N(s) = n]}{P[N(s) = n]} \\ &= P[N(t - s) = k] \\ &= e^{[-\lambda(t - s)]} \frac{[\lambda(t - s)]^k}{k!} \end{aligned} \tag{3}$$

Determine the expected value $E[N(t)N(s)]$. We have

$$\mathbb{E}[N(t)N(s)] = \mathbb{E}[N(t)]\mathbb{E}[N(s)] = (\lambda t)^2$$

Exercise 03

Messages arrive at a telegraph office as a Poisson process with mean rate of 3 messages per hour.

(a) What is the probability that no messages arrive during the morning hours 8:00 A.M. to noon?

$\lambda = 3$ (messages/ hour)

The morning hours 8:00 A.M. to noon = 4 hours

$$P[N(4) = 0] = e^{(-3 \cdot 4)} \frac{(3 \cdot 4)^0}{0!} = 6.1442 \times 10^{-6}$$

(b) What is the distribution of the time at which the first afternoon message arrives?

Let T be the time at which the first afternoon message arrives. The probability that no message arrives in the time interval of length $t - 12$

$$P(T > t] = P[N(t) - N(12) = 0] = e^{-\lambda(t-12)} = e^{-3 \cdot (t-12)}, t > 12$$

So the distribution of the time at which the first afternoon message arrives can be written as:

$$F_T(t) = 1 - e^{-3 \cdot (t-12)}, t > 12$$

which means that the time is exponentially distributed with parameter $\lambda = 3$

Exercise 04

Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $N(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities and conditional probabilities:

(a) $P[N(1) = 2]$

$$P[N(1) = 2] = e^{-2 \cdot 1} \frac{(2 \cdot 1)^n}{2!} = 2e^{-2}$$

(b) $P[N(3) = 6 | N(1) = 2]$

$$P[N(3)=6|N(1)=2] = \frac{P[N(1)=2, N(3)=6]}{P[N(1)=2]} = \frac{2^6}{3}e^{-6}\frac{2!}{e^{-2}\cdot 2^2} = \frac{2^5}{3}e^{-4} \tag{4}$$

(c) $P[N(1)=2 \text{ and } N(3)=6]$

$$P[N(1)=2 \text{ and } N(3)=6] = P[N(1)=2, N(3)-N(1)=6-2] = P[N(1)=2, N(2)=4] = e^{(-2\cdot 1)}\frac{(2\cdot 1)^2}{2!}e^{(-2\cdot 2)}\frac{(2\cdot 2)^4}{4!}$$

(d) $P[N(1)=2 \mid N(3)=6]$

$$P[N(1)=2 \mid N(3)=6] = \frac{P[N(1)=2, N(3)=6]}{P[N(3)=6]} = \frac{2^6}{3}e^{-6}\frac{6!}{e^{-6}6^6} = \frac{2^6}{3}\frac{6!}{6^6} = \frac{5\cdot 2^7}{6^5} \tag{5}$$

Exercise 05

Let $N(t); t \geq 0$ be a Poisson process having rate parameter $\lambda = 2$. Determine the numerical values to two decimal places for the following probabilities:

(a) $P[N(1) \leq 2]$

$$P[N(1) \leq 2] = \sum_{k=0}^{2} e^{-2t}\frac{(2t)^k}{k!}$$

(b) $P[N(1)=1 \text{ and } N(2)=3]$

$$P[N(1)=1 \text{ and } N(2)=3] = P[N(1)=1]P[N(2)=3]$$

(c) $P[N(1) \geq 2 \mid N(1) \geq 1]$

$$P[N(1) \geq 2 \mid N(1) \geq 1] = \frac{P[N(1) \geq 2, N(1) \geq 1]}{P[N(1) \geq 1]} = \frac{(1 - P[N(1) < 2])(1 - P[N(1) < 1])}{1 - (P[N(1) < 1])}$$

Exercise 06

Let $N(t); t \geq 0$ be a Poisson process having rate parameter $\lambda = 2$. Determine the following expectations:

(a) $\mathbb{E}[N(2)]$

$$\mathbb{E}[N(2)] = \lambda t = 2\lambda$$

(b) $\mathbb{E}[N(1)2]$

$$\mathbb{E}[N(1)2] = 2\mathbb{E}[N(1)] = 2\lambda$$

(c) $\mathbb{E}[N(1)N(2)]$

$$\mathbb{E}[N(1)N(2)] = \mathbb{E}[N(1)]\mathbb{E}[N(2)] = 2\lambda^2$$

Exercise 07

Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of $\lambda = 10$ customers per minute. Let M be the number of customers arriving between $9:00$ and $9:10$. Also, let N be the number of customers arriving between $9:30$ and $9:35$.

a) What is the distribution of $M + N$?

Note: W be the first odd time index with an arrival of $\mathbf{X}$, and Y be the first even time index with an arrival of $\mathbf{X}$. We have $X \perp\!\!\!\perp Y$

$\tau_1 = 10$ (9:00 - 9:10)

$\tau_2 = 5$ (9:30 - 9:35)

We have $M \sim \text{Poisson}(\lambda\tau_1) = \text{Poisson}(10 \cdot 10)$, $N \sim \text{Poisson}(\lambda\tau_2) = \text{Poisson}(10 \cdot 5)$. And due to $M \perp\!\!\!\perp N$ and the sum of two independent Poisson random variables is a Poisson random variable. So that

$$(M + N) \sim \text{Poisson}(\lambda\tau_1 + \lambda\tau_2) = \text{Poisson}(\lambda(\tau_1 + \tau_2))$$

b) Let $\widetilde{N}$ be the number of customers arriving between $9:10$ and $9:15$. What is the distribution of $M + \widetilde{N}$?

$\tau_3 = 5$ (9:10 - 9:15)

$\widetilde{N} \sim \text{Poisson}(\lambda\tau_3) = \text{Poisson}(50)$. Furthermore, $\widetilde{N}$ is also independent of N. Thus, the distribution of $M + N$ is the same as the $M + \widetilde{N}$

(c) Is it true if we say "the distribution of $M + N$ is the same as the distribution of $M + \widetilde{N}$".

$M + \widetilde{N}$ is the number of arrivals during an interval of length 15, and has therefore a Poisson distribution with parameter $10 \cdot 15 = 150$. In the previous case $M + N$ is the sum of two independent Poisson random variables and also has a Poisson distribution with parameter $10 \cdot 15 = 150$. So, it true if we say "the distribution of $M + N$ is the same as the distribution of $M + \widetilde{N}$".

In general, the probability of k arrivals during a set of times of total length τ is always given by $P(k, \tau)$, even if that set is not an interval. (In this example, we dealt with the set $[9:00, 9:10] \cup [9:30, 9:35]$, of total length 15).

Exercise 08

A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ particles per minute. What is the probability that the first particle appears after three minutes?

$$P[N(3) - N(0) = 0] = (2 \cdot 3)^0 \frac{e^{(-2 \cdot 3)}}{0!} = e^{-6}$$

Exercise 09

A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ particles per minute.

(a) What is the probability that the first particle appears some time after three minutes but before five minutes?

$$\begin{aligned} P(\text{the first particle appears after 3 minutes but before 5 minutes}) &= P[N(3) = 0, N(5) \geq 1] \\ &= P[N(3) = 0, N(5) - N(3) \geq 1] \\ &= P[N(3) = 0] P[N(5) - N(3) \geq 1] \\ &= P[N(3) = 0] P[N(2+3) - N(0+3) \geq 1] \\ &= P[N(3) = 0] P[N(2) - N(0) \geq 1] \\ &= P[N(3) = 0] P[N(2) \geq 1] \\ &= P[N(3) = 0] (1 - P[N(2) \leq 1]) \\ &= (2 \cdot 3)^0 \frac{e^{(-2 \cdot 3)}}{0!} (1 - e^{-4}) \\ &= e^{-6} - e^{-10} \end{aligned} \tag{6}$$

(b) What is the probability that exactly one particle is emitted in the interval from three to five minutes?

$$\begin{aligned} P(\text{exactly one particle is emitted between 3 and 5 minutes}) &= P[N(5) - N(3) = 1] \\ &= P[N(2+3) - N(0+3) = 1] \\ &= P[N(2) - N(0) = 1] \\ &= (2 \cdot 2)^0 \frac{e^{(-2 \cdot 2)}}{0!} \\ &= 4e^{-4} \end{aligned} \tag{7}$$

Exercise 10

Customers enter a store according to a Poisson process of rate $\lambda = 6$ per hour. Suppose it is known that but a single customer entered during the first hour. What is the conditional probability that this person entered during the first fifteen minutes?

We hav $\lambda = 6$ per hour = $1/10$ per minute.

$$\begin{aligned} P[\text{person entered during the first fifteen minutes} \mid \text{a single customer entered during the first hour}] &= P[N(15) = 0 \mid \\ &= \frac{P[N(60) \geq 1,}{P[N(60} \\ &= \frac{(1 - P[N(60)}{1 - P} \end{aligned}$$

Exercise 11

Let $N(t)$ be a Poisson process of rate $\lambda = 3$ per hour. Find the conditional probability that there were two events in the first hour, given that there were five events in the first three hours.

$$\begin{aligned} P[N(1) = 2 \mid N(3) = 5] &= \frac{P[N(3) = 5, N(1) = 2]}{P[N(3) = 5]} \\ &= \frac{P[N(3) - N(1) = 5 - 2, N(1) = 2]}{P[N(3) = 5]} \\ &= \frac{P[N(2) = 3, N(1) = 2]}{P[N(3) = 5]} \end{aligned} \tag{9}$$

Exercise 12

Bacteria are distributed throughout a volume of liquid according to a Poisson process of intensity $\lambda = 0.6$ organisms per mm^3 . A measuring device counts the number of bacteria in a $10mm^3$ volume of the liquid. What is the probability that more than two bacteria are in this measured volume?

Let X denotes the number of bacteria in a a $10mm^3$ volume of the liquid. Hence, X is a Poisson distribution with lambda parameter $\lambda = 0.6 \times 10 = 6$. So, the probability that more than two bacteria are in this measured volume can be calculated as:

$$\begin{aligned} P(\text{more than two bacteria are in this measured volume}) &= P[X \geq 2] \\ &= 1 - P[X \leq 2] \\ &= 1 - (P[X=0) + P[X=1) + P[X=2)) \\ &= 1 - \left(e^{-6}\frac{6^0}{0!} + e^{-6}\frac{6^1}{1!} + e^{-6}\frac{6^2}{0!}\right) \\ &= 1 - e^{-6}\left(1 + 6 + \frac{6^2}{2}\right) \\ &\approx 0.938 \end{aligned} \tag{10}$$

Exercise 13

Customer arrivals at a certain service facility follow a Poisson process of unknown rate. Suppose it is known that 12 customers have arrived during the first three hours. Let N_i be the number of customers who arrive during the i-th hour, $i = 1, 2, 3$. Determine the probability that $N_1 = 3$, $N_2 = 4$, and $N_3 = 5$.

Suppose that the Poisson process has rate λ. Then the conditional probability can be writen as:

$$\begin{aligned} P[N_1 = 3, N_2 = 4, N_3 = 5 \mid N = 12] &= \frac{P[N_1 = 3, N_2 = 4, N_3 = 5]}{P[N = 12]} \\ &= e^{-\lambda}\frac{\lambda^3}{3!}e^{-\lambda}\frac{\lambda^4}{4!}e^{-\lambda}\frac{\lambda^5}{5!}\frac{12!}{e^{-3\lambda}(3\lambda)^{1}2} \\ &= \frac{12!}{3!4!5!3^{12}} \\ &\approx 0.0522 \end{aligned} \tag{11}$$

Exercise 14

Patients arrive at the doctor's office according to a Poisson process with rate $\lambda = 1/10$ minute. The doctor will not see a patient until at least three patients are in the waiting room.

Let $N(t)$ be the Poisson process with mean λ. We know that

$$P[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-(\lambda t)}$$

(a) Find the expected waiting time until the first patient is admitted to see the doctor.

Let Z_n be the time between arrival of the n-th patient and the $n-1$-th patient. Z_n are iid exponential random variables with mean $\frac{1}{\lambda}$, donated by $\mathbb{E}[Z_n] = \frac{1}{\lambda}$.

Let T_n be the arrival time of n-th patient, which means

$$T_n = \sum_{j=1}^{n} Z_n$$

The expectation of T_n:

$$\mathbb{E}[T_n] = \mathbb{E}\left[\sum_{j=1}^{n} Z_n\right] = \sum_{j=1}^{n} \mathbb{E}[Z_n] = \sum_{j=1}^{n} \frac{1}{\lambda} = \frac{n}{\lambda}$$

The expected waiting time until the first patient is admitted to see the doctor:

$$\mathbb{E}[T_3] = \frac{3}{1/10} = 30$$

(b) What is the probability that nobody is admitted to see the doctor in the first hour?

$$\begin{aligned} P[\text{ nobody is admitted to see the doctor in the first hour}] &= P[\text{At most 2 patient arrive in first hour}] \\ &= P[X(t) \geq 2] \\ &= P[X(60) \geq 2] \\ &= P[X(60) = 0] + P[X(60) = 1] + P[X(60) = 2] \end{aligned} \tag{12}$$

Exercise 15

Let S_n , denote the time of the n-th event of a Poisson process $N(t)$ with rate λ. Suppose that one event has occurred in the interval $(0, t)$. Show that the conditional distribution of arrival time S_1 , is uniform over $(0, t)$.

Proof

According the hypothesis, one event has occurred in the interval $(0, t)$. Let S_1 denotes the first event in the interval $[t_1, t_2]$, where $t_1 = 0$, and $t_2 = t$, then for $s \in [t_1, t_2] = (0, t)$, we find the conditional probability

$$\begin{aligned} P[S_1 \leq s \mid N(t) = 1] &= \frac{P[S_1 \leq s, N(t) = 1]}{P[N(t) = 1]} \\ &= \frac{P[N(s) = 1]P[N(t) - N(s) = 0]}{P[N(t) = 1]} \\ &= e^{-\lambda s}\frac{\lambda s}{1!}e^{-\lambda(t-s)}\frac{\lambda(t-s)}{0!}\frac{1!}{e^{-\lambda s}\lambda s} \\ &= \frac{s}{t}, s < t \end{aligned} \tag{13}$$

which is the CDF of the Uniform $(0, t)$ distribution. $\square$