General Information

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Stochastic Process - Some proof for chapter 01

If $X(t)$ is a stationary independent increments and if $X(0) = 0$, then

$$\mathbb{E}[X(t)] = \mu_1 t$$

$$Var[X(t)] = \sigma_1^2 t$$

$$Var[X(t) - X(s)] = \sigma_1^2 (t-s), t > s$$

$$K_X(s,t) = Cov[X(t), X(s)] = \sigma_1^2 \min(t,s)$$

where, $\mu_1 = \mu_X(1), \sigma_1^2 = Var[X(1)]$

Show:

$$\mathbb{E}[X(t)] = \mu_1 t$$

We have

$$E[X(t)] = E[X(t) - X(0)]$$

For any t, and s

$$\begin{aligned} \mathbb{E}[X(t+s)] &= \mathbb{E}[X(t+s) - X(s) + X(s) - X(0)] \\ &= \mathbb{E}[X(t+s) - X(s)] + \mathbb{E}[X(s) - X(0)] \\ &= \mathbb{E}[X(t) - X(0)] + \mathbb{E}[X(s) - X(0)] \\ &= \mathbb{E}[X(t)] + \mathbb{E}[X(s)] \end{aligned} \tag{1}$$

We have to solve

$$\begin{aligned} &\mathbb{E}[X(t+s)] = \mathbb{E}[X(t)] + \mathbb{E}[X(s)] \\ &\Leftrightarrow f(t+s) = f(t) + f(s) \text{ (Cauchy functional equation)} \end{aligned} \tag{2}$$

Solve above equation, we have:

$$\mathbb{E}[X(t)] = \mu_1 t, \mu_1 = \mu_X(1) = \mathbb{E}[X(1)]$$

Show:

$$Var[X(t)] = \sigma_1^2 t$$

For any t, and s

$$\begin{aligned} Var[X(t+s)] &= Var[X(t+s) - X(s) + X(s) - X(0)] \\ &= Var[X(t+s) - X(s)] + Var[X(s) - X(0)] \\ &= Var[X(t) - X(0)] + Var[X(s) - X(0)] \\ &= Var[X(t)] + Var[X(s)] \end{aligned} \tag{3}$$

We have to solve

$$\begin{aligned} &Var[X(t+s)] = Var[X(t)] + Var[X(s)] \\ &\Leftrightarrow f(t+s) = f(t) + f(s) \text{ (Cauchy functional equation)} \end{aligned} \tag{4}$$

Solve above equation, we have:

$$Var[X(t)] = \sigma_1^2 t, \sigma_1^2 = Var[X(1)]$$

Show:

$$Var[X(t) - X(s)] = \sigma_1^2 (t-s), t > s$$

$$\begin{aligned} Var[X(t)] &= Var[X(t) - X(s) + X(s) - X(0)] \\ &= Var[X(t) - X(s)] + Var[X(s) - X(0)] \\ &= Var[X(t) - X(s)] + Var[X(s)] \end{aligned} \tag{5}$$

$$Var[X(t) - X(s)] = Var[X(t)] - Var[X(s)] = \sigma_1^2 (t-s), t > s, \sigma_1^2 = Var[X(1)] \tag{6}$$

Show

$$K_X(s,t) = Cov[X(t), X(s)] = \sigma_1^2 \min(t,s)$$

$$\begin{aligned} Var[X(t) - X(s)] &= \mathbb{E}[[(X(t) - X(s)) - \mathbb{E}[X(t) - X(s)]]^2] \\ &= \mathbb{E}[((X(t) - \mathbb{E}[X(t)]) - (X(s) - \mathbb{E}[X(s)]))^2] \\ &= \mathbb{E}[(X(t) - \mathbb{E}[X(t)])^2 - 2(X(t) - \mathbb{E}[X(t)])(X(s) - \mathbb{E}[X(s)]) + (X(t) - \mathbb{E}[X(t)])^2] \\ &= Var[X(t)] + Var[X(s)] - 2Cov[X(t), X(s)] \end{aligned} \tag{7}$$

$$\begin{aligned} Cov[X(t), X(s)] &= \frac{1}{2}[Var[X(t)] + Var[X(s)] - Var[X(t) - X(s)]] = \frac{1}{2}\sigma_1^2(t + s - (t-s)) \\ &= \begin{cases} Cov[X(t), X(s)] = \frac{1}{2}\sigma_1^2(t+s-(t-s)) = \sigma_1^2 s, \text{ if } t > s \\ Cov[X(t), X(s)] = \frac{1}{2}\sigma_1^2(t+s-(s-t)) = \sigma_1^2 t, \text{ if } s > t \end{cases} \end{aligned} \tag{8}$$