HW_A1

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0.1 General Information

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[ ]:
```
import numpy as np
import matplotlib.pyplot as plt

from math import comb
from collections import Counter
```

0.2 Stochastic Process - Homework Assignment 01

0.2.1 Exercise 01

Let $X_1, X_2, ...$ be independent Bernoulli random variables with $P(X_n = 1) = p$ and $P(X_n = 0) = q = 1 - p$ for all n. The collection of random variables $\{x_n, n > 1\}$ is a stochastic process, and it is called a Bernoulli process.

(a) Describe the Bernoulli process

The Bernoulli process is a sequence of independent Bernoulli trials. For each trial in this sequence, it has binary outcome (success donated as 1, and failure donated as 0).

Formally, at each trial n, we have: - $P(\text{sucess}) = P(X_n = 1) = p$

- $P(\text{failure}) = P(X_n = 0) = q = 1 - p$

Examples: - Sequence of lottery wins/losses

- Arrivals (each second) to a bank
- Arrivals (at each time slot) to server

Properties of this sequence: - Expectation: $\mathbb{E}[X_n] = p \cdot 1 + (1 - p) \cdot 0 = p$

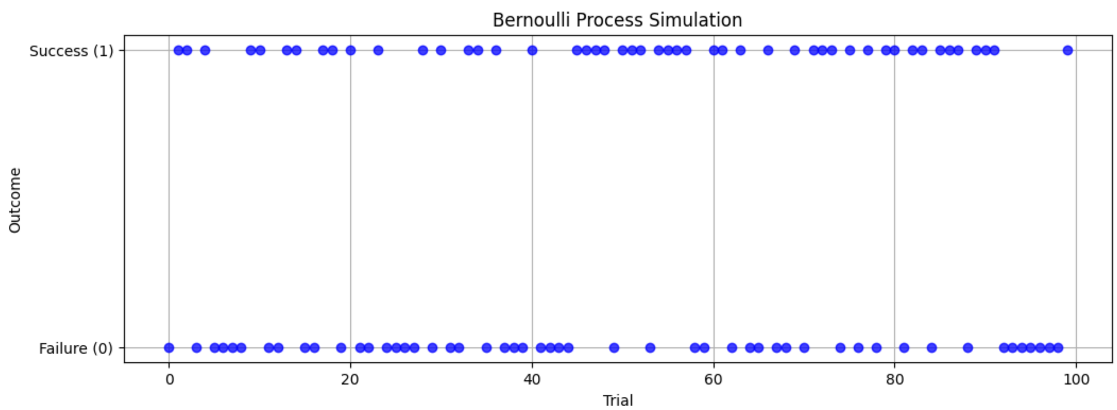
- Variance: $Var(X_n) = pq = p \cdot (1 - p)$
- Autocorrelation: $R_X(t, s) = \mathbb{E}[X_t X_s] = p \cdot p = p^2$
- Autocovariance: $K_X(t, s) = Cov[X_t, X_s] = \mathbb{E}\{(X_t - p)(X_s - p)\} = 0$

(b) Construct a typical sample sequence of the Bernoulli process.

[ ]:
```python
# Parameters for the Bernoulli process
p = 0.5          # probability of success
q = 1 - p        # probability of failure
n = 100          # number of trials
```

[ ]:
```python
# Generating a Bernoulli process
bernoulli_trials = np.random.binomial(1, p, n)
```

[ ]:
```python
# Visualizing the Bernoulli process
plt.figure(figsize=(12, 4))
plt.plot(bernoulli_trials, drawstyle='steps-mid', marker='o', color='blue',␣
↪alpha=0.75, linestyle='None')
plt.title('Bernoulli Process Simulation')
plt.xlabel('Trial')
plt.ylabel('Outcome')
plt.yticks([0, 1], ['Failure (0)', 'Success (1)'])
plt.grid(True)
plt.show()
```



(c) Determine the probability of occurrence of the sample sequence obtained in part (b)

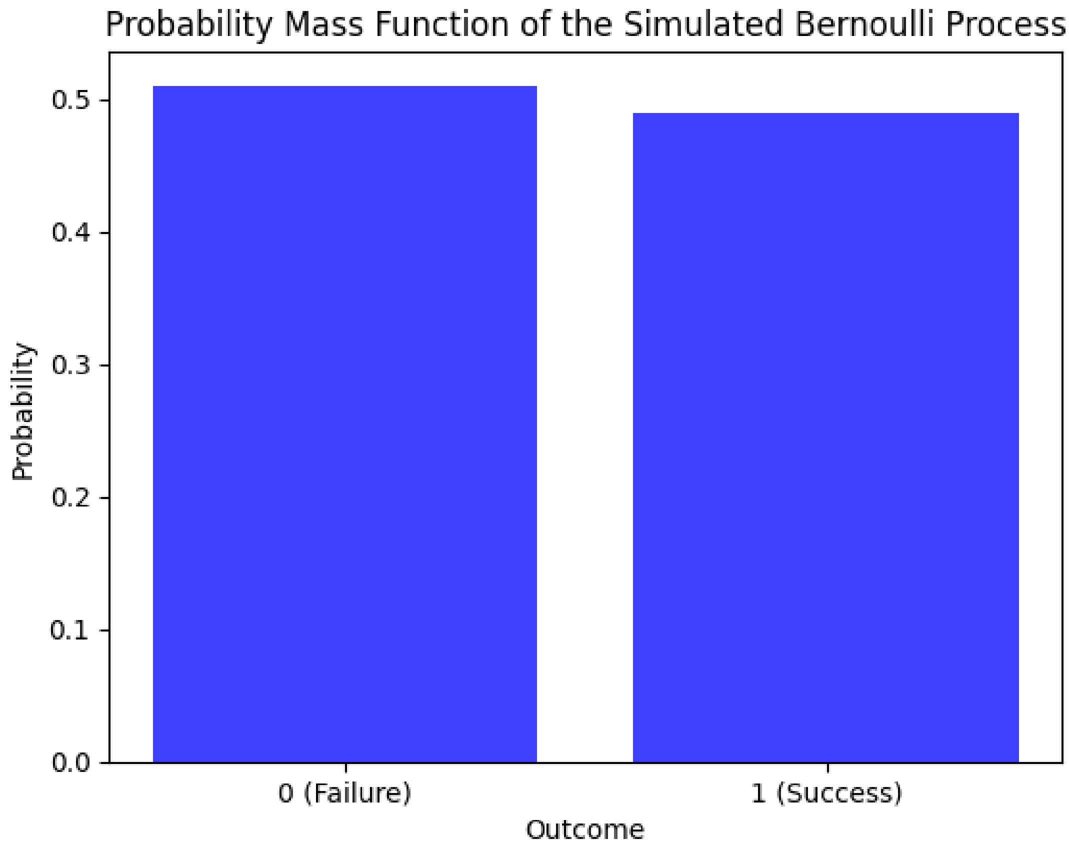
[ ]:
```python
# Count the frequency of each outcome in the simulated Bernoulli process
outcome_counts = Counter(bernoulli_trials)

# Calculate the probabilities for each outcome (0 and 1)
probabilities = {outcome: count / n for outcome, count in outcome_counts.
↪items()}

# Visualizing the Probability Mass Function (PMF)
plt.bar(probabilities.keys(), probabilities.values(), color='blue', alpha=0.75)
plt.xlabel('Outcome')
plt.ylabel('Probability')
plt.title('Probability Mass Function of the Simulated Bernoulli Process')
```

```
plt.xticks([0, 1], ['0 (Failure)', '1 (Success)'])
plt.show()

# Output the probabilities of each outcome
probabilities
```



[ ]: {0: 0.51, 1: 0.49}

0.2.2 Exercise 02

Let $Z_1, Z_2, ...$ be independent identically distributed random variables with $P(Z_n = 1) = p$ and $P(Z_n = -1) = q = 1 - p$ for all n. Let

$$X_n = \sum_{i=1}^{n} Z_i, n = 1, 2, ...$$

and $X_0 = 0$. The collection of random variables $\{X_n, n > 0\}$ is a stochastic process, and it is called the simple random walk X_n

(a) Describe the simple random walk X_n

Following the given definition, X_n is a stochastic process. At each time t in this process has the value that determined by the random variables (identically distributed random variables) Z_i can taken value 1 with probability p and -1 with probability $1-p$ (we can see Z_i is i.i.d. Bernoulli trial)

The outcome of this process is the cumulative sum of these random variables that can be represented as a path of moving steps (move up or down at each time t depdent on the value of random variable Z_i)

(b) Construct a typical sample sequence (or realization) of X_n

[ ]:
```
# Parameters for the simulation

## Probability that Z_n = 1
p = 0.5

# Number of steps in the random walk
n_steps = 100
```

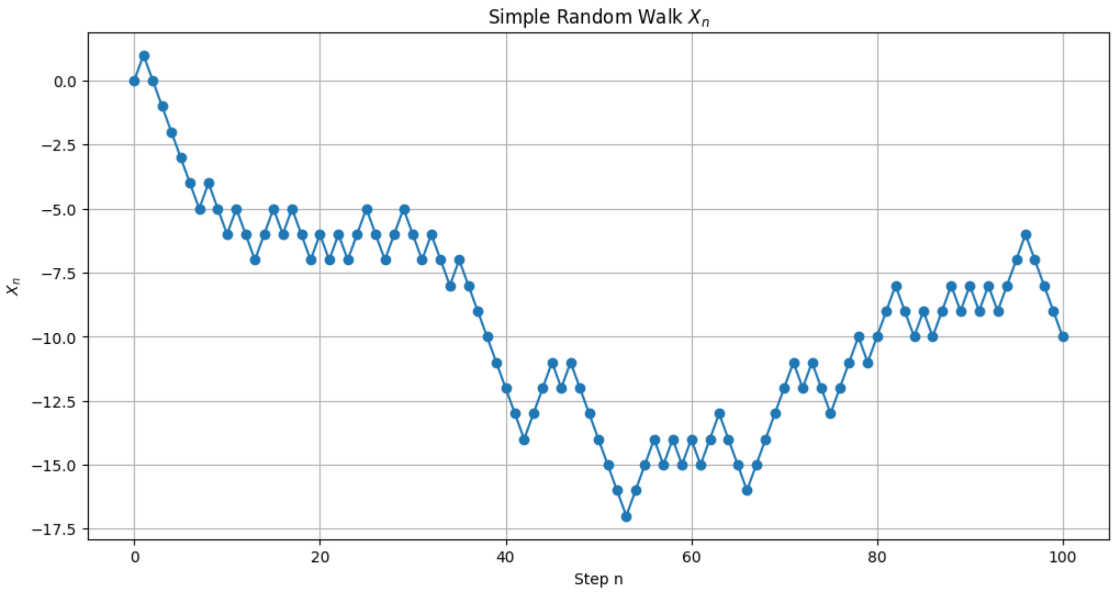
[ ]:
```
# Generating the Z_n values (-1 or 1) based on the probability p
Z_values = np.random.choice([1, -1], size=n_steps, p=[p, 1-p])
```

[ ]:
```
# Constructing the random walk X_n
X_n = np.cumsum(Z_values)  # Cumulative sum of Z_values
```

[ ]:
```
# Adding X_0 = 0 at the beginning of the sequence
X_n = np.insert(X_n, 0, 0)
X_n
```

[ ]:
```
array([  0,   1,   0,  -1,  -2,  -3,  -4,  -5,  -4,  -5,  -6,  -5,  -6,
        -7,  -6,  -5,  -6,  -5,  -6,  -7,  -6,  -7,  -6,  -7,  -6,  -5,
        -6,  -7,  -6,  -5,  -6,  -7,  -6,  -7,  -8,  -7,  -8,  -9, -10,
       -11, -12, -13, -14, -13, -12, -11, -12, -11, -12, -13, -14, -15,
       -16, -17, -16, -15, -14, -15, -14, -15, -14, -15, -14, -13, -14,
       -15, -16, -15, -14, -13, -12, -11, -12, -11, -12, -13, -12, -11,
       -10, -11, -10,  -9,  -8,  -9, -10,  -9, -10,  -9,  -8,  -9,  -8,
        -9,  -8,  -9,  -8,  -7,  -6,  -7,  -8,  -9, -10])
```

[ ]:
```
# Visualizing the simple random walk X_n
plt.figure(figsize=(12, 6))
plt.plot(X_n, marker='o')
plt.title('Simple Random Walk $X_n$')
plt.xlabel('Step n')
plt.ylabel('$X_n$')
plt.grid(True)
plt.show()
```



(c) Derive the first-order probability distribution of the simple random walk X_n

Considering the simple random walk X_n. At a fixed time t_1, which mean $n = 1$. So we can write the first-order probability distribution of the simple random walk X_n as:

$$X_1 = X_0 + \sum_{i=1}^{1} Z_i = X_0 + Z_1$$

As (a), we realize that Z_1 is i.i.d. Bernoulli trial that $P(Z_n = 1) = p$ and $P(Z_n = -1) = q = 1 - p$ for all n. So that the first-order probability distribution of the simple random walk X_n can be written as:

$$X_1 = X_0 + p^k(1-p)^{1-k}, k \in \{-1, 1\}$$

Due to $X_0 = 0$, so

$$X_1 = p^k(1-p)^{1-k}, k \in \{-1, 1\}$$

(d) Find the probability that $X_n = -2$ after four steps

[ ]:
```
# Parameters for part (d)
n_steps_d = 4  # Number of steps
target_sum_d = -2  # Target sum after n steps
```

[ ]:
```
# For X_n = -2 in 4 steps, we need 1 successes (+1) and 3 failures (-1)
# Probability for getting success is p, so Probability for getting failure is 1␣
↪- p
```

```
# Number of successes required
num_minus_d = (n_steps_d - target_sum_d) // 2
print(f"Number of success (+1) required (in any order in 4 steps) is␣
↪{num_minus_d}")
```

```
Number of success (+1) required (in any order in 4 steps) is 3
```

[ ]:
```
# Probability that X_n = -2 after four steps
probability_Xn_minus2 = comb(4,3) * p * (1 - p)**3
print(f"Probability that X_n = -2 after four steps is: {probability_Xn_minus2}")
```

```
Probability that X_n = -2 after four steps is: 0.25
```

(e) Find the mean and variance of the simple random walk X_n

Let $Z_i = 2Y_i - 1$, with Y_i is Bernoulli random variable. So that:

$$X_n = \sum_{i=1}^{n} Z_i = \sum_{i=1}^{n}(2Y_i - 1) = 2\sum_{i=1}^{n} Y_i - n$$

Mean of the simple random walk X_n

$$\mathbb{E}[X_n] = \mathbb{E}\left[2\sum_{i=1}^{n} Y_i - n\right] = 2\sum_{i=1}^{n}\mathbb{E}[Y_i] - n = 2pn - n = 2n(p-1)$$

Variance of the simple random walk X_n

$$Var[X_n] = Var\left[2\sum_{i=1}^{n} Y_i - n\right] = 4Var\left[\sum_{i=1}^{n} Y_i\right] = 4np(1-p)$$

[ ]:
```
# Mean and Variance of the simple random walk X_n
mean_Xn = n_steps * (2 * p - 1)
variance_Xn = 4 *n_steps * p * (1 - p)
print(f"Mean and variance of the simple random walk X_n:\n\tMean =␣
↪{mean_Xn}\n\tVariance = {variance_Xn}")
```

```
Mean and variance of the simple random walk X_n:
        Mean = 0.0
        Variance = 100.0
```

(f) Find the autocorrelation function $R_X(n,m)$ of the simple random walk X_n

As the following lecture, we have the formulation of the autocorrelation function $R_X(n,m)$ as

$$R_X(n,m) = \mathbb{E}[X_n X_m]$$

Set: $m = n + \tau$

$$R_X(n, n+\tau) = \mathbb{E}[X_n X_{n+\tau}]$$

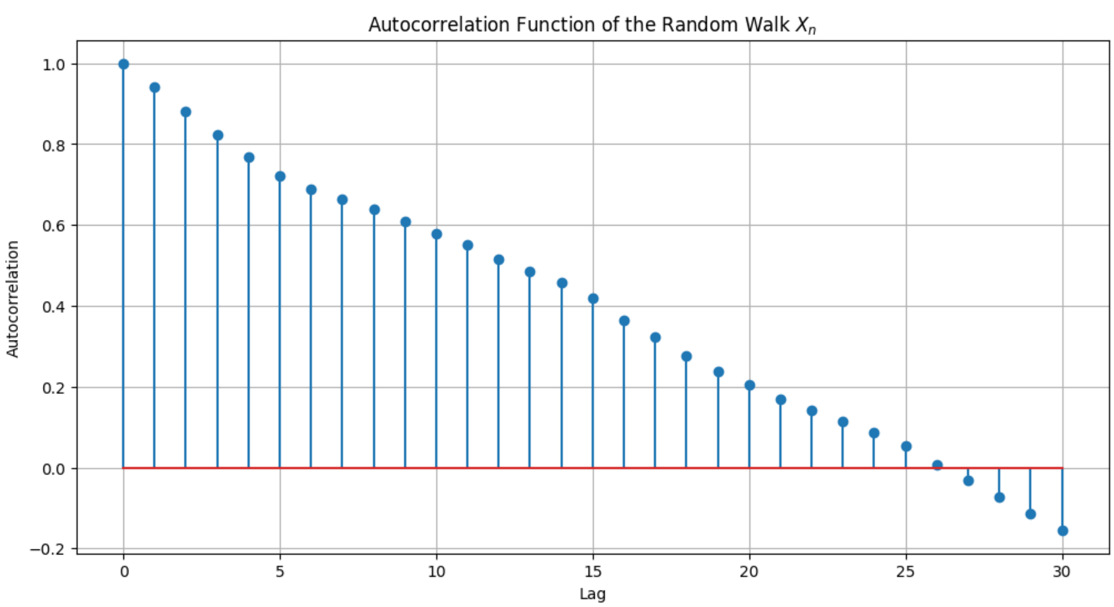
With the simple random walk, the autocorrelation can be written as:

$$R_X(\tau) = \frac{1}{n-\tau} \sum_{i=1}^{n-\tau} (X_i - \bar{X})(X_{i+\tau} - \bar{X})$$

[ ]:
```
def autocorrelation(x, lag):
    n = len(x)
    mean_x = np.mean(x)
    autocorr = np.sum((x[:n-lag] - mean_x) * (x[lag:] - mean_x)) / np.sum((x -␣
 ↪mean_x) ** 2)
    return autocorr
```

[ ]:
```
# Compute autocorrelation values for lags from 0 to max_lag
max_lag = 30
autocorr_values = [autocorrelation(X_n, lag) for lag in range(max_lag + 1)]

# Visualizing the autocorrelation function
plt.figure(figsize=(12, 6))
plt.stem(range(max_lag + 1), autocorr_values)
plt.title('Autocorrelation Function of the Random Walk $X_n$')
plt.xlabel('Lag')
plt.ylabel('Autocorrelation')
plt.grid(True)
plt.show()
```



(g) Now let the random process $X(t)$ be defined by

$$X(t) = X_n, n \leq t < n+1$$

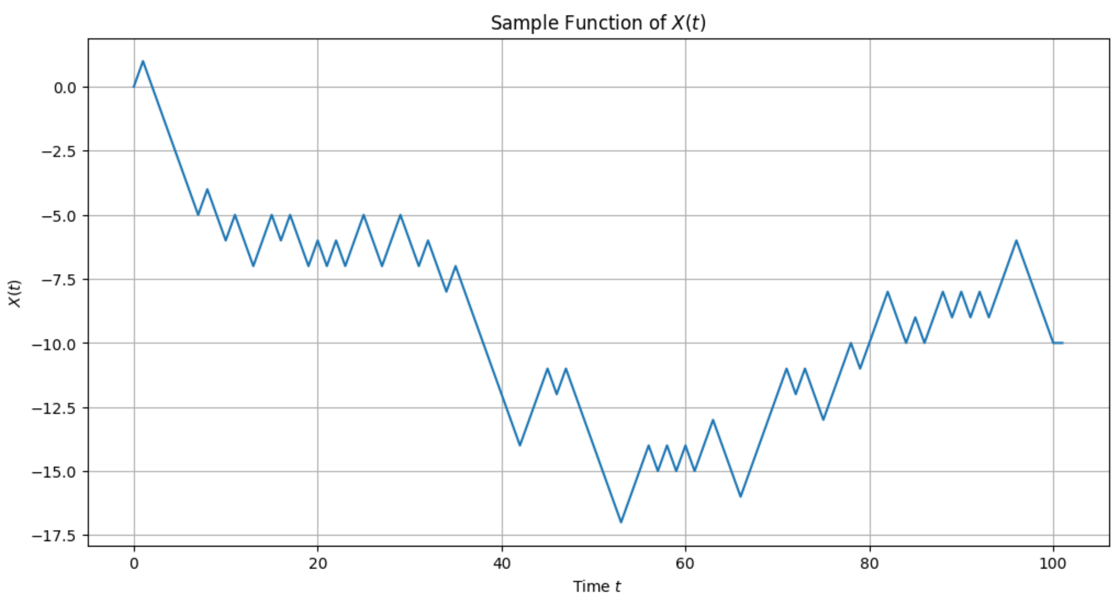
Describe X(t). Construct a typical sample function of $X(t)$

$X(t)$ is a piecewise constant function (hàm hằng từng phần) that defined by the simple random walk X_n with the constraint $n \leq t < n+1$

[ ]:
```
# Time points for visualization
t_values = np.arange(0, n_steps + 1, 0.01)
```

[ ]:
```
# Constructing the X(t) function as a piecewise constant function based on X_n
X_t = np.interp(t_values, np.arange(n_steps + 1), X_n)
```

[ ]:
```
# Visualizing the X(t) process
plt.figure(figsize=(12, 6))
plt.plot(t_values, X_t)
plt.title('Sample Function of $X(t)$')
plt.xlabel('Time $t$')
plt.ylabel('$X(t)$')
plt.grid(True)
plt.show()
```



0.2.3 Exercise 03

Consider a random process $X(t)$ defined by

$$X(t) = Y \cos \omega t, t \geq 0$$

where ω is a constant and Y is a uniform random variable over $(0, 1)$.

(a) Describe $X(t)$

[ ]:

```

```

(b) Sketch a few typical sample functions of $X(t)$

[ ]:

```
# Parameters for the simulation
omega = 2 * np.pi  # Set omega to 2 for a full cycle
t = np.linspace(0, 2*np.pi, 500)  # Time range for one full cycle
```

[ ]:

```
# Generate several values for Y uniformly distributed in (0, 1)
Y_values = np.random.uniform(0, 1, 10)  # 5 different values for Y
```

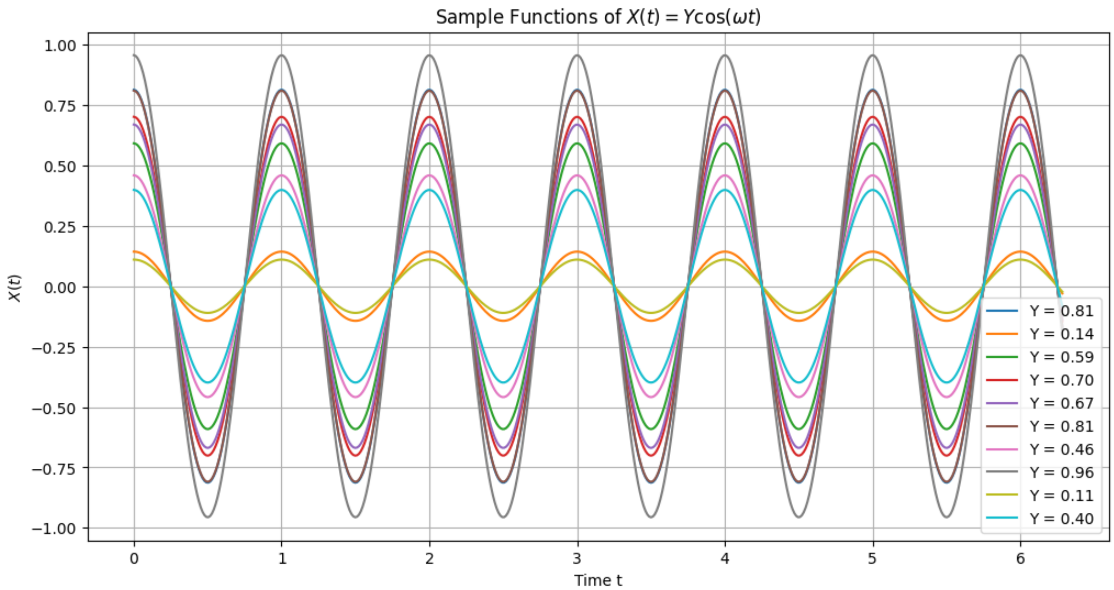
[ ]:

```
# Generating sample functions
sample_functions = [Y * np.cos(omega * t) for Y in Y_values]
```

[ ]:

```
# Visualizing the sample functions
plt.figure(figsize=(12, 6))
for i, Y in enumerate(Y_values):
    plt.plot(t, sample_functions[i], label=f'Y = {Y:.2f}')

plt.title('Sample Functions of $X(t) = Y\cos(\omega t)$')
plt.xlabel('Time t')
plt.ylabel('$X(t)$')
plt.legend()
plt.grid(True)
plt.show()
```



(c) Find $\mathbb{E}[X(t)]$

Considering the continuous random variable Y with probability density function is $f_Y(y), the expected value of a function $g(Y)$ given by:

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{+\infty} g(y) f_Y(y)$$

In this case, $g(Y) = Y \cos \omega t, t \geq 0$ and $f_Y(y)$ is the pdf of the uniform distribtion $(0, 1)$. So:

$$\mathbb{E}[X(t)] = \int_0^1 y \cos \omega t dy = \frac{1}{2} \cos(\omega t)$$

[ ]:
```
import sympy as sp
```

[ ]:
```
# Define the symbol for Y
y = sp.symbols('y')
```

[ ]:
```
# Define omega and t as constants for this calculation
omega = 2 * sp.pi
t = sp.symbols('t', real=True)
```

[ ]:
```
# Define the function Y*cos(omega*t)
function = y * sp.cos(omega * t)
```

[ ]:
```
# Compute the expected value (integral over 0 to 1)
expected_value = sp.integrate(function, (y, 0, 1))
```

```
print(f"The expected value (integral over 0 to 1) E[X(t)] = {expected_value} if␣
    ↪omega = {omega}")
```

```
The expected value (integral over 0 to 1) E[X(t)] = cos(2*pi*t)/2 if omega =
2*pi
```

(d) Find the autocorrelation function $R_X(t, s)$ of $X(t)$

As the following lecture, we have the formulation of the autocorrelation function $R_X(n, m)$ as

$$R_X(t, s) = \mathbb{E}[X_t X_s]$$

with $X_t = Y \cos \omega t$ and $X_s = Y \cos \omega s$, so that $X_t X_s = Y^2 \cos(\omega t)\cos(\omega s)$

$$\mathbb{E}[X_t X_s] = \mathbb{E}[Y^2 \cos(\omega t)\cos(\omega s)] = \int_0^1 y^2 \cos(\omega t)\cos(\omega s) dy = \frac{1}{3}\cos(\omega t)\cos(\omega s)$$

[ ]:
```
# Define the symbol for Y
y = sp.symbols('y')
```

[ ]:
```
# Define omega, t and s as constants for this calculation
omega = 2 * sp.pi
t = sp.symbols('t', real=True)
s = sp.symbols('s', real=True)
```

[ ]:
```
# Define the function Y*cos(omega*t)
function = y * y * sp.cos(omega * t) * sp.cos(omega * s)
```

[ ]:
```
# Compute the expected value (integral over 0 to 1)
autocorrelation_function = sp.integrate(function, (y, 0, 1))

print(f"The autocorrelation function = {autocorrelation_function} if omega =␣
    ↪{omega}")
```

```
The autocorrelation function = cos(2*pi*s)*cos(2*pi*t)/3 if omega = 2*pi
```

(e) Find the autocovariance function $K_X(t, s)$ of $X(t)$

Following the definition of autocovariance function, we have:

$$K_X(t, s) = Cov[X_t, X_s] = \mathbb{E}[X_t - \mu_X(t)][X_s - \mu_X(s)] = R_X(t, s) - \mu_X(t)\mu_X(s)$$

By using previous results, we have:

$$K_X(t, s) = \frac{1}{3}\cos(\omega t)\cos(\omega s) - \frac{1}{4}\cos(\omega t)\cos(\omega s)$$

0.2.4 Exercise 04

Consider a discrete-parameter random process $X(n) = \{X_n, n \geq 1\}$ where the X_i is iid random variable with common cdf $F_X(x)$, mean p, and variance σ^2.

(a) Find the joint cdf of $X(n)$

Due to X_i is iid random variable with common cdf $F_X(x)$, so the join cdf (cummulative distribued function) of $X(n)$ is

$$F_X(x_1, x_2, ..., x_k; t_1, t_2, ..., t_k) = P(X(t_1) \leq x_1, X(t_2) \leq x_2, ..., X(t_k) \leq x_k) = F_X(x_1) \times F_X(x_2) \times ... \times F_X(x_k)$$

(b) Find the mean of $X(n)$

Due to X_i is iid random variable with common cdf $F_X(x)$, so the mean of $X(n)$

$$\mathbb{E}[X(n)] = p$$

(c) Find the autocorrelation function $R_X(m, n)$ of $X(n)$

$$R_X(m, n) = \mathbb{E}[X_m, X_n] = \begin{cases} \mathbb{E}[X_n^2] = \sigma^2 + p^2, \text{ if } m = n \\ \mathbb{E}[X_m]\mathbb{E}[X_n] = p \cdot p = p^2, \text{ if } m \neq n \end{cases}$$

(d) Find the autocovariance function $K_X(m, n)$ of $X(n)$

$$K_X(m, n) = Cov[X_m, X_n] = \mathbb{E}[X_t - \mu_X(t)][X_s - \mu_X(s)] = R_X(t, s) - \mu_X(t)\mu_X(s) = \begin{cases} \sigma^2, \text{ if } m = n \\ 0, \text{ if } m \neq n \end{cases}$$