

Involutory Property of the Discrete Hartley Transform

Frank the Giant Bunny

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Given a column vector $x \in \mathbb{R}^n$, its *Discrete Hartley Transform* (DHT) is defined as another vector $y \in \mathbb{R}^n$ such that

$$y_j = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} x_i \operatorname{cas}\left(\frac{2\pi}{n}ij\right) \quad \text{for } j \in \{0, \dots, n-1\} \quad (1)$$

where the cas function is defined as $\operatorname{cas} \theta = \cos \theta + \sin \theta$. Interestingly, the DHT is an *involution*; that is, the DHT is the same as the inverse DHT.

$$x_i = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} y_j \operatorname{cas}\left(\frac{2\pi}{n}ij\right) \quad \text{for } i \in \{0, \dots, n-1\} \quad (2)$$

This paper proves the DHT is indeed an involution.

Target Equality To simplify (1) and (2), define an $n \times n$ symmetric matrix H whose (i, j) -entry is $\frac{1}{\sqrt{n}} \operatorname{cas}\left(\frac{2\pi}{n}ij\right)$. Then the DHT and the inverse DHT are

$$y = Hx \quad \text{and} \quad x = Hy$$

where $x = [x_0, \dots, x_{n-1}]^\top$ and $y = [y_0, \dots, y_{n-1}]^\top$. Then proving the involutory property of the DHT reduces to showing that $H^2 = I$ where I is an $n \times n$ identity matrix. This further reduces to showing that the rows in H are orthogonal; that is,

$$\langle h_i, h_{i'} \rangle = \begin{cases} 1, & \text{if } i = i'; \\ 0, & \text{otherwise;} \end{cases}$$

where $h_i = \frac{1}{\sqrt{n}} [\operatorname{cas}\left(\frac{2\pi}{n}i0\right), \dots, \operatorname{cas}\left(\frac{2\pi}{n}i(n-1)\right)]^\top$ is the i^{th} row in H . It may be written in terms of cas functions as

$$\sum_{j=0}^{n-1} \operatorname{cas}\left(\frac{2\pi}{n}ij\right) \operatorname{cas}\left(\frac{2\pi}{n}i'j\right) = \begin{cases} n, & \text{if } i = i'; \\ 0, & \text{otherwise;} \end{cases} \quad (3)$$

which is the target equality for the involutory property.

CAS Identity The proof of (3) begins with an identity about cas functions.

$$\operatorname{cas} \alpha \operatorname{cas} \beta = \sin(\alpha + \beta) + \cos(\alpha - \beta) \quad (4)$$

Proof of (4). A $\text{cas}(\cdot)$ can be simplified to $\cos(\cdot)$ as follows:

$$\text{cas } \theta = \cos \theta + \sin \theta = \sqrt{2} \left(\frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta \right) = \sqrt{2} \cos \left(\theta - \frac{\pi}{4} \right) \quad (*)$$

Then

$$\begin{aligned} \text{cas } \alpha \text{ cas } \beta &= \sqrt{2} \cos \left(\alpha - \frac{\pi}{4} \right) \cdot \sqrt{2} \cos \left(\beta - \frac{\pi}{4} \right) && \text{by } (*) \\ &= \cos \left(\alpha + \beta - \frac{\pi}{2} \right) + \cos(\alpha - \beta) && 2 \cos \theta \cos \phi = \cos(\theta + \phi) + \cos(\theta - \phi) \\ &= \sin(\alpha + \beta) + \cos(\alpha - \beta) && \cos \left(\theta - \frac{\pi}{2} \right) = \sin \theta \end{aligned}$$

which completes the proof. \square

Target Simplified The target equality (3) is decomposed into two summations by (4).

$$\sum_{j=0}^{n-1} \text{cas} \left(\frac{2\pi}{n} i j \right) \text{cas} \left(\frac{2\pi}{n} i' j \right) = \sum_{j=0}^{n-1} \sin \left(\frac{2\pi}{n} (i + i') j \right) + \sum_{j=0}^{n-1} \cos \left(\frac{2\pi}{n} (i - i') j \right)$$

The above will be interpreted as the real and imaginary parts in geometric progressions of complex numbers:

$$\sum_{j=0}^{n-1} \text{cas} \left(\frac{2\pi}{n} i j \right) \text{cas} \left(\frac{2\pi}{n} i' j \right) = \Im \left\{ \sum_{j=0}^{n-1} \omega^{(i+i')j} \right\} + \Re \left\{ \sum_{j=0}^{n-1} \omega^{(i-i')j} \right\} \quad (5)$$

where ω is the *primitive n^{th} root of unity* $\omega = \exp(\iota 2\pi/n) = \cos(2\pi/n) + \iota \sin(2\pi/n)$. The identity (5) is easily proved by the De Moivre's identity.

Summation Lemma The last puzzle to the involutory property proof is the summation of a geometric series:

$$\text{For an integer } k, \quad \sum_{j=0}^{n-1} (\omega^k)^j = \begin{cases} n, & \text{if } k \text{ is a multiple of } n; \\ 0, & \text{otherwise.} \end{cases}$$

This lemma ensures that the imaginary part of $\sum_{j=0}^{n-1} (\omega^k)^j$ is zero regardless of the integer value k . On the other hand, the real part is n if k is a multiple of n and zero otherwise. Plugging in these values to the RHS of (5) yields the desired identity (3), which completes the involutory property proof.