Determination of the Gravitational Constant with a Beam Balance

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The Newtonian gravitational constant G was determined by means of a novel beam-balance experiment with an accuracy comparable to that of the most precise torsion balance experiments. The gravitational force of two stainless steel tanks filled with 13 521 kg mercury, on 1.1 kg test masses was measured using a commercial mass comparator. A careful analysis of the data and the experimental error yields $G = 6.674\,07(22) \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$. This value is in excellent agreement with most values previously obtained with different methods.

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In 1998 the task group on fundamental constant of the Committee on Data for Science and Technology (CODATA) recommended for the Newtonian gravitational constant the value $6.673(10) \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ [1]. Surprisingly, the uncertainty was enlarged by a factor of 12 since the last adjustment in 1986. The reasons given for this choice can be summarized as follows: (1) The Physikalisch Technische Bundesanstalt (PTB) published in 1995 a value for G more than forty standard deviations higher than the 1986 recommendation [2]. (2) Up to now there has been no explanation for this discrepancy. (3) The historical difficulties, especially the Kuroda effect [3] should be considered. This effect contributes a systematic bias to certain types of torsion-balance experiments.

Since 1998 noteworthy results of two torsion-balance experiments were published with a relative uncertainty below 50 ppm. First, the experiment carried out by Gundlach and Merkowitz [4] was published in 2000 with a relative uncertainty of 13 ppm. Second, the experiment by Quinn et al. [5] gave a result with an uncertainty of 41 ppm. However these two results differed by more than four standard deviations.

Here we report the final result of our experiment with a beam balance. A preliminary result was published in 1998 with a relative uncertainty of 220 ppm [6]. This experiment differs substantially from the torsion-balance experiments in the following points: (1) The measurement was performed parallel to the local acceleration, (2) the gravitational attraction was measured at an effective distance of ≈ 1 m and (3) the gravitational force was several orders of magnitude larger due to large masses. Thus, the systematic uncertainties were considerably different than those of the torsion-balance experiments.

The principle of our experiment is shown in Fig. 1. The mass setup consists of two movable tanks, labeled field masses (FM) and two smaller masses, called test masses (TM). The FM's are hollow cylinders that can be moved vertically between two positions (A= apart and T= together) such that the TM's pass through the central hole. A device, called a mass exchanger, allows either one of the two TM's to be connected to the beam balance. In each FM position the weight difference of the two TM's is determined. The weight difference is the signal of interest. From the known mass distribution, the amplitude of the signal and the value of the local acceleration a value for the gravitational constant G can be determined.

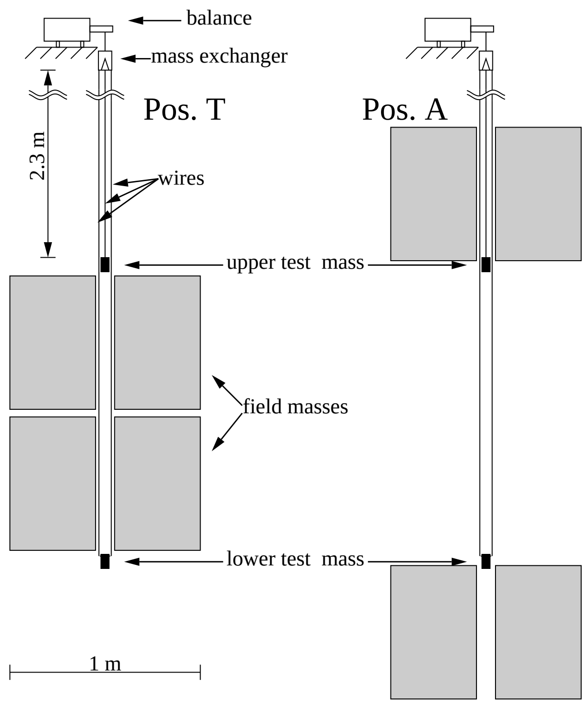

FIG. 1: The principle of the experiment. The two field masses are shown in the two positions together (T) and apart (A) used for the measurements. The apparatus is described in [7].

The experiment is located at the Paul Scherrer Institute (Villigen, Switzerland) in a 4.5 m deep pit. The pit has thick concrete walls, which provide the essential thermal and mechanical stability. The pit is divided into two parts by a working platform. The upper part houses the balance and the electronics and the lower part houses the

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FM's and the TM's. Both parts have their own temperature stabilization system. The temperature variations in both rooms were less than 0.1 K. To avoid forces due to buoyancy and convection, the TM's and the balance are inside a vacuum system (pressure $< 10^{-4}$ Pa).

A modified *Mettler-Toledo* AT1006 mass comparator was used to determine the gravitational force. The resolution of the balance, after several modifications, was 12.5 ng with a total measuring range of 1.7 g. The typical noise of the balance was 200 ng (integration time 40s). In order to calibrate the balance, two 100 mg wire weights were used. The weights were made from stainless steel wires with a thickness of 0.5 mm and 0.96 mm. The surface area of the wires differed by a factor of two, so that they could also be used to monitor sorption effects. Such effects can be ruled out at the level of the uncertainty given by the calibration. The weights were calibrated in air at metas (Swiss National Metrology Institute) before and after each series of measurements. The combined standard uncertainty of this calibration was 0.35 μg. An irreversible change of weight by 1.0 μg of one of the two calibration masses after the first measurement cycle was observed. After this, the mass of the calibration weights remained constant within the measured uncertainties. To convert the balance reading into a force, a knowledge of the precise value of the local acceleration is required. The value at the pan of the balance was 9.807 233 5(6) m/s^2 [8].

As FM's we use two stainless steel tanks filled with 6760.017 kg and 6760.618 kg of mercury. The sum of the mercury mass was known with an accuracy of 27 g. A liquid was selected in order to achieve a homogeneous mass distribution. Mercury was chosen due to its high density. The outer diameter of each tank was 1046 mm, the inner diameter was 100 mm and the height was 700 mm. The inner part of the tank was machined very carefully and measured with an uncertainty of 1 μm before assembling and filling the tanks. The filling with mercury changed slightly the shape of the tanks. The actual shape was measured with an uncertainty of 0.05 mm and was also calculated with the help of a finite element analysis. The two methods agreed within their uncertainties. The density of mercury samples taken from the upper and lower tanks were measured by the PTB. At a temperature of 20 ^{0}C and a pressure of 101325 Pa, a mercury density of 13 545.89(04) kg/m^3 for the upper FM and 13 545.95(04) kg/m^3 for the lower FM was obtained [9]. For the calculation of the gravitational force, a pressure and temperature correction has to be taken into account. The density obtained by the measurement at PTB agrees with the density calculated from the known mass of the mercury and the volume of the FM's as calculated from the precise determination of the dimensions.

Basically, the mass setup of the experiment has cylindrical symmetry. This simplifies the mass integration. For an ideal hollow cylinder with density ρ, height $2B$, inner radius R_1 and outer radius R_2, the z-component of the gravitational field along the axis of symmetry (z-axis) can be calculated [10] giving

$$\begin{aligned} g_z(0,0,z) = \frac{\partial \Phi}{\partial z} &= 2\pi\rho G\left(r_2^+ - r_2^- + r_1^- - r_1^+\right) \ , \quad (1) \\ \text{with } r_{1,2}^{\pm} &= \sqrt{R_{1,2}^2 + (z \pm B)^2} \ . \end{aligned}$$

The main parts of the FM's can be divided into hollow cylinders. The field at the positions of the TM's was calculated as the sum effects of these cylinders using the above formula. Additional small asymmetric components were approximated by point masses.

The force in the z-direction on a cylindrical mass m, height $2b$ and radius r in a gravitational field g can be calculated from the expansion

$$\begin{aligned} F_z &= m(g_z + \frac{\partial^2 g_z}{\partial z^2}(\frac{1}{6}b^2 - \frac{1}{8}r^2) \\ &\quad + \frac{\partial^4 g_z}{\partial z^4}(\frac{1}{120}b^4 - \frac{1}{48}r^2b^2 + \frac{1}{192}r^4) + \cdots) \ . \quad (2) \end{aligned}$$

The expansion factors can easily be obtained by integrating the Taylor expansion of a field over the volume of a cylinder. Again, the asymmetric components of the TM were treated as point masses. We also calculated the uncertainty in the mass integration for an off-axis and a tilted TM. Both were below 3 ppm for a 0.5 mm of axis discrepancy and a tilt angle of 0.017 rad. The tilt as well as the position of the TM with respect to the axis were measured and were below these values in all three final series of measurements mentioned below.

We have five position parameters to determine. They are (1) the separation of the two tanks in position T, (2) the separation of the two tanks in position A, (3) the distance between the two TM's, (4) the distance between the centers of the combined FM's in position A and position T, and (5) the distance between the center of mass of both TM's together and the combined FM's in position T. A detailed investigation of the mass integration shows that the dependence of the amplitude of the signal on the two asymmetry parameters (4) and (5) is $\approx$ 1 ppm/mm. By choosing different positions for the TM's, one can change the influence of the other three parameters. The dependence on these parameters is typically 100 ppm/mm. The position of TM's and FM's was measured with an accuracy of $\pm$0.035 mm using a leveling instrument. In addition the travel distances of the tanks were measured with an accuracy of $\pm$0.014 mm employing digital length-difference instruments.

For the first series of measurements, we used copper TM's. Each TM had a height $2b \simeq 77$ mm, a diameter $2r \simeq 45$ mm and a mass of $m \simeq 1.1$ kg. To prevent the copper from oxidizing, the TM's were gold plated. The second and third series were taken with tantalum TM's. With the higher density of tantalum it was possible to decrease the term $b^2/6 - r^2/8$ in Eq. (2). For

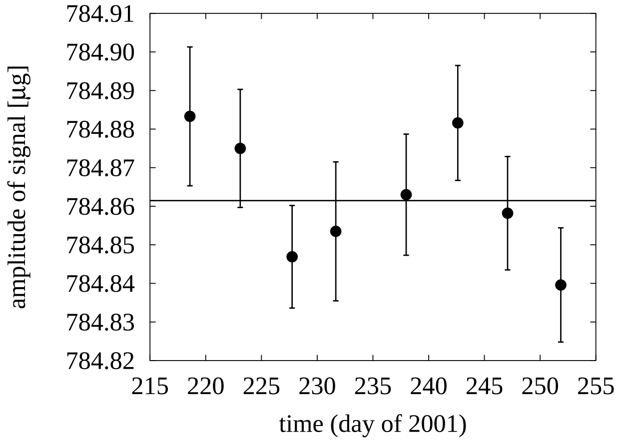

FIG. 2: The amplitude of the signal vs. time measured with the copper test masses. Each data point is averaged over approximately 240 measurements of the amplitude with different loads on the balance.

the tantalum TM's, $2b$ was approximately 39.9 mm and $2r$ approximately 45.8 mm. Hence the influence of the second derivative of the gravitational field in z-direction g_z is a factor of 234 times smaller than for a copper TM. As listed in Table I, the values for G agree well for the different TM's. This is an important consistency check of our mass integration.

By far the largest uncertainty in the determination of G reported in 1998 [6] was due to the assumed nonlinearity of the balance. Since we compare the amplitude of the signal (784 μg) to the mass of the calibration weights (both 100 mg weights together) any nonlinearity of the balance can produce a systematic error. A rough estimate of this nonlinearity error based on information given by *Mettler-Toledo* gave the upper limit of 200 ppm, used in [6]. In order to reduce this large error, we developed a new method which averages out the nonlinearity in situ. The amplitude of the signal was measured at many different working points within the calibration interval. By averaging the different readings, the influence of the nonlinearity of the balance is considerably reduced. A special mechanical handler allows the balance to be loaded automatically with two sets of 16 wire weights. Each wire in the first set has an average mass of 783 μg and 12 533 μg in the second set. With both sets the gravitational constant can be measured at 256 different working points within the calibration interval. The different working points were equally spaced with a standard deviation of 2 μg. During the experiment, a few of the small wires caused troubles due to sticking. These data points could easily be identified and were not used in the data analysis. Therefore some steps were missing. The remaining influence of the nonlinearity on the mean value was estimated by simulation.

The procedure of the measurement was as follows: The mass difference of the TM's following a ULU scheme (U= weighing the upper, L= the lower and U= the upper TM). The difference is calculated by taking the drift of the balance into account. Typically, the balance drift is about 10 μg/d. Each determination of one TM was done with four of the wire weights loaded on the balance. After one ULU cycle was finished, the next cycle was started with a new set of loadings and is now LUL. After eight such cycles (32 different loadings) the tanks were moved to the second position and the measurement was repeated. This was done a third time to retain an ATA measurement for the two FM positions A and T. This larger cycle was then repeated 8 times in order to cover all the possible 256 loading positions. In summary, the loading on the balance was increased in 783 μg steps up to the maximum of 200 mg; and in the next measurement, the loading was decreased to zero. We call the measurement of the signal amplitude over 256 steps a measurement cycle. Each cycle took 4.5 days to complete. The data points obtained for the Cu series (in total 8 cycles) are plotted in Fig. 2.

From the data, an upper bound for the nonlinearity of the balance can be deduced. The simplest approach for the dependence of the reading on the load is a linear function plus a sine function. The amplitude of the sine was found to be less than 10 ng. In order to obtain the uncertainty of the measurement due to the nonlinearity, we simulated characteristic curves with different periods of the sine and found an uncertainty of 20.7 ppm. As shown in Table I the relative signal amplitude between the three measurements differed by $\approx 10^{-3}$. The measured values of G are nevertheless in good agreement, supporting the assumption of a very small error due to nonlinearity.

TABLE I: Summary of the important quantities for the three series of measurements (Cu, Ta , Ta II).

	Cu	Ta I	Ta II
number of cycles	8	3[a]	6
signal amplitude (μg)	784.8615	789.6129	788.5834
statistical uncert. (ppm)	6.9	7.1	8.4
uncert. due position (ppm)	7.0	6.1	16.3
sorption correction (ppm)	20	3	5
$b^2/6 - r^2/8$ (mm^2)	368	1.57	1.57
G (10^{-11} m^{3}kg^{-1}s^{-2})	6.67403	6.67409	6.67410

[a]We have 130 additional cycles with only 16 loadings.

We have to correct our measurement for a sorption effect at the TM's. Moving the large FM's changes the temperature of the vacuum tube at the position of the TM's. For example in the Cu measurement, the temperature change at the upper and lower TM positions was 0.04 K and 0.01 K respectively. Due to this temperature change, water was desorbed from the inner wall of the vacuum tube and was adsorbed by the TM's. This gives a correlated change of the mass due to moving the tanks.

The correction for this effect was determined experimentally by heating the vacuum tube without moving the tanks. It depends on the material of the TM and the positions of the tanks. In the worst case (Cu), this correction was 20 ppm, which was known with an uncertainty of 50%.

Additionally, various systematic effects were investigated. To evaluate the influence of the magnetic forces on the TM's the magnetic field, its gradient and their change between the two FM positions were measured. From the measured magnetic susceptibility of the TM's, we calculated a magnetic effect for the copper TM's of 0.1 ppm and 1 ppm for the tantalum TM's. The influence on the balance temperature and the tilt of the balance were also investigated.

TABLE II: Uncertainty budget of the three measurement series. The first seven topics differ for the three different series.

source of uncertainty	contribution (parts in 10^6)
nonlinearity of the balance	20.7
position of TM's	9.6
sorption effect on TM's	6.7
statistical uncertainty	5.2
calibration weight	3.7
TM mass distribution	2.6
magnetic forces on the TM's	0.6
FM mass distribution	20.6
numerical integration	5.0
tilt of the balance	4.0
local gravity	0.1
total	32.8

We collected data from three different series of measurements, denoted as Cu, Ta I and Ta II (different field mass positions, compared to Ta I). The main results of these series are listed in Table I. We obtained values of G for the three series of measurements and averaged these values weighted with their uncertainties. As a result the final value of G was found to be

$$G = (6.674\,07 \pm 0.000\,22) \times 10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}\ .$$

The uncertainties of the individual measurements were added quadratically, taking into account the correlation of the different contributions. The uncertainty budget is listed in Table II. A comparison with recent published values of G is shown in Fig. 3.

We wish to thank the Paul Scherrer Institute for their hospitality and the assistance in the surveying work. We also express our gratitude to *metrotec* and *Mettler-Toledo* for their help in producing the calibration device and the small weights. We wish to thank Phillipe Richard from metas for calibrating the masses and for measuring the susceptibility of the TM's. We would like to thank E. Klingelé [8] and H. Bettin [9] for their measurements.

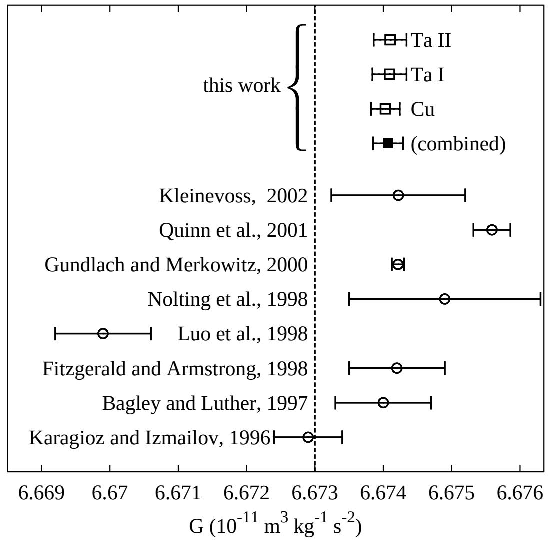

FIG. 3: A comparison of recent published values of G [4–6, 11–15]. The outlined rectangles indicates the three results determined with different mass setups. The dashed line represents the CODATA recommendation $G = 6.673(10) \times 10^{-11}$ m^3 kg^{-1} s^{-2} [1].

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