A Note on Stackelberg Games

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Abstract: Nash game and Stackelberg one are two basic games in game theory community. It is extremely important to further investigate them. Here we show that the leaders will benefit from the interactions in a game. The organization structure in economics is considered with game theory techniques. By game theory approaches, the Nash games and Stackelberg game (leader-follower games) are compared. In a Stackelberg game, if the leader acts as an average player in a Nash game, his/her payoff function value will be reduced. We therefore conclude that a player in a Nash game will benefit from the interactions if he/she transfers into the leader in the corresponding Stackelberg game and other players act as the followers.

Key Words: Stackelberg game, payoff function, game theory

1 INTRODUCTION

Modern game theory was born in 1928 when John von Neumann published his minimax theorem. As a branch of applied mathematics, game theory plays exceedingly important roles. On the other hand, as an important tool, game theory has pervaded almost all fields of sciences and social sciences, such as in biology field (Nowak et al., 2004; Taylor et al., 2004; Traulsen et al. 2005)and in economics (Ambler and Paquet, 1997; Flache, 2002;Nie, 2007). There exist many authors about game theory, see in (Fent et al., 2002; Feichinger et al., 2002; Fuderberg and Tirole, 2003) and the references mentioned therein.

There exist a number of papers to study the social policies with game theory (Fent et al., 2002; Feichinger et al., 2002). In (Fent et al., 2002; Feichinger et al., 2002), law enforce is modelled as game theory. In this paper, we will study a fundamental problem in the society. Namely, two crucial economic organizations, one being Nash game and the other Stackelberg game, are compared.

To simplify the problem, we always assume that all players are rational throughout this paper. In the economic structure, there are various models in game theory, which motivates us to consider the social organization structures by game theory approaches. In an unstructured organization, it is a Nash game between all members in this economic organization (http:// en.wikipedia. org/wiki/ Nash_equilibrium). About Nash games, in (Nash, 1951), the existence of Nash equilibrium is obtained. In (Hofbauer and Sigmund, 1988; Harsanyi, 1973), if Nash equilibriums (NE) are all regular, the number of NE must be odd for two player games. Under such cases, we assume that a member in the society is a player and all payers aim to maximize their objective functions or payoff functions. Furthermore, we assume that the information, which includes the payoff functions and the possible strategies of all players, is known to each player. All players therefore aim to find an optimal strategy for themselves according to their corresponding payoff functions.

Stackelberg games, which are also called leader-follower games, are initially proposed by Stackelberg in 1952 (von Stackelberg, 1952) based on some economic monopolization phenomena. Stackelberg games are recently attached importance (Nie et al., 2006; Nie, 2005) for dynamic version. In a Stackelberg game, one player acts as a leader and the rest as followers. The problem is then to find an optimal strategy for the leader, assuming that the followers react in such a rational way that they optimize their objective functions given the leader's actions. This is the static bilevel optimization model introduced by Von Stackelberg (von Stackelberg, 1952). Actually, there exist many social organizations with hierarchy structures. In (Keyhani, 2003), the problem of energy services is modelled as Stackelberg games. In (Ilies et al., 2005) and the mentioned references, there are a detail analysis about leader-follower games. There exist extensive analysis about Stackelberg games (Gal-Or, 1985; Ilies et al., 2005)

This paper is organized as follows: In the next section, based on game theory model, we find that the leader will benefit from the game, in which we compare the Stackelberg game with Nash game under the same condition. Namely, when a player in a Nash game is transferred into a leader while the other players act as followers, the game is transferred into a Stackelberg game. We find that the payoff function of this player will be correspondingly increased. Some remarks are given in the final section.

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2 THELEADERS BENEFIT FROM THE INTERACTIONS

An unstructured economic organization can be modelled as a Nash game and all members in this organization are all players of the corresponding Nash game. We assume that there are N players and the payoff function for the ith player to be $g_i(x)$ for $i = 1, 2, \cdots, N$, where $x = (x_1, x_2, \cdots, x_N)$ is the decision variables and x_i is the decision variables of the ith player for $i = 1, 2, \cdots, N$. The strategy space for the ith player is denoted $X_i \subset R^{n_i}$ and n_i is an integer. $g_i : R^{n_1+n_2+\cdots+n_N} \longrightarrow R$ for $i = 1, 2, \cdots, N$ and $x \in X = X_1 \times X_2 \times \cdots X_N$. The problem of unstructured organization is modelled as follows: For $i = 1, 2, \cdots, N$, we have

$$\max_{x_i \in X_i} \quad g_i(x). \tag{1}$$

This is a Nash game between N players and all players aim to maximize their corresponding objective functions.

We now assume that a player to be the leader. Without loss of generality, we assume that the first player is transferred into the leader and the rest players are all become followers. The variables and the payoff functions have the same meanings. We denote $X_{-1} = X_2 \times X_3 \times \cdots \times X_N$, $x_{-1} = (x_2, x_3, \cdots, x_N) \in X_{-1}$ and $g_{-1}(x) = (g_2(x), g_3(x), \cdots, g_N(x))^T$. The above Nash game is accordingly transferred into a Stackelberg game. The induced problem is formally given as follows:

$$\begin{array}{ll} \max_{x_1 \in X_1} & g_1(x) \\ x_{-1} \in & \arg\max\{g_{-1}(x) : x_{-1} \in X_{-1}\}. \end{array} \tag{2}$$

This is a Stackelberg game or a leader-follower game.

Let the optimal solution and the corresponding optimal value to (1) be $x^\star = (x_1^\star, x_2^\star, \cdots, x_N^\star)$, $g^\star = (g_1(x^\star), g_2(x^\star), \cdots, g_N(x^\star))^T$, respectively. On the other hand, denote the optimal solution and the corresponding optimal value to (2) to be $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N)$, $\bar{g} = (g_1(\bar{x}), g_2(\bar{x}), \cdots, g_N(\bar{x}))^T$, respectively.

According to the problems (1) and (2), we first show that $x^\star$, the optimal strategy to (1), is also a feasible strategy to (2). Namely, when the first player employs feasible strategy $x_1^\star$ according to (1) and the definition of Nash game, we also have $g_i(x^\star) \geq g_i(x_{-i}^\star, x_i)$ for all $x_i \in X_i$, where $x_{-i}^\star := (x_1^\star, \cdots, x_{i-1}^\star, x_{i+1}^\star, \cdots, x_N^\star)$ with $i = 2, 3, \cdots, N$. In other words, $x_{-1}^\star$ is an optimal response for other players to the first player's policy $x_1^\star$.

We then have that the optimal policy of (2.1) is a feasible strategy to (2.2) and we therefore obtain $\bar{g}_1 = g_1(\bar{x}) \geq g_1^\star = g_1(x^\star)$.

We immediately have the following conclusion: For the problems (1) and (2) with the corresponding solutions, we then have

$$\bar{g}_1 = g_1(\bar{x}) \geq g_1^\star = g_1(x^\star). \tag{3}$$

The result (3) indicates that the leader in general plays overwhelmingly advantageous position in game theory, which is an extremely interesting result and is the main rtesult of this paper.

Many social phenomena support this result. A vote, for example, is in general exceedingly drastic and many candidates hope to win so that they can play the leading position. As we know, a monopolization corporation will benefit from the monopolization position. To eliminate this unfairness, in many counties the corresponding laws are established to forbidden monopolization corporations.

3 CONCLUDING REMARKS

By quantitative techniques, we combine the game theory with economic organization structures. We conclude that the leader plays an advantageous role in a game, which is very popular social phenomena.

As a byproduct, we compare two games, Nash games and Stackelberg games, and an interesting result, reflected many social phenomena, is now obtained.We just consider pure strategy in this paper. When mixed strategies are considered, the results do not hold. For example, consider a normal form Nash game described by the following payoff matrix:

	L	R
T	(10, 4)	(3, 6)
B	(2, 7)	(9, 5)

Figure 1 A game with payoff matrix

It is obvious that the above game has no pure strategy equilibrium. But, it does have a mixed strategy equilibrium, which is: player 1 plays T and B with a probability of 50In the above example, a player's strategy space in a normal form Nash game is NOT the same as his or her strategy space when she/he becomes the leader in the corresponding Stackelberg game, which results the above example.

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