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Classical Mechanics

2020-21 class

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`Classical Mechanics, 2020-21` class

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To Kepler-Galileo-Newton-Leibnitz-Bernouli(s)-Euler-
Lagrange-Hamilton-Jacobi

– LAAN

These notes are not yet fully endorsed by myself and should be used with some caution. Note that lots of typos should be present. Also, please, do not hesitate to communicate such suspected typos or possible inaccuracies (Nov 2019)

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1.1 General

Dynamics of a body relates its position $\mathbf{r} = \mathbf{r}(t)$ and velocity $\mathbf{v} = \mathbf{v}(t)$ and acceleration $\mathbf{a} = \mathbf{a}(t)$ as a function of time (motion) with the forces that applied to it. The link between the kinematical state of the body and the forces that are applied on it is Newton's 2nd law. From mathematical point of view this 'link' is nothing else than differential equations in space and time. [1]

1: The Newton's 2nd law is a 2nd-order ordinary differential equation (ODE) in time, which given the force, $\mathbf{F}(\mathbf{r}, t)$, the position vector, $\mathbf{r}(t)$, is determined.

$$\ddot{r}(t) = \frac{1}{m}\mathbf{F}(\mathbf{r}, t). \qquad (1.1)$$

As concerns the theoretical formulation of mechanics, originated by Isaac Newton (1687), there are two main approaches followed so far. One is the so-called *vectorial dynamics* where the central concept is a 'directional' quantity (*force*) assumed that is exercised on the material bodies. The other approach is the *analytical dynamics* where the central concepts are scalar quantities, such kinetic and potential energy (or their sum which is the total energy), which the bodies is assumed that they 'posses' given their velocity and their position. The latter approach has proven more fruitful conceptually and in the vast majority much more powerful. More sophisticated developments of this line of thinking have led to the most modern physical theories, including the quantum mechanics.

In this chapter the two approaches will be presented and few simple examples will be worked out. In the classical mechanics context the analytical approach (energy) can be derived from the vectorial approach (force) and as such are completely equivalent. In addition, it is concluded that for an isolated physical system there is a quantity which remains constant during its lifetime, regardless how and what complicated physical processes are taking place. Moreover the well-known particular expressions for the kinetic energy $T = mv^2/2$ and the potential energy, $V(x)$, are also derived from the same route, namely the Newton's mechanical laws for forces. [2]

2: Therefore in the below the derived quantities of main interest, measured as, position in (m), velocity in m/s, acceleration in m/s^2, force in N=kg m/s^2, momentum in 'kg m/s' energy in J=N×m, power in W=J/s.

Finally, it should be mentioned that the validity of Newton's law is challenged as the masses of the bodies become too large (more accurately their mass density) and/or their speeds. Again, more accurately, after Einstein, the speed of material bodies can't overpass the speed of light in vacuum, conventionally denoted as c. In these cases, one should rely on Einstein's theories of special and general relativity (if one focuses exclusively on the gravity 'force'). That said, in the below, when numerical evaluation takes place, the SI system of units is used (kg,m,s) for the mass, length and time, respectively. In the SI system one of the universal constants is the gravitational constant, G, taken here:

$$\boxed{G \approx 6.6741 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{Kg\ s}^2}}$$

The same SI unit system is adopted for the involved electromagnetic quantities such as charge (Cb) and the vacuum dielectric, (permittivity, ε_0) and magnetic (permeability, μ_0) constants. These are defined such that,

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \simeq 3 \times 10^8 \text{ m/s}, \quad \mu_0 = 4\pi \times 10^{-7} \frac{\text{N}}{\text{m}^2}, \quad \varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{Cb}}{\text{Nm}^2}$$

1.2 Basics of (Cartesian) kinematics

In kinematics one is studying the properties of the position vector $\mathbf{r} = \mathbf{r}(t)$, velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ of a material body. Knowledge of the position vector $\mathbf{r} = \mathbf{r}(t)$ is sufficient to determine any kinematic quantity required, by taking its time derivatives. Depending on the physical problem the corresponding calculations are facilitated by a proper choice of the coordinate system (Cartesian, Spherical, Cylindrical etc). In the below the related definitions are given for the two of the most useful coordinate systems, namely the Cartesian (CCS) and the polar coordinate systems (PCS). [3]

3: In textbooks, frequently, the unit vector triad is denoted interchangeably as, $(\hat{\imath}, \hat{\jmath}, \hat{k}) = (\hat{x}, \hat{y}, \hat{z})$

The quantities of interest in kinematics are the *position*, $\mathbf{r}$, *velocity*, $\mathbf{v}$, and *acceleration*, $\mathbf{a}(t)$, generally defined in the 3-D space; however often the motion of a mechanical object is restricted in 1-D or 2-D space. In this case by properly choosing the coordinate system we can safely ignore one of the dimension(s). In order to concentrate on the methods, we'll work out the problems mostly using a Cartesian CS and later on we'll extend the methods to the polar CS.

Position Among the fundamental quantities in mechanics is the position vector, $\mathbf{r} = \mathbf{r}(t)$ used to determine the object's motion. *Ultimately one can say that the main task of a mechanical problem is to provide the means to evaluate $\mathbf{r} = \mathbf{r}(t)$.*

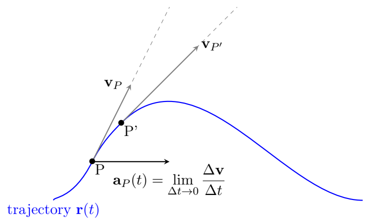


Figure 1.1: Particle's acceleration and velocity vector.

The time dependence of the position vector in a Cartesian Coordinate System (CCS) $(\hat{x}, \hat{y}, \hat{z})$ is expressed as:

$$\mathbf{r} = \mathbf{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \tag{1.2}$$

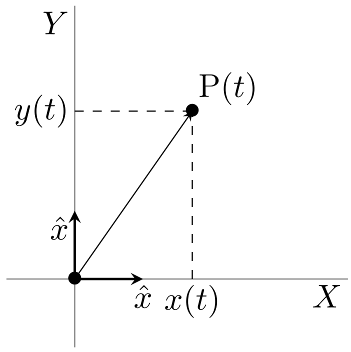


Figure 1.2: 2-D CCS, $\mathbf{r} = (x, y)$.

with $\hat{x}, \hat{y}, \hat{z}$, the mutually orthogonal vectors along the axes xyz, respectively. The magnitude of the position vector $\mathbf{r}$, being the distance from the axis origin is equal to:

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}, \tag{1.3}$$

Its matrix-representation is:

$$\mathbf{r}(t) = x(t)\begin{pmatrix}1\\0\\0\end{pmatrix} + y(t)\begin{pmatrix}0\\1\\0\end{pmatrix} + z(t)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix} \tag{1.4}$$

Velocity: is defined to be the first derivative in time (rate of change) of the position vector $\mathbf{r}(t)$:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t} \tag{1.5}$$

The velocity vector $\mathbf{v}(t)$ in CCS is as:

$$\mathbf{v}(t) = v_x(t)\hat{x} + v_y(t)\hat{y} + v_z(t)\hat{z} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \tag{1.6}$$

[4] with the magnitude of the velocity (speed) equal to:

$$v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2},$$

4: Note that, conventionally, time-derivatives are denoted by,

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t),$$
$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t)$$

and accordingly for their components.

Acceleration: is defined to be the first derivative in time of the velocity: [5]

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{v}(t+\delta t) - \mathbf{v}(t)}{\delta t} \tag{1.7}$$

5: Or equivalently the second derivative in time of the position vector

$$\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$$

The expression of the acceleration in a Cartesian CS is the following:

$$\mathbf{a}(t) = a_x(t)\hat{x} + a_y(t)\hat{y} + a_z(t)\hat{z} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} \tag{1.8}$$

with the magnitude of the acceleration a equal to:

$$a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{1.9}$$

1.3 Polar coordinates

An alternative system of coordinates is the so-called polar coordinates system where the position vector of the system is expressed by its radial distance and angle from a fixed point. For example, if we take this fixed point to be the origin of a Cartesian Oxyz system then any $P = P(x, y, z)$ arbitrary point can be represented by (r, θ, z) as,

$$\mathbf{r} = \begin{cases} x\hat{x} + y\hat{y} + z\hat{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{expressed in } \hat{x}, \hat{y}, \hat{z} \text{ basis} \\ r\hat{r} + \theta\hat{\theta} + z\hat{z} = \begin{pmatrix} r \\ \theta \\ z \end{pmatrix}, & \text{expressed in } \hat{r}, \hat{\theta}, \hat{z} \text{ basis} \end{cases} \tag{1.10}$$

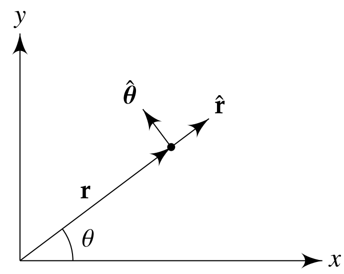


Figure 1.3: Polar coordinate system on the $x - y$ plane $z = 0$. Note that $\hat{r}$, $\hat{\theta}$ change direction with the point $P(x, y) = P(r, \theta)$.

Polar coordinates on the plane. A frequently occurring motion is motion on the plane. Then we may concentrate on the followings only on the $r = r(t), \theta = \theta(t)$ position radial and angle variables [6]. So assuming for example $z = 0$, the position vector, $\mathbf{r}(t)$ on the plane xy can be expressed via the polar coordinates (r, θ) which in terms of the $\hat{x}, \hat{y}$ basis are:

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

6: For example, motion in a gravitational or electrical field. Then for a suitable choice of the $Oxyz$ of the coordinate system the motion along the z−axis can be ignored (either the system does not move at all in this direction or it moves with constant velocity)

Our object is to separate the motion of the particle into radial and angular components. We do so by defining unit vectors in the directions of increasing r and increasing θ:

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

$$\hat{\theta} = -\sin\theta\hat{z} + \cos\theta\hat{y} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

So we see that $\mathbf{r}$ and $\hat{\theta}$ do depend on both $\hat{x}, \hat{y}$ but not on $\hat{z}$. Also of major importance is that the $\hat{r}$,$\hat{\theta}$ vary in direction in complete contrast with the fixed directions of $\hat{x}, \hat{y}$. This means that if the particle's position is represented in polar coordinate system then these unit vectors change direction continuously (as the particle moves); equivalently when we take time derivatives then we have to consider the following time derivatives of $\hat{r}, \hat{\theta}$:

$$\boxed{\dot{\hat{r}} = \dot{\theta}\hat{\theta}, \qquad \dot{\hat{\theta}} = -\dot{\theta}\hat{r}. \qquad (1.11)}$$

With this one can now express the position, velocity and acceleration in this new polar basis. The position is given by

$$\mathbf{r} = r\hat{r}. \tag{1.12}$$

Taking the derivative gives the velocity as

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}. \tag{1.13}$$

The acceleration is then calculated as

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

Summarizing we have in the polar system [7]:

position-velocity-acceleration

$$\mathbf{r} = r\hat{r}, \qquad \text{position} \tag{1.14}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}, \qquad \text{velocity} \tag{1.15}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \qquad \text{acceleration} \tag{1.16}$$

7: Where the components are:

$v_r = \dot{r}$ is the *radial velocity*
$v_\theta = \dot{\theta}$ is the *angular velocity*

and

$a_r = \ddot{r} - r\dot{\theta}^2$ is the *radial acceleration*
$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} =$ is the *angular acceleration*

Uniform motion in a circle

For a particle in a uniform circular motion (since $r = R$) $\dot{r} = 0$ and $\dot{\theta} = \omega =$ constant. So the angular velocity is

$$v_\theta = R\dot{\theta} = R\omega = \text{const}, \qquad \ddot{\theta}) = 0.$$

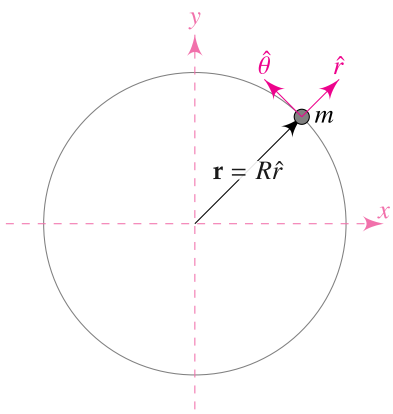
The speed is given by

$$v = |\dot{\mathbf{r}}| = R\omega = \text{const}$$

and the acceleration is

$$\ddot{\mathbf{r}} = -R\omega^2\hat{r} + 0\hat{\theta}.$$

Hence in order to make a particle of mass m move uniformly in a circle, we must supply a *centripetal force* mv^2/r towards the center.



1.4 Determining position and velocity

In the below we'll consider two cases, one where the acceleration is constant in time, $a = const$, while in the second one we'll assume that the acceleration varies with time, $a = a(t)$. In the former case the solution is trivial, while for the latter the solution's complexity varies depending on the problem. There might be solutions for the position and the velocity that can be expressed analytically (in terms of known functions) but it can also be the case where the body's path can only be traced only numerically.

1-D motion

Lets that at time t_0, body's position is x_0 and its velocity equal to v_0:

$$x(t_0) = x_0 \qquad v(t_0) = v_0,$$

The task now can be expressed as below:

Given the acceleration $a(t)$, our goal now is to determine position and velocity at any given time, namely the quantities, $x = x(t)$ and $v = v(t)$.

Calculation of $v = v(t)$
In this case, by using definition of the acceleration as the first time derivative of the velocity $a = dv/dt$, we evaluate the definite integral of $dv = a(t)dt$ from time $t = t_0$ up to t:

$$a(t)dt = dv(t) \qquad \rightarrow \qquad \int_{t_0}^{t} a(t')dt' = \int_{v_0}^{v} dv = v(t) - v(0)$$

to obtain,

$$v(t) = v_0 + \int_{t_0}^{t} a(t)dt. \tag{1.17}$$

This way we obtain the velocity as a function of acceleration (known quantity) and the initial velocity. Since both acceleration and initial velocity are given, then the time dependance of body's velocity is also known. The exact time-dependance $v = v(t)$ is determined when the exact time-dependance of acceleration $a = a(t)$ is given.
Calculation of $x = x(t)$
Having determined velocity one proceeds by recalling the definition of velocity ($v = dx/dt$) to evaluate the definite integral of $dx = v(t)dt$ from the initial time t_0 to an arbitrary time t. The result provides the body's position $x(t)$ as a function of time.
For example, if we set $\int_{t_0}^{t} dt'a(t') = v(t')$, then the above procedure gives the following expression for the position:

$$x(t) = x_0 + v_0(t - t_0) + \int_{t_0}^{t} dt'\, v(t'). \tag{1.18}$$

3-D motion

The above relations for the position, velocity and acceleration are generalized straightforwardly for 2-D and 3-D cases, in order to include more general cases of motion for a particle, e.g. projectile motion, planetary motion, collisions etc. Lets that at time t_0, body's position is $\mathbf{r}_0$ and its velocity equal to $\mathbf{v}_0$. Again, as in the one-dimension case, given the acceleration $\mathbf{a}(t)$, our final task is to determine the position, $\mathbf{r}(t)$ and velocity, $\mathbf{v}(t)$ at any given time. The method use is to start from the corresponding definitions for acceleration, first, and then for velocity:

Calculation of velocity $v = v(t)$
Recalling the definition of acceleration as $\mathbf{a} = d\mathbf{v}/dt$, we integrate $d\mathbf{v} = \mathbf{a}dt$, from time $t = t_0$ up to arbitrary time t. So by assuming that at time t the velocity of the particle is $\mathbf{v}(t)$ starting at time $t_0 = 0$ with velocity $\mathbf{v}(0) = \mathbf{v}_0$:

$$\mathbf{a}(t)dt = d\mathbf{v}(t) \qquad \rightarrow \qquad \int_{t_0}^{t} \mathbf{a}(t')dt' = \int_{\mathbf{v}_0}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}(t) - \mathbf{v}(0)$$

to arrive at,

$$\boxed{\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{a}(t')dt' \qquad (1.19)}$$

Most of than not, in relatively simple problems, this integral can be analyzed the vectors to cartesian coordinates so that to transform the integrals to one-dimensional integrals.
Calculation of position $\boldsymbol{r} = \boldsymbol{r}(t)$
Having determined velocity $\mathbf{v} = \mathbf{v}(t)$ from step (2), similarly, from the definition of velocity as, $\mathbf{v} = d\mathbf{r}/dt$, we integrate $d\mathbf{r} = \mathbf{v}(t)dt$ from the initial time t_0 up to arbitrary time t. The result provides position vector $\mathbf{r} = \mathbf{r}(t)$. Along similar lines with those we followed to evaluate $\int_{t_0}^{t} \mathbf{a}(t)dt = \mathbf{v}(t)$ one arrives at an expression for $\mathbf{r} = \mathbf{r}(t)$. At time t the position of the particle is $\mathbf{r}(t)$ starting off (at time $t_0 = 0$) from $\mathbf{r}(0) = \mathbf{r}_0$:

$$\mathbf{v}(t)dt = d\mathbf{r}(t) \qquad \rightarrow \qquad \int_{t_0}^{t} \mathbf{v}(t')dt' = \int_{\mathbf{r}(0)}^{\mathbf{r}} d\mathbf{r} = \mathbf{r}(t) - \mathbf{r}(0),$$

finally obtaining:

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(t')dt'. \qquad (1.20)}$$

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Example 1.4.1 (Constant acceleration) In the special case where $\mathbf{a} = const$ using (1.19) one obtains:

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}(t - t_0). \qquad (1.21)$$

8: For constant acceleration, $\mathbf{a} = cnst$ practical relations for position, velocity and acceleration are:

$$v^2 = v_0^2 + 2\mathbf{a}(\mathbf{r}(t) - \mathbf{r}_0)$$

$$\mathbf{r}(t) - \mathbf{r}_0 = \frac{\mathbf{v}(t) + \mathbf{v}_0}{2}(t - t_0)$$

Substituting the above expression in (1.20), again integrating, we arrive at:

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t - t_0) + \frac{1}{2}\mathbf{a}(t - t_0)^2 \tag{1.22}$$

Example 1.4.2 (Time varying acceleration) *For motion in 1-D acceleration is $a(t) = 3t^2 + 1$. It is given that a time $t_0 = 1$ sec the particle is at position $x(t_0) = 1$ m and starts at rest ($v(t_0) = 0$ m/s). Find its position $x(t)$ and evaluate for $t = 2$ sec.*

The task here is from knowledge of $a = a(t)$ to obtain $x = x(t)$.

Using (1.19) with the help of initial conditions we arrive at,

$$v(t) = t^3 + t - 2. \qquad \text{check that indeed } v(1) = 0 \tag{1.23}$$

Substituting the above expression in (1.20), again integrating, and using the initial condition $x(1) = 1$ m we obtain:

$$x(t) = \frac{1}{4}t^4 + \frac{1}{2}t^2 - 2t + \frac{9}{4}. \qquad \text{check that indeed } x(1) = 1. \tag{1.24}$$

1.5 Force and the Newton's mechanics laws

Force is the central concept in Newtonian mechanics. As described by Newton's laws of motion, forces are what causes objects to accelerate, according to the celebrated Newton's 2nd law $F = ma$, where the acceleration, a is due to force, F.

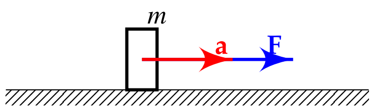


Figure 1.4: Sketch for the Newton's 2nd law, $F = ma$. Acceleration of a body is proportional to the force applied. The proportionality factor is its mass, m.

Newton's 1st law:
If the net force that acts on a body is zero, then the body moves with constant velocity. Mathematically this expressed as:

$$\mathbf{F} = 0 \quad \Longrightarrow \quad \mathbf{v} = const. \tag{1.25}$$

Newton's 2nd law:
If $\mathbf{F}$ is the net force that acts to a body of mass m and $\mathbf{p} = m\mathbf{v}$ its momentum, then the following fundamental law holds:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}). \tag{1.26}$$

Few comments are worth to be mentioned at this point in relation with the Newtons laws; first, that the 1st Newton's law is obtained as a *special* case of the 2nd Newton's law when $\mathbf{F} = 0$.

Second that it is important to bear in mind that the familiar (simpler) expression $\mathbf{F} = m\mathbf{a}$ is valid only when the object's *mass is constant*, $\dot{m} = 0$.

$$m = const. \longrightarrow \frac{dm}{dt} = 0 \;\Rightarrow$$
$$\mathbf{F} = \frac{d}{dt}(mv) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}(t),$$

with $\mathbf{a}(t)$ body's acceleration.

Finally, it is also important to emphasize that the two forces $\mathbf{F}_{12}, \mathbf{F}_{21}$ are applied to the *different* bodies 'A' and 'B', respectively.

Newton's 2nd law in a CCS ($Oxyz$, $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$) according Eqns (1.26) and (1.8) is written as:

$$\mathbf{F} = m\frac{dv_x}{dt}\hat{x} + m\frac{dv_y}{dt}\hat{y} + m\frac{dv_z}{dt}\hat{z} = m\frac{d^2x}{dt^2}\hat{x} + m\frac{d^2y}{dt^2}\hat{y} + m\frac{d^2z}{dt^2}\hat{z}$$
$$= ma_x\hat{x} + ma_y\hat{y} + ma_z\hat{z},$$

Therefore the components along the CCS axes are as,

$$F_x = m\ddot{x}, \qquad F_y = m\ddot{y}, \qquad F_z = m\ddot{z}$$

1.6 Examples

Example 1.6.1 (Free motion-no force, $\mathbf{F} = 0$.) This is nothing else but the Newton's 1st law.

Let's assume the case where initially ($t = 0$) the object (of mass m) is found at the origin of the chosen CS, so that $\mathbf{r}_0 = 0$ (*It is let as an exercise the case where* $\boldsymbol{r}(0) = \boldsymbol{r}_0 \neq 0$). From Newton's 2nd law for $\mathbf{F} = 0$ we have:

$$m\frac{d\mathbf{v}}{dt} = 0 \qquad \Longrightarrow \qquad \mathbf{v}(t) = \mathbf{v}_0 = cnst.$$

Now, from the definition of the velocity, $\mathbf{v}$ we have:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_0 \qquad \Longrightarrow \qquad \boxed{\mathbf{r}(t) = \mathbf{v}_0 t,} \tag{1.27}$$

which ensures a uniform motion (constant velocity).

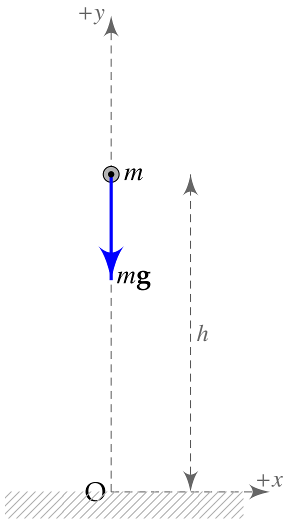


Figure 1.5: Vertical free-fall, $F = -mg\hat{y}$. No initial velocity.

Example 1.6.2 (Free fall in 1-D, $\mathbf{F} = cnst.$) Let's assume a body falling under the Earth's gravitational acceleration, with no initial velocity ($v_0 = 0$) and starting off from an altitude, $h > 0$ above Earth's ground plane (approximated as a flat plane). We assume that the gravitational acceleration is constant and equal to $g = 9.81\ \mathrm{m/s^2}$,

(i) *Coordinates*
It is convenient to choose a Cartesian CS with its $x - z$ plane to coincide with Earth's horizontal plane, and the Oy having direction upwards (opposite the direction of the gravitational acceleration). Newton's second law analyzed in the 3 Cartesian components gives:

$$\begin{aligned} \mathbf{F} = m\mathbf{a} \quad &\rightarrow \quad (F_x, F_y, F_z) = m(a_x, a_y, a_z) \\ &\rightarrow \quad (0, -mg, 0) = m(0, a_y, 0). \end{aligned}$$

Note that the '-' sign in for the y–component due to opposite directions of the y–axis (upwards) and the gravitational acceleration (downwards). The acceleration components are as below,

$$a_y(t) = \ddot{y}(t) = -g, \qquad a_x = a_z = 0$$

with initial conditions,

$$x(0) = z(0) = 0, \quad y(0) = h, \qquad \text{and} \qquad \mathbf{v}(0) = (0,0,0) \tag{1.28}$$

(ii) *Equations of Motion (EOM):* Direct integration of the above equations for the y–component give:

$$v(t) = -gt, \tag{1.29}$$

$$y(t) = h - \frac{1}{2}gt^2. \tag{1.30}$$

Folllowing the same procedure we also end up to,

$$x(t) = z(t) = 0.$$

which tells us that since the initial velocities along the x, z axes were zero with no force components along the same axes we could have completely ignored these 2 directions and worked with the $y-$component exclusively. So, effectively, although the object leaves in a 3-D space, in practice we have an 1-D problem. This it would be true even if $v_y \neq 0$.

So, in mechanical problems where we have zero force and zero initial velocity components along the same direction, it is convenient to choose one of the axis of our CS along this direction. In this case we can exclude from our analysis this axis from the outset.

For example, if for some problem $F_z = 0$ and $v_z(0) = 0$ then we can ignore the dynamics along this component (since it always be $z(0) = z_0 = cnst$). The example below treats this case.

Example 1.6.3 (Constant force, $\mathbf{F} = c$.) Let's study the fall of a particle in the gravity field with some initial position and speed, $\mathbf{r}_0$ and $\mathbf{v}_0$, respectively.

(i) *Coordinate system:*
It is convenient to choose the initial time such that $t_0 = 0$ and the orientation and position of the CCS such that $\mathbf{r}_0 = (0, h, 0)$ and $\mathbf{v}_0 = (v_{x0}, v_{y0}, 0)$. In this case,

$$\mathbf{F} = -mg\hat{y}$$

(ii) *Equations of Motion (EOM)*
The EOM are provide by the Newton's 2nd law:

$$m\frac{d}{dt}\mathbf{v}(t) = \mathbf{F} \quad \longrightarrow \quad m\frac{d}{dt}(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}) = -mg\hat{y}.$$

Analyzing the above vector equation in its 3 components:

$$m\frac{dv_x(t)}{dt} = 0 \quad \Longrightarrow \quad v_x(t) = v_{x0} \tag{1.31}$$
$$m\frac{dv_y(t)}{dt} = -mg \quad \Longrightarrow \quad v_y(t) = v_{y0} - g(t - t_0) \tag{1.32}$$
$$m\frac{dv_z(t)}{dt} = 0 \quad \Longrightarrow \quad v_z(t) = v_{z0} \tag{1.33}$$

$$\begin{aligned} x(t) &= x_0 + v_{x0}(t - t_0) \\ y(t) &= y_0 + v_{y0}(t - t_0) - \frac{1}{2}g(t - t_0)^2, \\ z(t) &= z_0 + v_{z0}(t - t_0) \end{aligned}$$

At this point one may substitute the initial values for the particle's position and velocity. Then the equations above specialize to,

$$x(t) = v_{x0}t \quad y(t) = h + v_{y0}t - \frac{1}{2}gt^2, \quad z(t) = 0$$

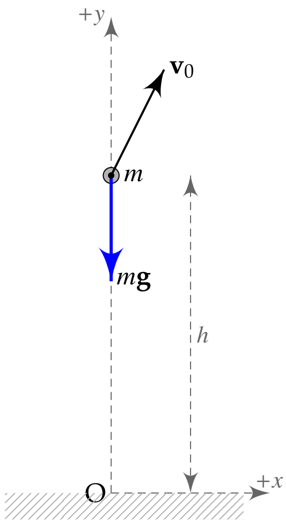


Figure 1.6: Vertical free-fall, $F = -mg\hat{y}$

Note that the initial values for the velocity, v_{x0}, v_{y0}, v_{z0} could have substituted in the equations for $v_x(t), v_y(t), v_z(t)$ in Eqs (1.31)-(1.33) instead, and then integrate! This would have simplified the calculations.

The above equations of motions corresponds to the case of motion with constant acceleration (y-axis) and under zero acceleration along $x, z-$ axes. Elimination of time from the above equations $x = x(t)$ and $y = y(t)$ provides us with the path equation of the body.

Example 1.6.4 (Trajectory equation in projectile motion) Let's at initial time $t_0 = 0$ body's position is $\mathbf{r}_0 = 0$ and this body is ejected with initial velocity $\mathbf{v}_0$. In this case, we define the CCS $Oxyz$ so that the motion to be in plane xy with $x-$axis being the horizontal axis and the y-axis being the vertical axis, along the gravity acceleration. The direction of $y-$axis is defined to be in opposite direction with gravity's acceleration vector. Analyzing the acceleration, $\mathbf{a}$, to its Cartesian components $\mathbf{a} = (a_x, a_y, 0)$ we have (the motion along the z-axis formally is $z = z(t) = 0$):

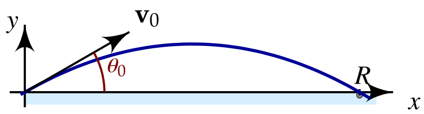


Figure 1.7: Projectile motion. The body's trajectory in a constant field is a parabola. Its details are determined by the initial velocity and the field's magnitude. In the special case of gravity the parabola is independent on the mass m of the body.

$$\mathbf{a} = 0\hat{x} + (-g)\hat{y} = -g\hat{y} \quad \Longrightarrow \quad (a_x, a_y) = (0, -g)$$

Thus, along $x-$axis we have motion with zero (constant) acceleration ($a_x = 0$) and along $y-$axis we again have motion with constant acceleration but non-zero ($a_y = -g$). By analyzing initial position and velocity vectors to their components we have:

$$\begin{aligned}\mathbf{r}(0) &= \mathbf{r}_0 = (x_0, y_0) = (0, 0)\\ \mathbf{v}(0) &= \mathbf{v}_0 = (v_{0x}, v_{0y}) = (v_0 \cos\theta, v_0 \sin\theta)\end{aligned}$$

where the ejection angle θ_0 is with respect to the horizontal axis ($x-$axis). Analyzed in components we have:

$$x\text{-axis}: \quad a_x = 0 \quad v_x = v_0 \cos\theta_0 \quad x = (v_0 \cos\theta_0)t \tag{1.34}$$

$$y\text{-axis}: \quad a_y = 0 \quad v_y = v_0 \sin\theta_0 - gt \quad y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \tag{1.35}$$

The distance from the axis-origin and body's speed at any given time are given by,

$$r(t) = \sqrt{x^2 + y^2}, \qquad v(t) = \sqrt{v_x^2 + v_y^2},$$

respectively. From the above equations, by eliminating the time dependence we obtain the body's *trajectory* [9] :

$$\boxed{y = \tan\theta_0\, x - \frac{g}{2v_0^2 \cos^2\theta_0} x^2.} \tag{1.36}$$

9: **Time elimination:** From (1.34) one solves for $t = x(t)/v_0 \cos\theta_0$ and substitutes to $y = y(t)$, (1.35), to obtain $y = y(x)$, (1.36).

So it is concluded that in a constant gravitational field the motion of any body is a parabola ($y = bx - cx^2$), the exact details of which are determined at initial time t_0, through the initial velocity v_0 and the gravity's acceleration, g.

Note that at any given time θ_0 is given by:

$$\tan\theta_0 = \frac{v_{0x}}{v_{0y}}$$

1.7 linear forces: $F(q) = -kq$

[10]

This is the case of harmonic oscillator and represents one of the most important cases as numerous physical systems exhibit analogous behaviour. Generally most low-energy bound systems always resemble approximately an harmonic oscillator.

10: Examples are mass-spring, pendulum, RLC electrical circuits, material/-electromagnetic waves, molecular vibrations,etc

Example 1.7.1 (Simple mass-sping and pendulum systems) Two particular examples of such systems are the simple cases of a mass attached in a massless sping or in hanged horizontally by a massless rod (simple pendulum).

These particular systems are used to serve as a simple mechanical models to illustrate the theory of harmonic oscillator systems; They are discussed separately, in detail, in a separate chapter (Chapter 4).

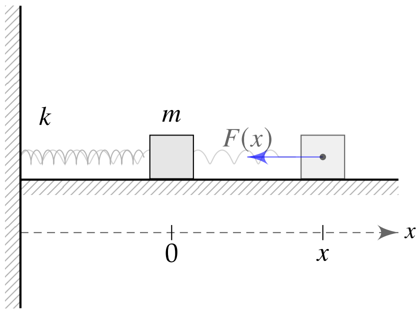


Figure 1.8: Mass-spring simple harmonic oscillator, $F(x) = -kx$

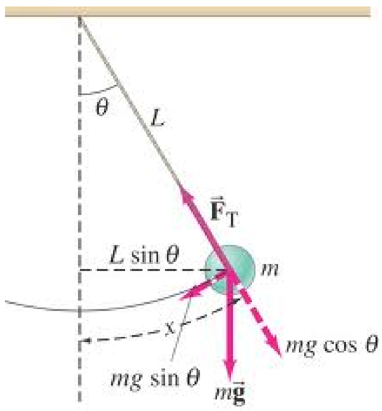


Figure 1.9: Simple pendulum cartoon. The angle θ should be assumed such that $\theta << 1$ so that this system to behave as a SHO.

1.8 Inverse-square forces, $F(x) \sim \pm k/x^2$

The most notable examples of such forces are the gravity and electric Coulombic force. Both have the following relation with the distance from the origin of a coordinate system:

$$\mathbf{F}(r) = \pm \frac{k}{r^2}\hat{r}, \tag{1.37}$$

The constant k is always negative for gravity while for electrostatic Coulomb forces it can be positive or negative. It will be shown that these forces give rise to motion having the shape of conical sections (ellipse,circle,parabola,hyperbola) depending on the initial conditions. They are responsible for the solar system planetary motion, collisions of charged particles and many other. Their main characteristic is that they are *central* forces in the sense that their strength depends on the distance alone and that they are always point to the origin of the force. These type of forces will be discussed in more detail in a separate chapter where the Kepler laws are derived.

Gravity

The (attractive) gravitational force was introduced by Newton to explain the observed orbits of the planets and was able to derive the Kepler's three laws.

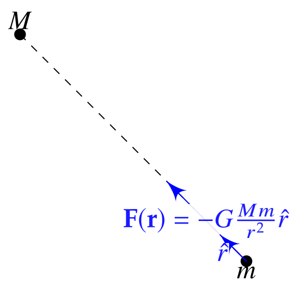


Figure 1.10: Particle m experiences the attractive force from particle M.

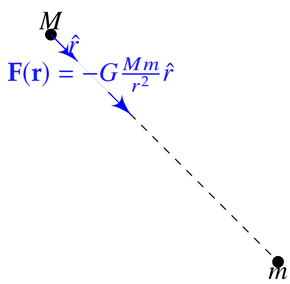


Figure 1.11: Particle M experiences the attractive force from particle m.

Newton's law of gravitation If a particle of mass M is fixed at a origin, then a second particle of mass m experiences a gravitational force (and vice-versa)

$$\mathbf{F}(\mathbf{r}) = -G\frac{Mm}{r^2}\hat{r}, \tag{1.38}$$

where $G \approx 6.6741 \times 10^{-11}\ \mathrm{m}^3/(\mathrm{Kg\ s}^2)$ is the *gravitational constant*. As the force is negative the particle is attracted to the origin. We say that the particle m moves in the force-field generated by the particle, M. Since there is nothing special with the particle, M, we can also say that the particle m also generates a force-field experienced by the particle M.

Coulomb's electrostatic force

Apart from their mass particles are also characterized by another physical quantity, namely their charge. Particles can be neutral or charged. The charge can be negative or positive. Likewise the gravity force-field generated by a massive particle, particles can generate an electrostatic force-field which is experienced only by other charged particle.

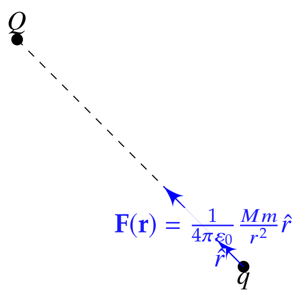


Figure 1.12: Charged particle q experiences the force from the charged particle q. The force can be attractive ($Qq < 0$) or repulsive ($Qq > 0$).

Assuming a particle of charge Q, fixed at origin, generates a force-field which is experienced by a particle of charge q at position, $\mathbf{r}$:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{r}, \tag{1.39}$$

where the electric constant ε_0 is the *electric constant* or *vacuum permittivity* or *permittivity of free space*.

1.9 Questions

Question 1. Using (1.17) and the definition of the acceleration for a particle prove (1.18)

Question 2. Constant acceleration in 1-D. *Use the method represented by* (1.17) *and* (1.18) *to show the following relations for constant acceleration in 1-D motion*:

$$v(t) = v_0 + a(t - t_0), \qquad x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 \tag{1.40}$$

$$v^2 = v_0^2 + 2a(x(t) - x_0), \qquad x(t) - x_0 = \frac{v(t) + v_0}{2}(t - t_0) \tag{1.41}$$

Question 3. *Integrate* (1.19) *and* (1.20) *for constant acceleration* $\boldsymbol{a}$ *=const. to prove the relations* (1.21) *for* $\boldsymbol{r}(t), \boldsymbol{v}(t)$ *and* $\boldsymbol{a}$.

Question 4. Time varying acceleration *For motion in a 2-D plane acceleration is* $a_x(t) = 3t^2 + 1, a_y(t) = 2t$. *It is given that a time* $t_0 = 1$ *sec the particle is at position* $(x(t_0), y(t_0)) = (1, 0)$ *m and starts at rest* $(v_x(t_0), v_y(t_0)) = (0, 0)$ *m/s). Find its position in plane at* $t = 2$ *sec*. What is the magnitude of its velocity at this time?

Question 5. Newton's laws *A point-like object of mass* m *is moving horizontally on a flat plane due to a constant force. The object experiences a friction* F_T *from the ground's surface. A Cartesian coordinate system is chosen such that the object (of mass* $m = 1$*kg) at time* $t = 1$ *s has the position* $\boldsymbol{r} = (2, 0, 0)$ *m and the total force exerted on it is* $\boldsymbol{F} = (1, 0, 0)$ *N.*

(i) Express the Newton's laws for the system (object-surface)
(ii) *Make a sketch and draw all the existing forces according to Newton's theory.*
(iii) *Find its position and the speed at the later time* $t = 3$ *sec.*
(iv) *What was its position and velocity at the earlier time* $t = 0.5$?

Question 6. Free Motion, $V(x) = c$.

For a freely moving object, $(F(x) = 0)$, show that $V(x) = constant$.

Use the method of (2.4) to derive a solution for the object's position $x = x(t)$ if initially the object's speed is $v_0 = 2$ m/s and its position $x_0 = 1$ m. Is it consistent with the Newton's 1st law?

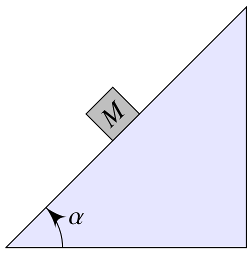


Figure 1.13: Figure for Question (9).

Question 7. Free fall, $V(x) \sim x$

An object is falling the gravity field. It's potential energy is approximated by $V(x) = mg$, where $m = 2$ sec is its mass and $g = 9.81$ m/s. Initially the object was at altitude, $h = 50$ m and thrown with initial velocity $v_0 = 1$ m/s.

(a) Use the method of (2.4) to derive a solution for the object's position $x = x(t)$, when

(i) the object was kicked downwards.

(ii) the object was kicked upwards with $v_0 = 1$ m/s.

(c) For each of the above cases, a(i) and a(ii) calculate the time will take the object to reach the ground level.

Question 8. Hooke's law (harmonic oscillator), $V(x) \sim x^2$

Assume an horizontal mass-spring m, k with no friction forces. The mass is moving under the potential $V(x) = kx^2/2$. Use the method of (2.4) to derive a solution for the mass' position $x = x(t)$ when initially $x(0) = 0$, $v(0) = 2$ m/sec. The spring's constant is $k = 8$ N/m and the object's mass is $m = 2$ Kg.

Question 9. Simple inclined plane *Consider the inclined plane of angle* $\alpha = 45^o$ *and an object of mass* $M = 1$ *kg sliding under the gravity field* ($g = 9.81$ *m/s*^{2}) *[see Fig* (1.13)*]. Assume no friction between the object and the plane's surface.*

(a) *Define a proper coordinate system and derive the equations of motions for the object.*

(b) *If initially the object starts from the top of the plane (height 1.5 m) from rest how long it will take to reach the ground level?*

2 Newtonian Mechanics - energy

Modern considerations of the dynamics of fundamental physical processes do not use the force as the central concept in their description, rather they utilize symmetry properties (time,translational,rotational, mirror, etc) to unveal quantities that remain constant during the even under question. Like manner classical mechanics, a quantity with this property is known as *energy* and in fact, the introduction of a *potential* field is what is considered to be fundamental, with the force concept being a derived. [12]

12: Yet, there are certain processes that cannot be described in terms of potentials fields and the use of forces is the proper one. Friction forces are an example.

2.1 Equivalence of energy and force formulations

To define the concept of energy and its main property, consider a particle of mass m moving in a 1-D space (straight line) found at time t_0 at position x_0, thus $x_0 = x(t_0)$. The particle moves under a force $F = F(x)$ (depends on the position only) and found at time t at position $x(t)$. For convenience it is assumed that the particle's mass is constant. The task for this motion is to show that there is a quantity that remains constant during this translation. This constant is what we'll be calling *energy* of the particle. To start, we do now that the system is described from the Newton' 2nd law. Then we have,

$$\begin{aligned} m\frac{dv}{dt} = F(x) \quad &\longrightarrow \quad mv\frac{dv}{dt} = vF(x) \quad \longrightarrow \quad mv dv = dx F(x) \\ &\longrightarrow \quad \int_{v_0}^{v} mv' dv' = \int_{x_0}^{x} dx' F(x') \\ &\longrightarrow \quad \left[\frac{1}{2}mv^2\right]_{v_0}^{v} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} dx' F(x') \end{aligned} \tag{E1}$$

Let's now define the kinetic and potential as follows. We say that the particle posseses energy due to causes; one is because of its kinetic state, namely its *velocity* alone, and is equal to,

$$T = \frac{1}{2}mv^2. \tag{2.1}$$

Additionaly, it also posseses energy because of its *position* in the force field, $F(x)$. This energy is called *potential* energy and is fully defined through its space variation properties as,

$$V(x) - V(x_0) = -\int_{x_0}^{x} dx' F(x') \tag{2.2}$$

Given the above definitions (E1) above is written as,

$$\frac{1}{2}mv_0^2 + V(x_0) = \frac{1}{2}mv^2 + V(x)$$

Noting that the times t_0, t were chosen arbitrarily the above relation holds for any values of t_0 and t and as such it is correct to say that the quantity defined as *mechanical energy*, [13]

$$\boxed{E(x,p) = \frac{1}{2}mv^2 + V(x) = \textbf{constant!}} \tag{2.3}$$

13: If $E = mv^2/2 + V(x)$ then its time-derivative is zero:

$$\begin{aligned} \frac{dE}{dt} &= mv\dot{v} + \frac{dV}{dx}v \\ &= v\left(m\dot{v} + \frac{dV}{dx}\right) = 0 \end{aligned}$$

Energy conservation law and dynamics

Conservation of energy leads to full determination of the particle's dynamics as below:

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

Since E is a constant and $dxt = dx/t$ one obtains,

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]} \qquad \longrightarrow \qquad \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}[E - V(x')]}} = \pm \int_{t_0}^{t} dt'$$

$$\boxed{t - t_0 = \pm \int \frac{dx}{\sqrt{\frac{2}{m}[E - V(x)]}}}. \tag{2.4}$$

Performing the integral leads to an expression for $x(t)$. In the general case this is not possible by analytical methods, but one can approximate the solution by numerical methods.

2.2 Potential energy

While the kinetic energy definition is a kind of 'universal' as it has the same form ($mv^2/2$) for all mechanical systems, potential energy it uniquely depends on the system. In lay language it is its identity. Different variations of the potential energy values as the system moves into the space indicates different systems and different force experienced. Within the current 1-D formulation it is easily proven that for a given potential energy field $V(x)$ we have [14],

$$V(x) = V(x_0) - \int_{x_0}^{x} dx' F(x') \qquad \longrightarrow \qquad F(x) = -\frac{dV}{dx} \tag{2.5}$$

So the potential energy $V(x)$ is always defined relative to a reference value $V(x_0)$. Needless to say, that suitable choices of x_0 and $V(x_)$ simplify the relevant algebra without sacrificing the correcteness of the method.

14: Usually $V(x_0) = 0$. The proper x_0 depends on the particular dependence of the potential $V(x)$ on the position x. Generally, for

$$V(x) \sim x^n = \begin{cases} x_0 = 0, & n > 0 \\ x_0 = \infty & n < 0 \end{cases}$$

Qualitative considerations of motion in a potential

One need not to rely on the complete determination of $x = x(t)$ to get understanding of the dynamics of the motion. For example, let's take a particle of mass m moving in the region of a potential energy field, $V(x)$.

Supposing the particle is released *from rest* at $x = x_0$. Therefore the initial conditions $x(t_0) = x_0$ and $v(t_0) = 0$). Then $E = V(x_0)$. Its subsequent motion depend on the initial position x_0. One can start from determining the so-called **equilibrium points**; these are points in space where the particles have the tendency to either stand still or to move around those points. From mathematical point of view the equilibrium points are the stationary points of the potential energy: [15]

15: $V'(x) = \frac{dV(x)}{dx}$

$$V(x_0) = 0 \qquad \rightarrow \qquad m\ddot{x}(t) = -V(x_0) = 0, \qquad \text{No force present}$$

- $x_0 = \pm 1$: For these positions $dV(x_0/dx = 0$ and the particle stays there for all t. These points are called *equilibrium* points (Note that the particle experiences no force at these points since $F(x_0) = 0$).
- $-1 < x_0 < 2$: The particle will oscillates back and forth in the potential well. The force it experiences will always point to the origin since $F(x) = -dV/dx$.
- $x_0 < -1$: The particle eventually will fall to $x = -\infty$. The force will lead the particle to negative directions.
- $x_0 > 2$: The particle has enough energy to oveshoot the highest potential energy at $x_0 = -1$ and will escape to the negative direction $x(t = \infty) = -\infty$. [16]

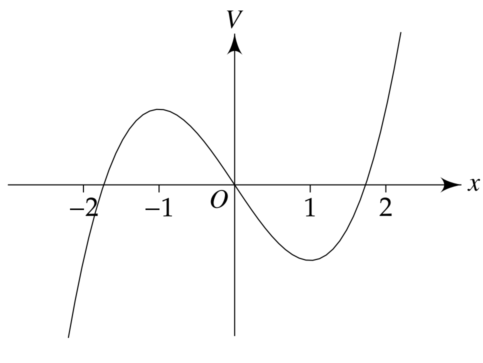


Figure 2.1: Potential $V(x)$

16: What about $x_0 = -2$?

2.3 Examples

In the below few examples are presented where the potential energy method is applied to solve the dynamics problem, e.g. to determine $x = x(t)$. The first step is from the given force to find the potential energy using (2.2) and then to solve (2.4).

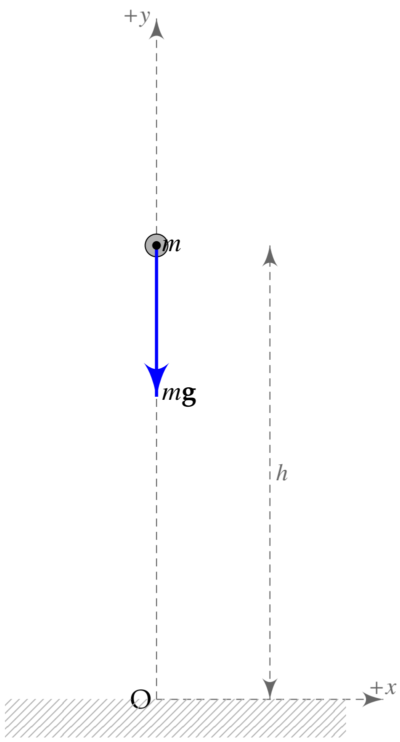


Figure 2.2: Vertical free-fall, $F(x) = mg\hat{y}$

Example 2.3.1 (Free fall under constant gravitational field) Let's assume a particle of mass m placed initially at $y(0) = h$ and let it to fall from rest. The latter means that its initial velocity is zero $v(0) = v_0 = 0$. By a suitable choice of the CCS $(Oxyz)$ so that the weihgt force to be expressed as $F(y) = -mg\hat{y}$ and setting the reference potential value to vanish at the origin $V(0) = 0$ one may obtain for the potential energy

$$V(y) = V(0) - \int_0^y dy'(-mg) = -\int_0^y dy'(-mg) = mgy$$

Since the gravity field is conservative the motion constant E (mechanical energy) is given by:

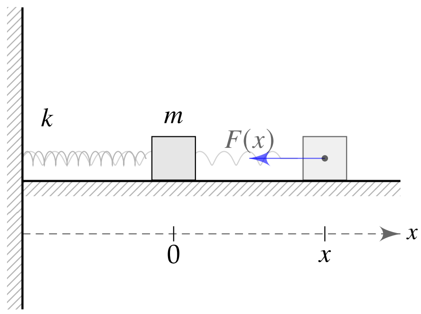
$$E = \frac{1}{2}mv_0^2 + mgy_0 = 0 + mgh = mgh.$$

From relation (2.4) we have:

$$\int_{y_0}^{y} \frac{dy'}{\sqrt{mgh - mgy'}} = \sqrt{\frac{2}{m}}t \quad \rightarrow \quad \int_h^y \frac{dy'}{\sqrt{h - y'}} = \sqrt{2g}t$$
$$\left[-2\sqrt{h - y'}\right]_h^y = \sqrt{2g}t \quad \rightarrow \quad 2\sqrt{h - y} = \sqrt{2g}t,$$

from where by squaring and solving for $y(t)$ we obtain the free-fall Gallileo's equation:

$$y(t) = h - \frac{1}{2}gt^2 \qquad (2.6)$$



Example 2.3.2 (Harmonic oscillator) Similar considerations hold for a mass-spring system, (m, k) obeying the Hooke's law $F = -kx$. For zero fiction forces ($F_T = 0$) the mass posseses a potential energy due to the stretching of the spring. Physically, no force is applied on the object when the spring has its natural length. It is convenient to choose as origin of the coordinate system this point. The object either it is displaced or it is kicked off or both. This way energy has been introduced in the system and its subsequent motion can be found either by applying the Newton's second law for $F(x) = -kx$ [17] or by relying in the conservation of energy. Energy will have the same initial value at all later times. Again applying the (2.2) we have,

$$V(x) = V(0) - \int_0^x dx'(-kx') = \frac{1}{2}kx^2. \qquad (2.7)$$

One then proceeds to substitute in (2.4) and perform the integration. This is left as a problem for the reader.

17: Reminder: From Newton's 2nd law the equation of motion is $m\ddot{x} = -kx.$, where its general solutions is given by

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

with $\omega = \sqrt{k/m}$ and A and B are constants determined by the initial position and velocity by $x(0) = A, \dot{x}(0) = \omega B$.

Newtonian Mechanics in 2- and-3 dimensions) | 3

The Newton's formulation for the particle's motion is of course naturally applied to objects moving in the 3-D space. All the concepts of the kinetic, potential and mechanical energy are directly applicable to the physical three-dimensional space. In the below these necessary clarifications for this extension are discussed. It should always bear in mind that from practical point of view, depending on the physical problem under question the goal is always to reduce the dimensionality of the problem. For example the motion of an object moving on a flat plane is essentially a 2-D problem, or as we'll see later the planetary motion in the solar system can be reduced to a planar (2-D) problem. Such kind of considerations greatly simplify the complexity of the equations of motions. Here we would like to extend the concept of energy (kinetic and potential) so to include motion in the 3-D physical space. The starting point it will again be the Newton's 2nd law:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{r}}{dt^2},$$

where in the particular cases we assume that the object's mass is constant. So the Newton's 2nd law in the two main coordinate systems gives for the forces along the corresponding axes:

Since the position vector may be expressed in the two coordinate systems as [19] ,

$$\begin{aligned} \mathbf{r} &= x\hat{x} + y\hat{y} + z\hat{z} \qquad \text{Cartesian} \\ &= r\hat{r} + \theta\hat{\theta} + z\hat{z} \qquad \text{Polar} \end{aligned}$$

19: With the elementary displacements

$$\begin{aligned} \mathbf{dr} &= dx\hat{x} + dy\hat{y} + dz\hat{z} \qquad \text{Cartesian} \\ &= dr\hat{r} + rd\theta\hat{\theta} + dz\hat{z} \qquad \text{Polar} \end{aligned}$$

we may write for the force components:

Cartesian coordinates:

$$\mathbf{F}(\mathbf{r}) = F_x\hat{x} + F_y\hat{y} + F_z\hat{z} = (m\ddot{x})\hat{x} + (m\ddot{y})\hat{y} + (m\ddot{z})\hat{z}$$

and the corresponding components

Cartesian force components

$$F_x = m\ddot{x}, \qquad \text{x-axis} \tag{3.1}$$

$$F_y = m\ddot{y}, \qquad \text{y-axis} \tag{3.2}$$

$$F_z = m\ddot{z}, \qquad \text{z-axis} \tag{3.3}$$

Polar coordinates: Similarly, if the force $\mathbf{F}(\mathbf{r})$ expressed in polar coordinates

$$\mathbf{F}(\mathbf{r}) = F_r\hat{r} + F_\theta\hat{\theta} + F_z\hat{z} = (ma_r)\hat{r} + (ma_\theta)\hat{\theta} + (ma_z)\hat{z}$$

with components:

Polar force components

$$F_r = m(\ddot{r} - r\dot{\theta}^2), \qquad \text{radial-axis} \qquad (3.4)$$
$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}), \qquad \text{angular axis (tangential)} \qquad (3.5)$$
$$F_z = m\ddot{z}, \qquad \text{z-axis} \qquad (3.6)$$

Work function The elementary work done from a force $\mathbf{F}$ on body which displaces it by $\mathbf{dr}$ is given by:

$$dW = \mathbf{dr} \cdot \mathbf{F}. \qquad \text{Elementary work} \qquad (3.7)$$

In other words, the work produced is the inner product $\mathbf{F}$ times the elementary displacement $\mathbf{dr}$. The above definition implies tha the force should be considered constant between the positions $\mathbf{r}$ and $\mathbf{r} + \mathbf{dr}$. In case the displacement is not infinitesimally small and given that the force is not constant (in contrast in the general case is function of the position), then the total work done during the displacement is given by the following *line* integral of force:

$$W_{ab} = \int_C \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}) = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}) \qquad \text{Total Work,} \qquad (3.8)$$

The rate of the generates/consumed work associated with a force is also a useful quantity and is known as the *power* generated/consumed by the force on the object:

$$P = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}, \qquad \text{Power} \qquad (3.9)$$

Cartesian coordinates: If[20] the force $\mathbf{F}(\mathbf{r})$ and the elementary displacement $\mathbf{dr}$ are expressed in CCS ($\mathbf{dr} = dx\hat{x} + dy\hat{y} + dz\hat{z}$) then the elementary work is calculated as: The inner product in the integral simplifies in the given coordinate system

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \equiv \mathbf{F}^T \cdot \mathbf{dr} = \begin{pmatrix} F_x, F_y, F_z \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dy \end{pmatrix} = F_x dx + F_y dy + F_z dz.$$

20: For example, the work produced from the gravitational force when a mass is displaced from a position with height y_a to a position having height y_b and for $\mathbf{F} = (0, -mg, 0)$ we find:

$$\begin{aligned} W_{ab} &= \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \\ &= \int_{y_a}^{y_b} (-mg\hat{y}) \cdot (dy\hat{y}) \\ &= \int_{y_a}^{y_b} (-mg) dy \\ &= mg(y_a - y_b) \end{aligned}$$

and the total work in Caresian CS,

$$W_{ab} = \int_{x_a,y_a,z_a}^{x_b,y_b,z_b} F_x dx + F_y dy + F_z dz$$

Polar coordinates: Similarly, if the force $\mathbf{F}(\mathbf{r})$ expressed as $\mathbf{F}(\mathbf{r}) = F_r\hat{r} + F_\theta\hat{\theta} + F_z dz$ and $\mathbf{dr} = dr\hat{r} + rd\theta\hat{\theta} + dz\hat{z}$ then the work in PCS is calculated as:

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \equiv \mathbf{F}^T \cdot \mathbf{dr} = \begin{pmatrix} F_r, F_\theta, F_z \end{pmatrix} \cdot \begin{pmatrix} dr \\ rd\theta \\ dz \end{pmatrix} = F_r dr + F_\theta r d\theta + F_z dz$$

$$W_{ab} = \int_{r_a,\theta_a}^{r_b,\theta_b} F_r dr + F_\theta r d\theta + F_z dz. \tag{3.10}$$

Kinetic Energy

From purely mathematical manipulations the following relation for the work done by a force on a body of mass m displaced from the point $A(\mathbf{r}_A)$ to $B(\mathbf{r}_B)$:

$$W_{ab} = T_b - T_a, \tag{3.11}$$

where T

$$T \equiv \frac{1}{2}m\mathbf{v}^2$$

is named as *kinetic energy* of the body. In other words the work produced by the force $\mathbf{F}$ is equal to the change of its kinetic energy $T_{ba} = T_b - T_a$ between the two positions A, B:

$$W_{ab} = \Delta T_{ba} = T_b - T_a \tag{3.12}$$

Potential energy.

The physical quantity $V = V(\mathbf{r})$ that posseses the above properties is called body's *potential energy* and in general depends on body's position in space. The potential energy differs for each fundamental force (gravitational,electromagnetic, spring, ..). The exact form is determined through the relation (3.18). From the latter relation (3.18) the total work produced by the force during the displacement of the body from the position A and B is equal to the variation of its potential energy as below:

$$W_{ab} = \Delta V_{ab} = V(\mathbf{r}_a) - V(\mathbf{r}_b) = V_a - V_b. \tag{3.13}$$

Mechanical energy theorem

From Eqns (3.12) and (3.13) we obtain the following relations:

$$W_{ab} = \Delta T_{ba} \qquad \text{and} \qquad W_{ab} = \Delta V_{ab}$$

$$\Rightarrow T_b - T_a = V_a - V_b \quad \longrightarrow \quad T_a + V_a = T_b + V_b$$

Last relation is the *theorem of mechanical energy conservation* for a motion in a conservative field (or equivalently in a conservative field force). We then define as *mechanical energy* the following quantity:

$$E = T + V(\mathbf{r}) = \text{ constant.} \tag{3.14}$$

which has the important property that is a *constant* through the body's motion.

It's important to emphasize that the definitions of *kinetic*, *potential* energy and the *work* didn't came 'out of the blue' but are based on the Newton's 2nd law as was demonstrated in the case of the 1-D case for a force $F = F(x)$. In the 3-D dimension we have [21] :

$$\begin{aligned}
\mathbf{F}(\mathbf{r}) = m\frac{d\mathbf{v}}{dt} \quad &\rightarrow \quad \mathbf{F}(\mathbf{r})\cdot\mathbf{v} = m\mathbf{v}\frac{d\mathbf{v}}{dt} \rightarrow \quad \mathbf{F}(\mathbf{r})\cdot\frac{d\mathbf{r}}{dt} = m\frac{d}{dt}(\frac{1}{2}\mathbf{v}^2) \\
\rightarrow \quad \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = d(\frac{1}{2}m\mathbf{v}^2) \quad &\rightarrow \quad \int_{\mathbf{r}_a}^{\mathbf{r}_b}\mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_{\mathbf{v}_a}^{\mathbf{v}_b} d(\frac{1}{2}m\mathbf{v}^2) \\
\rightarrow \quad W_{ab} = T_b - T_a \quad &\rightarrow \quad V(\mathbf{r}_a) - V(\mathbf{r}_b) = T_b - T_a \\
\rightarrow \quad V(\mathbf{r}_a) + T_a = V(\mathbf{r}_b) + T_b,&
\end{aligned}$$

with the last equation being the total energy conservation theorem. In the above the 'definitions' for W_{ab} and T were used.

21: Show that for any time-dependent vector $v = v(t)$, the following relations hold:

$$\mathbf{v}\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\frac{1}{2}\mathbf{v}^2)$$

and

$$\mathbf{v}^2 = v^2$$

As a conclusion we end up to the following statement [22] :

A mechanical system, where the force is not explicitly time-dependent, is characterized by a constant-of-motion quantity, called *energy* which is the sum of two different in nature quantities, the kinetic, T and the potential energy, $V(\mathbf{r})$, *all being scalar*. The *kinetic* energy depends on the system's speed, $\mathbf{v}$ and has a universal form for all systems, while the *potential* energy depends on the system's position, $\mathbf{r}$ and is characteristic of the system. The potential energy is intimately associated with the force exerted on the system.

$$E = T + V(\mathbf{r}) = \text{ constant.} \tag{3.15}$$

$$T = \frac{1}{2}m\mathbf{v}^2 \tag{3.16}$$

$$V(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}_b} d\mathbf{r}\cdot\mathbf{F}(\mathbf{r}) \quad \longleftrightarrow F(\mathbf{r}) = -\nabla V(\mathbf{r}) \tag{3.17}$$

To solve a mechanical problem, namely to calculate its motion, $\mathbf{r} = \mathbf{r}(t)$ we may use either the *conservation-of-total-energy theorem* or the *Newton's 2nd law*. For complicated cases (i.e. 3-D problems, multiparticle, rotational motion etc) the former is more convenient as no vectorial quantities are involved in the formulation.

22: *Explicitly and implicitly time-dependent forces*: The force

$$F(x,t) = -kx - F_0\sin(\omega t)$$

is *explicitly* time-dependent. In contrast, the force

$$F(x) = -kx$$

is *implicitly* time-dependent, since the force is time-dependent via the particle's position $x = x(t)$. So more accurate is to write,

$$F(x,t) = F(x(t)) = -kx(t).$$

In the case of a explicitly time-dependent force (i.e. driven harmonic oscillator) we can still define the energy of the system as $E = T + V(x)$ but it will not be a constant-of-motion, any more. The energy will be time-dependent, $E = E(t)$. Nevertheless, still the concept of energy/kinetic/potential are still of fundamental importance.

Conservative force fields The forces $\mathbf{F}(\mathbf{r})$ which actually are the gradient of a scalar field $V(\mathbf{r})$ their line integral is *independent of the actual* *integration curve taken*. Such forces are called *conservative* [23] :

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot \mathbf{dr} = V(\mathbf{r}_a) - V(\mathbf{r}_b) \tag{3.18}$$

These forces (*conservative*) are then written as:

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) = -\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}. \tag{3.19}$$

with the nabla operator defined as a spatial derivative vector and $V(\mathbf{r})$ some scalar function of position. [24]

23: A quick practical test to check whether a given force is conservative or not is to calculate its curl.

$$\nabla \times \mathbf{F}(\mathbf{r}) = 0 \rightarrow \text{conservative force}$$

24: In other words, for such fields, only the end points are important for the evaluation of the (line) integral and not the particular path.

3.1 Central forces

An important class of force fields occurring in nature are spherically symmetric. In other words they show no any directional preference in space. The general form of such forces is the following[25]

While in general a spherical coordinate system would be a suitable choice to express the dynamical Newton's laws it is also convenient to choose a polar coordinate system (r, θ, z). We then choose the central force to lie in the xy plain. In this case the motion along the $z-$axis it would be a uniform motion with constant velocity or if there is no initial velocity in the $z-$direction then the object will never leave the xy plane. So, all the object's dynamics takes place on this plane.

25:

$$\mathbf{F}(\mathbf{r}) = -f(r)\hat{r} \tag{3.20}$$

Accordingly such forces are associated with a potential energy field $V(r)$ expressed as,

$$f(r) = -\frac{dV}{dr} \quad \rightarrow \quad V(r) = -\int_{r_0}^{r} dr f(r)$$

We then say that a particle posses potential energy equal to $V(r)$ which depends only on its distance from the origin of the coordinate system.

Motion in a central force field

To this end for $\mathbf{F} = f(r)\hat{r}$ we have [26] for the Newton's 2nd law in polar coordinates $(r, \theta, \hat{z})$:

$$m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = f(r)\hat{r}.$$

26: You are asked to apply all the below for the important case where,

$$f(r) = -\frac{k}{r^2}$$

corresponding to the gravitational force between two particles ($k = Gm_1m_2 > 0$) and the electrostatic force when the particles are charged ($k = q_1q_2/4\pi\varepsilon_0$).

Central force components

$$F_r(r) = m(\ddot{r} - r\dot{\theta}^2) = f(r)$$
$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0.$$

Therefore we see that the θ component of the central force is zero, $F_\theta(r) = 0$, so the relevant equation gives,

$$m(r\ddot{\theta} - 2\dot{r}\dot{\theta}) = 0 \quad \rightarrow \quad \frac{1}{r}\frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \rightarrow \quad mr^2\dot{\theta} = cnst = L. \tag{3.21}$$

where we named $mr^2\dot{\theta} = L$. Now let's evaluate the particle's angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$: This is equal to the z component angular momentum $\mathbf{L}$:

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = m(r\hat{\mathbf{r}}) \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = 0 + mr^2\dot{\theta}\ \hat{r} \times \hat{\theta} = mr^2\dot{\theta}\ \hat{\mathbf{z}}.$$

From the above we then arrive at the following general conclusion for motion under the influence of central forces:

The above result combined with (3.21) proves that L is constant, which is the *conservation of angular momentum* principle.

Angular momentum as an extra constant-of-motion

Particles who move n field with spherical symmetry have their angular momentum constant. This is equal to,

$$\mathbf{L} = mr^2\dot{\theta}\ \hat{z} \tag{3.22}$$

Mechanical energy. Conservation of angular momentum allows the reduction of the 2D problem to an 1D problem. This is done as follows,

The radial (r) component of the equation of motion is,

$$m(\ddot{r} - r\dot{\theta}^2) = f(r).$$

Substitution of $\dot{\theta}$ by use of $r^2\dot{\theta} = L/m$ gives

$$m\ddot{r} = f(r) + \frac{L^2}{mr^3} = V_{eff}(r). \tag{3.23}$$

where the *effective* radial potential $V_{eff}(r)$ was defined [27] and recalling the procedure followed to determine the energy constant in the 1-D case we can conclude immediately that the total energy of the particle is constant-of-motion and equal to: [28]

27:

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2rm^2}, \quad f(r) = -\frac{dV}{dr}$$

28: Note that $\hat{r}$ and $\hat{\theta}$ are orthogonal,

$$\dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\theta}.$$

and $\dot{\theta} = L/mr^2$

$$\begin{aligned} E &= \frac{1}{2}m|\dot{r}^2 + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{L}{2mr^2} + V(r) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r). \end{aligned}$$

We then have arrived at the following expressions for the total energy and angular momentum of the particle,

Constants-of-motion in spherical symmetric fields

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = cnst \tag{3.24}$$

$$L = mr^2\dot{\theta} = cnst'. \tag{3.25}$$

together with the conclusion that these are constants of motion.

Dynamics in a central force field

In principle, one can determine the radial distance $r(t)$ by integrating the energy equation

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) \qquad \rightarrow \qquad t = \pm\sqrt{\frac{m}{2}}\int_{r_0}^{r} \frac{dr'}{\sqrt{E - \frac{L^2}{2mr'^2} - V(r')}}$$

for $t_0 = 0$ and $r(0) = r_0$. Unfortunately, for the potential fields of interest is not so easy to evaluate the integral. An alternative approach is instead to find the time evolution of $r(t), \theta(t)$ is to calculate the shape $r(\theta)$ of the orbit.

3.2 Examples

Example 3.2.1 (Projectile motion in a gravity field) (i) *Coordinates* Let's first consider a CCS $(Oxyz)$ with the $y-$axis in parallel with gravity's force and opposite direction. We place the origin of the CS at the Earth's ground level (see figure (3.1)). In this case the position vector, $\mathbf{r}$, and the gravitational force, $\mathbf{F}$ are expressed as

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}, \qquad \mathbf{F} = -mg\hat{y}.$$

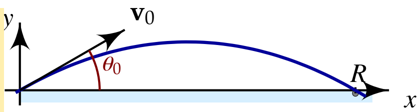


Figure 3.1: Projectile motion in the potential $U(y) = mgy$

(ii) *Potential Energy* The potential enegy, $V(\mathbf{r})$, of the projectile in the Earth's field, can be calculated by evaluating the below (line) integral,

$$V(\mathbf{r}) = V(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}).$$

The inner product in the integrand simplifies in the given coordinate system

$$\mathbf{dr} = dx\hat{x} + dy\hat{y} + dz\hat{z} = \begin{pmatrix} dx \\ dy \\ dy \end{pmatrix} \qquad \rightarrow$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \equiv \mathbf{F}^T \cdot \mathbf{dr} &= \left(F_x, F_y, F_z\right) \cdot \begin{pmatrix} dx \\ dy \\ dy \end{pmatrix} = dx \cdot 0 + dy(-mg) + dz \cdot 0 \\ &= -mgdy\end{aligned}$$

The reference point for the potential energy is chosen at the ground's level $(V(\mathbf{r}_0 = 0) = 0)$. Then we find:

$$\begin{aligned}V(\mathbf{r}) = V(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} &= 0 - \int_{(0,0,0)}^{(x,y,z)} mgdy' \\ &= \left[\ mgy\ \right]_{(0,0,0)}^{(x,y,z)} = mgy.\end{aligned}$$

(iii) *Energy conservation and EOM.*
We can now use the energy consevation property. Initially the energy of the projectile is totally kinetic

$$E = \frac{1}{2}mv_0^2 + V(0) = \frac{1}{2}mv_0^2$$

Since the energy is a constant of motion the following holds:

$$E = \frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 \qquad \rightarrow \qquad v^2 = v_0^2 - 2gy. \tag{3.26}$$

The v^2 can be analyzed in its three components:

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

From Newton's 1st law we know that when no forces are present then the velocity of an objects stays constant. Here we have force only along the y−axis which means that the velocities along the x, z axes will remain the same. So given the initial conditions we have,

$$v_x(t) = v_0 \cos\theta_0, \qquad v_z = 0$$

Then it results that,

$$v^2(t) = v_0^2 \cos^2\theta_0 + v_y^2(t)$$

Replacing the latter expression in (3.26) we obtain, a differential equation for the $y(t)$ component as below:

$$\begin{aligned}
&v_0^2\cos^2\theta_0 + v_y^2(t) = v_0^2 - gy \\
&\qquad\qquad \longrightarrow \quad v_y^2(t) = v_0^2\sin^2\theta_0 - 2gy(t) \\
&\left[v_y(t) \equiv \frac{dy}{dt}\right] \quad \longrightarrow \quad \frac{dy}{dt} = \pm\sqrt{v_0^2\sin^2\theta_0 - 2gy} = \sqrt{2g}\sqrt{a-y}, \\
&\left[a = \frac{v_0^2\sin^2\theta_0}{2g}\right] \quad \longrightarrow \quad \frac{dy}{\sqrt{a-y}} = \pm\sqrt{2g}\,dt,
\end{aligned}$$

This latter equation is integrated as below:

$$\begin{aligned}
\pm t\sqrt{2g} &= \sqrt{2g}\int_0^t dt' = \int_0^y \frac{dy'}{\sqrt{a-y'}} = -\int_0^y \frac{d(a-y')}{\sqrt{a-y'}} = \int_0^{a-y} \frac{du}{\sqrt{u}} \\
&= \left[2\sqrt{u}\right]_a^{a-y} = 2(\sqrt{a-y} - \sqrt{a}).
\end{aligned}$$

From the latter solving for $y = y(t)$ we have,

$$\begin{aligned}
(a-y) &= \left(\sqrt{a} \pm t\sqrt{\frac{g}{2}}\right)^2 \qquad \longrightarrow \\
y(t) &= \left[a - \left(\sqrt{a} \pm t\sqrt{\frac{g}{2}}\right)^2\right] = a - a \pm t\sqrt{2ga} - \frac{1}{2}gt^2,
\end{aligned}$$

from where by replacing the value of $2ga = v_0^2\sin^2\theta_0$ we end up to the same equation as in (1.35),

$$\boxed{y(t) = (v_0\sin\theta_0)t - \frac{1}{2}gt^2} \tag{3.27}$$

The negative sign (in front of the $v_0\sin\theta_0$) was ignored since for the initial velocity along the y−axis as $v_{0y} = v_0\sin\theta_0$.

Example 3.2.2 (Projectile motion using the Newton's law for forces) So we have three unknown functions, $(x(t), y(t), z(t)$ to determine and only one differential equation. We need two more equations. For these we can use the Newton's 2nd law expressed in terms of the potential energy, $\mathbf{F} = -\nabla V(\mathbf{r})$:

$$m\frac{d\mathbf{v}}{dt} = -\nabla V(\mathbf{r}) \quad \rightarrow \quad m\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix} = -\begin{pmatrix} \frac{\partial}{\partial x}(mgy) \\ \frac{\partial}{\partial y}(mgy) \\ \frac{\partial}{\partial z}(mgy) \end{pmatrix} = \begin{pmatrix} 0 \\ mg \\ 0 \end{pmatrix}$$

$$\rightarrow \quad \boxed{v_x = v_0 \cos\theta_0, \quad v_y = v_0 \sin\theta_0 - gt, \quad v_z = 0}$$

Now from (3.26) we get, since $v_0^2 = v_{0x}^2 + v_{0y}^2$ [29]

Example 3.2.3 (Work of a mass-spring system (Harmonic oscillator)) [30] In this case we consider a CCS ($Oxyz$) with the $x-$axis in parallel with spring's force and in opposite direction. In this case the gravitational force is expressed as $\mathbf{F} = -kx\hat{x}$ (Hooke's law). The work produced from the spring's force when the body is displaced from the position x_a to a position x_b is as follows:

$$\begin{aligned} W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_{x_a}^{x_b} (-kx\hat{x}) \cdot (dx\hat{x} + 0\hat{y} + 0\hat{z}) \\ &= \int_{x_a}^{x_b} (-kx)dy = \frac{1}{2}kx_a^2 - \frac{1}{2}kx_b^2 \end{aligned} \tag{3.28}$$

Example 3.2.4 Uniform circular motion-centripetal force

One can think a bead constraint to moving along a circular wire with constant angular velocity, ω.

(i) *Coordinate system*
The obvious choice is the polar CS with its origin placed at the center of the particle's circular orbit. The circular constraint is expressed by setting,

$$r = R \quad \longrightarrow \quad \dot{r} = 0, \quad \text{and} \quad \ddot{\theta} = 0$$

Therefore the velocity is along the $\hat{\theta}$ direction:

$$\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = \dot{\theta}\hat{\theta} = R\dot{\theta}\hat{\theta}$$

(ii) Equations of Motion (EOM)
These are obtained by applying Newton's 2nd law: When $r = R \;\longrightarrow\; \dot{r} = 0$ and $v = R\dot{\theta}$,

$$m\frac{d}{dt}\mathbf{v}(t) = \mathbf{F} \quad \longrightarrow \quad m(\overset{0}{\cancel{\ddot{r}}} - r\dot{\theta}^2)\hat{r} + m(r\overset{0}{\cancel{\ddot{\theta}}} + 2\dot{r}\dot{\theta})\hat{\theta} = \mathbf{F}.$$

Analyzing the above vector equation in its 3 components:

$$F_r = -mR\dot{\theta}^2 = -m\frac{v^2}{R} \quad F_\theta = mR\ddot{\theta} = m\frac{dv}{dt} = 0 \quad F_z = m\ddot{z} = 0$$

Also,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(R\hat{r} \times \mathbf{v}) = mR^2\hat{r} \times (\dot{\theta}\hat{\theta}) = mR^2\dot{\theta}\hat{z} = \text{cnst.}$$

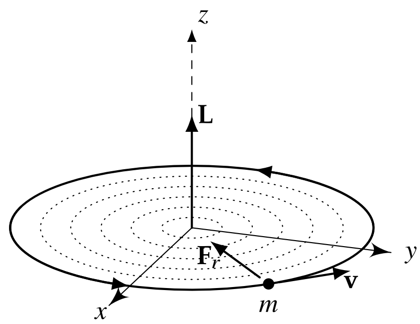
29:

$$v_{0x} = v_0 \cos\theta_0,$$
$$v_{0y} = v_0 \sin\theta_0$$

30: Note that in accordance with (3.18) we have for $U(x)$:

$$U(x) = \frac{1}{2}kx^2 + C,$$

where again $C = const.$ is a constant that is determined as long as the reference point for the potential energy is determined. For example, if *we define* the body's position at $C, x_c = 0$ such that the potential energy is zero we find $C = 0$.



3.3 Questions

Question 1. Newton's equation in polar coordinates

(a) Express the Newton's 2nd law in a polar 2-D coordinate system (ignore the z–coordinate).

(b) Specialize the EOMs for a central force like:

$$f(r) = -\frac{k}{r^2}\hat{r}$$

(c) What is the potential energy corresponding to this force?

(d) Write down the expression for the total mechanical energy, that is the sum of the kinetic and the potential energy.

Linear forces, $F(x) = -kx$ Simple Harmonic Oscillator 4

We say that a physical quantity of a physical system exhibits a *simple* harmonic behavior (SHM) when it is oscillatory in time with a constant period (T_0) and constant amplitude (A_0).

The physical quantity that exhibits an oscillatory behaviour, depends on the actual physical system. To mention a few, for example, a SHO can be a mass-spring system (k, m) with the position (x) of the mass (m) performing an oscillatory motion under the action of the spring force $F = -kx$. [32] A yet another example of a SHO system is a string-mass pendulum (l, m) system with the oscillatory quantity being the angle (θ) of the string with the normal, under the gravity's action. Other examples of SHO systems may include molecular vibrational motions, radiative atomic systems, earthquakes, etc.... Although the above mentioned systems can be completely different in their nature, these systems share similar properties as regards their evolution in time. This common behaviour in time is what is modeled by the concept of the simple harmonic oscillator system.

32: Another case of SHO system is an LC-circuit (capacitor-inductor electrical circuit, (L, C)) with the capacitor's polarity (q) oscillating in time as current flows through the inductor

Below we'll work out two very common physical systems that represent an *harmonic oscillator system*, namely the ideal spring-mass and pendulum systems.

4.1 Mass-spring system

Within the classical mechanical theory the position of a physical object of mass m subject to a force F satisfies the 2-nd Newton's equation,

$$\frac{d^2}{dt^2}x(t) = \frac{1}{m}F(x,t), \tag{4.1}$$

The whole subject of classical mechanics is the mathematical solution of the above differential equation. Generally, the above partial differential equation is so complicated that an analytic solution (solution expressed in a closed form in terms of known mathematical functions) is impossible. This is the general rule. The system under consideration should be oversimplified in order to have analytical solutions of the 2nd Newtons' law. In the present case simplification of the problem consists (a) 1-D motion, (b) the object occupies no space (point-like object) and (c) the force is linearly proportional to the object's position. Assumed the above we are in a position where analytical solutions of (4.1) can be found.

The above assumptions are expressed by the Hooke's law which states that the spring exerts a force (Nt) to the mass equal to:

$$F(x) = -kx,$$

where x (meter) represents the displacement of mass from its equilibrium position in a Cartesian coordinate system Ox. The spring's constant k ($Nt/meter$) being the restoring constant of the massless spring, and m *(Kg)* is the mass of the pointlike object attached to the spring.

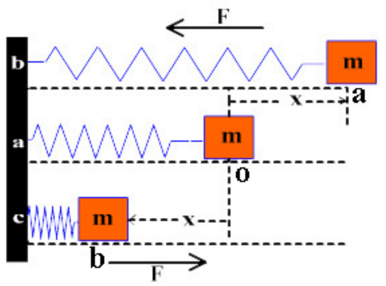


Figure 4.1: Mass-spring system without friction forces. This system is known to behave as simple harmonic oscillator. The mass will oscillate around its equillibrium position ($x = 0$ at all times.

The above definition implies that the equilibrium position is defined to be the position where the spring has its natural length, so that at this position, the spring exerts no force to the mass. In all the followings we set this equilibrium position as the zero of our Cartesian system, $x \equiv 0$,

> *Equilibrium position = Spring's natural length*
>
> $$x \equiv 0 \Longleftrightarrow F(0) \equiv 0.$$

Newton's 2nd law: The motion of the mass (m), namely its distance time evolution $x = x(t)$ can be found by the use of the Newton's 2nd law*:

$$\begin{aligned} F = m\tfrac{d^2}{dt^2}x(t) &\quad\Longrightarrow\quad -kx(t) = m\ddot{x}(t) \quad\Longrightarrow \\ m\ddot{x}(t) + kx(t) = 0 &\quad\Longrightarrow\quad \ddot{x}(t) + \frac{k}{m}x(t) = 0. \end{aligned}$$

* From now on the derivatives in time will be denoted by a dot at the top of the symbol, i.e. $\dot{x}(t) = d/dt[x(t)] = v(t)$. This is a convention that is generally followed by the majority of the scientific community both in the research and education sector. Better familiarize yourself with this.

Therefore, we end up to the following differential equation (ODE) for the mass-spring system:

$$\boxed{\ddot{x}(t) + \omega_0^2 x(t) = 0, \qquad \omega_0 = \sqrt{\frac{k}{m}}.} \tag{4.2}$$

where ω_0 is called *eigenfrequency* of the system, an important physical property that fully characterizes the given system. The general solution of this equation is a linear combination of the periodic functions.

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{4.3}$$

Given this expression the velocity is calculated by,

$$v(t) = -A\omega_0 \sin\omega_0 t + B\omega\cos\omega_0 t. \tag{4.4}$$

The quantities A, B are constants (in time) that are determined by specifying the initial conditions (the system has to start evolving). This means that the values of A, B are dependent on the particular method that the system was started. There are not many ways that one can *'fire'* the system. Either one initially displaces the mass or gives an velocity or both simultaneously. *Therefore different initial conditions result to different motion for the object attached to the spring $x(t)$. Nevertheless the basic charactheristic remain the same in all cases. This is the fact that the motion is periodic with frequency ω_0.*

Eigenfrequency. The *angular eigenfrequency* ω_0 (rad/sec): is the basic parameter that fully characterizes the SHO:

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0, \qquad (rads/sec) \tag{4.5}$$

where T_0 (*sec*) is the oscillation's period and f_0 is the frequency measured in $\sec^{-1}$ (Hz).

Initial conditions. The *initial conditions*, namely, the initial position and velocity of the mass at a particular time[33]:

$$x(0) = x_0 \qquad\qquad v(0) = \dot{x}(0) = v_0, \tag{4.6}$$

Substitution of the initial conditions into (4.7) and (4.8) give

$$\begin{aligned} x(0) = x_0 \quad &\longrightarrow \quad A = x_0 \\ v(0) = v_0 \quad &\longrightarrow \quad B\omega_0 = v_0. \end{aligned}$$

Finally we arrive at the folllowing expressions for the position and the velocity of the mass:

33: We take this time as the start of our time ($t_0 = 0$). This choice is only made for convenience. The motion of the system can be completely determined even if we know the initial conditions at a different time than zero $t_0 \neq 0$).

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t \tag{4.7}$$
$$v(t) = v_0 \cos \omega_0 t - x_0 \omega_0 \sin \omega_0 t \tag{4.8}$$

where the velocity is obtained by differentiating Eq. (4.7). The acceleration may be obtained by double differentiation of Eq. (4.7) [34].

34: Alternatively:

$$a(t) = \ddot{x}(t) = -\omega_0^2 x(t)$$

Mechanical Energy. The mechanical energy is a constant of motion and it is given by:

$$E = T + V = \frac{1}{2} m v^2(t) + \frac{1}{2} k x^2(t) = \text{Constant in time,} \tag{4.9}$$

where $V(x) = kx^2/2$ is the potential energy of the spring, while $T = mv^2(t)/2$ is the kinetic energy of the mass.

4.2 Simple pendulum

A *simple pendulum* is a model consisting of a point mass (m) suspended by a massless, unstretchable string of length l. No other forces are taken into account (e.g. air resistance at Earth's atmosphere). It is known that when the point mass is pulled to one side of its straight-down equilibrium position and released, it swings ('oscillates' is the proper scientific term) about the equilibrium position due to the gravity force, $W = mg$ (considered constant).

Application of the Newton's 2nd law ($\mathbf{F} = m\ddot{\mathbf{r}}$) for the pendulum's mass starts by choosing the Oxy Cartesian coordinate system such that the axis y lies along the vertical direction and the x−axis along the horizontal direction of the figure (5.3). The y−axis is taken to be positive to the downward direction while the x−axis is taken positive to the right-direction. In this case the Newton's 2nd law for the x, y components takes the form:

$$m(\ddot{x}\hat{x} + \ddot{y}\hat{y}) = F_x\hat{x} + F_y\hat{y} \qquad \rightarrow \qquad \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}.$$

By taking the components separately we obtain two ODEs for the $x(t), y(t)$ in terms of the polar angle θ and the magnitute of the tension F_T:

$$\ddot{x} = -F_T \sin\theta, \tag{4.10}$$

$$m\ddot{y} = mg - F_T \cos\theta \tag{4.11}$$

Note that in principle, F_T is time-dependent and an unknown of the problem. The other unknown is the theta angle, $\theta(t)$ which however is related with $x(t)$ and $y(t)$:

$$x(t) = L\sin\theta, \qquad y(t) = L\cos\theta. \tag{4.12}$$

It is beneficial (and simpler) to look for the differential equation of $\theta(t)$ since by knowing $\theta(t)$ we can find $x(t), y(t)$ at once from the above equations. To this end we have to express $\ddot{x}, \ddot{y}$ in terms of the angle $\theta(t)$. So using (4.12) we have,

$$\begin{aligned} \dot{x}(t) &= L\sin\theta &\rightarrow \qquad \ddot{x}(t) &= -L\dot{\theta}\cos\theta - L\ddot{\theta}\sin\theta \\ \dot{y}(t) &= L\cos\theta &\rightarrow \qquad \ddot{y}(t) &= -L\dot{\theta}\sin\theta + L\ddot{\theta}\cos\theta \end{aligned}$$

Substitution of the above expressions into (4.10)-(4.11) after a straightforward algebra (with the reader called to confirm) we arrive at the celebrated *pendulum equation*:

$$\boxed{\ddot{x}(t) + \frac{g}{L}\sin\theta(t) = 0, \qquad \text{pendulum equation} \qquad (4.13)}$$

The above equation generally is *non-linear* and generally is not amenable to an analytical solution[35] Nevertheless, this equation approximately turns to an easily soluble problem, provided that $\theta(t)$ takes only small values. Below we examine this case.

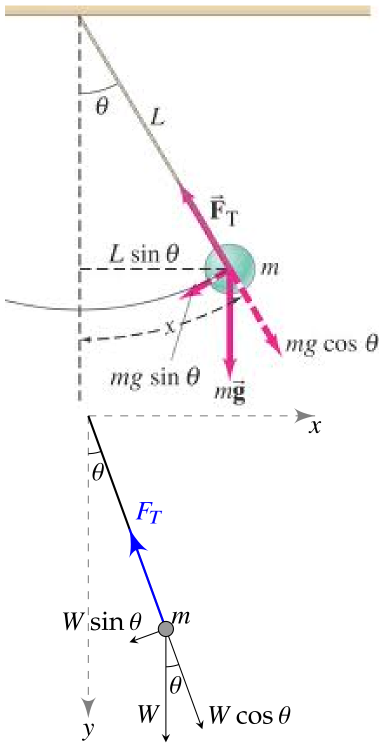


Figure 4.2: Idealized simple pendulum sketch. The point mass moves under the weight force, $W = mg$. At the bottom figure the coordinates axes are shown explicitly

35: A solution which can be written in terms of standard known functions, e.g. sin, cos, tan etc... Non-linearity is due to the presence of the sin terms which in principle it includes higher powers of θ, if one recalls the series expansion of the sin function.

Small-angle approximation $\theta \ll 1$. At this point we can employ the *small-angle* approximation which consists to ask solutions for $\theta \ll 1$ [36]. Then,

$$\theta \ll 1 \quad \longrightarrow \quad \sin\theta \sim \theta, \qquad \tan\theta \sim \theta, \qquad \sin\theta \sim 1 - \frac{\theta^2}{2} \tag{4.14}$$

Essentially, we assume (in the small angle approximation) that the pendulum has no vertical motion (ergo its acceleration along the $y-$axis is zero). This should result that the component of the vertical tension F_T opposes completely the weight of the mass. Eventually, we arrive at an equation satisfied by the angle of the pendulum:

$$\boxed{\ddot{\theta}(t) + \omega_0^2\theta(t) = 0, \qquad \omega_0 = \sqrt{\frac{g}{L}}, \qquad \theta \ll 1.} \tag{4.15}$$

We note that this equation for the pendulum's angle θ that is identical with that of an harmonic oscillator system[37]. The above equation should be supplemented with the initial conditions for the pendulum's dynamics can be fully predicted:

$$\theta(0) = \theta_0, \qquad \dot{\theta}(0) = \Omega_0, \tag{4.16}$$

where θ_0 and Ω_0 are the initial angle (with the normal direction) and angular velocity of the pendulum.

Following the standard procedure (*assuming that* $\theta(t) = A\cos\omega_0 t + B\sin\omega_0 t$ *and after applying the initial conditions)* the time-evolution of the pendulum's angle $\theta = \theta(t)$ is found to be [38] :

$$\boxed{\theta(t) = \theta_0\cos\omega_0 t + \frac{\Omega_0}{\omega_0}\sin\omega_0 t,} \tag{4.17}$$

The angular velocity of the pendulum can be found by taking the first derivative in time of the angle θ, namely if we define, $\Omega(t) \equiv \dot{\theta}$ we obtain,

$$\Omega(t) = \Omega_0\cos\omega_0 t - \omega_0\theta_0\sin\omega_0 t, \tag{4.18}$$

The period T_0 of the pendulum is equal to:

$$\boxed{T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}}.} \tag{4.19}$$

Note that the period of the pendulum is independent on the mass of the attached object (for point-like objects) and of the initial angle θ_0.

36: In principle, for anglues less than 22^o this would be a good approximation to apply.

37: To convince yourself for the validity of this approximation, consider the Taylor expansion for the sine and cosine,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...,$$

and

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^5}{5!} + ...$$

and then keep the firs terms since in the small angle approximation $\theta \ll 1$ and as such $\theta^k \ll \theta^{k+1}, k = 1, 2, ..$

38: An alternatively shortcut to arrive at the small-angle equation for the pendulum (without going through (4.13)) is to employ the approximation into transformation equations (4.12) directly:

$$x(t) = L\sin\theta(t) \sim L\theta(t)$$
$$y(t) = L\cos\theta(t) \sim L$$

Then substitution of the above expressions into at the initial system (4.10)-(4.11) results directly to (4.14). The drawback of this approach is that then we would have lost an essential part of the underlying physics involved in (4.13).

4.3 Damped harmonic oscillator

We say that a physical system represents a damped harmonic oscillator (DHO) when we insert a damping force in a simple harmonic oscillator system. This damping force represents the influence of the environment on the motion of the simple harmonic oscillator. In general the overall result of the damping force is to extract energy from the SHO. The SHO cannot oscillate for ever and eventually it will stop oscillating. In most cases, the total energy of the SHO dissipates to the environment as a heat.

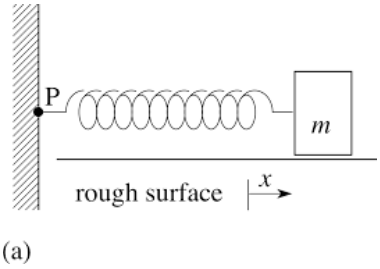


Figure 4.3: Sketch of mass-spring system with damping, namely the damped harmonic oscillator (DH0)

Spring-mass system Assuming an ideal mass-spring system, and as usual if x represents the displacement of the mass from it's equilibrium position we can discuss the basic properties of a DMO. The forces acting on the mass are the force from the spring (Hooke's law), $F = -kx$ and the friction force due to the contact of the mass with the ground. This latter force is opposed the motion of the mass and model it as $-bv$. Taking into the account the above expressions for the forces we write for the 2nd Newton's law:

$$\begin{aligned}
\sum_i F_i = m\frac{d^2}{dt^2}x(t) \quad &\Longrightarrow \quad -kx(t) - bv(t) = ma(t) \\
&\Longrightarrow \quad -kx(t) - b\dot{x}(t) = m\ddot{x}(t) \\
&\Longrightarrow \quad m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0 \\
&\Longrightarrow \quad \ddot{x}(t) + \frac{b}{m}\dot{x}(t) + \frac{k}{m}x(t) = 0
\end{aligned}$$

Therefore, we obtain the following differential equation (ODE) for a damped harmonic oscillator system:

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = 0, \qquad \omega_0 = \sqrt{\frac{k}{m}}, \qquad \gamma = \frac{b}{m}. \tag{4.20}$$

Solutions for the DHO: The solution of the above differential equation for the DHO depends on the relation between the eigenfrequency ω_0 and the damping parameter γ. The details of the derivation can be found at the appendix. We can distinguish among the three different solutions as follows:

(i) **Light damping**: $\omega_0 > \gamma/2$. In this case the motion is again oscillatory, however when compared with the undamped harmonic oscillator ($\gamma = 0$) two important points should be emphasized: (a) the amplitude $A(t)$ of the oscillations decays exponentially and (b) the frequency ω of the oscillation is decreased by an amount that depends on the value of the damping parameter γ. In this case the oscillation of the mass is expressed as below:

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1 \cos \omega t + A_2 \sin \omega t), \qquad \omega_0 > \frac{\gamma}{2} \tag{4.21}$$

$$\omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}, \qquad \omega = \frac{2\pi}{T} = 2\pi f, \tag{4.22}$$

where the *Angular eigenfrequency* ω (rad/sec) characterizes the system's oscillations. Related characteristic quantity are the oscillation's period, T and the frequency, f measured in $\sec^{-1}$ (Hz).

(ii) **Heavy damping**: $\omega_0 < \gamma/2$. In this case there are *no oscillations* at all and the mass returns to it's equilibrium position at large times:

$$x(t) = e^{-\frac{\gamma}{2}t}\left(A_1 e^{\alpha t} + A_2 e^{-\alpha t}\right), \qquad \omega_0 < \frac{\gamma}{2} \tag{4.23}$$

$$\alpha = \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2}, \tag{4.24}$$

where α (Hz) it is a characteristic quantity of the system which contributes to the system's damping properties.

(iii) **Critical damping**: $\omega_0 = \gamma/2$, Again, there are *no oscillations* and the mass returns to it's equilibrium position without passing it. The exact motion is given by the following expression:

$$x(t) = e^{-\frac{\gamma}{2}t}\left(A_1 + A_2 t\right), \qquad \omega_0 = \frac{\gamma}{2} \tag{4.25}$$

In all the above cases A_1, A_2 are constants (independent on time) that are determined through the initial conditions. As initial conditions should be understood the following equations, which provide the position ($x(0)$) and the velocity $\dot{x}(t)$) at the initial time $t_0 = 0$:

$$x(0) = x_0 \qquad v(0) = \dot{x}(0) = v_0, \tag{4.26}$$

Mechanical Energy of a DHO and Quality factor The energy for a DHM is *not a constant* of the motion as it decreases as a function of the time. Eventually all the energy of the DHO will be dissipated into the surrounding environment. A measure of the rate that energy is dissipating is expected to be the γ parameter which describes the strengh of the system-environment interaction.

In the present case, we'll examine the case of light damping of a good harmonic oscillator by assumming $\omega_0 >> \gamma$. The mechanical energy of the mass-sping system has be defined to be the sum of the kinetic and the potential energy:

$$E(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2, \tag{4.27}$$

For the case of light damping we have for the position $x(t)$:

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1 \cos \omega t + A_2 \sin \omega t),$$

Taking the derivative of the position we find for the velocity $v(t) = \dot{x}$:

$$\begin{aligned}
v(t) &= e^{-\frac{\gamma}{2}t}(\omega A_2 - \frac{\gamma}{2}A_1)\cos\omega t - (A_2\frac{\gamma}{2} + \omega A_1)\sin\omega t) \\
&\simeq e^{-\frac{\gamma}{2}t}(\omega A_2 + 0)\cos\omega t - (0 + \omega A_1)\sin\omega t) \\
&= e^{-\frac{\gamma}{2}t}\omega(A_2\cos\omega t - A_1\sin\omega t),
\end{aligned}$$

where from going from the first to the second line we used that $\omega_0 >> \gamma$ and ignored the terms containing γ by setting it zero. Substituting the above expressions into the expression for the energy [(Eq.(4.27))] we get:

$$\begin{aligned}
E &= \frac{1}{2}e^{-\gamma t}\left[k(A_1\cos\omega t + A_2\sin\omega t)^2 + m\omega^2(A_2\cos\omega t - A_1\sin\omega t)^2\right] \qquad (\omega \to \omega_0) \\
&\simeq \frac{1}{2}e^{-\gamma t}\left[k(A_1\cos\omega_0 t + A_2\sin\omega_0 t)^2 + m\omega_0^2(A_2\cos\omega_0 t - A_1\sin\omega_0 t)^2\right] \qquad (k = m\omega_0^2) \\
&= \frac{1}{2}e^{-\gamma t}k\left[(A_1\cos\omega_0 t + A_2\sin\omega_0 t)^2 + (A_2\cos\omega_0 t - A_1\sin\omega_0 t)^2\right] \\
&= \frac{1}{2}e^{-\gamma t}k\left[(A_1^2 + A_2^2)\cos^2\omega_0 t + (A_1^2 + A_2^2)\sin^2\omega_0 t + 2A_1A_2\cos\omega_0 t\sin\omega_0 t - 2A_1A_2\cos\omega_0 t\sin\omega_0 t\right] \\
&= \frac{1}{2}e^{-\gamma t}k(A_1^2 + A_2^2)(\cos^2\omega_0 t + \sin^2\omega_0 t) = \frac{1}{2}e^{-\gamma t}kA_0^2.
\end{aligned}$$

where A_0 is the amplitude of the motion *at initial time*. In going from the first line to the second we assumed that $\omega \simeq \omega_0$ which constitutes a good approximation if we recall that $\omega = \sqrt{(}\omega_0 - (\gamma/2)^2)$ and $\gamma << \omega_0$. In addition from the second line to the third we used the relation $\omega_0^2 = \sqrt{(}k/m) \to k = m\omega_0^2$. We may summarize the result by setting $E_0 = kA_0^2/2$ and write for the mechanical energy of this underdamped DHO:

$$E(t) \simeq E_0 e^{-\gamma t}, \qquad E_0 = \frac{1}{2}kA_0^2 = \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2. \tag{4.28}$$

Therefore we see that eventually the energy of the DHO will vanish ($E(t \to \infty) \to 0$). The decay rate of the energy (energy dissipation) is given by the γ.

Quality factor A quantity useful to characterize the DHO is the *quality factor*, which is a measuere of the relative loss of energy of the DHO within a oscillation period:

$$Q = \frac{\omega_0}{\gamma}.$$

To see the rationale that underlies this definition, first assume the energy of the DHO at a particular time t:

$$E(t) = E_0 e^{-\gamma t}.$$

Then let's ask for the energy at a time after a period of oscillation $t = t + T_0$, where $T_0 = 2\pi/\omega_0$:

$$E(t + T_0) = E_0 e^{-\gamma(t+T_0)} = E_0 e^{-\gamma t}e^{-\gamma T_0} = E(t)e^{-\gamma T_0}$$

Now let's form the loss of energy within this period and divide by $E(t)$. The absolute value of it is[39]

$$\left|\frac{\Delta E(t)}{E(t)}\right| \equiv \frac{E(t)-E(t+T_0)}{E(t)} = 1-e^{-\gamma T_0} = 1-e^{-\gamma(\frac{2\pi}{\omega_0})}$$
$$= 1-[1-(2\pi\frac{\gamma}{\omega_0})+\frac{1}{2!}(2\pi\frac{\gamma}{\omega_0})^2-...] \simeq \boxed{2\pi(\frac{\gamma}{\omega_0}) = \frac{2\pi}{Q}}$$

From the above relation we can see that a good oscillator ($|\Delta E/E| << 1$) requires high-Q! Therefore, *the higher the quality factor the better the DHO*. Using other words, we can also say that high quality factor of an harmonic oscillator suggests for weak interaction with it's environment and therefore low rate energy loss .

39:

$$\left|\frac{\Delta E(t)}{E(t)}\right| \simeq \frac{2\pi}{Q}$$

4.4 Examples

Example 4.4.1 (Simple mass-spring system) For a mass-spring system with $m = 2$ Kgr and $k = 2$ N/m we assume that initially the mass is found at the equilibrium position and is released with initial velocity equal to $v_0 = -2$ m/s. Let's assume the following form for the solution:

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

(One can also start by assuming the alternative forms for $x(t)$. The final expressions and values must be the same). The system's eigenfrequency is calculated by:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\,N/m}{2\,kgr}} \qquad \rightarrow \qquad \boxed{\omega_0 = 1\,\mathrm{Hz}}$$

Next we apply the initial conditions ($x(0) = 0$) and $v(0) = -2$ m/sec to the above solution for the simple harmonic oscillator to obtain for $x(t)$:

$$x(t) = \frac{v_0}{\omega_0} \sin \omega_0 t = -2 \sin(\omega_0 t + \pi)$$

Finally, using $x = x(t)$ we may obtain the other physical quantities of interest:

$$\begin{aligned} v(t) &= \dot{x}(t) = -2 \cos \omega_0 t & a(t) &= \ddot{x}(t) = -\omega_0^2 x(t) = 2 \sin \omega_0 t \\ T(t) &= \frac{1}{2} m v^2 = 4 \cos^2(t) & U(t) &= \frac{1}{2} k x^2 = 4 \sin^2(t) \\ E &= T(t) + U(t) = \frac{1}{2} m A_0^2 \omega_0^2 = \frac{1}{2} k A^2 = 4\,\mathrm{Joules}, \end{aligned}$$

In Fig. (4.4) one can see the relevant plots.

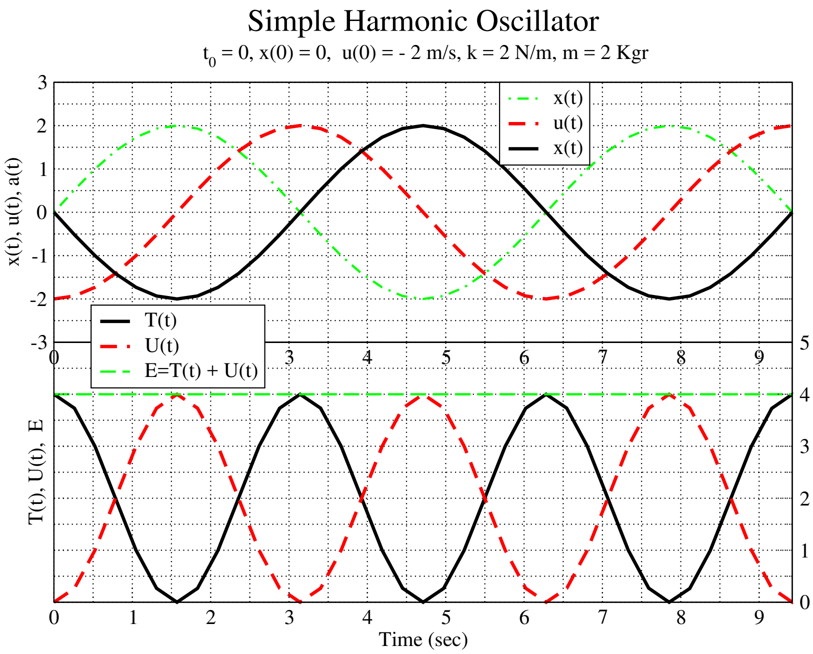
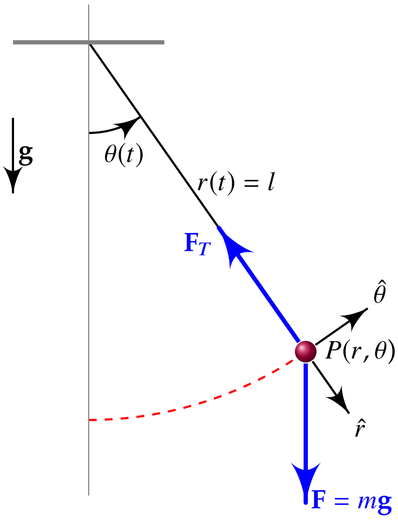


Figure 4.4: Plot of position, velocity, acceleration, kinetic and potential energy as a function of time for a spring mass-system. Parameters of the system are $m = 2$ Kgr, $k = 2$ N/m, $x(0) = 0$ and $u(0) = 2$ m/s. The period of the oscillations are $T = 2\pi = 6.282$ sec.

Example 4.4.2 (Simple pendulum in the small angle approximation, $\theta_0 \ll 1$) Consider the simple pendulum where an object of mass m is hanged by a string and moves in a Earth's gravity field (acceleration is g). The string applies tension T on the object. We raise the object initially at an angle θ_0 and leave it to evolve. The task is to find its subsequent motion, $\theta = \theta(t)$ at all later times, when $\theta_0 \ll 1$ radians.



One may use Newton's 2nd law for forces using either a CCS or a PCS system. In the present case one is benefited if a PCS system is employed given that the string's length is constant, equal to l. So effectively, the problem is reduced from a 2-D to a 1-D for the angle $\theta(t)$. Following the same lines of thinking as in the problem of the simple harmonic oscillator in the previous chapter, elimination of the r variable results to a potential energy which depends only on the θ angle.

$$V(\theta) = V(\theta_0) - \int_{\theta_0}^{\theta} d\theta'(-mgl\sin\theta') = -mgl\cos\theta, \qquad V(0) = -mg\ell. \tag{4.29}$$

$$E = T + V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta.$$

Therefore $V \propto -\cos\theta$. The stable equilibrium are at $\theta = 0$, and unstable equilibrium at $\theta = \pi$. [40]

(i) If initially $E > mg\ell$, then $\dot{\theta}(t)$ never vanishes and the pendulum makes full circles.
(ii) If $0 < E < mg\ell$, then $\dot{\theta}(t)$ vanishes at $\theta = \pm\theta_0$ for some $0 < \theta_0 < \pi$ i.e. $E = -mg\ell\cos\theta_0$. The pendulum oscillates back and forth.

Conservation of energy allows to apply (2.4). By defining the oscillation period T as the time that takes to the object to revisit its position

40: Points where $V'(\theta) = 0$. Stability depends on the sign of $V''(\theta)$.

then it follows

$$\int_0^{\theta_0} \frac{d\theta'}{\sqrt{\frac{2E}{m\ell^2} + \frac{2g}{\ell}\cos\theta'}} = \int_0^{T_0/4} dt = \frac{T_0}{4}.$$

The pendulum's energy is given by the initial condition as $E = -mg\ell\cos\theta_0$.

$$\frac{T_0}{4} = \sqrt{\frac{\ell}{2g}} \int_0^{\theta_0} \frac{d\theta'}{\sqrt{\cos\theta' - \cos\theta_0}}.$$

In general, this integral is not trivial to evaluate in terms of familiar analytical functions, but assuming a small initial displacement $\theta_0 \ll$ [41] one obtains

41: $\cos\theta \approx 1 - \frac{1}{2}\theta^2,\ \theta \ll 1$

$$T_0 \approx 4\sqrt{\frac{\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} = 2\pi\sqrt{\frac{\ell}{g}}\ !$$

which is the result obtained in (4.19). Note that the oscillation period is independent of the amplitude θ_0 (and of the mass m).

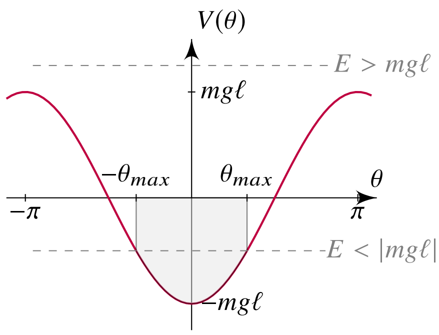


Figure 4.5: Simple pendulum. Force field sketch and potential energy.

Example 4.4.3 (Underdamped solutions of DHO in terms of x_0 and v_0) For the case of light dampong ($\omega_0 > \gamma$) calculate the constants A_1, A_2 of Eq. (4.21) in terms of the initial conditions $x(0) = x_0$ and $v(0) = v_0$.

(a) Give the solutions in terms of v_0 if $x_0 = 0$.

(b) Give the solutions in terms of x_0 and if $v_0 = 0$.

(c) Take the case that both x_0 and v_0 are non-zero. However set $\gamma = 0$. Does the result coincide with the simple harmonic solution?

For the case of light damping we have for the position $x(t)$:

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1 \cos \omega t + A_2 \sin \omega t),$$

Taking the derivative of the position we find for the velocity $v(t) = \dot{x}$:

$$v(t) = e^{-\frac{\gamma}{2}t}\left[(\omega A_2 - \frac{\gamma}{2}A_1)\cos \omega t - (A_2\frac{\gamma}{2} + \omega A_1)\sin \omega t\right]$$

Now we use the initial conditions and we find:

$$\begin{aligned} x(0) = x_0 &\Longrightarrow 1 \cdot (A_1 \cdot 1 + A_2 \cdot 0) = x_0 \Longrightarrow & A_1 &= x_0, \\ v(0) = v_0 &\Longrightarrow 1 \cdot ((\omega A_2 - \frac{\gamma}{2}A_1) + 0) = v_0 \Longrightarrow & (\omega A_2 - \frac{\gamma}{2}A_1) &= v_0, \end{aligned}$$

Solving for the above system in terms of A_1 and A_2 we obtain:

$$A_1 = x_0, \qquad A_2 = \frac{v_0 + x_0\gamma\omega/2}{\omega}$$

That results to the following exressions for $x(t)$ and $v(t)$:

$$\boxed{\begin{aligned} x(t) &= e^{-\frac{\gamma}{2}t}(x_0 \cos \omega t + \frac{v_0 + x_0\gamma\omega/2}{\omega}\sin \omega t), \\ v(t) &= e^{-\frac{\gamma}{2}t}(v_0 \cos \omega t - \frac{x_0\omega_0^2 + v_0\gamma\omega/2}{\omega}\sin \omega t), \end{aligned}}$$

where the relation $\omega^2 = \omega_0^2 - (\gamma/2)^2$ was used to introduce ω_0 into the expressions. Having the general solution for the light-damping case we can consider the special cases:

(a) If $x_0 = 0$ we have $A_1 = 0$ and $A_2 = v_0/\omega$. Then $x(t)$ and $v(t)$ are written as:

$$\begin{aligned} x(t) &= \frac{v_0}{\omega}e^{-\frac{\gamma}{2}t}\sin \omega t \\ v(t) &= e^{-\frac{\gamma}{2}t}\left(v_0 \cos \omega t - \frac{v_0\gamma}{2\omega}\sin \omega t\right) \end{aligned}$$

(b) if $v_0 = 0$ we have $A_1 = x_0$ and $A_2 = (\gamma/2\omega)x_0$. Then $x(t)$ and $v(t)$

are written as:

$$x(t) = x_0 e^{-\frac{\gamma}{2}t}\left(\cos\omega t + \frac{\gamma}{2\omega}\sin\omega t\right)$$
$$v(t) = -\frac{x_0\omega_0^2}{\omega} e^{-\frac{\gamma}{2}t}\sin\omega t$$

(c) Setting $\gamma = 0$, we end up with $A_1 = x_0$ and $A_2 = v_0/\omega$ and $\omega = \omega_0$. The decaying factor $e^{-\gamma t/2} = 1$ is becoming unity and we obtain the oscillatory solutions for the SHO:

$$x(t) = x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t,$$
$$v(t) = v_0\cos\omega_0 t - x_0\omega_0\sin\omega_0 t,$$

We left to the reader to confirm that these relations are indeed the solutions for the SHO when both initial conditions $x(0) = x_0$ and $v(0) = v_0$ are both different than zero.

Alternative forms for the light-damping DHO Show that the position $x(t)$ of a DHO can also be expressed as below:

$$x(t) = A_0 e^{-\frac{\gamma}{2}t}\cos(\omega t - \phi), \tag{4.30}$$

where A_0 is the amplitude of the motion and ϕ is known as the initial phase. Find A_0 and ϕ in terms of the constants A_1, A_2 of Eq. (4.21).

Eq. (4.21) gives for $x(t)$:

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1\cos\omega t + A_2\sin\omega t)$$

If we set,

$$A_1 = A_0\cos\phi, \qquad A_2 = A_0\sin\phi \tag{4.31}$$

we may rewrite the equation for $x(t)$:

$$x(t) = e^{-\frac{\gamma}{2}t}A_0(\cos\phi\cos\omega t + \sin\phi\sin\omega t) = A_0 e^{-\frac{\gamma}{2}t}\cos(\omega t - \phi)$$

A_0 and ϕ can be found from Eqns (4.31) by first adding A_1^2 and A_2^2:

$$A_1^2 + A_2^2 = A_0^2(\cos\phi + \sin^2\phi) = A_0^2, \quad \Longrightarrow \boxed{A_0 = \sqrt{A_1^2 + A_2^2}}. \tag{4.32}$$

while by dividing A_2 and A_1 we obtain:

$$\boxed{\tan\phi = \frac{A_2}{A_1}} \tag{4.33}$$

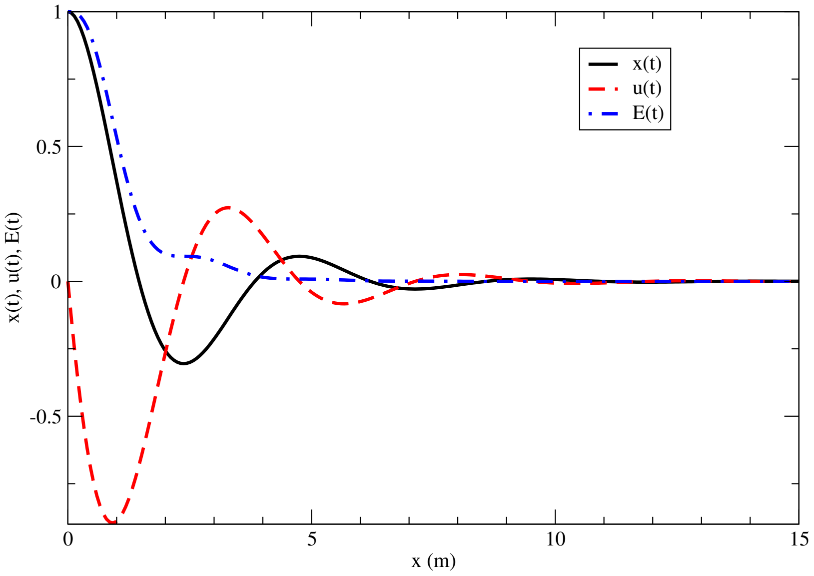


Figure 4.6: Plot of position, velocity and energy as a function of time for the spring mass-system of the application of the theory. Parameters of the system are $m = 1$ Kgr, $k = 2$ N/m, $b = 1$ kgr/m $x(0) = 1$ m and $u(0) = 0$ m/s. The period of the oscillations are $T = 2\pi/\omega = 4.71$ sec.

Example 4.4.4 (Damped mass-spring system) For a underdamped mass-spring system with $m = 1$ Kgr and $k = 2$ N/m we assume that initially the mass is found at the equilibrium position ($x = 1$ m) and is released from rest. We assume that the friction of the surface (that the spring-mass system is located) is opposite to the direction of the velocity of the mass and proportional to it's velocity. We express this force as $F_T = -bv(t)$, with $b = 1$ kgr/sec.

(a) Determine the motion of the mass at any later time by considering the following form for the position $x(t)$:

$$x(t) = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi),$$

(b) What is the energy as a function of time $E = E(t)$?

(c) Plot the $x = x(t), v = v(t)$ and $E = E(t)$.

First we determine the eigenfrequency of the system ω_0 and the damping parameter γ:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\,N/m}{1\,kgr}} = \sqrt{2}\,\text{Hz}$$

$$\gamma = \frac{b}{m} = \frac{1\,Kgr/sec}{1\,kgr} = 1\,\text{sec}^{-1} = 1\ \text{Hz}$$

Since $\omega_0 > \gamma$ then we we have the case of *light damping* oscillators. We can use the following form for the position $x(t)$:

$$x(t) = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi), \tag{4.34}$$

The frequency of the oscillatory (but decaying) motion is equal to:

$$\omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2} = \sqrt{2 - 1/4} = \sqrt{7/4} = 1.3229\,\text{Hz}$$

At this stage it remains to calculate A_0 and the phase angle ϕ. To this end we are going to use the initial conditions. We always need to calculate the velocity of the mass, which is given as:

$$v(t) = \frac{dx}{dt} = \dot{x}(t) = -A_0 e^{-\frac{\gamma}{2}t}(\omega \sin(\omega t + \phi) + \frac{\gamma}{2}\cos(\omega t + \phi) \qquad (4.35)$$

Applying the initial conditions to Eqns. (4.34) and (4.35) we have:

$$\begin{aligned}
x(0) &= 1 \\
&\Longrightarrow A_0 e^{-\frac{\gamma}{2}\cdot 0}\cos(\omega \cdot 0 + \phi) = 1 \\
&\Longrightarrow A_0 \cos\phi = 1 \Longrightarrow A_0 = \frac{1}{\cos\phi} \\
v(0) &= 0 \\
&\Longrightarrow -A_0 e^{-\frac{\gamma}{2}\cdot 0}\left[\omega \sin(\omega \cdot 0 + \phi) + \frac{\gamma}{2}\cos(\omega \cdot 0 + \phi)\right] = 0 \\
&\Longrightarrow \omega \sin\phi = -\frac{\gamma}{2}\cos\phi, \\
&\Longrightarrow \tan\phi = -\frac{\gamma}{2\omega}
\end{aligned}$$

Finally we end up to the following two equation for the amplitude and the phase:

$$\begin{aligned}
\tan\phi &= -\frac{\gamma}{2\omega} = -\frac{1}{2 \cdot 1.3229} \\
&\Longrightarrow \phi = \tan^{-1}(-0.377957) = -20.696^o = -0.3612 \text{ rad} \\
A_0 &= \frac{1}{\cos\phi} \\
&\Longrightarrow A_0 = 1/\cos(-0.3612) = 1/0.9354 = 1.067\,\text{m}
\end{aligned}$$

Therefore the position and the velocity of the mass are given as:

$$\begin{aligned}
x(t) &= 1.067e^{-0.5t}\cos(1.3229t - 0.3612) \\
v(t) &= -1.067e^{-0.5t}(1.3229 \sin(1.3229\,t - 0.3612) + 0.5\cos(1.3229\,t - 0.3612)) \\
E(t) &= \frac{1}{2}kx^2(t) + \frac{1}{2}mv^2(t) = x^2(t) + 0.5v^2(t)
\end{aligned}$$

In Figure (4.6), the relevant plots are shown.

4.5 Appendix: Solving the DHO equation

Assume* the DHO equation supplemented with the initial conditions at time $t_0 = 0$:

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = 0, \qquad \omega_0 = \sqrt{\frac{k}{m}}, \qquad \gamma = \frac{b}{m}$$
$$x(0) = x_0, \qquad v(0) = v_0.$$

Eq. (4.20) represents a 2nd-order ordinary differential equation (ODE) with constant coefficients. Standard theory of ODEs proves that the general solution is written as a linear combination of two (independent) solutions as:

$$x(t) = A_1 x_1(t) + A_2 x_2(t) \tag{4.36}$$

with the constants A_1, A_2 determined through the values of the $x(t_0)$ and it's derivative $dx(t_0)/dt$ at a specified time t_0. Usually, and inn the present case we take them as the initial conditions of the position $x(0) = x_0$ and the velocity $v(0) = v_0$ at time $t_0 = 0$ (for simplicity).

A standard procedure to calculate $x_1(t), x_2(t)$ is to seek for solutions of the form

$$x_i(t) = A_i e^{\sigma_i t}.$$

Substituting the above expression into Eq (4.20) we end up to the following

$$(\sigma_i^2 + \gamma\sigma_i + \omega_0^2)A_i e^{\sigma_i t} = 0 \quad \Longrightarrow \quad \sigma_i^2 + \gamma\sigma_i + \omega_0^2 = 0,$$

Heavy damping (overdamped): $\gamma/2 < \omega_0$: Since neither A_i nor the exponent can be zero (for finite values of time) we end up to the second-order algebraic equations, which in general provides two solution for the possible σ_i:

$$\sigma_i^2 + \gamma\sigma_i + \omega_0^2 = 0 \quad \Longrightarrow \quad \sigma_{1,2} = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right)$$
$$= -\frac{\gamma}{2} \pm \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2}, \qquad \frac{\gamma}{2} > \omega_0.$$

The above values of $\sigma_{1,2}$ are acceptable only in the case of $\gamma/2 > \omega_0$. In this case, we arrive to two solutions which are independent each other and the general solution Eq.(4.36) can now be re-written as,

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1 e^{at} + A_2 e^{-at}), \qquad a = \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2}, \qquad \frac{\gamma}{2} > \omega_0. \tag{4.37}$$

* Complex arithmetics is used in this section.

representing the *overdamped solution*.

Light damping (underdamped): $\gamma/2 < \omega_0$: In that case we work as follows: we start from the roots of the algebraic equation for the σ's, and rewrite them as:

$$
\begin{aligned}
\sigma_{1,2} &= -\frac{\gamma}{2} \pm \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2} = -\frac{\gamma}{2} \pm \sqrt{-[\omega_0^2 - (\frac{\gamma}{2})^2]} \\
&= -\frac{\gamma}{2} \pm \sqrt{\imath^2[\omega_0^2 - (\frac{\gamma}{2})^2]} = -\frac{\gamma}{2} \pm \imath\sqrt{\omega_0^2 - (\frac{\gamma}{2})^2},
\end{aligned}
$$

where going from the first line to the second we used the well known property of the imaginary unit $\imath^2 = -1$ and from the second to the third line we used $\sqrt{\imath^2} = \imath$. In this case we obtain the general solution of Eq. (4.36) as a linear combination of two complex exponentials ($e^{\pm\imath\omega t}$)

$$
x(t) = e^{-\frac{\gamma}{2}t}(A_1' e^{\imath\omega t} + A_2' e^{-\imath\omega t}), \qquad \omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}, \qquad \frac{\gamma}{2} < \omega_0. \tag{4.38}
$$

We can reexpress the above solution in terms of the real function $\sin\omega t$ and $\cos\omega t$ to obtain:

$$
\boxed{x(t) = e^{-\frac{\gamma}{2}t}(A_1 \cos\omega t + A_2 \sin\omega t), \quad \omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}, \quad \frac{\gamma}{2} < \omega_0.} \tag{4.39}
$$

To the above end one needs to utilize the other very well known property of a complex exponential $e^{\imath x} = \cos x + \imath \sin x$. This last step is left as an exercise to the reader.

Critical damping (underdamped): $\gamma/2 = \omega_0$: In this case we obtain only one root for the σ, namely $\sigma = -\gamma/2$ and thus with this method we can obtain only one from the required two (independent) solutions:

$$
x(t) = A_1 e^{-\frac{\gamma t}{2}} + x_2(t)
$$

In this case the second solution $x_2(t)$ should be found using a different method. In the present case, an educated guess is to seek for solutions of the type:

$$
x(t) = u(t)e^{-\gamma t/2}
$$

Substitution of this expression into the DHO equation results to $\ddot{u}(t) = 0$ and therefore $u(t) = ct$. So we can assume as the second (independent) solution $x_2(t) = A_2 t e^{-\frac{\gamma t}{2}}$ to arrive at:

$$
\boxed{x(t) = e^{-\frac{\gamma}{2}t}(A_1 + A_2 t), \qquad \omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}, \qquad \frac{\gamma}{2} = \omega_0.} \tag{4.40}
$$

4.6 Questions

Question 1. Assume the solution for the mass-spring system with initial conditions at $t_0 = x_0, v_0$:

$$x(t) = x_0 \sin \omega_0 t + \frac{v_0}{\omega_0} \cos \omega_0 t$$

Prove that for this SHO the total mechanical energy is a constant of motion:

$$E(t) = \frac{1}{2} k x^2(t) + \frac{1}{2} m v^2(t) = \textit{constant}. \tag{4.41}$$

What is the value of this constant if $x_0 = 0.5$ m $v_0 = 0.5$ m/sec and $k = 1$ N/m, $m = 1$ Kg.

Question 2. [42] Confirm that the displacement of an SHO in a mass-spring system can be also expressed as,

$$x(t) = A_0 \sin(\omega_0 t + \phi_0),$$

where the amplitude A_0 and the phase ϕ are given in terms of the initial position (x_0) and velocity (v_0) as,

$$A_0 = \sqrt{x_0^2 + (\frac{v_0}{\omega_0})^2}, \qquad \phi_0 = \tan^{-1}(\frac{x_0 \omega_0}{v_0})$$

(*hint*: start from the expression $A \cos \omega_0 t + B \sin \omega_0 t$ and utilize the identities for the $\cos(a+b) = \cos a \cos b - \sin a \sin b$ or $\sin(a+b) = \sin a \cos b + \sin b \cos a$ to arrive at the desired form.)

42: The purpose of the last two questions is to show that there are alternative expressions, though equivalent forms to represent the motion of a HO $x(t), v(t)$. These conclusions hold generally and do not apply only for the mass-spring system.

Question 3. A point-like object attached to a spring completes one oscillation every 2.4 sec. At $t = 0$ is released from rest at a distance $x_0 = 0.1$ m from it's equilibrium position.

(a) What is the eigenfrequency ω_0 of this simple harmonic oscillator system?

(b) what is the position of the object at time $t = 0.3$ sec after it's release?

(c) For the problem of question 1 what is it's acceleration at time $t = 0.3$ sec?

Question 4. Vertical mass-spring system. Assume the vertical mass-spring system where a massless spring of constant k, connected with a point-like mass of mass m is placed on a gravitational field and is hanged from the free side of the spring.

Prove that this system exhibits a simple harmonic motion. What is the period of the motion if $m = 2$ Kg and $k = 4$ N/m? Would the period of this spring differ if this system was placed on Moon?

Question 5. A mass-spring system with $m = 2$ Kgr and $k = 8$ N/m is perturbeb in the following manner: At initial time $t = 0$ it is displaced by $x_0 = \sqrt{2}$ m from it's equilibrium position and it is kicked with initial velocity $v_0 = -2$ m/s. Which one from the expressions below gives the position as a function of time?

(a) What is the velocity at time $t = 1.26$ sec?

(b) What is the total mechanical energy at time $t = 0.12$ sec?

Question 6. A pendulum completes a full oscillation in 4 seconds on earth. What is the corresponding time if we instead place the pendulum on the moon?

Question 7.

Let's define as *ideal sping-mass* the idealized model where a point mass $m > 0$ is attached to a massless spring characterized by it's restoring strength through the constant $k > 0$. Assume that the general solution for position of the mass as a function of time of this harmonic oscillator is given by

$$x(t) = A_0 \sin(\omega_0 t + \phi), \tag{4.42}$$

with A_0 being the amplitude (the maximum displacement of the mass) of the motion, ω_0 its eigenfrequency and ϕ its phase angle.

(a) Assuming that at initial time $t = 0$, the displacement of the mass is x_0 and its velocity is v_0 provide the amplitude of the motion A_0, and the phase angle ϕ in terms of x_0 and v_0 as:

$$A_0 = \sqrt{x_0^2 + (v_0/\omega_0)^2}, \qquad \tan\phi = \frac{\omega_0 x_0}{v_0} \tag{4.43}$$

(b) What is the expression for the velocity of the mass as a function of time?

Question 8.

A mass (0.5 Kgr) attached to a massless spring (k= 200 N/m) is released from rest at distance 20 cm from its equilibrium position

(a) Making use of the initial conditions determine the exact form for the position of the mass as a function of time

(b) Find the maximum and minimum speed of the mass

(c) What is the maximum acceleration that the mass achieves?

(d) What is the time that the mass is halfway to the center from its original position. Having determined the time, find the velocity, acceleration, energy, kinetic and potential energy of the mass at this instant of its motion

Question 9. Assume a *simple pendulum* consisting of a massless, unstretchable string of length $l = 0.25$ meters. Given that pendulum is located at a place where the gravity acceleration is equal to $g = 9.81$ m/s^2:

(a) If initially the mass pulled to an angle $\theta_0 = 3.6^o$ degrees and released with no initial speed, give the angle of the pendulum as a function of time ($\theta = \theta(t)$).

(b) What is the period of oscillation of this pendulum?

(c) What the length of the simple pendulum should be, if we wanted to define the time interval of 1 sec by a full oscillation of it?

(d) What is the circular frequency in Hz in this case?

Question 10. Starting from the standard solution for the position $x(t) = A_1 \cos \omega t + A_2 \sin \omega t$ show that it can be rewritten as:

$$x(t) = A_0 e^{-\gamma t/2} \cos(\omega t + \phi)$$

Give A_0 and ϕ in terms of A_1 and A_2. (hint: express A_1 and A_2 in a terms of A_0 ϕ in a similar way as shown in the application section (where a similar problem is solved).

Question 11.

Starting from the expression $E = mv^2/2 + kx^2/2$ for the damped mass-spring system (damping parameter γ) show that the energy decreases as:

$$\frac{d}{dt}E(t) = -\gamma m v^2. \tag{4.44}$$

The fact that the above expression can only be negative allows to conclude that for a DHO there is net energy loss from the DHO system to its environment. In the mass-spring system the energy is transformed to heat due to the friction (contact) forces.

(Hint: Take the first derivative in time of the mechanical energy given above, $E(t)$, and then replace the derived term that includes $-kx$ using the DHO equation ($m\ddot{v} = -kx(t) - bv(t)$).)

Question 12. For the spring-mass system of example 1, determine the motion and derive the relevant plots for $x(t)$ and $v(t)$ when the damping constant $b = 4$ Kgr/sec.

Question 13. Spring-mass system 2 For the spring-mass system of example 1, determine the motion and derive the relevant plots for $x(t)$ and $v(t)$ when the damping constant $b = 2\sqrt{2}$ Kgr/sec.

Question 14. Spring-mass system 3 An object of mass (0.5 Kg) attached to a massless spring (k= 200 N/m) with a damping parameter $b = 10$ at initial time $t_0 = 0$ is found at its equilibrium position with velocity $v(0) = 2$ m/sec.

(a) Making use of the initial conditions calculate the functional dependance of the position of the mass on time.

(b) Find the maximum and minimum speed of the mass.

(c) What is the maximum acceleration achieved by the mass?

5 Lagrangian Mechanics

In this section it is introduced an alternative description of the dynamics of mechanical systems, due to a number of people, including the names of Leibniz, Bernoulli, Euler, Lagrange etc.. to mention a few. This method it has been proven practically considerably more useful than Newton's method (especially for more complicated mechanical systems). Most importantly, the concepts introduced and worked out here are directly extensible to other domains of physics, not strictly restricted to mechanical systems. A very brief exposition, without digging the deeper principles is presented below.

5.1 Lagrangian method - 1-D

Before we start, it is convenient to represent the space variables not by the conventional x variable but by q as a reminder that the dependence variable can be of any nature (one-dimensional space, angle, radial distance, r, cylindrical distance, or some combination of them). In this case we define the *generalized coordinate* q and its corresponding *generalized velocity*, by, [44]:

$$q, \qquad \dot{q} = \frac{dq}{dt}. \tag{5.1}$$

44: Again, for example:

$(x, \dot{x}) \to (q, \dot{q}$ or
$(\theta, \dot{\theta}) \to (q, \dot{q})$

Having defined the proper (generalized) coordinates to describe the particle's motion a quantity we first construct the system's **Lagrangian** which is the difference between its kinetic and potential energy:

Lagrangian

$$\mathscr{L}(q, \dot{q}, t) = T(q, \dot{q}) - V(q) = \frac{1}{2}m\dot{q}^2 - V(q), \quad \textbf{Lagrangian} \tag{5.2}$$

Then one forms the system's time-integral between two time points, t_1 and t_2. This called *action* (integral):

Action integral

$$S[q, \dot{q}] = \int_{t_1}^{t_2} dt \mathscr{L}(q(t), \dot{q}(t); t). \tag{5.3}$$

By use of calculus-of-variations the stationary point of its time-integral one may derive the system's equation(s) of motion (EOM):

Principle of least-action (or Hamilton principle)

The path of a mechanical system is the stationary functional point of the system's *action*:

$$\delta S[q, \dot{q}] = 0, \qquad \delta q(t_1) = \delta q(t_2) = 0. \quad \textbf{Hamilton's principle} \tag{5.4}$$

The above principle results to the following E-L EOMs:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) - \frac{\partial \mathscr{L}}{\partial q} = 0, \qquad \textbf{Euler–Lagrange equations} \tag{5.5}$$

Before to proceed to specific examples demonstrating this method let's see whether the E-L equations are consistent with the Newton's 2nd law.

First note that, if the kinetic energy depends only on the velocities, $\dot{q}$ then,

$$\frac{\partial \mathscr{L}}{\partial q} = \frac{\partial}{\partial q}(T(\dot{q}) - V(q)) = 0 - \frac{\partial V}{\partial q} = F(q).$$

Lagrange	Newton	Generalized
$\dfrac{\partial \mathcal{L}}{\partial \dot{q}}$	$m\dot{q} = mv = p$	Momentum
$\dfrac{\partial \mathcal{L}}{\partial q}$	$-\dfrac{\partial V}{\partial q} = F$	Force

Table 5.1: A comparison of the central quantities in the Euler-Lagrange and Newton's theories for a mechanical system

The latter component is called the *generalized force* 'acting' on the particle [45]. Accordingly,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} (T(\dot{q}) - V(q)) = m\dot{q} - 0 = p$$

is equal to mass times its (generalized) velocity, also known as its **generalized momentum**. With these definitions the Euler-Lagrange equations say simply that,

$$\frac{dp}{dt} = F.$$

The time derivative of momentum is force, which is the Newton's 2nd law!

45: It coincides with the usual force only when $q = x$. When, for example, $q = \theta$ then the generalized force corresponds to the torque.

Extension to more dimensions One of the strengths of the Lagrangian mechanics it is the easiness of its generalization. If the Lagrangian contains more than one position variables (e.g. a particle moving in 2,3 dimensions or two (or more) particles in 1-dimension (or more)) then the corresponding Euler-Lagrange equations are separable in nature. We can obtain the EOMs for each of the position variables by applying (5.7) separately. More specifically, if we have $q_i(t)$ position variables (e.g. x, y, z or (r, θ, ϕ) or $\mathbf{r}, \theta, z$) and the corresponding velocities, $\dot{q}_i = dq_i/dt$ then the Lagrangian in general will have the form:

Lagrangian for several variables

$$\begin{aligned} \mathcal{L}(q_1, q_2, \cdots, \dot{q}_1, \dot{q}_2, \cdots, t) &= T(\dot{q}_1, \dot{q}_2, \cdots) - V(q_1, q_2, \cdots) \qquad (5.6)\\ &= \sum_{i=1,2,} \frac{1}{2} m\dot{q}_i^2 - V(q_1, q_2, \cdots), \quad \textbf{Lagrangian} \end{aligned}$$

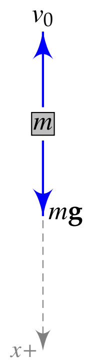
In this case of a multivariable Lagrangian the equation(s) of motion (EOMs) are derived by,

Euler-Lagrangian EOMs

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}, \qquad i = 1, 2, \cdots \qquad (5.7)$$

5.2 Examples

Example 5.2.1 (Free-fall body) *Our purpose here to derive the EOMs using the E-L equations. We need to construct the Lagrangian of the system, which requires an expression for its kinetic and potential energy.*



(i) First a suitable coordinate system to describe the motion needs to be set. For this, a vertical 1-dimensional axis (x) is defined with its positive axis pointing downwards.

(ii) In this case, the particle's position is $x(t)$, its velocity is $v(t) = \dot{x}$. Then its kinetic energy is $T = mv^2/2$. The potential energy is, when its position is $x(t)$ above Earth's ground,

$$V(x) = V(0) - \int_0^x dx(mgx) = -mgx, \qquad V(0) = 0.$$

(iii) The particle's Lagrangian is,

$$\mathscr{L}(x, \dot{x}, t) = T(\dot{x}) - V(x) = \frac{1}{2}m\dot{x}^2 + mgx \tag{5.8}$$

(iv) We need the partial derivatives of $\mathscr{L}$. For, $q = x$ we have,

$$\frac{\partial}{\partial \dot{x}}\mathscr{L} = m\dot{x} + 0 = m\dot{x}, \qquad \frac{\partial}{\partial x}\mathscr{L} = 0 + mg = mg$$

(v) Then applying the E-L equations we find:

$$\frac{d}{dt}(m\dot{x}) = mg \qquad \longrightarrow \qquad \ddot{x} = g$$

Example 5.2.2 (The Atwood Machine) *Consider a frictionless cylindrical pulley (of radius R) with two masses, m_1 and m_2, hanged with a string of length L:*

The task her is to quantify the motion of the masses. To this end, we assume a coordinate system placed horizontally at the center of the pulley, with its y–axis pointing downwards. Since the pulley is a cylinder of radius R we may represent the masses position by:

$$m_1: \qquad \mathbf{r}_1(t) = (x_1, y_1, z_1) = (-R, y_1(t), 0)$$

and

$$m_2: \qquad \mathbf{r}_2(t) = (x_2, y_2, z_2) = (R, y_2(t), 0)$$

The kinetic energy is

$$T(\dot{y}_1, \dot{y}_2) = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2,$$

whereas the potential energy is given by,

$$T(\dot{y}_1, \dot{y}_2) = -m_1 g y_1 - m_2 g y_2$$

since the two masses are moving under the constant gravity field of acceleration, $\mathbf{g} = g\hat{y}$.

So we have for the Lagrangian:

$$\begin{aligned} \mathcal{L}(y_1, y_2, \dot{y}_1, \dot{y}_2) &= T(\dot{y}_1, \dot{y}_2) - T(\dot{y}_1, \dot{y}_2) \\ &= \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + m_1 g y_1 + m_2 g y_2 \end{aligned} \tag{5.9}$$

An important note here is that the position variables y_1 and y_2, are constrained by $y_1(t) + y_2(t) + \pi R = L$[46].

$$y_1(t) + y_2(t) = \ell, \qquad \ell = L - \pi R = \text{const.}$$

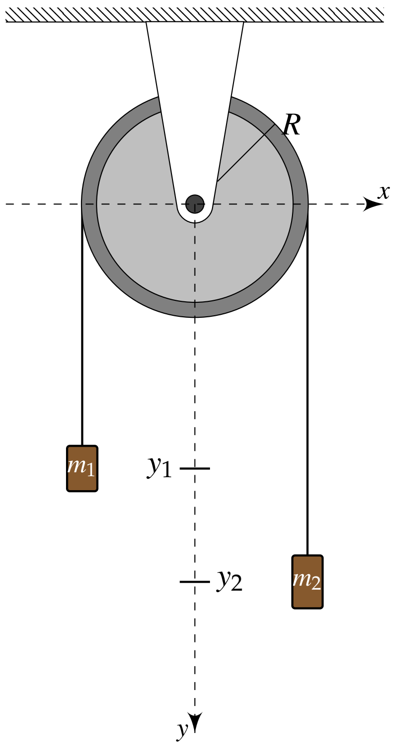


Figure 5.1: Pulley with two masses, m_1 and m_2. The string has a length L.

46: πR is the half circle arc length of the pulley.

At this point we have two different ways to proceed. Either, we express everything in terms of one of the coordinates (say $y_1(t)$), determine the EOMs and solve for $y_1(t)$ and then use the latter equation to find $y_2(t)$, or we continue like the two coordinates are independent each other and have a two-coordinates problem to find the EOMs and solve for $y_1(t), y_2(t)$.

The former approach is certainly simpler as it is 1-dimensional problem, while the second one is slightly more complicated (but not that much because the constraint $y_1(t) + y_2(t) = \ell$ can always be employed at any stage of the solution.

Set: $y_2(t) = \ell - y_1(t)$

(i) *Coordinates* All the involved quantities below will be written in terms of the mass '1' position and speed $y_1(t), \dot{y}_1(t)$:

(ii) *Lagrangian:*
Noting that,

$$\dot{y}_2 = 0 - \dot{y}_1 \quad \longrightarrow \quad \dot{y}_2^2 = \dot{y}_1^2$$

The kinetic and potential energy for the system become:

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_1^2 = \frac{1}{2}(m_1+m_2)\dot{y}_1^2$$
$$V = -m_1 g y_1 - m_2 g(\ell - y_1) = -(m_1 - m_2)g y_1 - m_2 g\ell$$

Then the Lagrangian is given by:

$$\boxed{\mathscr{L}(y_1, \dot{y}_1) = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)g y_1 + m_2 g\ell} \tag{5.10}$$

The last expression for the (reduced) Lagrangian is the working one that we'll be based on for the derivation of the E-L EOMs for the system. In fact, we'll produce for the position variable, y_1 and then utilizing the relation $y_1 + y_2 = \ell$ will derive the EOMs for the y_2.

(iii) E-L equations
To employ the E-L equations we calculate the partial derivatives of $\mathscr{L}$ [47]:

$$\frac{\partial}{\partial \dot{y}_1}\mathscr{L} = (m_1+m_2)\dot{y}_1, \qquad \frac{\partial}{\partial y_1}\mathscr{L} = (m_1-m_2)g$$

Then the E-L equations give:

$$\frac{d}{dt}((m_1+m_2)\dot{y}_1) = (m_1-m_2)g \quad \longrightarrow \quad \ddot{y}_1(t) = \frac{m_1-m_2}{m_1+m_2}g$$

So this describes a falling object in a downwards gravitational acceleration

$$a = \left(\frac{m_1-m_2}{m_1+m_2}\right)g$$

Finally by using $y_1(t) + y_2(t) = \ell$ we end up to have determined the acceleration for both the masses:

$$\ddot{y}_1(t) = \frac{m_1-m_2}{m_1+m_2}g, \tag{5.11}$$
$$\ddot{y}_2(t) = \frac{m_2-m_1}{m_1+m_2}g. \tag{5.12}$$

It is straightforward to integrate the expression for $\ddot{y}_1(t)$ twice (with the proper initial conditions) to obtain the complete solution to the motion $y_1(t)$ and $\dot{y}_2(t)$ [48].

47: Note the following special cases:

$$m_1 = m_2 \quad \rightarrow \quad \ddot{y}_1 = 0$$

and

$$m_2 = 0 \quad \rightarrow \quad \ddot{y}_1 = g$$

as expected.

48: You can do it yourself for example when,

$$y_1(0) = 0, \qquad \dot{y}_1(0) = 0$$

Example 5.2.3 (Simple horizontal mass-spring system 1-D) *Assume an horizontal mass-spring system, with no-friction, characterized by the mass m and the spring's constant, k. We want to derive the EOM for this system. We expect this to be that of the harmonic oscillator.*

We need to construct the Lagrangian of the system, which requires an expression for its kinetic and potential energy.

(i) First a suitable coordinate system to describe the motion needs to be set. For this, an horizontal 1-dimensional axis (x) with its origin placed at the position where the spring has its natural length.

(ii) As usual, the particle's position is $x(t)$, its velocity is $v(t) = \dot{x}$. Then its kinetic energy is $T = mv^2/2$. The potential energy is, when its position is $x(t)$ above Earth's ground,

$$V(x) = \frac{1}{2}kx^2$$

(iii) The particle's Lagrangian is,

$$\mathscr{L}(x, \dot{x}, t) = T(\dot{x}) - V(x) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

(iv) We need the partial derivatives of $\mathscr{L}$. For, $q = x$ we have,

$$\frac{\partial}{\partial \dot{x}}\mathscr{L} = m\dot{x} + 0 = m\dot{x}, \qquad \frac{\partial}{\partial x}\mathscr{L} = 0 - kx = -kx$$

(v) Then applying the E-L equations we find:

$$\frac{d}{dt}(m\dot{x}) = -kx \qquad \rightarrow \qquad \ddot{x} + \frac{k}{m}x = 0.$$

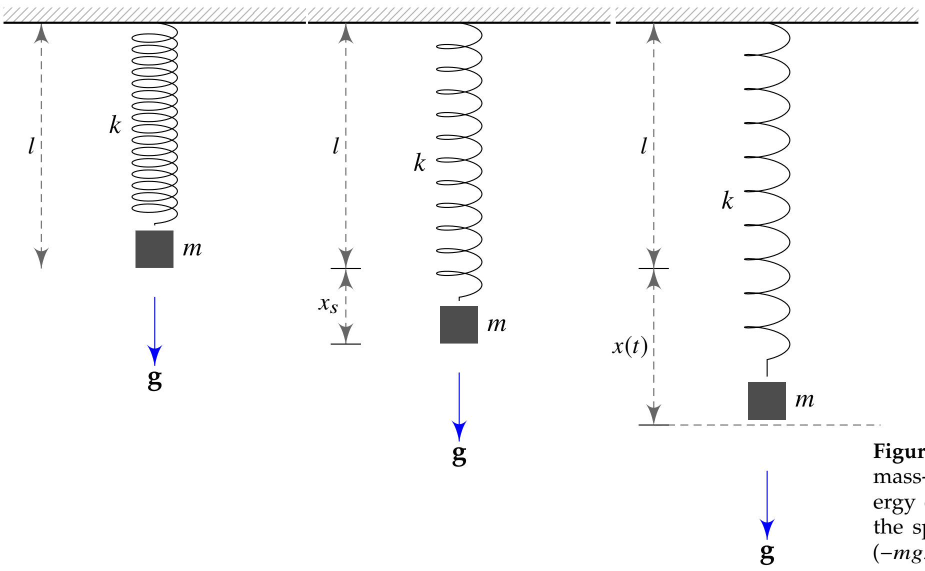


Figure 5.2: The case of a vertical mass-spring system. The potential energy of the mass-spring is the sum of the spring's ($kx^2/2$) and the gravity's ($-mgx$) potential energy.

Example 5.2.4 (Simple vertical mass-spring system 1-D) *Assume a vertical mass-spring system, in the Earth's gravity field, characterized by the mass m, the spring's constant, k and the acceleration g. At time $t = 0$ the mass is at rest and is kicked-off downwards. We want to find the EOM.*

The new feature in this problem, relative to the corresponding horizontal mass-spring problem is that the mass has an additional potential energy due to the Earth's gravitational field. So the total potential energy will be the sum of the energy due to the interaction with the spring and the Earth's field:

$$V = V_{ho} + V_{gravity}.$$

Derivation of the EOM: Now we proceed to construct the system's Lagrangian by expressing the kinetic and potential energy in suitable position and velocity variables. The system's motion is entirely restricted in 1-dimension (vertical) along the direction of the Earth's gravity direction. So,

A vertical 1-dimensional axis (x) is defined with its positive axis pointing downwards. It's origin is placed at the lower side of the spring (with the spring having its natural length). Furthermore it is convenient to set a common reference point for the spring's and the Earth's potential energy. We'll set this reference point (where the potential energy is zero at the origin of the chosen coordinate system).

Kinetic energy:

As usual, the particle's position is $x(t)$, its velocity is $v(t) = \dot{x}$. Then its kinetic energy is

$$T(\dot{x}) = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2.$$

Potential energy:

Then the potential energy is, when its position is $x(t)$ away from the origin we have [49] ,

$$V(x) = \frac{1}{2}kx^2 - mgx$$

49: Note that,

$$F(x) = -\frac{dV}{dx} = -kx + mg$$

Lagrangian: Now the Lagrangian can be formed: The particle's Lagrangian is,

$$\mathscr{L}(x, \dot{x}, t) = T(\dot{x}) - V(x) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + mgx$$

Euler-Lagrange equations and EOM: We need the partial derivatives of $\mathscr{L}$. For, $q = x$ we have,

$$\frac{\partial}{\partial \dot{x}}\mathscr{L} = m\dot{x} + 0 = m\dot{x}, \qquad \frac{\partial}{\partial x}\mathscr{L} = 0 - kx + mg = -kx + mg$$

Then applying the E-L equations we find:

$$\frac{d}{dt}(m\dot{x}) = -kx + mg \qquad \rightarrow \qquad m\ddot{x} + kx = mg.$$

Dividing by m we obtain the EOM for an HO, albeit with a non-homogeneous term:

$$\ddot{x} + \omega_0^2 x = g, \qquad \omega_0 = \sqrt{\frac{k}{m}} \tag{5.13}$$

Solution of the EOM: [50] It's not very difficult to guess the general solution:

$$x(t) = \frac{g}{\omega_0^2} + A\cos\omega_0 t + B\sin\omega_0 t. \tag{5.14}$$

50: The solution of a non-homogeneous 2nd-order differential equation of the type:

$$y''(t) + py'(t) + qy(t) = f(t)$$

is the sum of the general solution $y_0(x)$ of the corresponding homogeneous problem:

$$y_h''(t) + py_h'(t) + qy_h(t) = 0$$

and a special solution $y_s(x)$:

$$y(t) = y_s(t) + y_h(t)$$

In the present case we have $p = 0, q = \omega_0^2$ and $f(t) = g$. The special solution is $y_s(t) = g/\omega_0^2$ and $y_h(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

The last two terms represent the solution for the simple HO (horizontal mass-spring system), say, $x_h(t)$:

$$\ddot{x}_h(t) + \omega_0^2 x_h(t) = 0.$$

So if the solution of (6.23) is represented as, $x(t) = a + x_h(t)$, where a is a constant, we have,

$$\frac{d^2}{dt^2}(a + x_h) + \omega_0^2(a + x_h) = g,$$

immediately results to $a(t) = g/\omega_0^2$.

The constants, A and B are determined from the initial conditions. Since the mass is at rest initially, this means that the mass has the position determined by the special solution $a = g/\omega_0^2$, or:

$$x(0) = \frac{mg}{k}.$$

Substituting this value to (5.14) we obtain,

$$\frac{mg}{k} = \frac{mg}{k} + A \qquad \longrightarrow \qquad A = 0.$$

The B constant is calculated by considering the initial velocity $v(0) = v_0$. From the general solution (5.14) we have,

$$v(t) = B\omega_0 \cos \omega_0 t.$$

Therefore:

$$v_0 = B\omega \qquad \longrightarrow \qquad B = \frac{v_0}{\omega}.$$

Summarizing the solution for the given initial conditions is found to be

$$x(t) = \frac{g}{\omega_0^2} + \frac{v_0}{\omega_0} \sin \omega_0 t$$
$$v(t) = v_0 \cos \omega_0 t$$

Example 5.2.5 (Simple pendulum) The task of this example is to derive the pendulum equation using the Euler-Lagrangian's methodology. Again, the most important part of the solution consists of constructing a suitable Lagrangian for the system. As usual, we choose first (a) a convenient coordinate system (b) we write down the kinetic and potential energy in terms of these coordinates and finally (c) we take the E-L equations in order to find the EOMs:

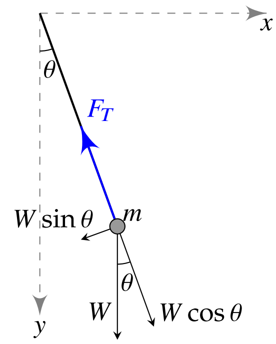


Figure 5.3: Idealized simple pendulum sketch. The point mass moves under the weight force, $W = mg$. At the bottom figure the coordinates axes are shown explicitly

Coordinates. Looks like the polar coordinate system is the proper one (the motion takes place in a plane normal to the ground). Then we choose the plane such that the z−axis is normal to the pendulum's plane. In this case in the below we may ignore the z−axis since $z = 0$. Therefore we have,

$$x = L\sin\theta, \qquad y = L\cos\theta.$$

An important simplification it has already taken place here since it is assumed that the distance of the mass is equal to $r = L$ (or equivalently $\dot{r} = 0$). This would mean,

$$\dot{x} = L\dot{\theta}\cos\theta, \qquad \dot{y} = -L\dot{\theta}\sin\theta$$

Kinetic and potential energy,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mL^2\dot{\theta}^2$$

For the potential energy we have,

$$V(y) = mgy = mgL(1 - \cos\theta),$$

where the zero-potential energy level is at the mass's lowest point ($\theta = 0$).

Lagrangian The system's Lagrangian now is given by,

$$\mathscr{L}(\theta, \dot{\theta}, t) = T - V = \frac{1}{2}mL^2\dot{\theta}^2 - mgL(1 - \cos\theta)$$

Partial derivatives and E-L equations:

$$\frac{\partial}{\partial\dot{\theta}}\mathscr{L} = mL^2\dot{\theta} - 0 = mL^2\dot{\theta} \qquad \frac{\partial}{\partial\theta}\mathscr{L} = 0 - mgL(0 + \dot{\theta}\sin\theta) = -mgL\dot{\theta}\sin\theta$$

By applying the E-L equations we find:

$$\frac{d}{dt}(mL^2\dot{\theta}) = -mgL\dot{\theta}\sin\theta$$
$$\rightarrow \qquad \ddot{\theta} + \frac{g}{L}\sin\theta = 0, \qquad \text{pendulum equation!}$$

The last equation is the same as the one we found using the Newtonian method (4.13). From this point on the effort is on the solution of this differential equation. Everything that was discussed at the relevant chapter apply fully (e.g. the small-angle approximation).

5.3 Questions

Question 1. Projectile motion Using the Lagrangian approach derive the EOMs for the problem of projectile motion, where a mass is moving under the Earth's gravitational field. (See figure (3.1)).

Question 2. Equation of motion for central fields

Assume an object of mass m having a potential energy which is spherically symmetric in polar coordinate system, namely $V = V(r)$.

(a) Apply the Lagrangian method in a polar coordinate system (ignore the $z-$coordinate) and derive the EOMs for the r, θ position variables and show that

$$m\ddot{r} = -\frac{dV}{dr} + \frac{L^2}{mr^3}, \qquad mr^2\dot{\theta} = \text{constant},$$

(b) Starting from the definition of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ show that the constant in the latter equation is the magnitude of $\mathbf{L}$.

Question 3. Isotropic harmonic oscillator

Assume a point-like object of mass m attached to a spring constant k, free to move on a plane (2-dimensional). Assuming a coordinate system where the $x - y$ plane resides on the plane and ignoring the $z-$axis the system is isotropical on the plane in the sense that its potential energy is given by

$$V(r) = \frac{1}{2}k(x^2 + y^2) = \frac{1}{2}kr^2.$$

where $r^2 = x^2 + y^2$.

(a) Using the Lagrangian method provide the EOM for the mass. You may use either the polar or the Cartesian CS to solve the problem.

(b) Calculate the force exerted on the mass from the spring. Apply the Newton's 2nd law and show that you arrive at the same EOM coming from the Lagrangian method

Hamiltonian Mechanics | 6

On yet an alternative method of the dynamics of mechanical systems is introduced in this section; this particular method, bearing the name of Hamilton, has been proven to be of profound significance for the development of the modern physical theories including statistical physics, chaos theory, quantum mechanics and quantum field theory. It's characteristic feature is its direct link with the energy of the system, which in the case of isolated systems represents a constant of the motion. From mathematical viewpoint it deviates from both the Newtonian and Lagrangian methods in that it derives EOMs which are first-order differential equations in time, albeit double in number.

6.1 Hamiltonian method in a nutshell

In order to introduce the method let's recall the simple HO problem in the form of a mass-spring system. So, let's assume an horizontal, frictionless, mass-spring system characterized by the constants m, k. The simple HO can be worked out either via its force or the potential energy (which we now know that they are equivalent descriptions):

$$F(x) = -kx = -\frac{dV(x)}{dx}, \qquad V(x) = \frac{1}{2}kx^2.$$

Once more you should bear in mind that the ultimate question in mechanics is the determination of the mass position in time, namely, on mathematical terms the function of $x = x(t)$. Newton's and Euler-Lagrange's methods derive the EOM's of $x(t)$ which are 2nd-order ODEs with the special solution determined by the initial conditions for $x(0), \dot{x}(0) = v(0)$.

In both last cases a certain relation (in the form of ODES) between the position, $x(t)(t)$ and its velocity, $\dot{x}(t)$ is established:

$$\frac{d}{dt}\dot{x}(t) = F(x,t),$$
$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}(x,\dot{x},t)}{\partial \dot{x}}\right) = \frac{\partial \mathscr{L}(x,\dot{x},t)}{\partial x}$$

As has been shown in the past chapters both they end up to the well-known HO equation

$$\ddot{x} + \omega^2 x(t) = 0, \qquad \omega_0^2 = \frac{k}{m}.$$

Here, we'll be deriving a new set of EOMs for the mass-spring system, but not via relations that link position with velocities for every instant but between *generalized* position and momentum, x, p, defined via the system's Lagrangian. As we'll see in the present particular case the generalized momentum coincides with the conventional momentum ($p = m\dot{x}$), given that the Lagrangian has been shown to be:

$$\mathscr{L}(x,\dot{x},t) = T(\dot{x}) - V(x) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Then the *generalized* momentum is defined to be the partial derivative of the Lagrangian over the 'velocity', $\dot{x}$:

$$p = \frac{\partial \mathscr{L}(x,\dot{x},t)}{\partial \dot{x}} = m\dot{x}.$$

The Hamiltonian approach for the simple HO. In order to form the Lagrangian we take the difference between the kinetic and potential energy of the system. Instead, let's now take the sum of these two energies:

$$\mathscr{H}(x,\dot{x},t) = T(\dot{x}) + V(x) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2$$

The above expression is still expressed in terms of x and $\dot{x}$; however in view of the simple relation, $p = m\dot{x}$ we can re-express it in terms of x and p instead. So, simple manipulations results to the following:

$$\mathcal{H}(x, p) = T(p) + V(x) = \frac{p^2}{2m} + \frac{1}{2}kx^2, \tag{6.1}$$

Now let's take the partial derivatives of the above Hamiltonian in terms of x, p (in Lagrangian, we had to take partial derivatives of the Lagrangian over $x, \dot{x}$ instead). Quite straightforwardly we find,

$$\frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}, \qquad \frac{\partial \mathcal{H}}{\partial x} = kx$$

Look's like that we have ended up to nothing as what we actually want is to establish relations between, x, p which is not the case here. We notice, however, for the first of these equations that since $p = m\dot{x}$ we may rewrite it as,

$$p = m\dot{x} \qquad \rightarrow \qquad \boxed{\frac{\partial \mathcal{H}}{\partial p} = \dot{x}}.$$

and a relation between, x, p (via $\mathcal{H}(x, p)$ is established. For the second one we need to rely on the results obtained via the Newtonian or Lagrangian mechanics. We know that

$$kx = -F = \frac{dp}{dt} = \dot{p} \qquad \rightarrow \qquad \boxed{-\frac{\partial \mathcal{H}}{\partial x} = \dot{p}}.$$

Summarizing what we have found is the below,

> The EOMs of the SHO may be derived by the sum of its kinetic and potential energy (named Hamiltonian) which depends on the mass's position, x and generalized momentum, p as:
>
> $$\mathcal{H}(x, p) = T(p) + V(x) \qquad \rightarrow \qquad \dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \dot{x}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x}$$

Thus we have succeeded in formulating a method where a certain function of x, p is capable of deriving the EOMs for the simple HO. These equations now instead of one of second order it 'splits' to 2 of first-order:

$$\ddot{x} + \omega_0^2 x = 0, \qquad \leftrightarrow \qquad \dot{x} = \frac{p}{m}, \qquad \dot{p} = -kx$$

Question now is what's new? What's is the point working this way? Apart from the standard argument is that the Newton's method is based on false assumptions (albeit approximately working fine) there are also practical issues for more complex mechanical systems (multi-particle, multidimensional, etc); one of the remarkable features of the Hamilton's method is that is applicable to any mechanical systems [52] and for any number and kind of variables and particles.

52: And even more, applies generally to dynamical systems beyond the realm of the classical mechanics.

6.2 Hamiltonian formulation of mechanics

As in the case of the Euler-Lagrangian method, it is convenient to represent the space variables by $q(t)$ [53]. Accordingly an associated *generalized momentum* is defined, via the system's Lagrangian:

53: With the agreement that it can represents any kind of position variable in one-dimensional or multidimensional space, angle, radial distance, r, cylindrical distance, or any combination of them

$$\mathscr{L}(q,\dot{q},t) \equiv T(q,\dot{q}) - V(q) = \frac{1}{2}m\dot{q}^2 - V(q), \quad \textbf{Lagrangian} \tag{6.2}$$

by the following scheme:

$$q(t) \quad \rightarrow \quad \dot{q} = \frac{dq}{dt}, \quad \rightarrow \quad p(t) = \frac{\partial \mathscr{L}(q,\dot{q},t)}{\partial \dot{q}}. \tag{6.3}$$

Having defined the proper (generalized) variables (position,velocity,momentum), then the *Hamiltonian* function can be defined by the following way:

$$\mathscr{H}(q,\dot{q},t) \equiv p\dot{q} - \mathscr{L}(q,\dot{q},t) \tag{6.4}$$

Why one chooses the Hamiltonian this particular way (amongst infinite number of combinations)? The reason is that when the Lagrangian is not dependent explicitly in time, that is $\mathscr{L} = \mathscr{L}(q,\dot{q})$ then it turns out that the Hamiltonian it is not dependent explicitly on time and furthermore is a constant-of-motion equal to the sum of the potential and kinetic energy:

$$\mathscr{L}(q,\dot{q}) = T(q,\dot{q}) - V(q) \quad \rightarrow \quad \mathscr{H}(q,p) = T(q,\dot{q}) + V(q) = \text{const.} \tag{6.5}$$

We'll go and demonstrate the latter statement later; for now we'll state the Hamiltonian EOMs by,

$$\dot{q} = \frac{\partial \mathscr{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathscr{H}}{\partial q} \tag{6.6}$$

Extension to more dimensions Before to proceed to specific examples demonstrating this method we'll also formulate the Hamiltonian method for multidimensional problems (as usual either one-particle in 2 or 3 dimensions or a multiparticle system) As in the Lagrangian method, if we have $i = 1, 2, \cdots, N$ position variables $q_i(t)$ (e.g. x, y, z or (r, θ, ϕ) or $\mathbf{r}, \theta, z)$ and the corresponding velocities, $\dot{q}_i = dq_i/dt$ then the Lagrangian generalized momentum, p_i is defined via the multivariable Lagrangian (**??**):

$$\begin{aligned} \mathscr{L}(q_1, q_2, \cdots, \dot{q}_1, \dot{q}_2, \cdots, t) &= T(\dot{q}_1, \dot{q}_2, \cdots) - V(q_1, q_2, \cdots) \\ p_i &= \frac{\partial}{\partial \dot{q}_i}\mathscr{L}(q_1, q_2, \cdots, \dot{q}_1, \dot{q}_2, \cdots, t), \\ i &= 1, 2, \cdots, N \end{aligned}$$

For such multivariable system the Hamiltonian EOMs are derived then by,

$$\begin{aligned}\dot{q}_i &= \frac{\partial}{\partial p_i}\mathcal{H}(q_1, q_2, \cdots, p_1, p_2, \cdots),\\ \dot{p}_i &= -\frac{\partial}{\partial q_i}\mathcal{H}(q_1, q_2, \cdots, p_1, p_2, \cdots),\\ i &= 1, 2, \cdots, N\end{aligned}$$

Hamiltonian function and energy conservation. Let's now see how we can arrive at the Hamiltonian function via the Lagrange function. For simplicity we take the case of one variable and calculate the total differential of the Lagrangian, $\mathcal{L}(q, \dot{q}, t)$; Differentiation rules for multivariable functions gives,

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(q, \dot{q}, t) &= \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{dq}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial \mathcal{L}}{\partial t}\frac{dt}{dt}\\ &= \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial \mathcal{L}}{\partial t},\end{aligned} \tag{6.7}$$

We can rewrite the second term of the last equation using the partial rule for the derivatives,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d\dot{q}}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q}\right) - \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q}$$

The last term of the above equation cancels with the first term of the right-hand-side of (6.7) to give:

$$\frac{d}{dt}\mathcal{L}(q, \dot{q}, t) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q}\right) + \frac{\partial}{\partial t}\mathcal{L}(q, \dot{q}, t) = \frac{d}{dt}(p\dot{q}) + \frac{\partial}{\partial t}\mathcal{L}(q, \dot{q}, t)$$

where $\partial\mathcal{L}/\partial\dot{q}$ was replaced by the generalized momentum p. We then end up to

$$\frac{d}{dt}(p\dot{q} - \mathcal{L}) = -\frac{\partial}{\partial t}\mathcal{L}(q, \dot{q}, t) \tag{6.8}$$

Now, generally, we define the Hamiltonian of the system as,

$$\mathcal{H}(p, q, t) = p\dot{q} - \mathcal{L}(q, \dot{q}, t) \tag{6.9}$$

It is not difficult to show that in the case of a non-explicit time-dependent Lagrangian, $\mathcal{L} = \mathcal{L}(q, \dot{q})$ the Hamiltonian function, $\mathcal{H}$, becomes constant-of-motion and coincides with the system's *energy*, E.

Inspection of (6.8) says that the quantity $p\dot{q} - \mathcal{L}$ is a constant-of-motion if the Lagrangian is not explicitly dependent on time, that is,

$$\frac{\partial}{\partial t}\mathcal{L}(q, \dot{q}) = 0 \qquad \rightarrow \qquad p\dot{q} - \mathcal{L} = \text{const} \equiv E$$

In the case where the kinetic energy depends only on the velocity, $\dot{q}$, quadratically and non-explicit time dependence of the Lagrangian, that is,

$$\mathcal{L}(q, \dot{q}) = T(\dot{q}) - V(q) = \frac{1}{2}m\dot{q}^2 - V(q)$$

the product $\dot{q}p$ is (after substituting the generalized momentum, (6.3)),

$$\dot{q}p = \dot{q}\frac{\partial \mathcal{L}}{\partial q} = \dot{q}m\dot{q} = m\dot{q}^2 = 2T.$$

and eventually we have,

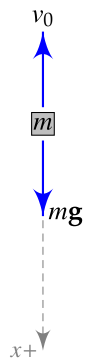
$$E = p\dot{q} - \mathcal{L} = 2T - (T - V) = T(\dot{q}) + V(q) = \text{const!}$$

The latter relation is nothing else than the expression for the familiar physical quantity known as *mechanical energy*. This coincides with the one derived using the Newtonian method.

Summarizing our findings we may say that:

> The Hamiltonian of a system is generally given by $\mathcal{H} = p\dot{q} - \mathcal{L}$. When there is not explicit time-dependence either in the Lagrangian or in the Hamiltonian then the Hamiltonian coincides with the energy of the system which is a constant-of-motion and it is equal to the sum of the potential and kinetic energy:
>
> $$\mathcal{H} = T(q, \dot{q}) + V(q) = \mathrm{E} = \text{const.}$$

6.3 Examples



Example 6.3.1 (Free-fall body) *Our purpose here to derive the EOMs using the Hamiltonian method. This problem was solved using the Lagrangian method and need not to repeat all the steps in detail here.*

(i) First, we need to construct the Lagrangian of the system in order to determine the generalized momentum. From Eq. (5.8) we have

$$\mathcal{L}(x, \dot{x}, t) = T(\dot{x}) - V(x) = \frac{1}{2}m\dot{x}^2 + mgx$$

Then the generalized momentum is,

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}.$$

The Lagrangian is not explicitly time-dependent therefore the Hamiltonian is the sum of the potential and kinetic energy [54] ,

54: You may check yourself that the formal definition of the Hamiltonian as, $H = p\dot{x} - L$ indeed gives $T + v$.

$$T(\dot{x}) + V(x) = \frac{1}{2}m\dot{x}^2 + mgx$$

But now we have to replace $\dot{x}$ with the momentum, $o = m\dot{x}$, to obtain,

$$\mathcal{H}(x, \dot{x}) = T(p) + V(x) = \frac{p^2}{2m} - mgx. \tag{6.10}$$

(ii) The 1st Hamiltonian EOMs is:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$

The 2nd Hamiltonian EOMs is:

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -(-mg) = mg$$

Gathering together the HEs we have:

$$\dot{x} = \frac{p}{m}, \qquad \text{(HE1)}, \qquad \dot{p} = mg \qquad \text{(HE2)} \tag{6.11}$$

We see from the above that the 1st HE (HE1) is just the definition of the relation between the particle's momentum and the velocity while the 2nd-one (HE2) is nothing else than the Newton's 2nd law (NE2):

$$\dot{p} = mg \qquad \longleftrightarrow \qquad \frac{dp}{dt} = F$$

The solution of the HEs is obtained following the same procedure as in the Newtonian case and initial conditions should be supplied for the momentum and position (equivalently for the velocity, since $p = m\dot{x} = mv$.

For example, from (HE2) we have,

$$p(t) = p(0) + mgt = p_0 + mgt \tag{6.12}$$

By substituting this value in (HE1) we have,

$$\dot{x} = \frac{p_0}{m} + gt = v_0 + gt \qquad \longrightarrow \qquad x(t) = x(0) + v_0 t + \frac{1}{2}gt^2. \tag{6.13}$$

The last equation is simply the well known solution for uniformly accelerating particle. It's derivative in time is[55]

$$v(t) = v_0 + gt$$

55: The below can also be obtained by (6.12) and $\dot{x} = v(t) = p(t)/m$

Example 6.3.2 (The Atwood Machine - The Hamiltonian method) Again we want to derive the EOMs for the system using the Hamiltonian approach where the generalized position and momentum are the relevant dynamic quantities to describe the system's dynamics. The reader at this point should revisit the corresponding problem which was treated within the Lagrangian method (see also Fig. (5.1). The problem is multivariable since the particles are two and as such the system's Hamiltonian should be expressed in terms of y_1, p_1, y_2, p_2 where p_1, p_2 are the corresponding generalized momenta:

$$\mathscr{H} = \mathscr{H}(y_1, y_2, p_1, p_2).$$

Nevertheless exploiting the relation $y_1(t) + y_2(t) = \ell$ it can be reduced to single-variable, y_1 as was demonstrated in the case of the Lagrangian method. So taking advantage of this reduction the Lagrangian was expressed as,

Then the Lagrangian is given by:

$$\mathscr{L}(y_1, \dot{y}_1) = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2)g y_1 + m_2 g \ell \tag{6.14}$$

The above Lagrangian describes a particle of mass $M = m_1 + m_2$ subject to a potential field $V(y_1) = -mgy_1 - m_2 g\ell$ where $M' = m_1 - m_2$[56]. The associated generalized momentum is (we denoted it with capital P_1 as a constant reminder that the particle of mass, M, described is fictitious):

$$P_1 = \frac{\partial}{\partial \dot{y}_1}\mathscr{L} = (m_1 + m_2)\dot{y}_1 = M\dot{y}_1, \tag{6.15}$$

In addition not explicit time-dependence in the Lagrangian is present, so we can write the Hamiltonian as the sum of the potential and the kinetic energy of this fictitious particle,

$$T(\dot{y}_1) + V(y_1) = \frac{1}{2}M\dot{y}_1^2 - M' g y_1 - m_2 g \ell$$

we also need to substitute the variable $\dot{y}_1$ in favor of P_1 using (6.15)[57]

$$\mathscr{H} = \frac{P_1^2}{2M} - M' g y_1 - m_2 g \ell \tag{6.16}$$

The Hamilton EOMs now are,

$$\dot{y}_1 = \frac{\partial \mathscr{H}}{\partial P_1} = \frac{P_1}{M} \qquad \dot{P}_1 = -\frac{\partial \mathscr{H}}{\partial y_1} = -(-M' g + 0) = M' g$$

Gathering together the above we have for the HEs:

$$\dot{y}_1 = \frac{P_1}{M} \quad \text{(HE1)}, \qquad \dot{P}_1 = M' g \quad \text{(HE2)} \tag{6.17}$$

By taking the derivative of (HE1), followed by use of (HE2) it's straightforward to show to obtain Eq. (5.11) for the acceleration of

56: Note that $M' = m_1 - m_2$ is not necessarily positive, as it's not actually represents the mass of any particle. It is simply a short hand notation.

57: For Cartesian coordinates the generalized momentum coincides with the familiar definition of momentum as the product of mass and velocity.

y_1. Finally, by use of $y_1(t) + y_2(t) = \ell$, Eq. (5.12) is also reproduced:

$$\ddot{y}_1 = \frac{\dot{P}_1}{M} = \frac{M'}{M} g = \frac{m_1 - m_2}{m_1 + m_2} g$$

We arrive to a set of EOMs for the two masses that are identical with those produce with the Lagrangian method (and no doubt with the Newtonian method).

One may stop here, since the original problem can be solved (that is calculation of $y_1(t), y_2(t)$). However, for completeness we must note the following:

Note added:

However, in the present case of the Hamiltonian method we need to derive the EOMs for the m_1 and m_2 mass in terms of y_1, y_2, p_1, p_2. To this we need to find the generalized momenta p_1, p_2. These now (quite quickly) can be derived via the original Lagrangian (**??**) to be,

$$p_i = \frac{\partial \mathcal{L}}{\dot{y}_i} = m_i \dot{y}_i, \qquad i = 1, 2. \tag{6.18}$$

Using the above and Eq. (6.15) we find that,

$$P_1 = \frac{M}{m_1} p_1$$

This will turn the (HE1) and (HE2) into the following EOMs in terms of y_1 and p_1:

$$\boxed{\dot{y}_1 = \frac{p_1}{m_1} \qquad \text{(HE1-1)}', \qquad \dot{p}_1 = m_1 \frac{M'}{M} g \qquad \text{(HE2-1)}'} \tag{6.19}$$

The corresponding EOMs for the mass m_2 can be produced by noticing that the original Lagrangian (5.9) is symmetric for the two masses provided the following exchanges:

$$(m_1, y_1) \quad \longleftrightarrow \quad (m_2, y_2)$$

So, having this property in mind and applied in $(HE1-1)'$ and $(HE2-1)'$ EOMs we obtain: we immediately arrive a

$$\boxed{\dot{y}_2 = \frac{p_2}{m_2} \qquad \text{(HE1-2)}', \qquad \dot{p}_2 = -m_2 \frac{M'}{M} g \qquad \text{(HE2-2)}'} \tag{6.20}$$

Summarizing the 4 HE EOMs, (HE1-1)′,(HE2-1)′,(HE1-2)′,(HE2-2)′ represent the Hamiltonian EOMs expressed in the dynamical variables y_1, y_2, p_1, p_2 are required by the Hamiltonian approach. The initial values for these dynamical variables must be supplied to find a specific solution.

Example 6.3.3 (Simple vertical mass-spring system 1-D) *Assume a vertical mass-spring system, in the Earth's gravity field, characterized by the mass m, the spring's constant, k and the acceleration g. At time $t = 0$ the mass is at rest and is kicked-off downwards (see also Fig. (5.2). We want to find the EOM with the Hamiltonian method.*

This problem has been treated with the Lagrangian method in the past chapter. So we can be quite quick and skip repetition. This system's Lagrangian is,

$$\mathscr{L}(x, \dot{x}) = T(\dot{x}) - V(x) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + mgx$$

which gives for the particle's generalized momentum:

$$p = \frac{\partial}{\partial \dot{x}} L(x, \dot{x}) = m\dot{x}.$$

There is not explicit time-dependence which means we can write for the system's Hamiltonian (also applying the required substitution $\dot{x} \to p$):

$$\mathscr{H}(x, p) = T(\dot{x}) + V(x) = \frac{p^2}{2m} + \frac{1}{2}kx^2 - mgx \tag{6.21}$$

$$\boxed{\dot{x} = \frac{p}{m}, \qquad \text{(HE1)}, \qquad \dot{p} = -kx + mg, \qquad \text{(HE2)}} \tag{6.22}$$

We have thus arrived at the Hamilton EOMs for this system. We can easily confirm that these end up to the same Euler-Lagrange EOMs (and Newton's equations). This is done as usual, that is by taking the derivative of (HE1) and using (HE2):

$$\ddot{x} = \frac{\dot{p}}{m} = \frac{-kx + mg}{m} \qquad \to \qquad \ddot{x} + \omega_0^2 x = g, \qquad \omega_0 = \sqrt{\frac{k}{m}} \tag{6.23}$$

Example 6.3.4 (Simple pendulum (Angle-dependent force)) The task of this example is to derive the pendulum equation using the Hamiltonian methodology. This problem has been treated in detail using both the Newtonian and the Lagrangian approach. So you should be advice the relevant discussion (also have a look at Fig. (5.3). Again, the most important part of the solution consists of constructing a suitable Lagrangian for the system, which has been done. The most convenient coordinates are the polar coordinates (r, θ), so one normally should formulate a Hamiltonian of the type:

$$\mathcal{H} = \mathcal{H}(r, \theta, p_r, p_\theta),$$

where p_r, p_θ are the generalized momenta for r and θ, respectively. However since the radial distance of the mass is constant ($r = L$) the number of position variables required can be reduced to one, namely, the θ angle. Following the same lines of thinking as in the Lagrangian method and using the reduced Lagrangian (in polar coordinates),

$$\mathcal{L}(\theta, \dot{\theta}) = T(\dot{\theta}) - V(\theta) = \frac{1}{2} mL^2 \dot{\theta}^2 - mgL(1 - \cos\theta)$$

we first find the generalized momentum for the angle variable, θ:

$$p_\theta = \frac{\partial \mathcal{L}(\theta, \dot{\theta})}{\partial \dot{\theta}} = mL^2 \dot{\theta}. \tag{6.24}$$

Again, as the Lagrangian is not explicit time-dependent, we may write down the Hamiltonian immediately (after having made the change from $\dot{\theta} \to p_\theta$:

$$\begin{aligned} \mathcal{H}(\theta, p_\theta) &= T(\dot{\theta}) + V(\theta) = \frac{1}{2} mL^2 \dot{\theta}^2 + mgL(1 - \cos\theta) \\ &= \frac{p_\theta^2}{2mL^2} + mgL(1 - \cos\theta) \end{aligned} \tag{6.25}$$

At this point we are ready to derive the Hamiltonian EOMs as,

$$\dot{\theta} = -\frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mL^2}, \qquad \dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = -mgL\sin\theta$$

$$\boxed{\dot{\theta} = \frac{p_\theta}{mL^2} \qquad \text{(HE1)}, \qquad \dot{p}_\theta = -mgL\sin\theta \qquad \text{(HE2)}.} \tag{6.26}$$

Once more the (HE1) represents the relation between the generalized momentum of θ and the θ variable while. We arrive at the EOMs produced via the Newton/Lagrangian methods by taking the time-derivative of (HE1) and then replace with (HE2):

$$\begin{aligned} &\ddot{\theta} = \frac{\dot{p}_\theta}{mL^2} = \frac{-mgL\sin\theta}{mL^2} = -\frac{g}{L}\sin\theta \\ &\ddot{\theta} + \frac{g}{L}\sin\theta = 0. \end{aligned}$$

Thus, we have confirmed that HEs arrive at the same EOMs we found using either the Newtonian method or the Lagrangian method. From this point on the effort is on the solution of this differential equation. Everything that was discussed at the relevant chapter apply fully (e.g. the small-angle approximation).

6.4 Questions

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58: *Hint: Note that* $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, *were* E *= const. Compare this with the mathematical expression for an ellipse* $x^2/a^2 + y^2/b^2 = 1$ *and proceed from there accordingly.*

Question 1. Simple Harmonic Oscillator and phase diagram

(a) For a simple one-dimensional HO of mass m and spring constant k, write down the Hamiltonian in terms of the x, p dynamical variables.

(b) Write down the EOMs and confirm that these lead to the Newton's 2nd law EOMs.

(c) Since the energy of the HO is constant-of-motion show that the plot of x versus p represents an ellipse.

The physical meaning of this plot is that the position of the mass, $x(t)$ and the velocity $v(t)$ are constrained in the sense at any given time, the point $P(t) = (x(t), p(t))$ on such plot is located on the corresponding ellipse and no where else.

Give the major and semi-major axes in terms of m, E, k. In what case the ellipse becomes a circle?

(d) Specialize the plot for for m = 1 kg, k =1 Nt/m and $x(0)$ = 1 m $p(0)$ = 2 kg m/s.

Question 2. Projectile motion

(a) Write the Hamiltonian for the projection motion of a particle in the Cartesian,(x, y) and polar coordinates, $((r, \theta)$.

(b) Write down (but do not solve) the corresponding Hamiltonian equations.

Question 3. Hamiltonian EOMs for central fields $V = V(r)$

Assume an object of mass m having a potential energy which is spherically symmetric, $V = -k/r,\ k > 0$.

(a) Apply the Lagrangian method in a polar coordinate system (ignore the z–coordinate) and write down in terms of $r, \theta, \dot{r}, \dot{\theta}$ position and velocity variables:,

$$\mathscr{L} = \mathscr{L}(r, \theta, \dot{r}, \dot{\theta})$$

Using the above Lagrangian, define the generalized momenta, associated with the r and θ position variables. Name them p_r and p_θ.

(b) Write down the Hamiltonian in terms of $r, \theta, p_r, p_t heta$:

$$\mathscr{H} = \mathscr{H}(r, \theta, p_r, p_t heta)$$

(c) Derive the Hamiltonian equations for the r, p_r, θ, p_θ pairs (but do not solve).

(d) Show that p_θ is a constant-of-motion. In other words,

$$p_\theta = mr^2\dot{\theta} = \text{const.}$$

Is the generalized momentum, p_θ related with any familiar physical quantity in the context of rotational motion?

Question 4. Isotropic harmonic oscillator The isotropic harmonic oscillator is defined as a particle of mass m, with potential energy

$$V(r) = \frac{1}{2}kr^2.$$

where $r^2 = x^2 + y^2 + z^2$.

In Cartesian (x, y, z) and polar (r, θ, z) coordinate systems write down the Hamiltonian and the corresponding Hamilton EOMs.

7 Coupled Harmonic Oscillators

So far we dealt with the problem of a single harmonic oscillator under different physical situations (simple and damped). In this chapter we'll be examining the case of two harmonic oscillators coupled with each other. An outline of the present chapter is as follows: First we'll derive using Newtonian, Lagrangian and Hamiltonian methods the equations-of-motion (EOM) for the problem of two mass-spring systems coupled via a spring. To demonstrate the methods will ignore friction forces. Having convinced ourselves that all these three methods lead to the same EOMs we'll proceed with its general solution and study two special cases corresponding to two different initial conditions.

7.1 Newtonian Method

We start by defining the 1-D Cartesian coordinate system Oy that will be used for the mathematical description of the masses' motion. We take the positive direction of the y-axis to the right of O. Our task is to find the position of the masses '1' and '2' as a function of time, namely the functions $y_1(t), y_2(t)$. For simplicity we consider that masses and the springs are identical characterized by m and k. For convenience we'll name this system as *simple coupled-mass system* in contrast to the more general case where m_1, m_2 and k_1, k_2, k_3 take arbitrary values.

Simple coupled-mass system (m, k)

The spring forces follow the Hooke's law $F = -kx_i,\ i = 1, 2$ where x_i represent the displacement of the masses from their respective equilibrium positions at positions l and $2l$ [see figure (7.1)]. Due to the latter property of the spring forces, while one can develop the present discussion in terms of the coordinates y_1 and y_2 (which provide the position of the masses relative to the origin of the coordinate system Oy) it is more convenient to measure the position of the masses relative to their equilibrium positions. That is because the Hooke's law is expressed in terms of the displacement of the mass from their equilibrium positions x_1, x_2 rather than their absolute positions y_1, y_2. So, if the natural length of the springs are l, then the following relations for the masses' locations hold:

$$x_1 = y_1 - l \qquad x_2 = x_2 - 2l \tag{7.1}$$

Application of the Newton's second law for the two masses '1','2' gives:

$$m\frac{d^2}{dt^2}x_1'(t) = -kx_1 + k(x_2 - x_1)$$
$$m\frac{d^2}{dt^2}x_2'(t) = -kx_2 - k(x_2 - x_1)$$

From Eq. (7.1), since l is constant, we see that $\ddot{x} = \ddot{x}'$ and direct substitution provides,

$$m\ddot{x}_1(t) = F_{11} + F_{21} = -kx_1 + k(x_2 - x_1) \tag{7.2}$$
$$m\ddot{x}_2(t) = F_{22} + F_{32} = -k(x_2 - x_1) - kx_2, \tag{7.3}$$

where the forces F_{ij} represent the force exerted by the spring 'i' on the particle 'j'. After re-arranging the various terms in the above differential equations we end up to the following coupled-system of ordinary-differential equation (ODE) for the $x_1(t), x_2(t)$:

$$m\ddot{x}_1(t) + 2kx_1 = kx_2, \qquad m\ddot{x}_2(t) + 2kx_2 = kx_1. \tag{7.4}$$

Lagrangian method

In order to derive the EOMs via the Euler-Lagrange Equations (ELEs) we have to construct the system's Lagrangian. To this we have (a) to

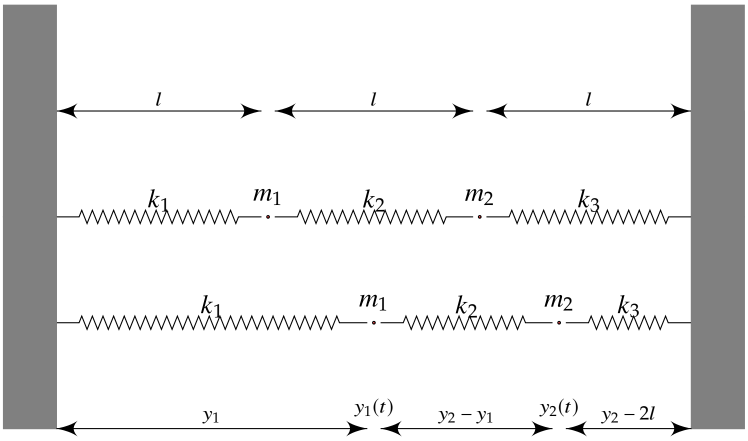

Figure 7.1: A simple coupled-mass (m, k) system. y_1 and y_2 represent the position of the masses '1' and '2'. The origin of the coordinate system is placed at the left wall. In the present demonstration we take the particles with the same mass $m_1 = m_2 = m$ and the springs with the same stiffness $k_1 = k_2 = k_3 = k$.

choose the coordinate variables (b) and find the kinetic and potential energy of the systems in terms of the chosen variables.

Obviously the problem is 1-D and the coordinate axis is self-evident to be along the line joining the masses (our model is that there is nothing else in the universe than the three springs and the two masses). To complete with the coordinate system we need to set the origin. The origin can be set anywhere in the joining line and we choose to put at the very left of the system. Let's name this system as Oy. With this choice, the position of the masses when the system is at equilibrium is l for the mass m_1 and $2l$ for the mass m_2. Let's now distort the system (in other words introduce energy) by either displacing the masses and/or kicking them with some initial speed (left or right). The corresponding mathematical expressions are,

$$y_1(0) = y_{10}, \qquad \dot{y}_1(0) = v_{10}, \qquad \text{particle '1'} \tag{7.5}$$

$$y_2(0) = y_{20}, \qquad \dot{y}_2(0) = v_{20}, \qquad \text{particle '2'.} \tag{7.6}$$

Following this initial distortion the system will evolve in time according to the ELEs as demanded by the *principle of least action*.

Let's now formulate the kinetic and potential energy in terms of y_1, y_2 and $\dot{y}_1, \dot{y}_2$ following the Lagrangian approach. The system's kinetic energy is relatively straightforward to express in these Cartesian coordinates as,

$$\boxed{T(\dot{y}_1, \dot{y}_2) = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2. \tag{7.7}}$$

The potential energy requires a bit more consideration. We should focus on the following property of the potential energy stored in a spring. Provided that a spring of restoring constant, k, has a natural length l then the potential energy stored in it when either squeezed or stretched and obtains length, y, is equal to,

$$V(y) = \frac{1}{2}k(y - l)^2.$$

Therefore it's potential energy is determined by its *length relative to its natural length*. With this rule in mind we can form the potential energy of the coupled HOs at an arbitrary time 't' the particles '1' and '2' have positions $y_1(t)$ and $y_2(t)$ respectively.

Spring '1':
Length is y_1 and therefore distortion relative to its natural length l is $y_1 - l$.
It's potential energy is,

$$V_1(y_1) = \frac{1}{2}k(y_1 - l)^2$$

Spring '2':
Length is $y_2 - y_1$ and the distortion relative its natural length l is: $y_2 - y_1 - l$.
It's potential energy is,

$$V_2(y_1, y_2) = \frac{1}{2}k(y_2 - y_1 - l)^2$$

Spring '3':
Length is $3l - y_2$ and distortion relative its natural length l is: $3l - y_2 - l$.
It's potential energy is,

$$V_3(y_2) = \frac{1}{2}k(3l - y_2 - l)^2$$

Then immediately we have for the potential energy

$$V(y_1, y_2) = \frac{1}{2}k(y_1 - l)^2 + \frac{1}{2}k(y_2 - y_1 - l)^2 + \frac{1}{2}k(y_3 - 2l)^2 \qquad (7.8)$$

Therefore we have for the Lagrangian function:

$$\begin{aligned} L(y_1, y_2, \dot{y}_1, \dot{y}_2) &= T(\dot{y}_1, \dot{y}_2) - V(y_1, y_2) \\ &= \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}m\dot{y}_2^2 - \frac{1}{2}k(y_1 - l)^2 - \frac{1}{2}k(y_2 - y_1 - l)^2 - \frac{1}{2}k(y_2 - 2l)^2 \end{aligned}$$

At this point it's convenient to apply a simple change of coordinates by setting:

$$x_1(t) = y_1(t) - l, \qquad x_2(t) = y_2(t) - 2l$$

Then we can see that,

$$\begin{aligned} \dot{y}_1 = \dot{x}_1, &\qquad \dot{y}_2 = \dot{x}_2, \\ y_2 - y_1 - l &= x_2 - x_1 \end{aligned}$$

In terms of these new variables $x_1, x_2, \dot{x}_2, \dot{x}_2$ the Lagrangian takes a simpler form:

$$
\begin{aligned}
\mathcal{L}(x_1, x_2, \dot{x}_2, \dot{x}_2) &= T(\dot{x}_2, \dot{x}_2) - V(x_1, x_2) \qquad (7.9) \\
&= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}kx_2^2
\end{aligned}
$$

Now it only remains to apply the ELEs equations in order to derive the EOMs for the system:

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}) - \frac{\partial \mathcal{L}}{\partial x_i} = 0, \qquad i = 1, 2.$$

For example, the generalized momentum is for the particles are,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m\dot{x}_1, \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m\dot{x}_2,$$

and

$$\frac{\partial \mathcal{L}}{\partial x_1} = -kx_1 + k(x_2 - x_1), \qquad \frac{\partial \mathcal{L}}{\partial x_1} = -kx_2 - k(x_2 - x_1)$$

It's now simple matter to utilize Eq. () and see that we obtain identical EOMS as those derived by the Newtonian method in (7.2), (7.3) (or (7.4)).

7.2 Solution for the coupled HOs

The above system can be further re-written, by dividing with the mass $m \neq 0$, as:

$$\ddot{x}_1 + \omega_0^2 x_1 = \omega_1^2 x_2 \tag{7.10}$$
$$\ddot{x}_2 + \omega_0^2 x_2 = \omega_1^2 x_1, \tag{7.11}$$

where

$$\omega_0 = \sqrt{\frac{2k}{m}}, \qquad \omega_1 = \sqrt{\frac{k}{m}}. \tag{7.12}$$

While there are general methods of solving this first-order ordinary differential equation (ODE) system here we'll be using a much simpler method, which does not require extra mathematics than those that are required for the solution of the simple HO oscillator of the first chapters. To this end, we add and subtract the equations (7.10) and (7.11) to obtain:

$$\ddot{x}_1 + \ddot{x}_2 + \frac{k}{m}(x_1 + x_2) = 0$$
$$\ddot{x}_1 - \ddot{x}_2 + \frac{3k}{m}(x_1 - x_2) = 0,$$

Finally, by defining,

$$u_+ = x_1 + x_2 \tag{7.13}$$
$$u_- = x_1 - x_2 \tag{7.14}$$

we end up to the following ODE for the new variables $u_\pm(t)$:

$$\ddot{u}_+ + \omega_+^2 u_+ = 0 \tag{7.15}$$
$$\ddot{u}_- + \omega_-^2 u_- = 0, \tag{7.16}$$

$$\omega_+ = \sqrt{\frac{k}{m}}, \qquad \omega_- = \sqrt{\frac{3k}{m}}. \tag{7.17}$$

With the above transformation $u_\pm = x_1 \pm x_2$ we have succeed to turn the coupled ODE system for x_1, x_2 to an **uncoupled** ODE system for the u_+ and u_- with eigenfrequencies ω_+ and ω_- respectively. As known, from the previous chapters, the general solution for the latter ODE* [Eqns (7.15) and (7.16)] are expressed as follows:

$$u_+(t) = A_+ \cos\omega_+ t + B_+ \sin\omega_+ t,$$
$$u_-(t) = A_- \cos\omega_- t + B_- \sin\omega_- t.$$

We can obtain the expressions for the positions $x_1(t)$ and $x_2(t)$ are by solving backwards the Eqns (7.13), (7.14) to obtain:

$$x_1(t) = \frac{1}{2}\left[u_+(t) + u_-(t)\right], \qquad x_2(t) = \frac{1}{2}\left[u_+(t) - u_-(t)\right].$$

* We do not need to consider initial conditions at this stage. The initial conditions are given in terms of the physical positions and velocities of the masses, namely the $x_1(0)$,$x_2(0)$ and $v_1(0)$, $v_2(0)$.

In this case the final expressions for $x_1(t)$ and $x_2(t)$ are given by*,

$$x_1(t) = \frac{1}{2}\left[A_+ \cos\omega_+ t + B_+ \sin\omega_+ t + A_- \cos\omega_- t + B_- \sin\omega_- t\right] \tag{7.18}$$

$$x_2(t) = \frac{1}{2}\left[A_+ \cos\omega_+ t + B_+ \sin\omega_+ t - A_- \cos\omega_- t - B_- \sin\omega_- t\right]. \tag{7.19}$$

In the above expressions the constants $A_\pm, B_\pm$ are calculated by considering the corresponding *initial conditions* for the initial positions and velocities of the masses m_1 and m_2:

initial conditions

$$x_1(0) = x_{10}, \qquad x_2(0) = x_{20} \tag{7.20}$$

$$v_1(0) = v_{10}, \qquad v_2(0) = v_{20} \tag{7.21}$$

Provided that $x_1(t)$ and $x_2(t)$ are known the velocities of the masses are obtained by taking the corresponding first-order time-derivatives:

$$v_1(t) = \frac{1}{2}\left[\omega_+(B_+ \cos\omega_+ t - A_+ \sin\omega_+ t) + \omega_-(B_- \cos\omega_- t - A_- \sin\omega_- t)\right] \tag{7.22}$$

$$v_2(t) = \frac{1}{2}\left[\omega_+(B_+ \cos\omega_+ t - A_+ \sin\omega_+ t) + \omega_-(A_- \sin\omega_- t - B_- \cos\omega_- t)\right] \tag{7.23}$$

Different modes of motion exist for the masses of this system, depending on the initial conditions (also known as *excitation mode*). In the below we'll demonstrate an example where the initial conditions are explicitly specified.

Example 7.2.1 (Coupled oscillators: Lowest mode excitation) Assume an two equal-masses coupled-mass-spring system with m, k known [see Fig. (7.1)]. Let's examine the case where the initial displacements are equal and point to the same direction. In other words the above initial conditions correspond to the case where we displace both masses by an equal amount from their respective equilibrium positions and then we simple release them. It is our purpose to find the subsequent motion of the masses m_1 and m_2.

$$x_1(0) = x_2(0) = x_0, \qquad v_1(0) = v_2(0) = 0.$$

Since the positions for the masses m_1 and m_2 are given by $x_1(t)$ and $x_2(t)$ in Eqns (7.18) and (7.19) it only remains to apply the initial conditions in order to determine the constants $A_\pm$ and $B_\pm$. Followin the standard procedure, by applying the initial conditions (7.20) to

* If we want the expressions for $x'_i(t), i = 1, 2$ then we have to use the transformation relations (7.1).

the expressions (7.18-7.19) we obtain:

$$x_1(0) = x_0 \implies A_+ + A_- = 2x_0$$
$$x_2(0) = x_0 \implies A_+ - A_- = 2x_0$$

Solving the above 2×2 algebraic system for A_+ and A_- gives:

$$A_+ = 2x_0, \qquad A_- = 0$$

Then we have for the positions:

$$x_1(t) = \frac{1}{2}\left[2x_0 \cos\omega_+ t + B_+ \sin\omega_+ t + B_- \sin\omega_- t\right]$$
$$x_2(t) = \frac{1}{2}\left[2x_0 \cos\omega_+ t + B_+ \sin\omega_+ t - B_- \sin\omega_- t\right].$$

By taking the derivative of the above quantities we find for the velocities:

$$v_1(t) = \frac{1}{2}\left[-2x_0\omega_+ \sin\omega_+ t + \omega_+ B_+ \cos\omega_+ t + \omega_- B_- \cos\omega_- t\right]$$
$$v_2(t) = \frac{1}{2}\left[-2x_0\omega_+ \sin\omega_+ t + \omega_+ B_+ \cos\omega_+ t - \omega_- B_- \cos\omega_- t\right].$$

Now applying the initial conditions (7.21) to the above equations we obtain the required algebraic equations for the remaining constants $B_\pm$:

$$v_1(0) = 0 \implies \omega_+ B_+ + \omega_- B_- = 0$$
$$v_2(0) = 0 \implies \omega_+ B_+ - \omega_- B_- = 0.$$

The above system gives for the B_+, B_- constants the following values:

$$B_+ = B_- = 0.$$

Therefore the final expressions for the positions and the velocities of the masses m_1 and m_2 are as below:

$$\boxed{\begin{array}{ll} x_1(t) = x_0 \cos\omega_+ t & x_2(t) = x_0 \cos\omega_+ t, \\ v_1(t) = -x_0\omega_+ \sin\omega_+ t, & v_2(t) = -x_0\omega_+ \sin\omega_+ t. \end{array}}$$

One can easily check that the above solutions are consistent with the initial conditions. From the above form of the solutions we see that the masses oscillate in phase around their respective equilibrium positions with the lowest (fundamental) frequency ω_+. So, at all times we have

$$x_1(t) = x_2(t), \qquad \omega_+ = \sqrt{\frac{k}{m}}$$

Effectively the two masses oscillate 'independently' each other as if the connecting spring k_2 was missing.

7.3 Questions

Question 1. Hamiltonian of simple coupled Oscillators

Assume a two-coupled-mass-spring system. The masses are equal (m) and the spring-constant known, k. Write down the Hamiltonian of the systems and the Hamiltonian EOMs. Confirm that these are equivalent with the EOMs derived via the Newton/Lagrangian methods, (7.4)).

Question 2. Special solution of simple coupled oscillators

Assume a two-coupled-mass-spring system. The masses are equal (m) and the spring-constant known, k. The two masses are initially released from rest but displaced such that $x_1(0) = -x_2(0) = a$. Find the the subsequent motion of the masses in terms of the known quantities of the problem, m, k, a.

Question 3. General coupled spring-mass HO system I: Newton's method The problem of the two coupled masses can be generalized to include the case where all the two masses have different masses $m_1 \neq m_2$ and the three springs have all different spring constants $k_1 \neq k_2 \neq k_3$.

Show that Newton's law for $x_1(t)$ and $x_2(t)$ ends to the following system of differential equations:

$$\ddot{x}_1 + \frac{k_1 + k_2}{m_1} x_1 = \frac{k_2}{m_1} x_2, \tag{7.24}$$

$$\ddot{x}_2 + \frac{k_3 + k_2}{m_2} x_2 = \frac{k_2}{m_2} x_1. \tag{7.25}$$

Question 4. General coupled spring-mass HO system II: Lagrange's method The problem of the two coupled masses can be generalized to include the case where all the two masses have different masses $m_1 \neq m_2$ and the three springs have all different spring constants $k_1 \neq k_2 \neq k_3$.

(a) Derive the Lagrangian function, $\mathscr{L}$ of this generalized system and applying the ELEs show that the EOMs are identical to equations (7.24)) and (7.24).

Appendix: Maths basics I | 8

Physics is mainly a quantitative science. Measurements and calculation of various physical quantities, e.g. position, velocity, forces, momentum, charge, voltage, current, etc.. are integral part of the modern days Physics framework. Physical quantities are related each other. These relations are most accurately described through mathematical expressions. Below you'll find some basics on the mathematics that frequently used for practical calculations. Anyone with the goal of mastering Physics at the UG level cannot escape from the use of algebraic calculus of vector and matrices, trigonometry and integro-differential calculus of multi-variable functions. In the context of Classical Mechanics (CM) Matrix algebra and multivariable calculus.

There is no other way to learn the basics than to practice them.

Basic mathematics include:

(i) **Geometry and Trigonometry.** Geometry represents one of the most ancient mathematical theories and aims to describe the properties of geometric shapes in two-dimensional space. Literally means *measurements of earth*. **Trigonometry** concetrates in the study of right triangles and the Pythagorean theorem. Note that, as differential calculus was not developed before Newton and Leibniz, most of the Newton's physical statements were proven using the methodology of Geometry.

(ii) **Algebraic calculus.** Essentially represents the theory of solving linear equations of the form $ax + b = 0$, quadratic equations of the form $ax^2 + bx + c = 0$ and higher order. Algebra, was cutting-edge mathematics when it was being developed in Baghdad in the 9th century.

(iii) **Differential calculus.** The introduction of a function, dependent on the values of a (single) variable, represents the most fundamental new concept in this theory. The theory that describes the properties of this dependance between x and $f(x)$ is the subject of *differential calculus*. Concepts such as derivatives and integration are introduced in this mathematical formulation. A natural generalization is the function dependent on many different variables studied by the *calculus of multivariable functions.*

(iv) **Calculus (multivariable)** Multivariable calculus introduces functions of several variables f(x,y,z...), and students learn to take partial and total derivatives. The ideas of directional derivative, integration along a path and integration over a surface are developed in two and three dimensional Euclidean space.

(v) **Analytic Geometry.** The use of algebraic techniques in the theory of geometry has boosted significantly the understanding of the properties of geometrical shapes. Analytical geometry is introduced in the geometry when one defines a coordinate system in describing the space. Cartesian coordinate systems are the most known among all known (cylindrical, spherical,..)

(vi) **Linear Algebra.**
Linear algebra, deals mainly with methods of solving systems of linear equations of the form $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = c_i,\ i = 1, 2 \cdots n$. It is closely linked with the matrix algebra where concepts such as inverse, determinant, characteristic matrix equation, symmetric, antisymmetric, unitary or Hermitian matrices are introduced.

(vii) **Ordinary Differential Equations (ODE) and physical laws.** Physics laws, being the relation between physical variables, from mathematical point of view, are *partial differential equations* (PDE) in time and space [61]. The most prominent example is the Newton's 2nd law:
$$\frac{d^2x(t)}{dt^2} = \frac{1}{m}F(x(t), t),$$
where the relation between the position $x(t)$ of an object, characterized by its (inertial) mass m and an external force $F(x, t)$ is expressed using mathematics. The above equation is nothing else than a partial differential equation with the unknown function being the $x(t)$.

(viii) **Partial Differential Equations** For doing physics in more than one dimension, it becomes necessary to use partial derivatives and hence partial differential equations. The first partial differential equations students learn are the linear, separable ones that were derived and solved in the 18th and 19th centuries by people like Laplace, Green, Fourier, Legendre, and Bessel.

(ix) **Probability and statistics** Probability became of major importance in physics when quantum mechanics entered the scene. A course on probability begins by studying coin flips, and the counting of distinguishable vs. indistinguishable objects. The concepts of mean and variance are developed and applied in the cases of Poisson and Gaussian statistics.

61: For example for the ODE for an (idealized) harmonic oscillator system (e.g. mass-spring,pendulum) takes the form:
$$\ddot{x}(t) + \omega_0^2 x(t) = 0.$$

Most of the problems in physics can't be solved exactly in closed form. Therefore one has to rely on approximate calculations with the associated methods required to be developed, such as, power series expansions, saddle point integration, and small (or large) perturbations.

In the following sections some basic concepts of the required mathematical skills for this module are presented.

8.1 Elements of trigonometry

The sin and cos function of an angle θ are defined through a right angle triangle as (see Fig. 8.1) and the use of the Pythagorean theorem:

$$\tan\theta = \frac{y}{x}, \quad \sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad x^2 + y^2 = r^2$$

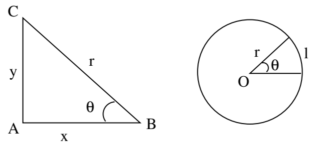

Figure 8.1: Right angle triangle

Standard properties of the sin and cos functions are:

$$\cos^2\theta + \sin^2\theta = 1 \quad \cos(-\theta) = \cos\theta \quad \sin(-\theta) = -\sin\theta$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

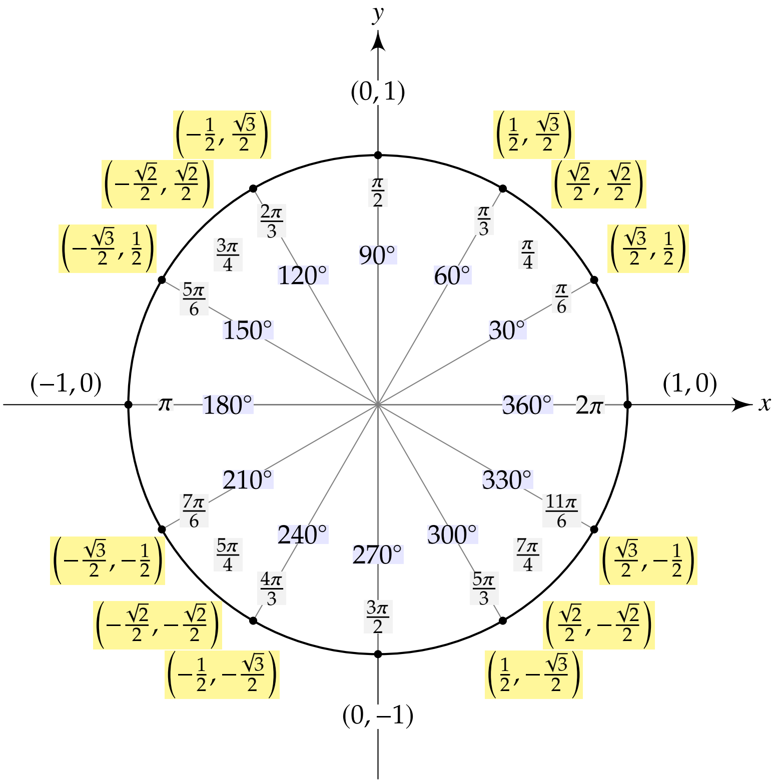

Figure 8.2: A set of values for $(x, y) = (\cos\theta, \sin\theta)$ for various θ angle. Angle θ is given in rads and in degrees (e.g. For $\theta = 5\pi/6 = (150^o) \rightarrow (\cos\theta, \sin\theta) = (-\sqrt{3}/2, 1/2)$).

Angles are measures on degrees (^{o}) or in radians:

$$360^o = 2\pi \text{ rad}$$

Finally for a cyclical disc of angle θ (see Fig. 8.1) the relation between the arc length (l), the angle and the radius r is as:

$$l = r\theta, \qquad \theta \text{ in radians.}$$

8.2 Vectorial calculus

In physics there are quantities which can be categorized as *scalar* or *vector* quantities. More specifically:

- **Scalars**: Physical quantities fully characterized from their magnitude. Basic scalar quantities in classical mechanics are time (sec) , length (m), mass (kg), work, kinetic and potential energy (Joule), temperature (^{o}K), etc.
- **Vectors** : Physical quantities fully characterized from (a) their magnitude and (b) their direction in space. Basic vector quantities in mechanics are position, velocity, acceleration, force, momentum, etc.

Vectors

Properties

$$\begin{aligned} \mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A}, \\ \mathbf{A}+(\mathbf{B}+\mathbf{C}) &= (\mathbf{A}+\mathbf{B})+\mathbf{C} \\ a(\mathbf{A}+(\mathbf{B}) &= a\mathbf{A}+a\mathbf{B}, \end{aligned}$$

Unit vectors Vectors with magnitude equal to unity. If a vector $\mathbf{A}$ is divided by its magnitude, then we get the unit vector $\hat{A} = \mathbf{A}/A$ which has the same direction as the vector $\mathbf{A}$, but with unit length.

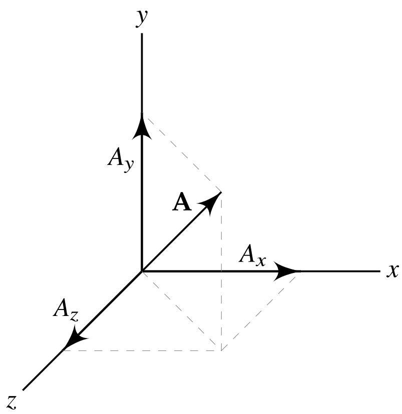

Figure 8.3: Cartesian components of a vector

Inner (or scalar) product The inner product between two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as:

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\Theta_{AB}, \qquad 0 \le \theta \le \pi, \tag{8.1}$$

where Θ_{AB} is the angle between the two vectors $\mathbf{A}, \mathbf{B}$.

Properties.

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= \mathbf{B}\cdot\mathbf{A} \\ \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{B})+\mathbf{A}\cdot\mathbf{C} \\ a(\mathbf{A}\cdot\mathbf{B}) &= (a\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(a\mathbf{B}) \end{aligned}$$

Outer (or cross or vector) product Outer product between two vectors **A, B** is defined as:

$$\mathbf{C} \;=\; \mathbf{A}\times\;\mathbf{B}\;\equiv \hat{u}AB\sin\Theta_{AB}, \qquad 0 \le \theta \le \pi, \tag{8.2}$$

where, $\hat{u}$ is along the direction of $\mathbf{C}$ and is normal to the plane defined by vectors $\mathbf{A}$ and $\mathbf{B}$ and direction such that a right-three-vector coordinate system to be defined.

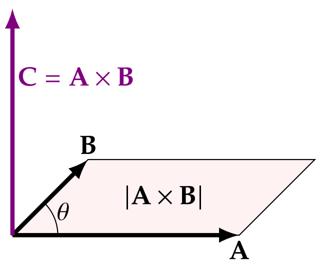

Figure 8.4: Cross product of two vectors is a vector.

Properties

$$
\begin{aligned}
\mathbf{A} \times \mathbf{B} &= -\mathbf{B} \times \mathbf{A} \\
\mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) + \mathbf{A} \times \mathbf{C} \\
a(\mathbf{A} \times \mathbf{B}) &= (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B})
\end{aligned}
\tag{8.3}
$$

Cartesian coordinate system

A Cartesian Coordinate Systems consists of three mutually $\hat{x}, \hat{y}, \hat{z}$ unit vectors, of fixed direction and origin, independent on the particle's position:

$$
\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0, \qquad \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1. \tag{8.4}
$$

A practical (and compact) representation of these unit vectors is to represent them as column:

$$
\hat{x} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{y} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{z} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
$$

We can then define the rows to be $\hat{x}^T, \hat{y}^T, \hat{z}^T$:

$$
\hat{x}^T \equiv \begin{pmatrix} 1, 0, 0 \end{pmatrix}, \quad \hat{y}^T \equiv \begin{pmatrix} 0, 1, 0 \end{pmatrix}, \quad \hat{z}^T \equiv \begin{pmatrix} 0, 0, 1 \end{pmatrix},
$$

In this case the inner-products above can be calculated using matrix algebraic techniques. For example,

$$
\hat{x} \cdot \hat{y} \equiv \hat{x}^T \cdot \hat{y} = \begin{pmatrix} 1, 0, 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 + 0 + 0 = 0
$$

and

$$
\hat{x} \cdot \hat{x} \equiv \hat{x}^T \cdot \hat{x} = \begin{pmatrix} 1, 0, 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 + 0 + 0 = 1
$$

Vector components In a Cartesian coordinate system $(Oxyz)$ a vector $\mathbf{A}$ is represented (and fully determined) by its three components along the axes (xyz), namely A_x, A_y, A_z in the following way:

$$
\mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \tag{8.5}
$$

with a magnitude equal to,

$$
A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}, \tag{8.6}
$$

In a matrix-representation $\mathbf{A}$ is represented as,

$$
\mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \equiv A_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + A_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + A_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.
$$

To perform any operation between two vectors **A** and **B** we can analyze the operations to their components.

Addition For example if we want to add **A** and **B** (see Fig. 1.2) the resulting vector **C** is given as:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y}$$

Inner product (scalar) If the vectors **A and B** are expressed through their components A_x, A_y and B_x, B_y,

$$\mathbf{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}, \qquad \mathbf{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z},$$

then inner product is given by:

$$\mathbf{A}\cdot\ \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{8.7}$$

Finally from Eqns (8.1) and (8.7) we find that the angle θ between the vectors equal to:

$$\cos\Theta_{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}$$

In a matrix-representation of **A** in the basis of $\hat{x}, \hat{y}, \hat{z}$ we have,

$$\mathbf{A}\cdot\mathbf{B} \equiv \mathbf{A}^T\cdot\mathbf{B} = \begin{pmatrix} A_x, A_y, A_z \end{pmatrix}\cdot\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = A_xB_x + A_yB_y + A_zB_z$$

As a special case, the square amplitude of **A** can also be calculated:

$$\mathbf{A}^2 \equiv \mathbf{A}^T\cdot\mathbf{A} = \begin{pmatrix} A_x, A_y, A_z \end{pmatrix}\cdot\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = A_xA_x + A_yA_y + A_zA_z = A_x^2 + A_y^2 + A_z^2.$$

Vector (cross/outer) product In an orthogonal coordinate system the cross product may be expresses as:

$$\mathbf{A}\times\ \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{8.8}$$

Triple vector product Consider three vectors **A, B, C**. If the vectors **A,B,C**, are analyzed into their components A_x, A_y, A_z , B_x, B_y, B_z and C_x, C_y, C_z, respectively, then the following vector products are defined as:

Scalar triple product:

$$\mathbf{A}\cdot(\ \mathbf{B}\ \times\ \mathbf{C}) = \begin{vmatrix} A_x & A_x & A_z \\ B_z & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{8.9}$$

Vectorial triple product: [62]

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \cdot \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} \tag{8.10}$$

62: It is worth noticing here that,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

8.3 2D Cartesian/polar coordinate system I

The vector **A** can be defined either through its components A_x, A_y on a Cartesian CS or through it's magnitude $A \equiv |\mathbf{A}|$ and the angle θ_a with the axis Ox on a polar coordinate system [63].

63: The unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ very frequently denoted as $\hat{x}, \hat{y}, \hat{z}$ instead.

$$\mathbf{A} = A_x \hat{\imath} + A_y \hat{\jmath} \tag{8.11}$$

For the components A_x, A_y simple trigonometry shows the following relations:

$$A_x = A\cos\theta_a, \quad A_y = A\sin\theta_a. \tag{8.12}$$

From the above relation we can also derive the following expressions:

$$A = \sqrt{A_x^2 + A_y^2}, \qquad \theta_a = \tan^{-1}(\frac{A_y}{A_x}) \tag{8.13}$$

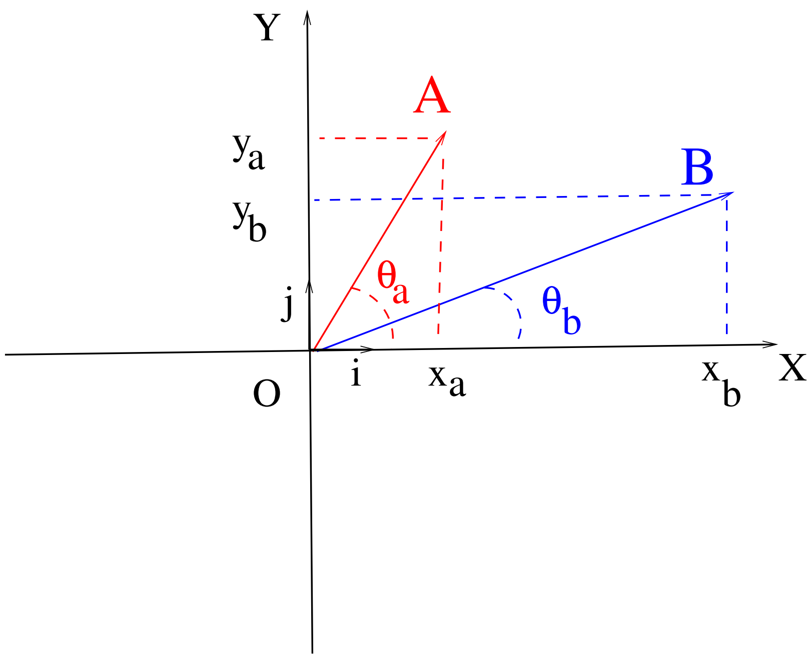

Figure 8.5: 2-D Cartesian/polar systems.

8.4 Algebraic calculus: Quadratic equation

$$ax^2+bx+c=(x-x_1)(x-x_2)=0, \quad \rightarrow \quad x_{1,2}=\frac{1}{2a}\begin{cases}-b\pm\sqrt{b^2-4ac}, & b^2-4ac>0\\ -b, & b^2=4ac\\ -b\pm\imath\sqrt{4ac-b^2}, & b^2-4ac<0\end{cases}$$

Also the following formulas hold for the roots:

$$x_1+x_2=-\frac{b}{a}, \qquad x_1x_2=\frac{c}{a}$$

8.5 Elements of differential calculus

Definition: The first derivative of a function $f(x)$ is defined by the following relation:

$$f'(x) = \frac{df(x)}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \tag{8.14}$$

The second derivative is defined as the first derivative of the first derivative $f''(x) = d(f'(x))/dx = d^2 f(x)/dx^2$. Similarly derivatives of higher order are defined:

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{n-1}(x)\right) \qquad n = 0, 1, 2, \ldots$$

Note that $f^{(0)}(x) = f(x)$.

Basic properties If f and g are functions of x then the following relations hold:

$$\frac{d}{dx}[f(x)g(x)] = f'g + fg' \qquad \text{(product rule)} \tag{8.15}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'g - fg'}{g^2} \qquad \text{(quotient rule)} \tag{8.16}$$

$$\frac{d}{dx}\left(F[g(x)]\right) = \frac{dF(g)}{dg}\frac{dg(x)}{dx}, \qquad \text{(chain rule)} \tag{8.17}$$

Derivatives of simple functions

$$\begin{array}{lll} (a)' = 0, & (x^a)' = ax^{a-1}, & a = \text{ a real number} \\ (\sin x)' = \cos x & (\cos x)' = -\sin x, & (\tan x)' = -1/\cos^2 x, \\ (e^x)' = e^x, & (\ln x)' = 1/x & \end{array}$$

Time derivatives Conventionally, derivatives of time-dependent functions, $f(t)$, are also (compactly) symbolized as:

$$\dot{f}(t) = \frac{df(t)}{dt}, \qquad \ddot{f}(t) = \frac{d^2 f(t)}{dt^2} = \frac{d}{dt}\left[\frac{df(t)}{dt}\right].$$

Examples of such derivatives in mechanics are velocity (first derivative of position) and acceleration (second derivative of position).

Spatial derivatives Conventionally, spatial derivatives of 1-dimensional functions (e.g. $f(x)$) are symbolized as:

$$f'(x) = \frac{df(x)}{dx}, \qquad f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left[\frac{df(x)}{dx}\right].$$

Examples of such derivatives are potential energies, forces etc...

Multivariable scalar functions

Let's assume a multivariate scalar function $f(x_1, x_2 \cdots, x_n)$. Then on can define partial derivatives where only one variable is allowed to vary will the other remain constant.

Consider a two-variable function $f(x, y)$. Then the partial derivatives in terms of variables (x, y) are given as :

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} = \frac{df(x, y = \text{const})}{dx}$$
$$\frac{\partial f(x,y)}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} = \frac{df(x = \text{const.}, y)}{dy}$$

Example: if $f(x, y) = e^{3x} \cos y$, then its partial derivatives are:

$$\frac{\partial f(x,y)}{\partial x} = \frac{df(x, y = \text{const.})}{dx} = 3e^{3x} \cos y$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{df(x = \text{const.}, y)}{dy} = -e^{3x} \sin y$$

8.6 Integration calculus

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The *indefinite* integral $\int f(x)dx$ of a function $f(x)$ is defined as

$$\int f(x)dx = g(x) + c \quad \Leftrightarrow \quad f(x) = \frac{d}{dx}g(x) \tag{8.18}$$

where $f(x) = dg(x)/dx$. Indefinite integral in this sense is the inverse of the derivative operation on a function.

The *definite* integral of the function $f(x)$ between a and b is defined as:

$$\int_a^b f(x)dx = \int_a^b \frac{dg(x)}{dx}dx = \int_a^b dg(x) = g(b) - g(a). \tag{8.19}$$

64: Indefinite integrals of some functions:

$$\int a dx = ax + C$$
$$\int x^a dx = \frac{x^{a+1}}{(a+1)} + C$$
$$\int \sin x dx = -\cos x + C$$
$$\int \cos x dx = \sin x + C$$
$$\int e^x dx = e^x + C$$
$$\int \frac{dx}{x} = \ln x + C$$

Basic properties If f and g are single-variable functions then the following relation holds:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \qquad \text{(integration by parts)} \tag{8.20}$$

Derivative of an integral (Leibniz integration rule* The following derivative for a general integral holds [65] ,

$$\boxed{\frac{d}{dx}\int_{\lambda_1(x)}^{\lambda_2(x)} dt F(x,t) = F(x,\lambda_2)\lambda_2' - F(x,\lambda_1)\lambda_1' + \int_{\lambda_1}^{\lambda_2} dt \frac{\partial F(x,t)}{\partial x}} \tag{8.21}$$

where $\lambda_i \equiv \lambda_i(x) \in [-\infty, \infty]$ and $\lambda_i'(x) \equiv \dfrac{d\lambda_i}{dx}, \quad i = 1, 2.$

65: In the special case where λ_1, λ_2 are independent on x we have,

$$\frac{d}{dx}\int_{\lambda_1}^{\lambda_2} dt F(x,t) = \int_{\lambda_1}^{\lambda_2} dt \frac{\partial F(x,t)}{\partial x}.$$

Therefore the derivative and the integral can be interchanged.

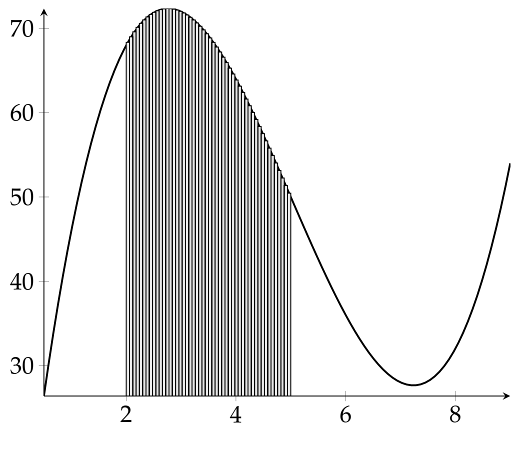

Figure 8.6: The geometrical meaning of integration is the area of the surface under $f(x) = -15(x-5) + (x-5)^3 + 50$: $A = \int_2^5 f(x)dx = 789/4$.

8.7 Extreme points of functions

Single-variable function To find the extreme point of a single-variable function we have to take the 1^{st} and 2^{nd} derivatives:

$$f'(x_m) = 0, \qquad f''(x_m) = \begin{cases} k > 0 & \text{minimum} \\ k < 0 & \text{maximum} \end{cases} \tag{8.22}$$

The point(s) x_m satisfying the above conditions are either a *maximum* or *minimum*.

Example. *Find the extrema of* $f(x) = -15(x-5) + (x-5)^3 + 50$.

$$f'(x) = -15 + 3(x-5)^2 = 3(x^2 - 10x - 30) = 0 \quad \rightarrow \quad \boxed{x_{1,2} = 5 \pm \sqrt{5}}$$

These two values will be the position of the extrema. Then we can check whether these values of x_1, x_2 are minimum or maximum of the functions. Either we directly substitute to obtain $f(x_1)$ and $f(x_2)$ or more generally we look the 2nd derivative:

$$f''(x) = 6(x-5) \qquad f''(5 \pm \sqrt{5}) = \mp 6\sqrt{5}$$

So if we name $x_2 = 5 + \sqrt{5}$ then $f''(x_2) = 6\sqrt{5} > 0$ and this point corresponds to *minimum*. For the remaining value we have, $x_1 = 5 - \sqrt{5}$ then $f''(x_1) = -6\sqrt{5} < 0$ and represents a maximum. And indeed this is the case as it is visually shown in Fig. (8.6) or if $f(x)$ directly evaluated:

$$f(x_1) = 10\left(5 + \sqrt{5}\right) = 10x_2 \simeq 72.36, \qquad \text{maximum}$$
$$f(x_2) = 10\left(5 - \sqrt{5}\right) = 10x_1 \simeq 27.639, \qquad \text{minimum}$$

Multi-variable functions* Along similar lines one can also search for the extrema of functions depending on more variables with the simplest extension of the method is to consider the extreme points of a two-variable function $f(x, y)$.

When looking for such extreme points we need to consider whether any explicit relationship between x and y is given, known as *constraint*; the mathematical formulation of such relationship (constraint) has the following form:

$$\phi(x, y) = 0. \tag{8.23}$$

When a constraint is present it essentially restricts the search for extremes only for those points of space which satisfies the constraint (or equivalently, satisfy the Eq.(8.23).

Therefore we take two different cases:

(i) *No constraint for x, y.*
The extreme points for this function are obtained by setting the partial derivatives with respect x, y equal to zero:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad \rightarrow \qquad \begin{cases} \frac{\partial}{\partial x} f(x_m, y_m) & = 0 \\ \frac{\partial}{\partial y} f(x_m, y_m) & = 0 \end{cases} \tag{8.24}$$

The above conditions define a *stationary point* with a value function $f(x_m, y_m)$; these two algebraic equations provides the extreme point (x_m, y_m). Whether this stationary point is maximum or minimum is determined by examining the second-order variation, $d^2 f$, with respect to x, y variables.

(ii) *Constraint is $\phi(x, y) = 0$*- Method of Lagrange multipliers
Let's assume the case where the x, y variables are not fully independent each other but they are related. Let's assume that their functional relation can be cast as $\phi(x, y) = 0$. In order to find the extreme points of $f(x, y)$ given this *constraint* we can work as follows:

- First we construct the *auxilliary* (augmented) function:

$$F(x, y, \lambda) = f(x, y) + \lambda\phi(x, y).$$

  This function now has 3 variables that is dependent on, namely (x, y, λ); where λ takes arbitrary values.
- Now we are looking the extreme points his new function $F(x, y, \lambda)$ in the usual way:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial \lambda}d\lambda = 0 \qquad \rightarrow \qquad \begin{cases} \frac{\partial F}{\partial x} & = 0 \\ \frac{\partial F}{\partial y} & = 0 \\ \frac{\partial F}{\partial \lambda} & = 0 \end{cases}$$

  The above system represents a system of three algebraic equations with three unknowns and its solution provides the extreme point (x_m, y_m) and the (now) useless λ.

8.8 Questions

(i) Write in scientific notation ($a \times 10^b$) the number 0.0123.
(ii) How many radians are 22.5^o degrees?
(iii) What is the angle in degrees of $\theta = \sin^{-1}(0.38261) = \arcsin(0.38261)$?
(iv) Assume a Cartesian coordinate system in two dimensions. A vector $\mathbf{A}$ is expressed as $\mathbf{A} = (1/\sqrt{2})\hat{\imath} + (1/\sqrt{2})\hat{\jmath} = (1/\sqrt{2}, 1/\sqrt{2})$. What is its magnitude?
(v) For the vector $\mathbf{A}$ of the previous problem what is the angle of the vector with the axis Ox?
(vi) In the same coordinate system of the last two problems, a vector $\mathbf{B}$ is defined to be as $\mathbf{B} = 3\hat{\imath} + 4\hat{\jmath} = (3, 4)$. What is the magnitude of the vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$?
(vii) For the vectors $\mathbf{A}$, $\mathbf{B}$ above, what is the angle of the vector $\mathbf{C} = \sqrt{2}\,\mathbf{A} - \mathbf{B}$ with the axis Ox?
(viii) What is the second derivative of function $\sin(x^2)$ and the first derivative of e^{sint}?
(ix) What is the indefinite integral of $\int (ax^2 + bx + c)dx$?
(x) What is the value of the definite integral of $\int_{-1}^{0} (x^2 - 3x + 1)dx$?
(xi) If $T = 2\pi/\omega$, show that:

$$\int_0^T dt \cos^2 \omega t = \int_0^T dt \sin^2 \omega t = \frac{T}{2},$$
$$\int_0^T dt \sin \omega t \cos \omega t = 0.$$

(xii) Assuming that A_1, A_2 are known, find the values of A and ϕ so that,
(a)
$$A_1 \cos \omega t + A_2 \sin \omega t = A \cos(\omega t + \phi)$$
(b)
$$A_1 \cos \omega t + A_2 \sin \omega t = A \sin(\omega t + \phi)$$

(xiii) Now assuming that A, ϕ are known, find the values of A_1 and A_2 so that the same expressions as above hold:
(a)
$$A_1 \cos \omega t + A_2 \sin \omega t = A \cos(\omega t + \phi)$$
(b)
$$A_1 \cos \omega t + A_2 \sin \omega t = A \sin(\omega t + \phi)$$

(xiv) Using the chain rule for the derivatives, show that,

$$\frac{d}{dt}[x(t)^2] = 2x(t)v(t), \qquad \text{where } v(t) = \dot{x}(t)$$

(xv) Find the extreme points of the following functions:

$$y(x) = x^2(t), \qquad y(x) = \frac{1}{1 + x^2}, \qquad y(x) = sin(2t)$$

(xvi) Find the extreme points of the following function, as well as their values at these points:

$$f(x) = \frac{1}{\sqrt{(x^2 - a^2)^2 + b^2}}$$

where, f_0, a, b known real, numbers.
(hint: Determine the extreme points from the solution of the equation $f'(x_m) = 0$. It might be easier to find the extreme points of the function under the square-root $((x^2 - a^2)^2 + b^2)$)

(xvii) Find the extreme points of the following function, as well as their values at these points:

$$f(x) = \frac{x}{(x^2 - a^2)^2 + b^2}$$

where, f_0, a, b known real, numbers.
(hint: Determine the extreme points from the solution of the equation $f'(x_m) = 0$.)