PS223: Introduction to methods of Classical Mechanics

L. A.A. Nikolopoulos

December 26, 2019

Contents

Chapter 1

Mathematical background

Physics is mainly a quantitative science. Measurements and calculation of various physical quantities, e.g. position, velocity, forces, momentum, charge, voltage, current, etc.. are integral part of the modern days Physics framework. Physical quantities are related each other. These relations are most accurately described through mathematical expressions. Below you'll find some basics on the mathematics that frequently used for practical calculations. Anyone with the goal of mastering Physics at the UG level cannot escape from the use of algebraic calculus of vector and matrices, trigonometry and integro-differential calculus of multi-variable functions. In the context of Classical Mechanics (CM) Matrix algebra and multivariable calculus.

There is no other way to learn the basics than to practise them.

Basic mathematics include:

1. **Geometry and Trigonometry.** Geometry represents one of the most ancient mathematical theories and aims to describe the properties of geometric shapes in two-dimensional space. Literally means *measurements of earth*. **Trigonometry** concetrates in the study of right triangles and the Pythagorean theorem. Note that, as differential calculus was not developed before Newton and Leibniz, most of the Newton's physical statements were proven using the methodology of Geometry.

2. **Algebraic calculus.** Essentially represents the theory of solving linear equations of the form $ax + b = 0$, quadratic equations of the form $ax^2 + bx + c = 0$ and higher order. Algebra, was cutting-edge mathematics when it was being developed in Baghdad in the 9th century.

3. **Differential calculus.** The introduction of a function, dependent on the values of a (single) variable, represents the most fundamental new concept in this theory. The theory that describes the properties of this dependance between x and $f(x)$ is the subject of *differential calculus*. Concepts such as derivatives and integration are introduced in this mathematical formulation. A natural generalization is the function dependent on many different variables studied by the *calculus of multivariable functions.*

4. **Calculus (multivariable)** Multivariable calculus introduces functions of several variables f(x,y,z...), and students learn to take partial and total derivatives. The ideas of directional derivative, integration along a path and integration over a surface are developed in two and three dimensional Euclidean space.

5. **Analytic Geometry.** The use of algebraic techniques in the theory of geometry has boosted significantly the understanding of the properties of geometrical shapes. Analytical geometry is introduced in the geometry when one defines a coordinate system in describing the space. Cartesian coordinate systems are the most known among all known (cylindrical, spherical,..)

6. **Linear Algebra.**

   Linear algebra, deals mainly with methods of solving systems of linear equations of the form $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = c_i, \ i = 1, 2 \cdots n$. It is closely linked with the matrix algebra where concepts such as inverse, determinant, characteristic matrix equation, symmetric, antisymmetric, unitary or Hermitian matrices are introduced.

7. **Ordinary Differential Equations (ODE) and physical laws.** Physics laws, being the relation between physical variables, from mathematical point of view, are *partial differential equations* (PDE) in time and space [1]. The most prominent example is the Newton's 2nd law:

$$\frac{d^2x(t)}{dt^2} = \frac{1}{m}F(x(t), t),$$

   where the relation between the position $x(t)$ of an object, characterized by its (inertial) mass m and an external force $F(x, t)$ is expressed using mathematics. The above equation is nothing else than a partial differential equation with the unknown function being the $x(t)$.

8. **Partial Differential Equations** For doing physics in more than one dimension, it becomes necessary to use partial derivatives and hence partial differential equations. The first partial differential equations students learn are the linear, separable ones that were derived and solved in the 18th and 19th centuries by people like Laplace, Green, Fourier, Legendre, and Bessel.

9. **Probability and statistics** Probability became of major importance in physics when quantum mechanics entered the scene. A course on probability begins by studying coin flips, and the counting of distinguishable vs. indistinguishable objects. The concepts of mean and variance are developed and applied in the cases of Poisson and Gaussian statistics.

Most of the problems in physics can't be solved exactly in closed form. Therefore one has to rely on approximate calculations with the associated methods required to be developed, such as, power series expansions, saddle point integration, and small (or large) perturbations.

In the following sections some basic concepts of the required mathematical skills for this module are presented.

[1] For example for the ODE for an (idealized) harmonic oscillator system (e.g. mass-spring,pendulum) takes the form:

$$\ddot{x}(t) + \omega_0^2 x(t) = 0.$$

1.1 Elements of trigonometry

The $\sin$ and $\cos$ function of an angle θ are defined through a right angle triangle as (see Fig. 1.1) and the use of the Pythagorean theorem:

$$\tan\theta = \frac{y}{x}, \quad \sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad x^2 + y^2 = r^2$$

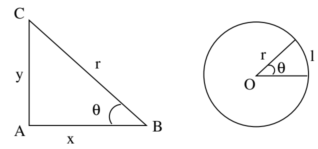


Fig 1.1: Right angle triangle

Standard properties of the $\sin$ and $\cos$ functions are:

$$\cos^2\theta + \sin^2\theta = 1 \quad \cos(-\theta) = \cos\theta \quad \sin(-\theta) = -\sin\theta$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

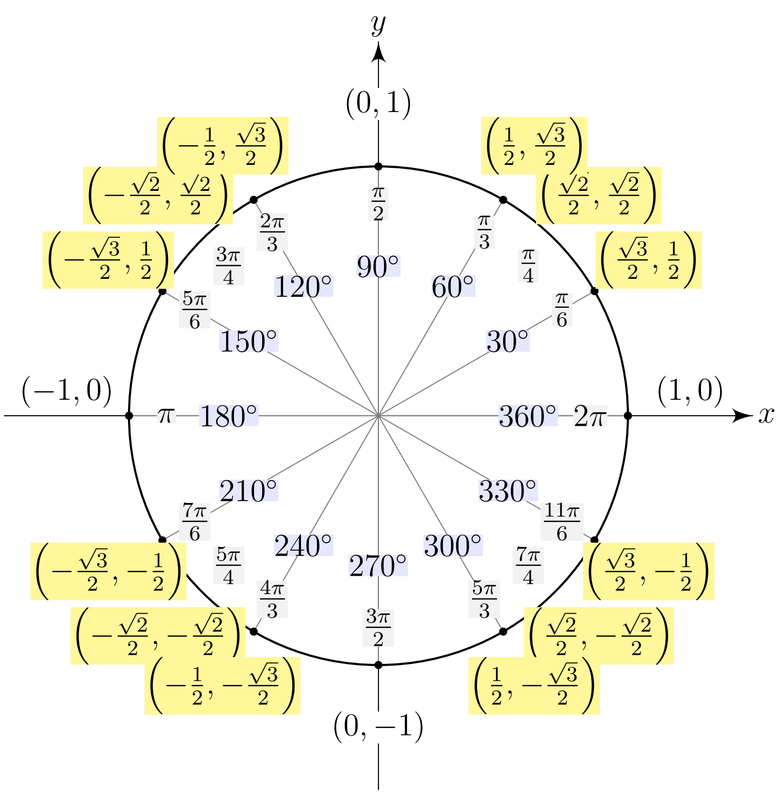


Fig 1.2. *A set of values for* $(x, y) = (\cos\theta, \sin\theta)$ *for various* θ *angle. Angle* θ *is given in rads and in degrees (e.g. For* $\theta = 5\pi/6 = (150^o) \quad \rightarrow \quad (\cos\theta, \sin\theta) = (-\sqrt{3}/2, 1/2)$*).*

Angles are measures on degrees (^{o}) or in radians:

$$360^o = 2\pi \text{ rad}$$

Finally for a cyclical disc of angle θ (see Fig. 1.1) the relation between the arc length (l), the angle and the radius r is as:

$$l = r\theta, \qquad \theta \text{ in radians.}$$

1.2 Vectorial calculus

In physics there are quantities which can be categorized as *scalar* or *vector* quantities. More specifically:

- **Scalars**: Physical quantities fully characterized from their magnitude. Basic scalar quantities in classical mechanics are time (sec) , length (m), mass (kg), work, kinetic and potential energy (Joule), temperature (^{o}K), etc.

- **Vectors** : Physical quantities fully characterized from (a) their magnitude and (b) their direction in space. Basic vector quantities in mechanics are position, velocity, acceleration, force, momentum, etc.

1.2.1 Vectors

Properties

$$
\begin{aligned}
\mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A}, \\
\mathbf{A}+(\mathbf{B}+\mathbf{C}) &= (\mathbf{A}+\mathbf{B})+\mathbf{C} \\
a(\mathbf{A}+(\mathbf{B}) &= a\mathbf{A}+a\mathbf{B},
\end{aligned}
$$

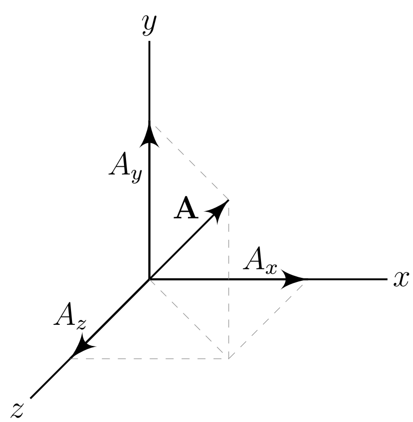


Fig 1.3: Cartesian components of a vector

Unit vectors Vectors with magnitude equal to unity. If a vector $\mathbf{A}$ is divided by its magnitude, then we get the unit vector $\hat{A}=\mathbf{A}/A$ which has the same direction as the vector $\mathbf{A}$, but with unit length.

Inner (or scalar) product The inner product between two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as:

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\Theta_{AB}, \qquad 0\le\theta\le\pi, \tag{1.1}$$

where Θ_{AB} is the angle between the two vectors $\mathbf{A},\mathbf{B}$.

Properties.

$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= \mathbf{B}\cdot\mathbf{A} \\
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{B})+\mathbf{A}\cdot\mathbf{C} \\
a(\mathbf{A}\cdot\mathbf{B}) &= (a\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(a\mathbf{B})
\end{aligned}
$$

Outer (or cross or vector) product Outer product between two vectors **A, B** is defined as:

$$\mathbf{C} \ = \ \mathbf{A}\times\ \mathbf{B} \ \equiv \hat{u}AB\sin\Theta_{AB}, \qquad 0\le\theta\le\pi, \tag{1.2}$$

where, $\hat{u}$ is along the direction of $\mathbf{C}$ and is normal to the plane defined by vectors $\mathbf{A}$ and $\mathbf{B}$ and direction such that a right-three-vector coordinate system to be defined.

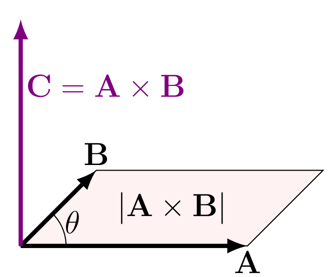


Fig 1.4: Cross product of two vectors is a vector.

Properties

$$
\begin{aligned}
&\mathbf{A}\times\ \mathbf{B} = -\,\mathbf{B}\times\ \mathbf{A} \\
&\mathbf{A}\times(\ \mathbf{B}+\ \mathbf{C}) = (\ \mathbf{A}\times\ \mathbf{B})+\ \mathbf{A}\times\ \mathbf{C} \\
&a(\ \mathbf{A}\times\ \mathbf{B}) = (a\ \mathbf{A})\times\ \mathbf{B} = \ \mathbf{A}\times(a\ \mathbf{B})
\end{aligned} \tag{1.3}
$$

1.2.2 Cartesian coordinate system

A Cartesian Coordinate Systems consists of three mutually $\hat{x}, \hat{y}, \hat{z}$ unit vectors, of fixed direction and origin, independent on the particle's position:

$$\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0, \qquad \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1. \tag{1.4}$$

A practical (and compact) representation of these unit vectors is to represent them as column:

$$\hat{x} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{y} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{z} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

We can then define the rows to be $\hat{x}^T, \hat{y}^T, \hat{z}^T$:

$$\hat{x}^T \equiv \begin{pmatrix} 1, 0, 0 \end{pmatrix}, \quad \hat{y}^T \equiv \begin{pmatrix} 0, 1, 0 \end{pmatrix}, \quad \hat{z}^T \equiv \begin{pmatrix} 0, 0, 1 \end{pmatrix},$$

In this case the inner-products above can be calculated using matrix algebraic techniques. For example,

$$\hat{x} \cdot \hat{y} \equiv \hat{x}^T \cdot \hat{y} = \begin{pmatrix} 1, 0, 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 + 0 + 0 = 0$$

and

$$\hat{x} \cdot \hat{x} \equiv \hat{x}^T \cdot \hat{x} = \begin{pmatrix} 1, 0, 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 + 0 + 0 = 1$$

Vector components In a Cartesian coordinate system $(Oxyz)$ a vector $\mathbf{A}$ is represented (and fully determined) by its three components along the axes (xyz), namely A_x, A_y, A_z in the following way:

$$\mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \tag{1.5}$$

with a magnitude equal to,

$$A = |\,\mathbf{A}\,| = \sqrt{A_x^2 + A_y^2 + A_z^2}, \tag{1.6}$$

In a matrix-representation $\mathbf{A}$ is represented as,

$$\mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \equiv A_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + A_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + A_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

To perform any operation between two vectors $\mathbf{A}$ and $\mathbf{B}$ we can analyze the operations to their components. For example if we want to add $\mathbf{A}$ and $\mathbf{B}$ (see Fig. 2.3) the resulting vector $\mathbf{C}$ is given as:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y}$$

Inner product (scalar) If the vectors **A** and **B** are expressed through their components A_x, A_y and B_x, B_y,

$$\mathbf{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}, \qquad \mathbf{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z},$$

then inner product is given by:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1.7}$$

Finally from Eqns (1.1) and (1.7) we find that the angle θ between the vectors equal to:

$$\cos\Theta_{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}$$

In a matrix-representation of **A** in the basis of $\hat{x}, \hat{y}, \hat{z}$ we have,

$$\mathbf{A} \cdot \mathbf{B} \equiv \mathbf{A}^T \cdot \mathbf{B} = \begin{pmatrix} A_x, A_y, A_z \end{pmatrix} \cdot \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = A_xB_x + A_yB_y + A_zB_z$$

As a special case, the square amplitude of **A** can also be calculated:

$$\mathbf{A} \cdot \mathbf{A} \equiv \mathbf{A}^T \cdot \mathbf{A} = \begin{pmatrix} A_x, A_y, A_z \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = A_xA_x + A_yA_y + A_zA_z = A_x^2 + A_y^2 + A_z^2.$$

Vector (cross/outer) product In an orthogonal coordinate system the cross product may be expresses as:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.8}$$

Triple vector product Consider three vectors **A, B, C**. If the vectors **A,B,C**, are analyzed into their components A_x, A_y, A_z , B_x, B_y, B_z and C_x, C_y, C_z, respectively, then the following vector products are defined as:

Scalar triple product:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_x & A_z \\ B_z & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.9}$$

Vectorial triple product: [2]

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \cdot \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} \tag{1.10}$$

[2] It is worth noticing here that,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

1.2.3 2D Cartesian/polar coordinate system

The vector **A** can be defined either through its components A_x, A_y on a Cartesian CS or through it's magnitude $A \equiv |\mathbf{A}|$ and the angle θ_a with the axis Ox on a polar coordinate system [3].

$$\mathbf{A} = A_x\hat{i} + A_y\hat{j} \tag{1.11}$$

[3] The unit vectors $\hat{i}, \hat{j}, \hat{k}$ very frequently denoted as $\hat{x}, \hat{y}, \hat{z}$ instead.

For the components A_x, A_y simple trigonometry shows the following relations:

$$A_x = A\cos\theta_a, \quad A_y = A\sin\theta_a. \tag{1.12}$$

From the above relation we can also derive the following expressions:

$$A = \sqrt{A_x^2 + A_y^2}, \qquad \theta_a = \tan^{-1}(\frac{A_y}{A_x}) \tag{1.13}$$

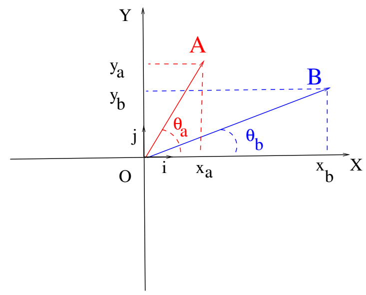


Fig 1.5: 2-D Cartesian/polar systems.

1.3 Algebraic calculus: Quadratic equation

Quadratic equation is defined as:

$$ax^2 + bx + c = 0, \qquad \longrightarrow \qquad x_{1,2} = \begin{cases} \frac{-b\pm\sqrt{b^2-4ac}}{2a}, & b^2 - 4ac > 0 \\ -\frac{b}{2a} & b^2 = 4ac \\ \frac{-b\pm\imath\sqrt{4ac-b^2}}{2a} & b^2 - 4ac < 0 \end{cases}$$

1.4 Elements of differential calculus

Definition: The first derivative of a function $f(x)$ is defined by the following relation:

$$f'(x) = \frac{df(x)}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \tag{1.14}$$

The second derivative is defined as the first derivative of the first derivative $f''(x) = d(f'(x))/dx = d^2 f(x)/dx^2$. Similarly derivatives of higher order are defined:

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{n-1}(x)\right) \qquad n = 0, 1, 2, ...$$

Note that $f^{(0)}(x) = f(x)$.

Basic properties If f and g are functions of x then the following relations hold:

$$\frac{d}{dx}[f(x)g(x)] = f'g + fg' \qquad \text{(product rule)} \tag{1.15}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'g - fg'}{g^2} \qquad \text{(quotient rule)} \tag{1.16}$$

$$\frac{d}{dx}\left(F[g(x)]\right) = \frac{dF(g)}{dg}\frac{dg(x)}{dx}, \qquad \text{(chain rule)} \tag{1.17}$$

Derivatives of simple functions

$$\begin{array}{lll} (a)' = 0, & (x^a)' = ax^{a-1}, & a = \text{ a real number} \\ (\sin x)' = \cos x & (\cos x)' = -\sin x, & (\tan x)' = -1/\cos^2 x, \\ (e^x)' = e^x, & (\ln x)' = 1/x & \end{array}$$

Time derivatives Conventionally, derivatives of time-dependent functions, $f(t)$, are also (compactly) symbolized as:

$$\dot{f}(t) = \frac{df(t)}{dt}, \qquad \ddot{f}(t) = \frac{d^2 f(t)}{dt^2} = \frac{d}{dt}\left[\frac{df(t)}{dt}\right].$$

Examples of such derivatives in mechanics are velocity (first derivative of position) and acceleration (second derivative of position).

Spatial derivatives Conventionally, spatial derivatives of 1-dimensional functions (e.g. $f(x)$) are symbolized as:

$$f'(x) = \frac{df(x)}{dx}, \qquad f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left[\frac{df(x)}{dx}\right].$$

Examples of such derivatives are potential energies, forces etc...

1.4.1 Vector functions (Vector fields)

These are functions that are defined in a multidimensional space. Their values determine 'magnitude' and 'direction'. Their arguments might be a single-variabled or a multivariate, i.e. If $\mathbf{F}$ is a vector-field then it could be such that $\mathbf{F} = \mathbf{F}(x_1)$ or $\mathbf{F}(x_1, x_2, .., x_n) = \mathbf{F}(\mathbf{x})$. In mechanics the arguments maybe time, $x_1 \to t$, distance, $x_1 \to x$ or position vector, $(x_1, x_2, x_3) \to (x, y, z)$ or a combinations of variables e.g. $(x_1, x_2, x_3, x_4) \to (x, y, z, t)$ or any other possible dependence. Examples of vector fields in mechanics are the position $\mathbf{r}$, velocity $\mathbf{v}$ acceleration $\mathbf{a}$, momentum $\mathbf{p} = m\mathbf{v}$ and force, $\mathbf{F} = m\mathbf{a}$. The derivative of a singly-varied vector function $\mathbf{F}(t)$ is defined as below: [4]

$$\mathbf{F}'(q) = \frac{d\,\mathbf{F}(q)}{dq} = \lim_{\delta q \to 0} \frac{\mathbf{F}(q + \delta q) - \mathbf{F}(q)}{\delta q} \tag{1.18}$$

[4] The q variable could represent, distance, angle, time or anything else.

The second derivative is defined as, $\mathbf{F}''(q) = d(\mathbf{F}'(q)/dq) = d^2\,\mathbf{F}(q)/dq^2$. In an orthogonal coordinate system $(Oxyz)$ the vector function can be expressed through its components as:

$$\mathbf{F}(q) = F_x(q)\hat{x} + F_y(q)\hat{y} + F_z(q)\hat{z} = \begin{pmatrix} F_x(q) \\ F_y(q) \\ F_z(q) \end{pmatrix}$$

then the first and second derivative are evaluated as:

$$\begin{aligned} \frac{d\,\mathbf{F}(q)}{dq} &= \frac{dF_x(q)}{dq}\hat{x} + \frac{dF_y(q)}{dq}\hat{y} + \frac{dF_z(q)}{dq}\hat{z} \\ \frac{d^2\,\mathbf{F}(q)}{dq^2} &= \frac{d^2F_x(q)}{dq^2}\hat{x} + \frac{d^2F_y(q)}{dq^2}\hat{y} + \frac{d^2F_z(q)}{dq^2}\hat{z} \end{aligned}$$

Basic properties If $\mathbf{F}(q), \mathbf{G}(q), g(q)$ are functions of one variable q then the following relations hold:

$$\begin{aligned} \frac{d}{dq}[\mathbf{F}(q)g(q)] &= \mathbf{F}'g + \mathbf{F}g' \\ \frac{d}{dq}[\mathbf{F}(q) \cdot \mathbf{G}(q)] &= \mathbf{F}'(q) \cdot \mathbf{G}\,(q) + \mathbf{F}(q) \cdot \mathbf{G}'(q) \end{aligned} \tag{1.19}$$

1.5 Integration calculus

[5]

The *indefinite* integral $\int f(x)dx$ of a function $f(x)$ is defined as

$$\int f(x)dx = g(x) + c \quad \Leftrightarrow \quad f(x) = \frac{d}{dx}g(x) \tag{1.20}$$

where $f(x) = dg(x)/dx$. Indefinite integral in this sense is the inverse of the derivative operation on a function.

The *definite* integral of the function $f(x)$ between a and b is defined as:

$$\int_a^b f(x)dx = \int_a^b \frac{dg(x)}{dx}dx = \int_a^b dg(x) = g(b) - g(a). \tag{1.21}$$

Basic properties If f and g are single-variable functions then the following relation holds:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \qquad \text{(integration by parts)} \tag{1.22}$$

Derivative of an integral (Leibnitz integration rule) The following derivative for a general integral holds [6] ,

$$\boxed{\frac{d}{dx}\int_{\lambda_1(x)}^{\lambda_2(x)} dtF(x,t) = F(x,\lambda_2)\lambda_2' - F(x,\lambda_1)\lambda_1' + \int_{\lambda_1}^{\lambda_2} dt\frac{\partial F(x,t)}{\partial x}} \tag{1.23}$$

where $\lambda_i \equiv \lambda_i(x) \in [-\infty,\infty]$ and $\lambda_i'(x) \equiv \dfrac{d\lambda_i}{dx} i = 1, 2$.

1.5.1 Integration of vector functions

Indefinite integral If $\mathbf{F}(t)$ is a single-variable vector function given by Eqn. (1.19), then its indefinite integral is given by:

$$\int \mathbf{F}(t)dt = \hat{x}\int F_x(t)dt + \hat{y}\int F_y(t)dt + \hat{z}\int F_z(t)dt \tag{1.24}$$

Definite integral If $\mathbf{G}(t)$ is a single-variable vector field such that $\mathbf{F}(t) = d\mathbf{G}(t)/dt$, then the *indefinite* integral of $\mathbf{F}(t)$ is equal to:

$$\int dt\, \mathbf{F}(t) = \int dt\frac{d}{dt}\,\mathbf{G}(t) = \mathbf{G}(t) + \mathbf{c}, \tag{1.25}$$

where $\mathbf{c}$ is a constant integration vector. The *definite* integral between the values a, b is:

$$\int_a^b dt\, \mathbf{F}(t) = \int dt\frac{d}{dt}\,\mathbf{G}(t) = [\mathbf{G}(t) + \mathbf{c}]_a^b = \mathbf{G}(t=b) - \mathbf{G}(t=a). \tag{1.26}$$

[5] Indefinite integrals of some functions:

$$\int a dx = ax + C$$
$$\int x^a dx = \frac{x^{a+1}}{(a+1)} + C$$
$$\int \sin x dx = -\cos x + C$$
$$\int \cos x dx = \sin x + C$$
$$\int e^x dx = e^x + C$$
$$\int \frac{dx}{x} = \ln x + C$$

[6] In the special case where λ_1, λ_2 are independent on x we have,

$$\frac{d}{dx}\int_{\lambda_1}^{\lambda_2} dtF(x,t) = \int_{\lambda_1}^{\lambda_2} dt\frac{\partial F(x,t)}{\partial x}.$$

Therefore the derivative and the integral can be interchanged.

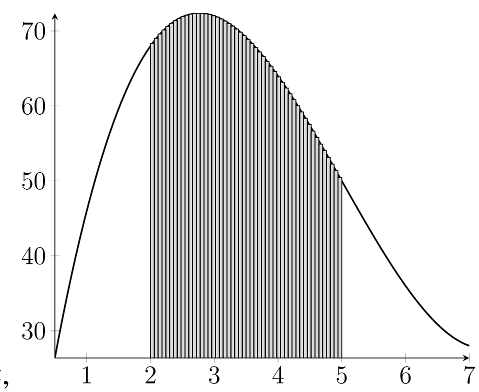


Fig 1.6: The geometrical meaning of integration is the area of the surface under $f(x)$: $\int_0^5 f(x)dx$

1.6 Differential calculus of multi-variable functions

Let's assume a multivariate scalar function $f(s_1, s_2, ..s_n)$. Then on can define partial derivatives where only one variable is allowed to vary will the other remain constant.

[7] Consider a two-variable function $f(x, y)$. Then the partial derivatives in terms of variables (x, y) are given as :

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} = \frac{df(x, y = \text{const})}{dx}$$
$$\frac{\partial f(x,y)}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} = \frac{df(x = \text{const.}, y)}{dy}$$

[7] **Example:** if $f(x, y) = e^{3x} \cos y$, then its partial derivatives are:

$$\frac{\partial f(x,y)}{\partial x} = \frac{df(x, y = \text{const.})}{dx} = 3e^{3x} \cos y$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{df(x = \text{const.}, y)}{dy} = -e^{3x} \sin y$$

1.6.1 Differential operator

Let's assume the vector field $\mathbf{F} = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z)$ and the scalar field $\phi(\mathbf{r}) = \phi(x, y, z)$. The vector field $\mathbf{F}(\mathbf{r})$, as known, in a orthogonal coordinate system may expressed through its components along the axes. For a $(Oxyz)$ coordinate system we thus have:

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z) = F_x(\mathbf{r})\hat{i} + F_y(\mathbf{r})\hat{j} + F_z(\mathbf{r})\hat{k}$$

A useful mathematical quantity, of a multi-variable vector field $\mathbf{F} = \mathbf{F}(\mathbf{r})$ is the *differential operator*, ∇ which has the role of vectorial first derivative. In Cartesian coordinates is expressed as:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{x}\frac{\partial}{\partial z}$$

Gradient of a scalar field Gradient of scalar field $\phi(\mathbf{r})$ is defined to be the following vectorial quantity:

$$\nabla\phi(\mathbf{r}) = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$

Global or total derivative Global derivative of scalar field $\phi(\mathbf{r})$ is the scalar quantity:

$$d\phi(r) = d\mathbf{r} \cdot \nabla\phi(\mathbf{r}) = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz, \tag{1.27}$$

where $\mathbf{dr} = \mathbf{r} - \mathbf{r}'$.

Divergence The *divergence* of a vector field $\mathbf{F}(\mathbf{r})$ is defined as the scalar quantity resulting from evaluating the inner product of the differential operator ∇ and the vector field $\mathbf{F}(\mathbf{r})$:

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Rotation/Curl The *rotation* (or *curl*) of a vector field $\mathbf{F}(\mathbf{r})$ is defined as the vector quantity resulting from evaluating the outer product of the differential operator ∇ and the vector field $\mathbf{F}(\mathbf{r})$:

$$\nabla \times \mathbf{F}(\mathbf{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial_x & \partial/\partial_y & \partial/\partial_z \\ F_x & F_y & F_z \end{vmatrix}$$

Line integral Of particular interest is the *line integral* of vector field $\mathbf{F}(\mathbf{r})$ along curve $C : \mathbf{r} = \mathbf{r}(t)$:

$$I_C = \int_C \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}) = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}) \tag{1.28}$$

In Cartesian coordinates the line integral is written as:

$$I_C = \int_C \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}) = \int_C F_x dx + F_y dy + F_z dz \tag{1.29}$$

Potential theorem

In the special case where the vector field can be written as the gradient of a scalar function $\phi(\mathbf{r})$, the line integral simplifies considerably [8],

$$\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r}) \qquad \rightarrow \qquad \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{dr} \cdot \nabla\phi(\mathbf{r}) = \phi(\mathbf{r}_b) - \phi_b(\mathbf{r}_a) \tag{1.30}$$

This is known as *potential theorem*.

To summarize, the differential operators in **Cartesian coordinates** are expressed as below [9]:

$$\nabla_{\mathbf{r}} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}, \tag{1.31}$$

$$\nabla_{\mathbf{r}} \cdot \mathbf{F}(\mathbf{r}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \tag{1.32}$$

$$\nabla_{\mathbf{r}} \times \mathbf{F}(\mathbf{r}) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z} \tag{1.33}$$

$$\nabla_{\mathbf{r}} f(\mathbf{r}) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}, \tag{1.34}$$

$$\nabla^2_{\mathbf{r}} f(\mathbf{r}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \tag{1.35}$$

$$\tag{1.36}$$

[8] To arrive to this expression (1.27) was used:

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{dr} \cdot \nabla\phi(\mathbf{r}) = \int_{\mathbf{r}_a}^{\mathbf{r}_b} d\phi(\mathbf{r}).$$

Its generalization to 2- and 3-dimensions are known as Stokes and Gauss theorems, respectively; All these theorems are of extremely important for the formulation of classical electrodynamics

[9] The subscript $\mathbf{r}$ at $\nabla_{\mathbf{r}}$ is most often than not suppressed when it is clear that the derivative variable is $\mathbf{r}$.

1.7 Extreme points of functions

Single-variable function To find the extreme point of a single-variable function we have to take the 1^{st} and 2^{nd} derivatives:

$$f'(x_m) = 0, \qquad f''(x_m) = \begin{cases} k > 0 & \text{minimum} \\ k < 0 & \text{maximum} \end{cases} \tag{1.37}$$

The point(s) x_m satisfying the above conditions are either a *maximum* or *minimum*.

Multi-variable function Let's consider the example of a two-variable function $f(x, y)$. We need to consider two cases:

1. *No constraint for x, y.*

   The extreme points for this function are obtained by setting

   $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad \to \qquad \begin{cases} \frac{\partial}{\partial x} f(x_m, y_m) & = 0 \\ \frac{\partial}{\partial y} f(x_m, y_m) & = 0 \end{cases} \tag{1.38}$$

   The above conditions define a *stationary point* with a value function $f(x_m, y_m)$; these two algebraic equations provides the extreme point (x_m, y_m). Whether this stationary point is extremum (maximum or minimum) is determined by the behaviour of the second variation, $d^2 f$, with respect to x, y.

2. *Constraint for x, y, $\phi(x, y) = 0$. Method of Lagrange multipliers*

   Let's assume the case where the x, y variables are not fully independent each other but they are related. Let's assume that their functional relation can be cast as $\phi(x, y) = 0$. In order to find the extreme points of $f(x, y)$ given this *constraint* we can work as follows:

   - First we construct the *auxilliary* (augmented) function:

     $$F(x, y, \lambda) = f(x, y) + \lambda\phi(x, y).$$

     This function now has 3 variables that is dependent on, namely (x, y, λ); where λ takes arbitrary values.

   - Now we are looking the extreme points his new function $F(x, y, \lambda)$ in the usual way:

     $$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial \lambda}d\lambda = 0 \qquad \to \qquad \begin{cases} \frac{\partial F}{\partial x} & = 0 \\ \frac{\partial F}{\partial y} & = 0 \\ \frac{\partial F}{\partial \lambda} & = 0 \end{cases}$$

     The above system represents a system of three algebraic equations with three unknowns and its solution provides the extreme point (x_m, y_m) and the (now) useless λ.

1.8 Examples

Example 1. Line integral along a circle

Imagine that we need the line integral of a function along a circle of radius R. Let's consider the case where the circle lies on the plane $xy(z = 0)$ and its center is at the origin of the coordinate system $(Oxyz)$. Our goal is to evaluate the line integral,

$$I_C = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr}$$

when the $\mathbf{r}$ lies along a circular path $r = R$. The simplest way to evaluate that integral is to use the polar coordinate system (r, θ). For $x = R\cos\theta$ and $y = R\sin\theta$ we have

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z} = R\cos\theta\hat{x} + Rsin\theta\hat{y},$$

Then **dr** is expressed as:

$$\mathbf{dr} = (-R\sin\theta\hat{x} + R\cos\theta\hat{y})d\theta \quad \longrightarrow \quad \begin{cases} dx & = -R\sin\theta d\theta, \\ dy & = +R\cos\theta d\theta \end{cases} \tag{1.39}$$

where θ is the angle with respect to the $x-$axis. Then the line integral is given by:

$$I_C = \oint_C F_x dx + F_y dy = \oint_{\theta_a}^{\theta_b} d\theta \left[-R\sin\theta F_x(r,\theta) + R\cos\theta F_y(r,\theta)\right] \tag{1.40}$$

Here the line integral has been expressed as a single-variable integral (θ), since $r = R = const.$

1.9 Questions

1. Write in scientific notation ($a \times 10^b$) the number 0.0123.

2. How many radians are 22.5^o degrees?

3. What is the angle in degrees of $\theta = \sin^{-1}(0.38261) = \arcsin(0.38261)$?

4. Assume a Cartesian coordinate system in two dimensions. A vector **A** is expressed as $\mathbf{A} = (1/\sqrt{2})\hat{\imath} + (1/\sqrt{2})\hat{\jmath} = (1/\sqrt{2}, 1/\sqrt{2})$. What is its magnitude?

5. For the vector **A** of the previous problem what is the angle of the vector with the axis Ox?

6. In the same coordinate system of the last two problems, a vector **B** is defined to be as $\mathbf{B} = 3\hat{\imath} + 4\hat{\jmath} = (3, 4)$. What is the magnitude of the vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$?

7. For the vectors **A**, **B** above, what is the angle of the vector $\mathbf{C} = \sqrt{2}\,\mathbf{A} - \mathbf{B}$ with the axis Ox?

8. What is the second derivative of function $\sin(x^2)$ and the first derivative of e^{sint}?

9. What is the indefinite integral of $\int (ax^2 + bx + c)dx$?

10. What is the value of the definite integral of $\int_{-1}^{0} (x^2 - 3x + 1)dx$?

11. If $T = 2\pi/\omega$, show that:

$$\int_0^T dt \cos^2 \omega t = \int_0^T dt \sin^2 \omega t = \frac{T}{2},$$

$$\int_0^T dt \sin \omega t \cos \omega t = 0.$$

12. Assuming that A_1, A_2 are known, find the values of A and ϕ so that,

    (a)
$$A_1 \cos \omega t + A_2 \sin \omega t = A \cos(\omega t + \phi)$$

    (b)
$$A_1 \cos \omega t + A_2 \sin \omega t = A \sin(\omega t + \phi)$$

13. Now assuming that A, ϕ are known, find the values of A_1 and A_2 so that the same expressions as above hold:

    (a)
$$A_1 \cos \omega t + A_2 \sin \omega t = A \cos(\omega t + \phi)$$

    (b)
$$A_1 \cos \omega t + A_2 \sin \omega t = A \sin(\omega t + \phi)$$

14. Using the chain rule for the derivatives, show that,

$$\frac{d}{dt}[x(t)^2] = 2x(t)v(t), \qquad \text{where } v(t) = \dot{x}(t)$$

15. Find the extreme points of the following functions:

$$y(x) = x^2(t), \qquad y(x) = \frac{1}{1+x^2}, \qquad y(x) = sin(2t)$$

16. Find the extreme points of the following function, as well as their values at these points:

$$f(x) = \frac{1}{\sqrt{(x^2 - a^2)^2 + b^2}}$$

where, f_0, a, b known real, numbers.

(hint: Determine the extreme points from the solution of the equation $f'(x_m) = 0$. It might be easier to find the extreme points of the function under the square-root $((x^2 - a^2)^2 + b^2)$)

17. Find the extreme points of the following function, as well as their values at these points:

$$f(x) = \frac{x}{(x^2 - a^2)^2 + b^2}$$

where, f_0, a, b known real, numbers.

(hint: Determine the extreme points from the solution of the equation $f'(x_m) = 0$.)

Chapter 2

Basics of kinematics

In kinematics one is studying the properties of the position vector $\mathbf{r} = \mathbf{r}(t)$, velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ of a material body. Knowledge of the position vector $\mathbf{r} = \mathbf{r}(t)$ is sufficient to determine any kinematic quantity required, by taking its time derivatives. Depending on the physical problem the corresponding calculations are facilitated by a proper choice of the coordinate system (Cartesian, Spherical, Cylindrical etc). In the below the related definitions are given for the two of the most useful coordinate systems, namely the Cartesian (CCS) and the polar coordinate systems (PCS). [10]

[10] In the below, the unit vector triad might be denoted interchangeably as, $(\hat{i}, \hat{j}, \hat{k}) = (\hat{x}, \hat{y}, \hat{z})$ and $(\hat{r}, \hat{\theta}) = (\hat{e}_r, \hat{e}_\theta)$

For a given vectorial physical quantity (position, velocity, acceleration, momentum, etc) the main idea here is based on the assumption that can be represented on an coordinate system by three scalar quantities which are its projection along the coordinate axes. So if $\mathbf{Q}$ is such a quantity and $\hat{i}, \hat{j}, \hat{k}$ defines an orthogonal coordinate system then we always can write, $\mathbf{Q}$ as,

$$\mathbf{Q} = \sum_{i=1-3} Q_i \hat{q}_i = Q_1\hat{q}_1 + Q_2\hat{q}_2 + Q_3\hat{q}_3 = (Q_1, Q_2, Q_3), \tag{2.1}$$

$$\hat{q}_i \cdot \hat{q}_i = 1, \qquad \hat{q}_i \cdot \hat{q}_j = 0 \tag{2.2}$$

The magnitude of this quantity in terms of its components is expressed by,

$$\mathbf{Q}^2 = \sum_{i=1-3} Q_q^2 = Q_1^2 + Q_2^2 + Q_3^2$$

In the present notes we'll concetrate mostly on two types of orthogonal coordinate systems, one suited better for translational motion while the other is more convenient to describe rotational motion of bodies. Those two will be referred as Cartesian coordinate system (CCS) while the other as polar coordinate system (PCS). So we have,

$$\text{Cartesian CS} \quad \longleftrightarrow \quad (\hat{q}_1, \hat{q}_2, \hat{q}_3) = (\hat{x}, \hat{y}, \hat{z}) \tag{2.3}$$

$$\text{Polar CS} \quad \longleftrightarrow \quad (\hat{q}_1, \hat{q}_2, \hat{q}_3) = (\hat{r}, \hat{\theta}, \hat{z}) \tag{2.4}$$

$$\tag{2.5}$$

As the vectorial quantities of interest for as are position, $\mathbf{r}$, velocity, $\mathbf{v}$ and acceleration $\mathbf{Q}$ will be replaced by one of these vectors. Moreover while in general the motion is taken place in the 3-D space we'll see that often the motion will be restricted in 1-D or 2-D space. In this case by properly chosing the coordinate system we can safely ignore one of the dimensions (conventionally this is the $z-$axis) and work out the problem either in the Cartesian $(\hat{x}, \hat{y})$ or in the polar $(\hat{r}, \hat{\theta})$ systems.

2.1 Kinematic quantities in mechanics

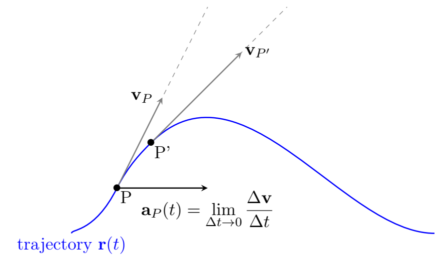


Fig 2.1: Particle's acceleration and velocity vector.

Among the fundamental quantities in mechanics is the position vector, $\mathbf{r} = r(t)$ used to determine the object's motion. Ultimately one can say that the main task of a mechanical problem is to provide the means to evaluate $\mathbf{r} = \mathbf{r}(t)$. *Velocity* is defined to be the first derivative in time (rate of change) of the position vector $\mathbf{r}(t)$

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t} \tag{2.6}$$

Acceleration is defined to be the first derivative in time of the velocity [11]:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{v}(t+\delta t) - \mathbf{v}(t)}{\delta t} \tag{2.7}$$

[11] Or equivalently the second derivative in time of the position vector

$$\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$$

Note that for convenience, conventionally, time-derivatives are denoted by,

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t), \qquad \mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t)$$

and accordingly for their components.

2.1.1 Cartesian coordinates

Position The time dependance of the position vector in a Cartesian Coordinate System (CCS) $(\hat{x}, \hat{y}, \hat{z})$ is expressed as:

$$\mathbf{r} = \mathbf{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \tag{2.8}$$

with $\hat{x}, \hat{y}, \hat{z}$, the mutually orthogonal vectors along the axes xyz, respectively. The magnitude of the position vector $\mathbf{r}$, being the distance from the axis origin is equal to:

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}, \tag{2.9}$$

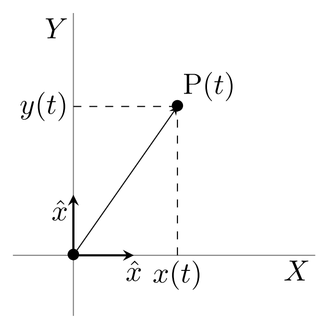


Fig 2.2: 2-D CCS, $\mathbf{r} = (x, y)$.

Its matrix-representation is:

$$\mathbf{r}(t) = x(t)\begin{pmatrix}1\\0\\0\end{pmatrix} + y(t)\begin{pmatrix}0\\1\\0\end{pmatrix} + z(t)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix} \tag{2.10}$$

Velocity The velocity vector $\mathbf{v}(t)$ in CCS is as:

$$\boxed{\mathbf{v}(t) = v_x(t)\hat{x} + v_y(t)\hat{y} + v_z(t)\hat{z} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} = \begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix}} \tag{2.11}$$

with the magnitude of the velocity (speed) equal to: $v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$,

Acceleration The expression of the acceleration in a Cartesian CS is the following:

$$\mathbf{a}(t) = a_x(t)\hat{x} + a_y(t)\hat{y} + a_z(t)\hat{z} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} \tag{2.12}$$

with the magnitude of the acceleration a equal to:

$$a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{2.13}$$

2.1.2 Polar CS

Position The time dependance of the position vector in a Polar Coordinate System (PCS) $(\hat{r}, \hat{\theta}, \hat{z})$ is expressed as:

$$\mathbf{r} = \mathbf{r}(t) = r(t)\hat{r} + z(t)\hat{z}, \tag{2.14}$$

with $\hat{r}, \hat{\theta}, \hat{z}$ unit vectors, mutually orthogonal each other:

$$\hat{r} \cdot \hat{\theta} = \hat{r} \cdot \hat{z} = \hat{\theta} \cdot \hat{z} = 0, \qquad \hat{z} \cdot \hat{z} = \hat{r} \cdot \hat{r} = \hat{\theta} \cdot \hat{\theta} = 1. \tag{2.15}$$

Its matrix-representation on the polar basis ($\hat{r}$,$\hat{\theta}$,$\hat{z}$):

$$\mathbf{r}(t) = r(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z(t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} r(t) \\ 0 \\ z(t) \end{pmatrix} \tag{2.16}$$

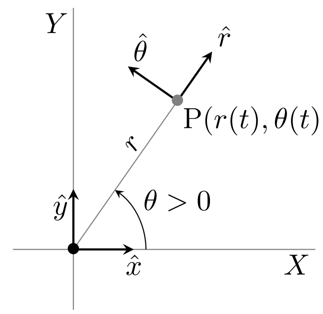


Fig 2.3: 2-D PCS $\mathbf{r} = (r, \theta)$.

Velocity The velocity vector in PCS is expressed as [12] :

$$\mathbf{v}(t) = v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z} = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ \dot{z} \end{pmatrix} \tag{2.19}$$

[12] Using (2.29) the time derivative for $\hat{r}, \hat{\theta}$ are:

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta} \tag{2.17}$$

$$\dot{\hat{\theta}} = -\dot{\theta}\hat{r} \tag{2.18}$$

So, in contrast with the cartesian unit vectors $\hat{x}, \hat{y}, \hat{z}$, the direction of the polar vector units $\hat{r}, \hat{\theta}$ change with time as the position vector tracks the point's path.

where v_r is the *radial* velocity and v_θ is the *tangential* (or angular) velocity. The radial velocity measures the rate of change of the distance of the body from the origin while the tangential velocity measures the rate of displacement of the body normal to its radial axis. The magnitude of the velocity in terms of the polar variables is:

$$v = |\mathbf{v}| = \sqrt{v_r^2 + v_\theta^2 + v_z^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}} \tag{2.20}$$

Acceleration The expression for the acceleration in a CCS ($\mathbf{r} = r\hat{e}_r$) is the following:

$$\mathbf{a}(t) = a_r\hat{r} + a_\theta\hat{\theta} + A_z\hat{z} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} + \ddot{z}\hat{z} = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ \ddot{z} \end{pmatrix} \tag{2.21}$$

where a_r is the *radial* acceleration and a_θ is the *tangential* acceleration. The magnitude of acceleration a in terms of (a_r, a_θ) is:

$$a = |\mathbf{a}| = \sqrt{a_r^2 + a_\theta^2 + \ddot{a}_z} \tag{2.22}$$

Planar polar CS

If the motion is taking place in a flat plane, then one may ignore the third axis (here $\hat{z}$) by setting $z = \dot{z} = \ddot{z} = 0$ in the above expressions. In this case we have for the position, velocity and acceleration in the planar polar CS:

$$\mathbf{r}(t) = r(t)\hat{r} \tag{2.23}$$
$$\mathbf{v}(t) = \dot{r}(t)\hat{r} + r\dot{\theta}(t)\hat{\theta} \tag{2.24}$$
$$\mathbf{a}(t) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \tag{2.25}$$

Cartesian to polar coordinates and vice versa The relationship of the position vector in the Cartesian CS and the polar CS is as:

$$\mathbf{r}(t) = x(t)\hat{x} + y(t)\hat{y} = r(t)\hat{r} \tag{2.26}$$

It is straightforward to show the following relations between (x, y) and (r, θ) variables:

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{2.27}$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \tag{2.28}$$

with the unit vectors $\hat{x}, \hat{y}$ related with the polar unit vectors $\hat{r}, \hat{\theta}$ through the following relations:

$$\begin{aligned} \hat{r} &= \hat{x}\cos\theta + \hat{y}\sin\theta, & \hat{x} &= \hat{r}\cos\theta - \hat{\theta}\sin\theta \\ \hat{\theta} &= -\hat{x}\sin\theta + \hat{y}\cos\theta & \hat{y} &= \hat{r}\sin\theta + \hat{\theta}\cos\theta \end{aligned} \tag{2.29}$$

2.2 Determining position and velocity

In the below we'll consider two cases, one where the acceleration is constant in time, $a = const$, while in the second one we'll assume that the acceleration varies with time, $a = a(t)$. In the former case the solution is trivial, while for the latter the solution's complexity varies depending on the problem. There might be solutions for the position and the velocity that can be expressed analytically (in terms of known functions) but it can also be the case where the body's path can only be traced only numerically.

2.2.1 1-D motion

Lets that at time t_0, body's position is x_0 and its velocity equal to v_0:

$$x(t_0) = x_0 \qquad v(t_0) = v_0,$$

The task now can be expressed as below:

Given the acceleration $a(t)$, our goal now is to determine position and velocity at any given time, namely the quantities, $x = x(t)$ and $v = v(t)$.

Calculation of $v = v(t)$

In this case, by using definition of the acceleration as the first time derivative of the velocity $a = dv/dt$, we evaluate the definite integral of $dv = a(t)dt$ from time $t = t_0$ up to t:

$$a(t)dt = dv(t) \qquad \rightarrow \qquad \int_{t_0}^{t} a(t')dt' = \int_{v_0}^{v} dv = v(t) - v(0)$$

to obtain,

$$v(t) = v_0 + \int_{t_0}^{t} a(t)dt. \tag{2.30}$$

This way we obtain the velocity as a function of acceleration (known quantity) and the initial velocity. Since both acceleration and initial velocity are given, then the time dependance of body's velocity is also known. The exact time-dependance $v = v(t)$ is determined when the exact time-dependance of acceleration $a = a(t)$ is given.

Calculation of $x = x(t)$

Having determined velocity one proceeds by recalling the definition of velocity ($v = dx/dt$) to evaluate the definite integral of $dx = v(t)dt$ from the initial time t_0 to an arbitrary time t. The result provides the body's position $x(t)$ as a function of time.

For example, if we set $\int_{t_0}^{t} dt' a(t') = v(t')$, then the above procedure gives the following expression for the position:

$$x(t) = x_0 + v_0(t - t_0) + \int_{t_0}^{t} dt'\, v(t'). \tag{2.31}$$

2.2.2 3-D motion

The above relations for the position, velocity and acceleration are generalized straightforwardly for 2-D and 3-D cases, in order to include more general cases of motion for a particle, e.g. projectile motion, planetary motion, collisions etc. Lets that at time t_0, body's position is $\mathbf{r}_0$ and its velocity equal to $\mathbf{v}_0$. Again, as in the one-dimension case, given the acceleration $\mathbf{a}(t)$, our final task is to determine the position, $\mathbf{r}(t)$ and velocity, $\mathbf{v}(t)$ at any given time. The method use is to start from the corresponding definitions for acceleration, first, and then for velocity:

Calculation of velocity $\mathbf{v} = \mathbf{v}(t)$

Recalling the definition of acceleration as $\mathbf{a} = d\mathbf{v}/dt$, we integrate $d\mathbf{v} = \mathbf{a}dt$, from time $t = t_0$ up to arbitrary time t. So by assuming that at time t the velocity of the particle is $\mathbf{v}(t)$ starting at time $t_0 = 0$ with velocity $\mathbf{v}(0) = \mathbf{v}_0$:

$$\mathbf{a}(t)dt = d\mathbf{v}(t) \qquad \rightarrow \qquad \int_{t_0}^{t} \mathbf{a}(t')dt' = \int_{\mathbf{v}_0}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}(t) - \mathbf{v}(0)$$

to arrive at,

$$\boxed{\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{a}(t')dt' \qquad (2.32)}$$

Most of than not, in relatively simple problems, this integral can be analyzed the vectors to cartesian coordinates so that to transform the integrals to one-dimensional integrals.

Calculation of position $\mathbf{r} = \mathbf{r}(t)$

Having determined velocity $\mathbf{v} = \mathbf{v}(t)$ from step (2), similarly, from the definition of velocity as, $\mathbf{v} = d\mathbf{r}/dt$, we integrate $d\mathbf{r} = \mathbf{v}(t)dt$ from the initial time t_0 up to arbitrary time t. The result provides position vector $\mathbf{r} = \mathbf{r}(t)$. Along similar lines with those we followed to evaluate $\int_{t_0}^{t} \mathbf{a}(t)dt = \mathbf{v}(t)$ one arrives at an expression for $\mathbf{r} = \mathbf{r}(t)$. At time t the position of the particle is $\mathbf{r}(t)$ starting off (at time $t_0 = 0$) from $\mathbf{r}(0) = \mathbf{r}_0$ [13]:

$$\mathbf{v}(t)dt = d\mathbf{r}(t) \qquad \rightarrow \qquad \int_{t_0}^{t} \mathbf{v}(t')dt' = \int_{\mathbf{r}(0)}^{\mathbf{r}} d\mathbf{r} = \mathbf{r}(t) - \mathbf{r}(0),$$

finally obtaining:

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(t')dt'. \qquad (2.34)}$$

[13] For the case where $\mathbf{a} = const$ we have:

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}(t - t_0) \quad (2.33)$$
$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t - t_0) + \frac{1}{2}\mathbf{a}(t - t_0)^2$$

Other useful expressions that relate position, velocity and acceleration are:

$$v^2 = v_0^2 + 2\mathbf{a}(\mathbf{r}(t) - \mathbf{r}_0)$$
$$\mathbf{r}(t) - \mathbf{r}_0 = \frac{\mathbf{v}(t) + \mathbf{v}_0}{2}(t - t_0)$$

2.3 Examples

Example 1. Free fall-kinematics

Let's assume a body falling under the Earth's gravitational acceleration, ($\sim g = 9.81$ m/s^2), with initial velocity v_0 and starting off from an altitude, $h > 0$ above Earth's grond plane (approximated as a flat plane).

1. *Coordinates*

   It is convenient to choose a Cartesian CS with its $x - z$ plane to coincide with Earth's horizontal plane, and the Oy having direction upwards (opposite the direction of the gravitational acceleration). Then the acceleration and the initial position are vectorially expressed as,

   $$a = -g\hat{y}, \qquad \mathbf{r}(0) = h\hat{y}, \qquad \mathbf{v}(0) = v_0\hat{y}.$$

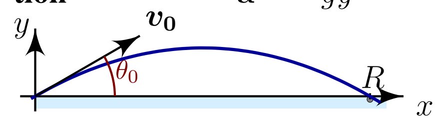
2. *Kinematic relations* In this case the kinematic equations give (along the axes x and z we have $x(t) = z(t) = 0$):

   $$a = -g, \tag{2.35}$$
   $$v(t) = v_0 - gt, \tag{2.36}$$
   $$y(t) = h + v_0 t - \frac{1}{2}gt^2. \tag{2.37}$$

Example 2. Projectile motion-kinematics

Let's at initial time $t_0 = 0$ body's position is $\mathbf{r}_0 = 0$ and this body is ejected with initial velocity $\mathbf{v}_0$. In this case, we define the CCS $Oxyz$ so that the motion to be in plane xy with $x-$axis being the horizontal axis and the y-axis being the vertical axis, along the gravity acceleration. The direction of $y-$axis is defined to be in opposite direction with gravity's acceleration vector. Analyzing the acceleration, $\mathbf{a}$, to its Cartesian components $\mathbf{a} = (a_x, a_y, 0)$ we have (the motion along the z-axis formally is $z = z(t) = 0$):

Projectile motion $\mathbf{a} = -g\hat{y}$



$$\mathbf{a} = 0\hat{x} + (-g)\hat{y} = -g\hat{y} \quad \Longrightarrow \quad (a_x, a_y) = (0, -g)$$

Thus, along $x-$axis we have motion with zero (constant) acceleration ($a_x = 0$) and along $y-$axis we again have motion with constant acceleration but non-zero ($a_y = -g$). By analyzing initial position and velocity vectors to their components we have:

$$\mathbf{r}(0) = \mathbf{r}_0 = (x_0, y_0) = (0, 0)$$
$$\mathbf{v}(0) = \mathbf{v}_0 = (v_{0x}, v_{0y}) = (v_0 \cos\theta, v_0 \sin\theta)$$

where the ejection angle θ_0 is with respect to the horizontal axis ($x-$axis). Given the above, kinematic equations (**??**)-(**??**), read:

$$x\text{-axis projection}: \quad a_x = 0 \tag{2.38}$$
$$v_x = v_0 \cos\theta_0 \tag{2.39}$$
$$x = (v_0 \cos\theta_0)t \tag{2.40}$$
$$y\text{-axis projection}: \quad a_y = 0 \tag{2.41}$$
$$v_y = v_0 \sin\theta_0 - gt \tag{2.42}$$
$$y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \tag{2.43}$$

Distance from the axis-origin and body's speed at any given time are given by,

$$r(t) = \sqrt{x^2 + y^2}, \qquad v(t) = \sqrt{v_x^2 + v_y^2},$$

respectively. From the above kinematic equations, by eliminating the time dependence we obtain the travelling path of the body: [14]

$$y = \tan\theta_0 \; x - \frac{g}{2v_0^2 \cos^2\theta_0} x^2. \tag{2.44}$$

So it is concluded that in a gravitational field the motion of any body is a parabola ($y = bx - cx^2$) the exact details of which are determined at initial time t_0, through the initial velocity v_0. Note that at any given time θ_0 is given by:

$$\tan\theta_0 = \frac{v_{0x}}{v_{0y}}$$

[14] **Time elimination:** From (2.40) one solves for $t = x(t)/v_0 \cos\theta_0$ and substitutes to $y = y(t)$, (2.43), to obtain $y = y(x)$, (2.44).

Example 3. Circular motion, r =**constant.** [15]
A suitable CS to describe circular motion (r=constant) [16]. Let's assume that the radius is R. Then we obtain:

$$r(t) = R \qquad \rightarrow \qquad \dot{r} = \frac{d}{dt}(R) = 0$$

and finally, from (2.23)-(2.25) for $\dot{r} = 0$:

$$\mathbf{r}(t) = R\hat{r}, \tag{2.45}$$
$$\mathbf{v}(t) = R\dot{\theta}\,\hat{\theta} \tag{2.46}$$
$$\mathbf{a}(t) = -R\dot{\theta}^2\hat{r} + R\ddot{\theta}\hat{\theta} \tag{2.47}$$

Therefore in circular motion we find for the *radial* acceleration $\mathbf{a}_r$ a direction towards the center of cycle and the *tangential* acceleration, $\mathbf{a}_\theta$, always normal to position vector $\mathbf{r}$,

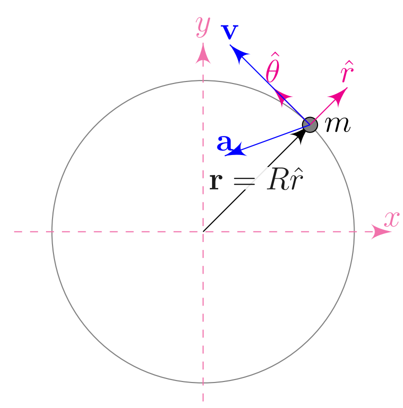
$$\mathbf{a}_r = -R\dot{\theta}^2\hat{r}, \qquad \mathbf{a}_\theta = R\ddot{\theta}\,\hat{\theta} \tag{2.48}$$

Accordingly the *tangential (linear)* velocity $\mathbf{v} = \mathbf{v}_\theta$, always normal to position vector $\mathbf{r}$ [17]

$$\mathbf{v} = \mathbf{v}_\theta = R\omega\hat{\theta}, \qquad \omega = \dot{\theta}. \tag{2.49}$$

[15] $R = r = const.$

[16] Note that the same problem becomes effectively 1-dimensional problem since it is only the angle θ that is time-dependent. This in contrast with a description in a Cartesian CS where two time-dependent variables, $x(t), y(t)$, are required for its description.



[17] $\omega = \dot{\theta}$ is the *angular velocity*

2.4 Questions

Question 1. Define a polar coordinate system (ignore the $z-$axis) as,

$$x(t) = r(t)\cos\theta(t), \qquad y(t) = r(t) = \sin\theta(t)$$

Prove the relations (2.29) between the Cartesian and Polar unit vectors.

Question 2. Using (2.29) prove equations (2.18).

Question 3. Using (2.18) prove equations (2.19) and (2.21).

Question 4. Using (2.30) and the definition of the acceleration for a particle prove (2.31)

Question 5. Constant acceleration a =const. Use the method represented by (2.30) and (2.31) to show that for constant acceleration:

(a)

$$v(t) = v_0 + a(t - t_0) \tag{2.50}$$

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 \tag{2.51}$$

(b) By properly manipulating the above relations show that:

$$v^2 = v_0^2 + 2a(x(t) - x_0) \qquad x(t) - x_0 = \frac{v(t) + v_0}{2}(t - t_0) \tag{2.52}$$

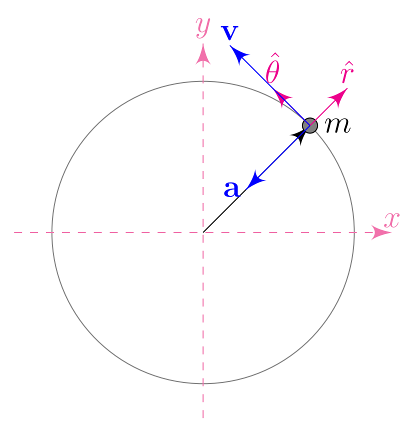


Fig 2.4: Circular motion with constant angular velocity, $\dot{\theta} = \omega$.

Question 6. Integrate (2.32) and (2.34) for constant acceleration $\mathbf{a}$ =const. to prove the relations (2.33) for $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $\mathbf{a}$.

Question 7. Circular motion with constant angular velocity $\dot{\theta}$ =const. Show that in the case of constant angular velocity the expressions for the position,velocity and acceleration simplify to,

$$\mathbf{r}(t) = R\hat{r} \tag{2.53}$$

$$\mathbf{v}(t) = R\omega\,\hat{\theta} = v_0\,\hat{\theta} \tag{2.54}$$

$$\mathbf{a}(t) = -R\dot{\theta}^2\hat{r} = -\frac{v_0^2}{R}\hat{r} \tag{2.55}$$

So, in the circular motion with constant angular velocity there is no tangential acceleration. Acceleration is radially directed pointing the center of the circular orbit!

Chapter 3

Newtonian dynamics I - forces

3.1 General

Dynamics of a body relates its position $\mathbf{r} = \mathbf{r}(t)$ and velocity $\mathbf{v} = \mathbf{v}(t)$ and acceleration $\mathbf{a} = \mathbf{a}(t)$ as a function of time (motion) with the forces that applied to it. The link between the kinematical state of the body and the forces that are applied on it is Newton's 2nd law. From mathematical point of view this 'link' is nothing else than differential equations in space and time. [18]

As concerns the theoretical formulation of mechanics, originated by Isaac Newton (1687), there are two main approaches followed so far. One is the so-called *vectorial dynamics* where the central concept is a 'directional' quantity (*force*) assumed that is exercised on the material bodies. The other approach is the *analytical dynamics* where the central concepts are scalar quantities, such kinetic and potential energy (or their sum which is the total energy), which the bodies is assumed that they 'posses' given their velocity and their position. The latter approach has proven more fruitful conceptually and in the vast majority much more powerful. More sophisticated developments of this line of thinking have led to the most modern physical theories, including the quantum mechanics.

In this chapter the two approaches will be presented and few simple examples will be worked out. In the classical mechanics context the analytical approach (energy) can be derived from the vectorial approach (force) and as such are completely equivalent. In addition, it is concluded that for an isolated physical system there is a quantity which remains constant during its lifetime, regardless how and what complicated physical processes are taking place. Moreover the well-known particular expressions for the kinetic energy $T = mv^2/2$ and the potential energy, $V(x)$, are also derived from the same route, namely the Newton's mechanical laws for forces.

Finally, it should be mentioned that the validity of Newton's law is challenged as the masses of the bodies become too large (more accurately their mass density) and/or their speeds. Again, more accurately, after Einstein, the speed of material bodies can't overpass the speed of light in vaccuum, convetionally denoted as c. In these cases, one should rely on Einstein's theories of special and general relativity (if one focuses exclusively on the gravity 'force'.) [19]

That said, in the below, when numerical evaluation takes place, the SI system of units is used (kg,m,s) for the mass, length and time, respectively. The same SI unit system is adopted for the involved electromagnetic quantities such as charge (Cb) and the vacuum dielectric, (permittivity, ε_0) and magnetic (permeability, μ_0)

[18] The Newton's 2nd law is a 2nd-order ordinary differential equation (ODE) in time, which given the force, $\mathbf{F}(\mathbf{r}, t)$, the position vector, $\mathbf{r}(t)$, is determined.

$$\ddot{\mathbf{r}}(t) = \frac{1}{m}\mathbf{F}(\mathbf{r}, t). \quad (3.1)$$

[19] Therefore in the below the derived quantities of main interest, measured as, position in (m), velocity in m/s, acceleration in m/s^2, force in N=kg m/s^2, momentum in Kgm/s energy in J=Nt$\times$, power in W=J/s.

constants. These are defined such that,

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \simeq 3\times10^8 m/s, \qquad \mu_0 = 4\pi\times10^{-7}\text{N/m}^2, \qquad \varepsilon_0 = 8.85\times10^{-12}\text{Cb/Nm}^2$$

3.2 Force and the Newton's mechanics laws

Force is the central concept in Newtonian mechanics. As described by Newton's laws of motion, forces are what causes objects to accelerate, according to the celebrated Newton's 2nd law $F = ma$, where the acceleration, a is due to force, F.

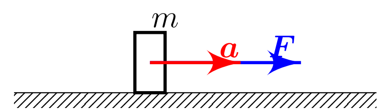


Fig 3.1: Sketch for the Newton's 2nd law, $F = ma$. Acceleration of a body is proportional to the force applied. The proportionality factor is its mass, m.

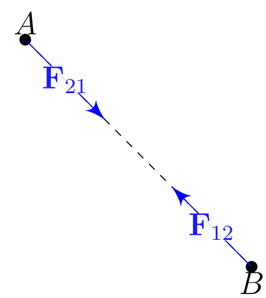


Fig 3.2: Newton's 3rd law

Newton's 1st law:

If the net force that acts on a body is zero, then the body moves with constant velocity. Mathematically this expressed as:

$$\mathbf{F} = 0 \qquad \Longrightarrow \qquad \mathbf{v} = const. \tag{3.2}$$

Newton's 2nd law:

If $\mathbf{F}$ is the net force that acts to a body of mass m and $\mathbf{p} = m\mathbf{v}$ its momentum, then the following fundamental law holds:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}). \tag{3.3}$$

Newton's 3rd law: If a body 'A' of mass m_1 applies a force $\mathbf{F}_{12}$ to a body 'B' of mass m_2, then body 'B' applies a force equal and opposite in the direction of $\mathbf{F}_{21}$ to the body 'A'. Mathematically this law is exressed as:

$$\mathbf{F}_{12} = -\mathbf{F}_{12}. \tag{3.4}$$

Few comments are worth to be mentioned at this point in relation with the Newtons laws; first, that the 1st Newton's law is obtained as a *special* case of the 2nd Newton's law when $\mathbf{F} = 0$.

Second that it is important to bear in mind that the familiar (simpler) expression $\mathbf{F} = m\mathbf{a}$ is valid only when the object's *mass is constant*, $\dot{m} = 0$.

$$m = const. \to \frac{dm}{dt} = 0 \;\Rightarrow$$
$$\mathbf{F} = \frac{d}{dt}(mv) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}(t),$$

with $\mathbf{a}(t)$ body's acceleration.

Finallly, tt is also important to emphasize that the two forces $\mathbf{F}_{12}, \mathbf{F}_{21}$ are applied to the *different* bodies 'A' and 'B', respectively.

Cartesian Coordinate System (CCS)

Newton's 2nd law in a CCS $(Oxyz,\ \mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z})$ according Eqns (3.3) and (2.12) is written as:

$$\begin{aligned}\mathbf{F} &= m\frac{dv_x}{dt}\hat{x} + m\frac{dv_y}{dt}\hat{y} + m\frac{dv_z}{dt}\hat{z} = m\frac{d^2x}{dt^2}\hat{x} + m\frac{d^2y}{dt^2}\hat{y} + m\frac{d^2z}{dt^2}\hat{z}\\ &= ma_x\hat{x} + ma_y\hat{y} + ma_z\hat{z},\end{aligned}$$

Therefore the components along the CCS axes are as,

$$F_x = m\ddot{x}, \qquad F_y = m\ddot{y}, \qquad F_z = m\ddot{z}$$

Polar Coordinate System (PCS)

Accordingly, in a PCS $(Or\theta, \mathbf{r} = r\hat{e}_r)$ again recalling Eqns (3.3) and (2.21) we obtain:

$$\begin{aligned}\mathbf{F} &= m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \qquad (3.5)\\ &= F_r(t)\hat{r} + F_\theta(t)\hat{\theta} + F_z\hat{z}\end{aligned}$$

where the components along the radial and tangential axes,

$$F_r = m(\ddot{r} - r\dot{\theta}^2), \qquad \text{radial force} \qquad (3.6)$$

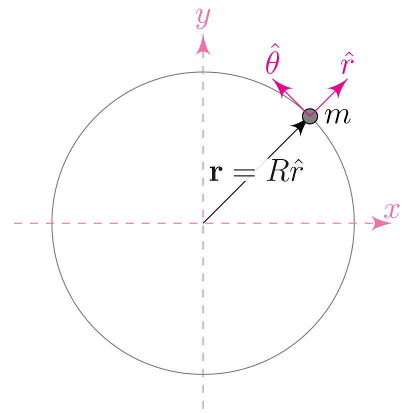
$$F_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad \text{tangential force.} F_z = m\ddot{z} \qquad (3.7)$$

3.3 Examples

Example 1. Free motion-no force, $\mathbf{F} = 0$ This is nothing else but the Newton's 1st law. It is let to the student to derive the position vector as a function of time, when $\mathbf{r}(0) = \mathbf{r}_0$ and $\mathbf{v}(0) = \mathbf{v}_0$:

$$m\frac{d\mathbf{v}}{dt} = 0 \qquad \Longrightarrow \qquad \mathbf{v}(t) = \mathbf{v}_0 \qquad \Longrightarrow \qquad \mathbf{r}(t) = \mathbf{r}_0 t,$$

which ensures that the motion in uniform (constant velocity).



Example 2. Circular motion-centripetal force

One can think a bead constraint to move along a circular wire.

1. *Coordinate system*

   The obvious choice is the polar CS with its origin placed at the center of the particle's circular orbit. The circular constraint is expressed by setting,

   $$r = R \qquad \to \qquad \dot{r} = 0, \qquad \text{where R is the circle's radius}$$

   Therefore the velocity is along the $\hat{\theta}$ direction:

   $$\mathbf{v} = \dot{r}\hat{r} + \dot{\theta}\hat{\theta} = \dot{\theta}\hat{\theta}$$

2. Equations of Motion (EOM)

   These are obtained by applying Newton's 2nd law: When $r = R \ \to \ \dot{r} = 0$ and $v = R\dot{\theta}$,

   $$m\frac{d}{dt}\mathbf{v}(t) = \mathbf{F} \qquad \to \qquad m\frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}) = \mathbf{F}.$$

   Analyzing the above vector equation in its 3 components:

   $$F_r = -mR\dot{\theta}^2 = -m\frac{v^2}{R}$$
   $$F_\theta = mR\ddot{\theta} = m\frac{dv}{dt}$$
   $$F_z = m\ddot{z}$$

Example 3. Constant force, $\mathbf{F} = \mathbf{const}$ [20]

Let's study the fall of a particle in the gravity field with some initial position and speed, $\mathbf{r}_0$ and $\mathbf{v}_0$, respectively.

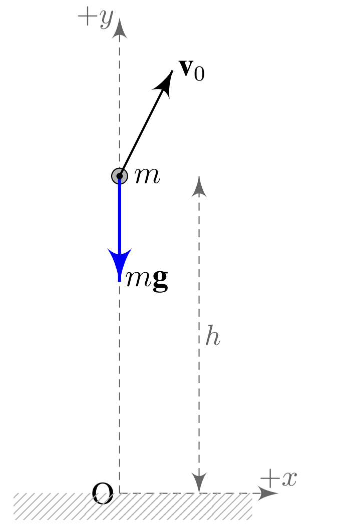


Fig 3.3: Vertical free-fall, $F = -mg\hat{y}$

1. *Coordinate system:*

   It is convenient to choose the initial time such that $t_0 = 0$ and the orientation and position of the CCS such that $\mathbf{r}_0 = (0, h, 0)$ and $\mathbf{v}_0 = (v_{x0}, v_{y0}, 0)$. In this case,

$$\mathbf{F} = -mg\hat{y}$$

2. *Equations of Motion (EOM)*

   The EOM are provide by the Newton's 2nd law:

$$m\frac{d}{dt}\mathbf{v}(t) = \mathbf{F} \qquad \to \qquad m\frac{d}{dt}(v_x\hat{x} + \mathbf{v}_y\hat{y} + v_z\hat{z}) = -mg\hat{y}.$$

   Analyzing the above vector equation in its 3 components:

$$\begin{aligned}
m\frac{dv_x(t)}{dt} &= 0 && \Longrightarrow \quad v_x(t) = v_{x0} && \Longrightarrow \quad x(t) = x_0 + v_{x0}(t - t_0) \\
m\frac{dv_y(t)}{dt} &= -mg && \Longrightarrow \quad v_y(t) = v_{y0} - g(t - t_0) && \Longrightarrow \quad y(t) = y_0 + v_{y0}(t - t_0) - \frac{1}{2}g(t - t_0)^2, \\
m\frac{dv_z(t)}{dt} &= 0 && \Longrightarrow \quad v_z(t) = v_{z0} && \Longrightarrow \quad z(t) = z_0 + v_{z0}(t - t_0).
\end{aligned}$$

3. *Initial values*

   At this point one may substitute the initial values for the particle's position and velocity. Then the equations above specialize to,

$$\begin{aligned}
x(t) &= v_{x0}t \\
y(t) &= h + v_{y0}t - \frac{1}{2}gt^2, \\
z(t) &= 0
\end{aligned}$$

The above equations of motions corresponds to the case of motion with constant acceleration (y-axis) and under zero acceleration along $x, z-$ axes. Elimination of time from the above equations $x = x(t)$ and $y = y(t)$ provides us with the path equation of the body.

[20] Examples are projectile motion, charge in a capacitor

3.4 linear forces: $F(q) = -kq$

[21]

This is the case of harmonic oscillator and represents one of the most important cases as numerous physical systems exhibit analogous behaviour. Generally most low-energy bound systems always resemble approximately an harmonic oscillator.

[21] Examples are mass-spring, pendulum, RLC electrical circuits, material/electromagnetic waves, molecular vibrations,etc

Example 1. Simple mass-sping and pendulum systems Two particular examples of such systems are the simple cases of a mass attached in a massless sping or in hanged horizontally by a massless rod (simple pendulum).

These particular systems are used to serve as a simple mechanical models to illustrate the theory of harmonic oscillator systems; They are discussed separately, in detail, in a separate chapter (Chapter 4).

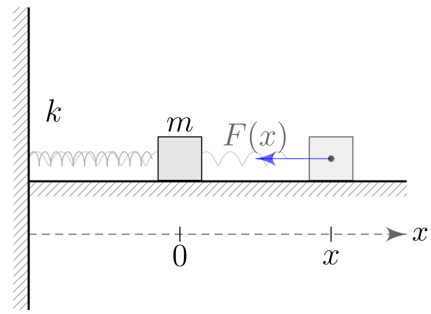


Fig 3.4: Mass-spring simple harmonic oscillator, $F(x) = -kx$

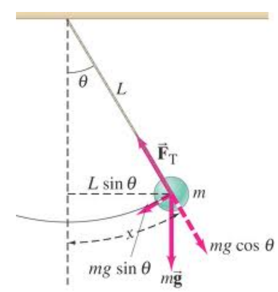


Fig 3.5: Simple pendulum cartoon. The angle θ should be assumed such that $\theta << 1$ so that this system to behave as a SHO.

3.5 Inverse-square forces, $F(x) \sim \pm k/x^2$

The most notable examples of such forces are the gravity and electric Coulombic force. Both have the following relation with the distance from the origin of a coordinate system:

$$\mathbf{F}(r) = \pm \frac{k}{r^2}\hat{r}, \tag{3.8}$$

The constant k is always negative for gravity while for electrostatic Coulomb forces it can be positive or negative. It will be shown that these forces give rise to motion having the shape of conical sections (ellipse,circle,parabola,hyperbola) depending on the initial conditions. They are responsible for the solar system planetary motion, collisions of charged particles and many other. Their main characteristic is that they are *central* forces in the sense that their strength depends on the distance alone and that they are always point to the origin of the force. These type of forces will be discussed in more detail in a separate chapter where the Kepler laws are derived.

3.5.1 Gravity

The (attractive) gravitational force was introduced by Newton to explain the observed orbits of the planets and was able to derive the Kepler's three laws.

Newton's law of gravitation If a particle of mass M is fixed at a origin, then a second particle of mass m experiences a gravitational force (and vice-versa)

$$\mathbf{F}(\mathbf{r}) = -G\frac{Mm}{r^2}\hat{r}, \tag{3.9}$$

where $G \approx 6.67 \times e^{-11}$ m^3/(Kg s^2) is the *gravitational constant*. As the force is negative the particle is attracted to the origin. We say that the particle m moves in the force-field generated by the particle, M. Since there is nothing special with the particle, M, we can also say that the particle m also generates a force-field experienced by the particle M.

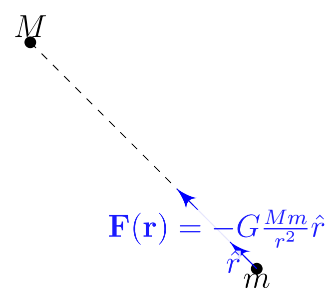


Fig 3.6: Particle m experiences the attractive force from particle M.

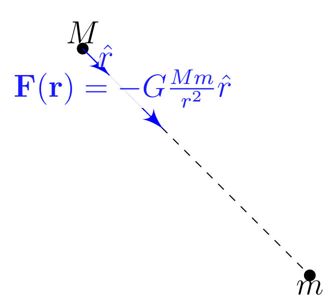


Fig 3.7: Particle M experiences the attractive force from particle m.

3.5.2 Coulomb's electrostatic force

Apart from their mass particles are also characterized by another physical quantity, namely their charge. Particles can be neutral or charged. The charge can be negative or positive. Likewise the gravity force-field generated by a massive particle, particles can generate an electrostatic force-field which is experienced only by other charged particle.

Assuming a particle of charge Q, fixed at origin, generates a force-field which is experienced by a particle of charge q at position, $\mathbf{r}$:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{r}, \tag{3.10}$$

where the electric constant ε_0 is the *electric constant* or *vacuum permittivity* or *permittivity of free space*.

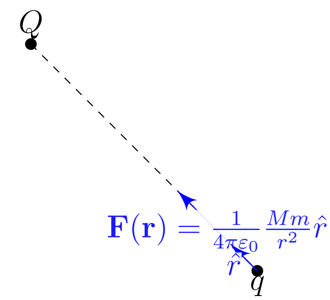


Fig 3.8: Charged particle q experiences the force from the charged particle q. The force can be attractive ($Qq < 0$) or repulsive ($Qq > 0$).

3.6 Questions

Question 1. Newton's laws

A point-like object of mass m is moving horizontally on a flat plane due to a constant force. The object experiences a friction F_T from the ground's surface. A Cartesian coordinate system is chosen such that the object (of mass $m = 1$kg) at time $t = 1$ s has the position $\mathbf{r} = (2, 0, 0)$ m and the *total* force exerted on it is $\mathbf{F} = (1, 0, 0)$ N.

1. Express the Newton's laws for the system (object-surface)
2. Make a sketch and draw all the existing forces according to Newton's theory.
3. Find its position and the speed at the later time $t = 3$ sec.
4. What is was its position and velocity at the earlier time $t = 0.5$?

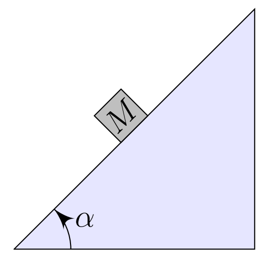


Fig 3.9: Figure for Question (2).

Question 2. Simple inclined plane

Consider the inclined plane of angle $\alpha = 45^o$ degrees and an object of mass $M = 1$ kg sliding under the gravity field ($g = 9.81$ m/s^2) [*see Fig* (3.10)*]*. Assume no friction between the object and the plane's surface.

(a) Define a proper coordinate system and derive the equations of motions for the object.

(b) If initially the object starts from the top of the plane (height 1.5 m) from rest how long it will take to reach the ground level?

Question 3. Inclined plane with a pulley

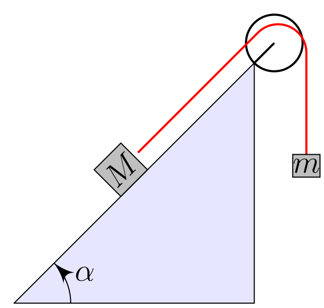


Fig 3.10: Figure for Question (3).

Consider the inclined plane of question (2) but now the mass M is pulled by mass $m = M/2$ through a pulley [*see Fig.* (3.11)]. Answer to the same (a) and (b) of question (2).

Question 4. Circular motion with constant angular velocity.

Show that in circular motion for constant angular velocity the force has only *radial* component (which is required for the directional *change* of the velocity). Find this force if the circle's radius is R and the angular velocity is ω.

Chapter 4

Linear forces, $F(x) = -kx$ Simple Harmonic Oscillator

We say that a physical quantity of a physical system exhibits a simple harmonic behaviour (SHM) when it is oscillatory with a constant period (T_0) and constant amplitude (A_0).

The physical quantity that exhibits an oscillatory behaviour, depends on the actual physical system. To mention a few, for example, a SHO can be a mass-spring system (k, m) with the position (x) of the mass (m) performing an oscillatory motion under the action of the spring force $F = -kx$. Another case of SHO system is an LC-circuit (capacitor-inductor electrical circuit, (L, C)) with the capacitor's polarity (q) oscillating in time as current flows through the inductor. A yet another example of a SHO system is a string-mass pendulum (l, m) system with the oscillatory quantity being the angle (θ) of the string with the normal, under the gravity's action. Other examples of SHO systems may include molecular vibrational motions, radiative atomic systems, earthquakes, etc.... Although the above mentioned systems can be completely different in their nature, these systems share similar properties as regards their evolution in time. This common behaviour in time is what is modeled by the concept of the simple harmonic oscillator system.

Below we'll work out three examples of very common physical systems that represent an *harmonic oscillator system*, namely the ideal spring-mass, the pendulum and the LC-circuit systems. All these system require a basic background in mechanics and electromagnestism as well as some familiarity with analytical, vectorial and differential mathematics.

4.1 Mass-spring system

Within the classical mechanical theory the position of a physical object of mass m subject to a force F satisfies the 2-nd Newton's equation,

$$\frac{d^2}{dt^2}x(t) = \frac{1}{m}F(x,t), \tag{4.1}$$

The whole subject of classical mechanics is the mathematical solution of the above differential equation. Generally, the above partial differential equation is so complicated that an analytic solution (solution expressed in a closed form in terms of known mathematical functions) is impossible. This is the general rule. The system

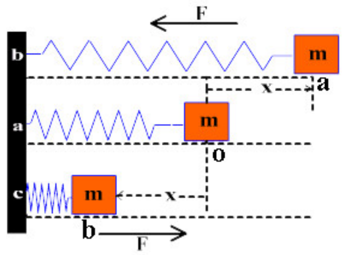


Fig 4.1. *Mass-spring system without friction forces. This system is known to behave as simple harmonic oscillator. The mass will oscillate around its equillibrium position ($x = 0$ at all times.*

under consideration should be oversimplified in order to have analytical solutions of the 2nd Newtons' law. In the present case simplification of the problem consists (a) 1-D motion, (b) the object occupies no space (point-like object) and (c) the force is linearly proportional to the object's position. Assumed the above we are in a position where analytical solutions of (4.1) can be found.

The above assumptions are expressed by the Hooke's law which states that the spring exerts a force (*Nt*) to the mass equal to:

$$F(x) = -kx,$$

where x (meter) represents the displacement of mass from its equilibrium position in a Cartesian coordinate system Ox. The spring's constant k (*Nt/meter*) being the restoring constant of the massless spring, and m *(Kg)* is the mass of the pointlike object attached to the spring.

The above definition implies that the equilibrium position is defined to be the position where the spring has its natural length, so that at this position, the spring exerts no force to the mass. In all the followings we set this equilibrium position as the zero of our Cartesian system, $x \equiv 0$,

$$\textit{Equilibrium position} = \textit{Spring's natural length}$$
$$x \equiv 0 \Longleftrightarrow F(0) \equiv 0.$$

Newton's 2nd law: The motion of the mass (m), namely its distance time evolution $x = x(t)$ can be found by the use of the Newton's 2nd law[1]:

$$\begin{aligned} F = m\tfrac{d^2}{dt^2}x(t) \quad &\Longrightarrow \quad -kx(t) = m\ddot{x}(t) \quad \Longrightarrow \\ m\ddot{x}(t) + kx(t) = 0 \quad &\Longrightarrow \quad \ddot{x}(t) + \frac{k}{m}x(t) = 0. \end{aligned}$$

Therefore, we end up to the following differential equation (ODE) for the mass-spring system:

[1]From now on the derivatives in time will be denoted by a dot at the top of the symbol, i.e. $\dot{x}(t) = d/dt[x(t)] = v(t)$. This is a convention that is generally followed by the majority of the scientific community both in the research and education sector. Better familiarize yourself with this.

$$\ddot{x}(t) + \omega_0^2 x(t) = 0, \qquad \omega_0 = \sqrt{\frac{k}{m}}. \tag{4.2}$$

where ω_0 is called *eigenfrequency* of the system, an important physical property that fully characterizes the given system. The general solution of this equation is a linear combination of the periodic functions.

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{4.3}$$

Given this expression the velocity can also be quantified as,

$$x(t) = -A\omega_0 \sin\omega_0 t + B\omega\cos\omega_0 t. \tag{4.4}$$

The quantities A, B are constants (in time) that are determined by specifying the initial conditions (the system has to start evolving). This means that the values of A, B are dependent on the particular method that the system was started. There are not many ways that one can *'fire'* the system. Either one initially displaces the mass or gives an velocity or both simultaneously. *Therefore different initial conditions result to different motion for the mass $x(t)$. Nevertheless the basic charactheristic remain the same in all cases. This is the fact that the motion is periodic with frequency ω_0*.

Eigenfrequency. The *angular eigenfrequency* ω_0 (rad/sec): is the basic parameter that fully characterizes the SHO:

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0, \qquad (\textit{rads/sec}) \tag{4.5}$$

where T_0 (*sec*) is the oscillation's period and f_0 is the frequency measured in sec^{-1} (Hz).

Initial conditions. The *initial conditions*, namely, the initial position and velocity of the mass at a particular time[22]:

$$x(0) = x_0 \qquad\qquad v(0) = \dot{x}(0) = v_0, \tag{4.6}$$

Substitution of the initial conditions into (4.7) and (4.8) give

$$x(0) = x_0 \qquad \rightarrow \qquad A = x_0$$
$$v(0) = v_0 \qquad \rightarrow \qquad B\omega_0 = v_0.$$

Finally we arrive at the folllowing expressions for the position and the velocity of the mass:

$$x(t) = x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t \tag{4.7}$$
$$v(t) = v_0\cos\omega_0 t - x_0\omega_0\sin\omega_0 t \tag{4.8}$$

where the velocity is obtained by differentiating Eq. (4.7) we obtain for the velocity
23

[22] We take this time as the start of our time ($t_0 = 0$). This choice is only made for convenience. The motion of the system can be completely determined even if we know the initial conditions at a different time than zero $t_0 \neq 0$).

[23]

$$a(t) = \ddot{x}(t) = -\omega_0^2 x(t)$$

Mechanical Energy. The mechanical energy is a constant of motion and it is given by:

$$E = T + V = \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) = \text{Constant in time,} \tag{4.9}$$

where $V(x) = kx^2/2$ is the potential energy of the spring, while $T = mv^2(t)/2$ is the kinetic energy of the mass. The concepts of kinetic/potential and total energy will be treated in more detail in a later chapter (Ch. 5) as it provides an alternative method to determined the dynamics of a mechanical system, amenable to be generalized to more complicated systems and beyond the context of classical mechanics (e.g. Electrodynamics, Thermodynamics, Quantum Mechanics, etc)

4.2 Simple pendulum

A *simple pendulum* is a model consisting of a point mass (m) suspended by a massless, unstretchable string of length l. No other forces are taken into account (e.g. air resistance at Earth's atmosphere). It is known that when the point mass is pulled to one side of its straight-down equilibrium position and released, it swings ('oscillates' is the proper scientific term) about the equilibrium position due to the gravity force, $W = mg$ (considered constant). Application of the Newton's 2nd law ($\mathbf{F} = m\ddot{\mathbf{r}}$) for the pendulum's mass starts by choosing the Oxy Cartesian coordinate system such that the axis y lies along the vertical direction and the $x-$axis along the horizontal direction of the figure (9.1). The $y-$axis is taken to be positive to the downward direction while the $x-$axis is taken positive to the right-direction. In this case the Newton's 2nd law for the x, y components takes the form:

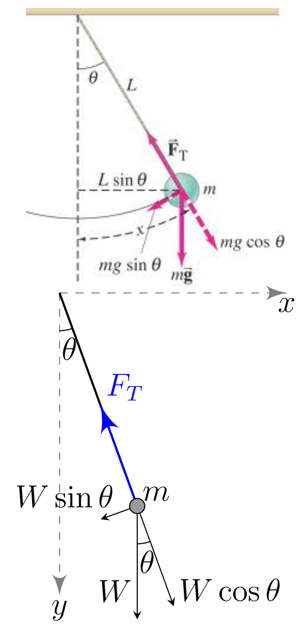


Fig 4.2: Idealized simple pendulum sketch. The point mass moves under the weight force, $W = mg$. At the bottom figure the coordinates axes are shown explicitly

$$m(\ddot{x}\hat{x} + \ddot{y}\hat{y}) = F_x\hat{x} + F_y\hat{y} \qquad \rightarrow \qquad \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}.$$

By taking the components separately we obtain two ODEs for the $x(t), y(t)$ in terms of the polar angle θ and the magnitute of the tension F_T:

$$\ddot{x} = -F_T \sin\theta, \tag{4.10}$$

$$m\ddot{y} = mg - F_T \cos\theta \tag{4.11}$$

Note that in principle, F_T is time-dependent and an unknown of the problem. The other unknown is the theta angle, $\theta(t)$ which however is related with $x(t)$ and $y(t)$:

$$x(t) = L\sin\theta, \qquad y(t) = L\cos\theta. \tag{4.12}$$

It is beneficial (and simpler) to look for the differential equation of $\theta(t)$ since by knowing $\theta(t)$ we can find $x(t), y(t)$ at once from the above equations. To this end we have to express $\ddot{x}, \ddot{y}$ in terms of the angle $\theta(t)$. So using (4.12) we have,

$$\dot{x}(t) = L\sin\theta \qquad \rightarrow \qquad \ddot{x}(t) = -L\dot{\theta}\cos\theta - L\ddot{\theta}\sin\theta$$

$$\dot{y}(t) = L\cos\theta \qquad \rightarrow \qquad \ddot{y}(t) = -L\dot{\theta}\sin\theta + L\ddot{\theta}\cos\theta$$

Substitution of the above expressions into (4.10)-(4.11) after a straightforward algebra (with the reader called to confirm) we arrive at the celebrated *pendulum equation*:

$$\ddot{x}(t) + \frac{g}{L}\tan\theta(t) = 0, \qquad \text{pendulum equation} \tag{4.13}$$

The above equation generally is *non-linear* and generally is not amenable to an analytical solution[24] Nevertheless, this equation approximately turns to an easily soluble problem, provided that $\theta(t)$ takes only small values. Below we examine this case.

Small-angle approximation $\theta \ll 1$. At this point we can employ the *small-angle* approximation which consists to ask solutions for $\theta \ll 1$ [25]. Then,

$$\theta \ll 1 \quad \rightarrow \quad \sin\theta \sim \theta, \qquad \tan\theta \sim \theta, \qquad \sin\theta \sim 1 - \frac{\theta^2}{2} \tag{4.14}$$

[24] A solution which can be written in terms of standard known functions, e.g. sin, cos, tan etc... Non-linearity is due to the presence of the sin terms which in principle it includes higher powers of θ, if one recalls the series expansion of the sin function.

[25] In principle, for anglues less than 22^o this would be a good approximation to apply.

Essentially, we assume (in the small angle approximation) that the pendulum has no vertical motion (ergo its acceleration along the $y-$axis is zero). This should result that the component of the vertical tension F_T opposes completely the weight of the mass. Eventually, we arrive at an equation satisfied by the angle of the pendulum:

$$\ddot{\theta}(t) + \omega_0^2\theta(t) = 0, \qquad \omega_0 = \sqrt{\frac{g}{L}}, \qquad \theta \ll 1. \tag{4.15}$$

We note that this equation for the pendulum's angle θ that is identical with that of an harmonic oscillator system[26]. The above equation should be supplemented with the initial conditions for the pendulum's dynamics can be fully predicted:

$$\theta(0) = \theta_0, \qquad \dot{\theta}(0) = \Omega_0, \tag{4.16}$$

where θ_0 and Ω_0 are the initial angle (with the normal direction) and angular velocity of the pendulum.

Following the standard procedure (*assuming that* $\theta(t) = A\cos\omega_0 t + B\sin\omega_0 t$ *and after applying the initial conditions)* the time-evolution of the pendulum's angle $\theta = \theta(t)$ is found to be [27] :

$$\theta(t) = \theta_0\cos\omega_0 t + \frac{\Omega_0}{\omega_0}\sin\omega_0 t, \tag{4.17}$$

The angular velocity of the pendulum can be found by taking the first derivative in time of the angle θ, namely if we define, $\Omega(t) \equiv \dot{\theta}$ we obtain,

$$\Omega(t) = \Omega_0\cos\omega_0 t - \omega_0\theta_0\sin\omega_0 t, \tag{4.18}$$

The period T_0 of the pendulum is equal to:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}}. \tag{4.19}$$

Note that the period of the pendulum is independent on the mass of the attached object (for point-like objects) and of the initial angle θ_0.

4.3 Coupled harmonic oscillators

In the previous chapters we dealt with the case of a single harmonic oscillator under different physical situations (simple, damped, driven and damped). In the present chapter we'll be examining the time evolution of two harmonic oscillators coupled each other. For simplicity we'll ignore interactions with the environment or external forces and consider two *simple* harmonic oscillators interacting via a given force.

An outline of the present chapter is as follows: First we'll take the usual mechanical case of two mass-spring system coupled via a sping. After we examine various specific cases, a brief discussion of the developed theory in the case of coupled electrical HOs (LC circuits) will be presented. The final section will consist of a generalization of the mechanical two-mass model to the case of N-coupled masses.

[26] To convince yourself for the validity of this approximation, consider the Taylor expansion for the sine and cosine,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...,$$

and

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^5}{5!} + ...$$

and then keep the firs terms since in the small angle approximation $\theta \ll 1$ and as such $\theta^k \ll \theta^{k+1}, k = 1, 2, ..$

[27] An alternatively shortcut to arrive at the small-angle equation for the pendulum (without going through (4.13)) is to employ the approximation into transformation equations (4.12) directly:

$$x(t) = L\sin\theta(t) \sim L\theta(t)$$
$$y(t) = L\cos\theta(t) \sim L$$

Then substitution of the above expressions into at the initial system (4.10)-(4.11) results directly to (4.14). The drawback of this approach is that then we would have lost an essential part of the underlying physics involved in (4.13).

4.4 Simple coupled-mass HO system

We start by defining the 1-D Cartesian coordinate system Ox' that will be used for the mathematical description of the masses' motion. We take the positive direction of the x'-axis to the right of O. Note that we have defined a *primed* coordinate system. Our task is to find the position of the masses m_1 and m_2 as a function of time, namely the functions $x_1'(t), x_2'(t)$. For simplicity we consider that masses and the springs are identical characterized by m and k. For convenience we'll name this system as *simple coupled-mass system* in contrast to the more general case where m_1, m_2 and k_1, k_2, k_3 take arbitrary values.

Simple coupled-mass system (m, k)

The spring forces follow the Hooke's law $F = -kx$ where x represents the displacement of the masses from their respective equilibrium positions O_1 and O_2 [see figure (4.3)]. Due to the latter property of the spring forces, while one can develop the present discussion in terms of the primed coordinates x_1' and x_2' (that provide the position of the masses relative to the origin of the coordinate system O) it is more convenient to measure the position of the masses relative to their equilibrium positions O_1 and O_2. That is because the Hooke's law is expressed in terms of the displacement of the mass from their equilibrium positions x_1, x_2 rather than their absolute positions x_1', x_2'. If the natural length of the spings is l, then the following relations for the masses' locations hold:

$$x_1' = l + x_1 \qquad x_2' = 2l + x_2 \tag{4.20}$$

Application of the Newton's second law for the two masses $'1', '2'$ gives:

$$m\frac{d^2}{dt^2}x_1'(t) = -kx_1 + k(x_2 - x_1)$$
$$m\frac{d^2}{dt^2}x_2'(t) = -kx_2 - k(x_2 - x_1)$$

From Eq. (4.36), since l is constant, we see that $\ddot{x} = \ddot{x}'$ and direct substitution provides,

$$m\ddot{x}_1(t) = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2(t) = -kx_2 - k(x_2 - x_1).$$

After re-arranging the various terms in the above differential equations we end up to the following coupled-system of ordinary-differential equation (ODE) for the $x_1(t), x_2(t)$:

$$m\ddot{x}_1(t) + 2kx_1 = kx_2$$
$$m\ddot{x}_2(t) + 2kx_2 = kx_1.$$

4.4.1 Method of solution

The above system can be further re-written, by dividing with the mass $m \neq 0$, as:

$$\ddot{x}_1 + \omega_0^2 x_1 = \omega_1^2 x_2 \tag{4.21}$$
$$\ddot{x}_2 + \omega_0^2 x_2 = \omega_1^2 x_1, \tag{4.22}$$

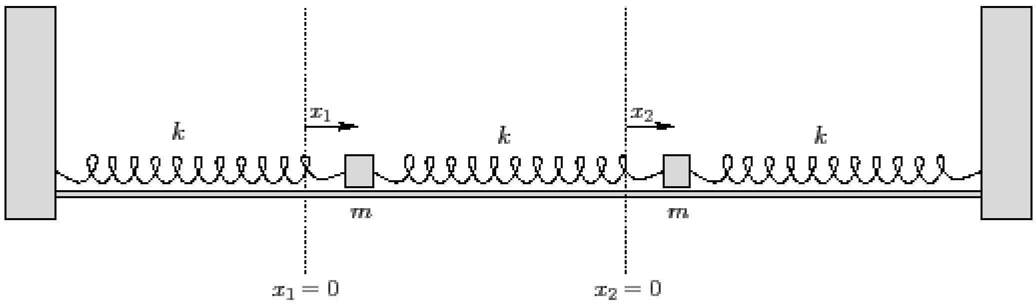


Fig 4.3. *A simple coupled-mass (m, k) system. x_1 and x_2 represent the position of the masses '1' and '2' relative to their respective equilibrium positions $O_1(x_1 = 0)$ and $O_2(x_2 = 0)$. The masses have the same mass m and all springs have the same stifness k.*

where

$$\omega_0 = \sqrt{\frac{2k}{m}}, \qquad \omega_1 = \sqrt{\frac{k}{m}}. \tag{4.23}$$

While there are general methods of solving this first-order ordinary differential equation (ODE) system here we'll be using a much simpler method, which does not require extra mathematics than those that are required for the solution of the simple HO oscillator of the first chapters. To this end, we add and subtract the equations (4.21) and (4.22) to obtain:

$$\ddot{x}_1 + \ddot{x}_2 + \frac{k}{m}(x_1 + x_2) = 0$$
$$\ddot{x}_1 - \ddot{x}_2 + \frac{3k}{m}(x_1 - x_2) = 0,$$

Finally, by defining,

$$u_+ = x_1 + x_2 \tag{4.24}$$
$$u_- = x_1 - x_2 \tag{4.25}$$

we end up to the following ODE for the new variables $u_\pm(t)$:

$$\ddot{u}_+ + \omega_+^2 u_+ = 0 \tag{4.26}$$
$$\ddot{u}_- + \omega_-^2 u_- = 0, \tag{4.27}$$

$$\omega_+ = \sqrt{\frac{k}{m}}, \qquad \omega_- = \sqrt{\frac{3k}{m}}. \tag{4.28}$$

With the above transformation $u_\pm = x_1 \pm x_2$ we have succeed to turn the coupled ODE system for x_1, x_2 to an **uncoupled** ODE system for the u_+ and u_- with eigenfrequencies ω_+ and ω_- respectively. As known, from the previous chapters, the general solution for the latter ODE[2] [Eqns (4.26) and (4.27)] are expressed as follows:

$$u_+(t) = A_+ \cos\omega_+ t + B_+ \sin\omega_+ t,$$
$$u_-(t) = A_- \cos\omega_- t + B_- \sin\omega_- t.$$

[2]We do not need to consider initial conditions at this stage. The initial conditions are given in terms of the physical positions and velocities of the masses, namely the $x_1(0)$,$x_2(0)$ and $v_1(0)$, $v_2(0)$.

We can obtain the expressions for the positions $x_1(t)$ and $x_2(t)$ are by solving backwards the Eqns (4.24), (4.25) to obtain:

$$x_1(t) = \frac{1}{2}\left[u_+(t) + u_-(t)\right], \qquad x_2(t) = \frac{1}{2}\left[u_+(t) - u_-(t)\right].$$

In this case the final expressions for $x_1(t)$ and $x_2(t)$ are given by[3],

$$x_1(t) = \frac{1}{2}\left[A_+ \cos\omega_+ t + B_+ \sin\omega_+ t + A_- \cos\omega_- t + B_- \sin\omega_- t\right] \tag{4.29}$$

$$x_2(t) = \frac{1}{2}\left[A_+ \cos\omega_+ t + B_+ \sin\omega_+ t - A_- \cos\omega_- t - B_- \sin\omega_- t\right]. \tag{4.30}$$

In the above expressions the constants $A_\pm, B_\pm$ are calculated by considering the corresponding *initial conditions* for the initial positions and velocities of the masses m_1 and m_2: *initial conditions*

$$x_1(0) = x_{10}, \qquad x_2(0) = x_{20} \tag{4.31}$$

$$v_1(0) = v_{10}, \qquad v_2(0) = v_{20} \tag{4.32}$$

Provided that $x_1(t)$ and $x_2(t)$ are known the velocities of the masses are obtained by taking the corresponding first-order time-derivatives:

$$v_1(t) = \frac{1}{2}\left[\omega_+(B_+ \cos\omega_+ t - A_+ \sin\omega_+ t) + \omega_-(B_- \cos\omega_- t - A_- \sin\omega_- t)\right] \tag{4.33}$$

$$v_2(t) = \frac{1}{2}\left[\omega_+(B_+ \cos\omega_+ t - A_+ \sin\omega_+ t) + \omega_-(A_- \sin\omega_- t - B_- \cos\omega_- t)\right] \tag{4.34}$$

Different modes of motion exist for the masses of this system, depending on the initial conditions (also known as *excitation mode*). In the below we'll demonstrate an example where the initial conditions are explicitly specified.

4.5 Examples

Example 1. Simple mass-spring system

For a mass-spring system with $m = 2$ Kgr and $k = 2$ N/m we assume that initially the mass is found at the equilibrium position and is released with initial velocity equal to $v_0 = -2$ m/s. Let's assume the following form for the solution:

$$x(t) = x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

(One can also start by assuming the alternative forms for $x(t)$. The final expressions and values must be the same)

[3]If we want the expressions for $x_i'(t), i = 1, 2$ then we have to use the transformation relations (4.36).

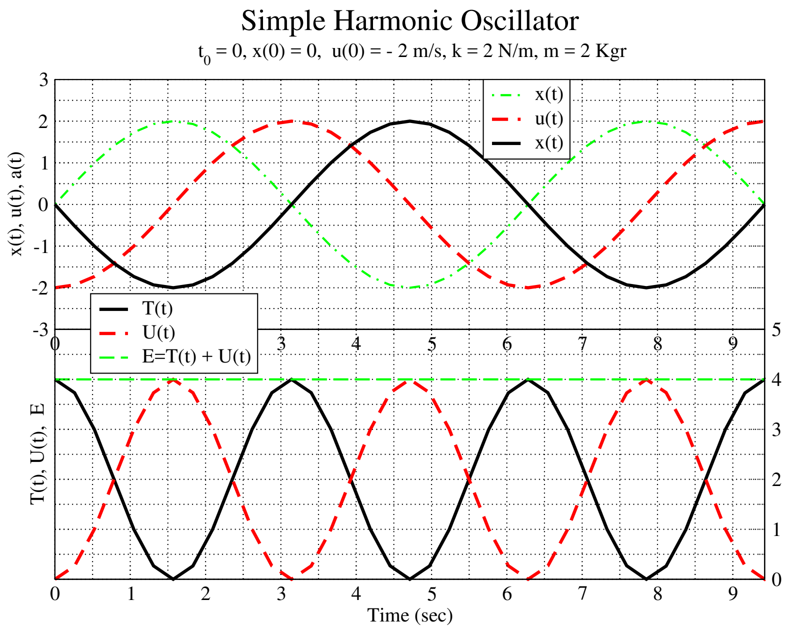


Fig 4.4. *Plot of position, velocity, acceleration, kinetic and potential energy as a function of time for a spring mass-system. Parameters of the system are $m = 2$ Kgr, $k = 2$ N/m, $x(0) = 0$ and $u(0) = 2$ m/s. The period of the oscillations are $T = 2\pi =$ 6.282 sec.*

First we calculate the eigenfrequency of the system:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\,N/m}{2\,kgr}} \Longrightarrow \omega_0 = 1\,\text{Hz}$$

Next we apply the initial conditions ($x(0) = 0$) and $v(0) = -2$ m/sec to the above solution for the simple harmonic oscillator to obtain for $x(t)$:

$$x(t) = \frac{v_0}{\omega_0} \sin \omega_0 t = -2 \sin(\omega_0 t + \pi)$$

Finally, knowning $x = x(t)$ we may obtain the other physical quantities of interest:

$$\begin{aligned}
v(t) &= \dot{x}(t) = -2 \cos \omega_0 t \\
a(t) &= \ddot{x}(t) = -\omega_0^2 x(t) = 2 \sin \omega_0 t \\
T(t) &= \frac{1}{2} m v^2 = 4 \cos^2(t) \\
U(t) &= \frac{1}{2} k x^2 = 4 \sin^2(t) \\
E &= T(t) + U(t) = \frac{1}{2} m A_0^2 \omega_0^2 = \frac{1}{2} k A^2 = 4 \text{ Joules,} \qquad (4.35)
\end{aligned}$$

In Fig. (4.4), the relevant plots are shown.

Example 2. Coupled oscillators: Lowest mode excitation

Assume an two equal-masses coupled-mass-spring system with m, k known [see Fig. (4.3)]. Let's examine the case where the initial displacements are equal and point to the same direction. In other words the above initial conditions correspond to the case where we displace both masses by an equal amount from their respective equilibrium positions and then we simple release them. It is our purpose to find the subsequent motion of the masses m_1 and m_2.

$$x_1(0) = x_2(0) = x_0, \qquad v_1(0) = v_2(0) = 0.$$

Since the positions for the masses m_1 and m_2 are given by $x_1(t)$ and $x_2(t)$ in Eqns (4.29) and (4.30) it only remains to apply the initial conditions in order to determine the constants $A_\pm$ and $B_\pm$. Followin the standard procedure, by applying the initial conditions (4.31) to the expressions (4.29-4.30) we obtain:

$$\begin{aligned} x_1(0) = x_0 \quad &\Longrightarrow \quad A_+ + A_- = 2x_0 \\ x_2(0) = x_0 \quad &\Longrightarrow \quad A_+ - A_- = 2x_0 \end{aligned}$$

Solving the above 2×2 algebraic system for A_+ and A_- gives:

$$A_+ = 2x_0, \qquad A_- = 0$$

Then we have for the positions:

$$\begin{aligned} x_1(t) =& \frac{1}{2}\left[2x_0 \cos\omega_+ t + B_+ \sin\omega_+ t + B_- \sin\omega_- t\right] \\ x_2(t) =& \frac{1}{2}\left[2x_0 \cos\omega_+ t + B_+ \sin\omega_+ t - B_- \sin\omega_- t\right]. \end{aligned}$$

By taking the derivative of the above quantities we find for the velocities:

$$\begin{aligned} v_1(t) =& \frac{1}{2}\left[-2x_0\omega_+ \sin\omega_+ t + \omega_+ B_+ \cos\omega_+ t + \omega_- B_- \cos\omega_- t\right] \\ v_2(t) =& \frac{1}{2}\left[-2x_0\omega_+ \sin\omega_+ t + \omega_+ B_+ \cos\omega_+ t - \omega_- B_- \cos\omega_- t\right]. \end{aligned}$$

Now applying the initial conditions (4.32) to the above equations we obtain the required algebraic equations for the remaining constants $B_\pm$:

$$\begin{aligned} v_1(0) = 0 \quad &\Longrightarrow \quad \omega_+ B_+ + \omega_- B_- = 0 \\ v_2(0) = 0 \quad &\Longrightarrow \quad \omega_+ B_+ - \omega_- B_- = 0. \end{aligned}$$

The above system gives for the B_+, B_- constants the following values:

$$B_+ = B_- = 0.$$

Therefore the final expressions for the positions and the velocities of the masses m_1 and m_2 are as below:

$$\boxed{\begin{aligned} x_1(t) &= x_0 \cos\omega_+ t \qquad & x_2(t) &= x_0 \cos\omega_+ t, \\ v_1(t) &= -x_0\omega_+ \sin\omega_+ t, \qquad & v_2(t) &= -x_0\omega_+ \sin\omega_+ t. \end{aligned}}$$

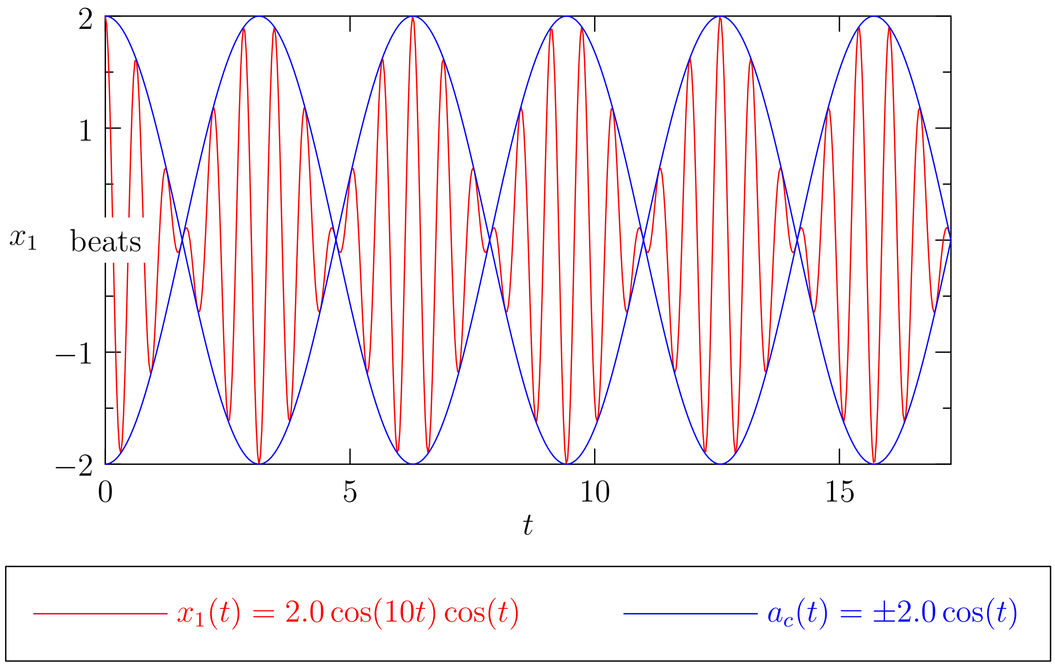


Fig 4.5. *Example of beating for the case where $x_0 = 2$, $\Omega_- = 1$ rad/sec and $\Omega_+ = 10$ rad/sec. With red is plotted the position of the the mass m_1 as a function of time, while with blue the slowly varying amplitude $a_c(t)$ is plotted.*

One can easily check that the above solutions are consistent with the initial conditions. From the above form of the solutions we see that the masses oscillate in phase around their respective equilibrium positions with the lowest (fundamental) frequency ω_+. So, at all times we have

$$x_1(t) = x_2(t), \qquad \omega_+ = \sqrt{\frac{k}{m}}$$

Effectively the two masses oscillate 'independently' each other as if the connecting spring k_2 was missing.

Example 3. Coupled-oscillators: periodic beats Another interesting physical situation, known as *beating*, appears by choosing a particular set of initial conditions where only one of the masses is perturbed initially

Let's take the initial conditions to be,

$$x_1(0) = x_0, \quad x_2(0) = 0, \qquad v_1(0) = v_2(0) = 0.$$

This means that initially the mass m_1 is pulled to some distance out of its equilibrium point O_1 and then released from rest. Again by following the standard procedure, we substituting the above initial conditions into the relevant expressions for $x_1(t)$ and $x_2(t)$ (4.29)-(4.34) we arrive at the constants $A_\pm$ and $B_\pm$[28]:

$$A_+ = A_- = x_0, \qquad B_+ = B_- = 0.$$

In this case we have for $x_1(t)$ and $x_2(t)$:

$$x_1(t) = \frac{x_0}{2}(\cos\omega_+ t + \cos\omega_- t)$$
$$x_2(t) = \frac{x_0}{2}(\cos\omega_+ t - \cos\omega_- t)$$

[28] Prove the above results for $A_\pm$, $B_\pm$.

An alternative form (and more intuitive) of the above expressions can be found if we use the trigonometric identities $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$[29]:

[29] Prove the above expressions for $x_1(t)$, $x_2(t)$.

$$x_1(t) = x_0 \cos(\Omega_- t) \cos(\Omega_+ t) = a_c(t) \cos(\Omega_+ t)$$
$$x_2(t) = x_0 \sin(\Omega_- t) \sin(\Omega_+ t) = a_s(t) \sin(\Omega_+ t)$$

where the Ω_- and Ω_+ frequencies are defined by:

$$\Omega_- = \frac{\omega_+ - \omega_-}{2}, \qquad \Omega_+ = \frac{\omega_+ + \omega_-}{2}.$$

By definition, the two frequencies Ω_- and Ω_+ differ such that $\Omega_- << \Omega_+$ allowing us to use a terminology of 'slow' (Ω_-) and 'fast' (Ω_+) frequency. Inspection of the solution $x_1(t) = a_c(t)\cos(\Omega_+ t)$ for the mass m_1, tell us that its motion is a reminiscent of the simple harmonic oscillator motion with frequency Ω_+ but with *varying amplitude* $a_c(t)$ rather than *constant*. For example the mass m_1 at time $t = 0$ has its maximum amplitude $a_c(0) = x_0$ while later on, at times circa $t_1 = \pi/2\Omega_-$ we have zero amplitude $a_c(t \sim t_1) \sim 0$. This means, that for times close to $t = 0$ the mass m_1 is very 'energetic' and oscillates quickly (with frequency Ω_+ around its equilibrium position O_1 while for times around t_1 it remains very close to O_1, almost motionless, and as such no energy is carried by this mass anymore since $x_1(t_1) \sim 0$ and $v_1(t_1) \sim 0$[4]. On the other hand the motion of the second mass m_2 (as seen from the expression for $x_2(t)$) is the same as the one just described (mass m_1), however with a time-difference equal to t_1. In other words, initially, the amplitude for m_2 is almost zero ($a_s(0) << x_0$) and almost motionless while at times close to t_1 the amplitude a_s takes its maximum value ($a_s(t_1) \sim x_0$). Therefore the energy, initially at m_1 mass, at time t_1 has transferred to the mass m_2. This transfer of energy from one mass to the other *(beating)* repeats with a period equal to $T_- = 2\pi/\Omega_-$.

[4]That the velocity of the mass '1' remains small for such times ($v_1(t \sim t_1 \sim 0)$ and therefore its kinetic energy, it can proven directly by taking the time-derivative of the expression for $x_1(t)$)

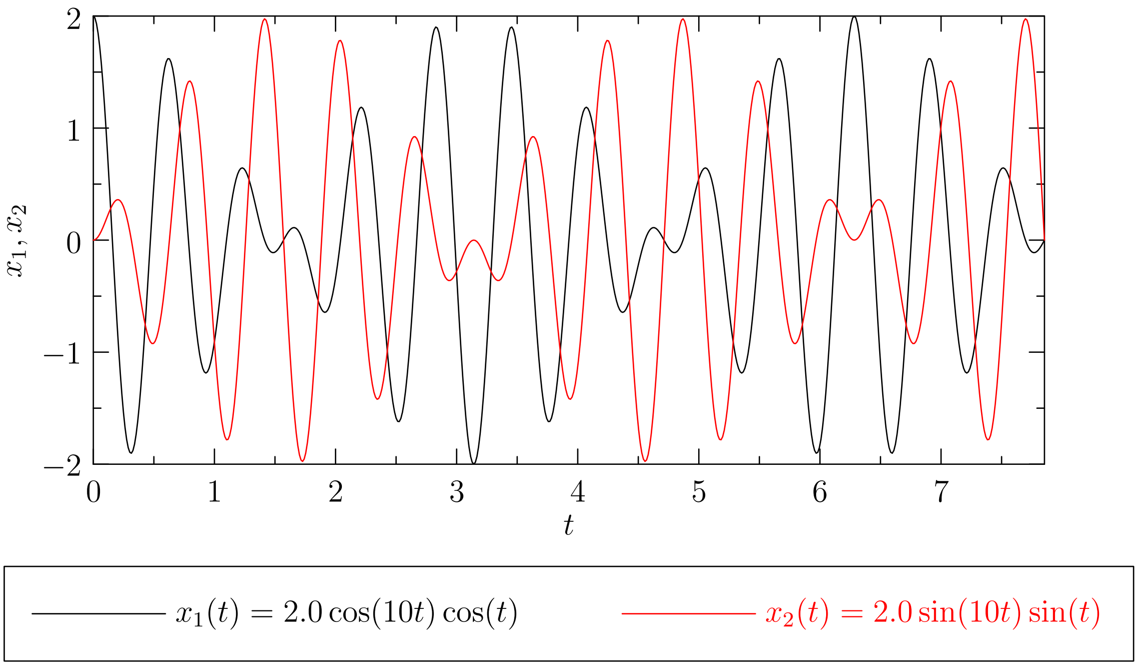


Fig 4.6. *Example of beating for the case where $x_0 = 2$, $\Omega_- = 1$ rad/sec and $\Omega_+ = 10$ rad/sec. With black colour is plotted the position of the the mass m_1 as a function of time, while with red colour the position of the mass m_2 is plotted.*

4.6 Waves as coupled harmonic oscillators***

In this section we'll develop a simple theory where the appearance of the travelling *wave* in matter (see Fig (4.7) is modeled as the infinite limit case of an N-coupled-mass system. Then naturally the wave will describe the transport properties of motion from one mass to the neighboring ones (from the mass 'A' located at the point x_0 and time t_0 to the mass 'B' at the point $x_t = v(t - t_0)$ and at time t). It might be worth to say few general things about waves in general, so that to have a better overview of this physical model used to describe transportation.

Assume a medium (e.g. air, string) and for simplicity we consider that it is constituted by particles at rest at their equilibrium position. If we disturb this medium at a certain point in space (say particle A) then we will displace this particle from its position which will vibrate around its equilibrium position. Since this particle A interacts with the other particles of the medium the externally induced vibrational motion will be transferred to the other particles as well. As a result of this interaction the initial distortion will cause a vibrational motion to all particles in the medium. Therefore, the initial distortion will be propagated through the medium. When this distortion propagation satisfies certain properties we call it 'wave'. It is important to note that the particles in the medium apart their vibrational motion around their equilibrium position are not travelling through the medium. It is their 'distortion' that travels through the medium and exactly this situation is what we call 'travelling wave'. In the ideal case of a travelling wave the initial 'supplied' energy into the particle 'A' is transferred (through the inter-particles forces) to another particle in the medium at a different time. Thus one, could safely say that it is also 'energy' that it is transferred when we have the conditions of a wave. In other words, travelling waves are characterized by the transfer of energy through space (see figure 1). A well known example from our daily life is sound and light (electromagnetic waves). The first one is a mechanical wave (a material is required for such a wave) while this is not the case for the light. Electromagnetic waves can be propagated through the space without the need of any material to be present. Needless to say however, that light can also be propagated through space where a material is present (e.g. glass).

Waves are also categorized as follows: (a) longitudinal and (b) transverse

(a) **Longitudinal waves**: when the displacement of the particles from their equilibrium position is along the propagation axis. Sound propagation is one example of longitudinal wave.

(b) **Transverse waves**: when the displacement of the particles from their equilibrium position is perpendicular to the propagation axis. String waves are one example of transverse wave.

As usual we start by defining the 1-D Cartesian coordinate system OX. The positive direction of the X-axis is taken to the right of the picture. We assume $N-$coupled identical masses m and springs all of the same stifness k and length l. We concentrate on the mass $'i'$ and we'll try to find the ODE that its displacement from the equilibrium position $\psi_i(t)$ satisfies. So in Fig (4.7) $x_i, x_{i\pm1}$ represent the equilibrium positions of masses $i, i\pm1$ while $\psi_i(t), \psi_{i\pm1}(t)$ represent the corresponding displacements.

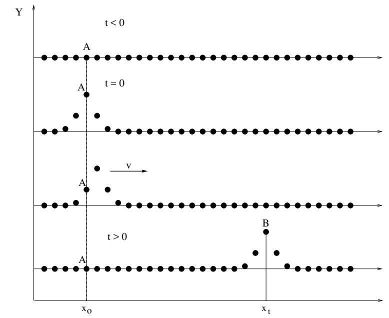


Fig 4.7: *Transverse travelling wave*: The initial distortion of particle A at $x_0 = x_A$ and at time $t = 0$, which displaced along the Y-axis (perpendicular to the $X-$axis) to a distance equal to $y_0 = y(x_A, 0)$, propagates along the $X-$axis to the right with a velocity v. While after a time $t = (x_B - x_A)/v$ the particle A returns to it's initial state $\psi(x_A, t) = 0$) the distortion propagated to the right at position $x_B = x_t$ and displaced the particle B perpendicular to the $X-$axis at a distance equal to $\psi_0 = \psi(x_B, t)$).

30

30 *N coupled-mass system* (m, k)

As usual, the spring forces follow the Hooke's law. Some careful analysis on the forces e exerted on mass i due to the coupling with the masses $m_{i\pm1}$ the following expression:

$$F_i = -k(\psi_i - \psi_{i-1}) - k(\psi_{i+1} - \psi_i) = -k(\psi_{i-1} - 2\psi_i + \psi_{i+1}). \tag{4.36}$$

Application of the Newton's second law for the $'i'$ mass gives:

$$m_i \frac{d^2}{dt^2}\psi_i(t) = F_i \quad \Longrightarrow \quad m_i \frac{d^2}{dt^2}\psi_i(t) = -k(\psi_{i-1} - 2\psi_i + \psi_{i+1}), \quad i = 1, 2, .., N$$

The above expression is the ODE for the position $\psi_i(t)$ of the mass i located (at rest) at x_i. So, each of the masses 'i' can be characterized uniquely by its equilibrium location x_i. This way there is one-to-one correspondence between i and x_i. We, then allowed to rewrite the N-coupled-mass ODE system as:

$$\frac{d^2}{dt^2}\psi(x_i, t) = \frac{k}{m(x_i)}\left[\psi(x_{i-1}, t) - 2\psi(x_i, t) + \psi(x_{i+1}, t)\right], \quad i = 1, 2, .., N \tag{4.37}$$

As we are interested only to formulate the governing equations for the motion of the masses rather than to solve them we'll not proceed further and discuss the required initial conditions that are associated with this ODE system. Instead we'll proceed further in order to establishe the governing equation when the above discrete system becomes so dense that only a continuous description is the appropriate one.

Wave equation as the limit of $N \to \infty$ and $\Delta_i = x_i - x_{i-1} \to 0$. In many cases, especially when the a macroscopic description of the physical phenomena is sufficient, matter is represented by a continuous medium of mass density $\rho = dm/dV$

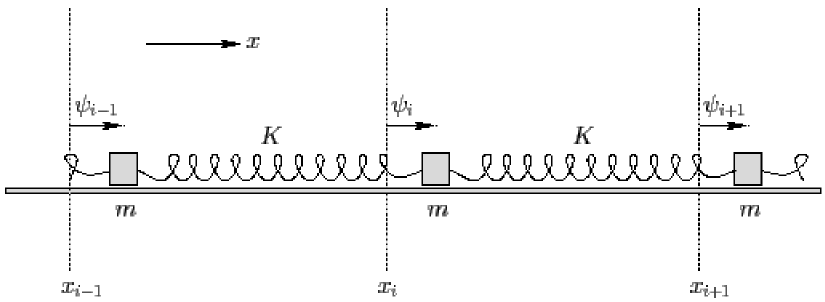


Fig 4.8. *A simple N-coupled-mass (m, k) system. x_i and $x_{i\pm1}$ represent the position of the masses $'i'$ and $'i \pm 1'$ relative to their respective equilibrium positions $(x_i = 0)$ and $(x_{i\pm1} = 0)$. The masses have the same mass m and all springs have the same stifness k.*

where dV represents the elementary spatial volume and dm the mass contained in it. In the present case where the medium is 1-D we write $\mu = dm/dx$, where x is the axis that defines the dimension of the medium. The medium now is characterized by its linear density μ which may be is constant or spatially dependent. For example, if the mass of a uniform string is $m = 2$ Kgr and it's length 10 m then it's linear density is:

$$\mu = \frac{2\ Kg}{10\ m} = 0.2 \text{ Kg/m.} \tag{4.38}$$

Based on the above clarifications we proceed to the continuous limit of Eq. (4.37) as below. We take $N \to \infty$ and infinintely close to each other $\Delta_i = x_i - x_{i-1} \to 0$. In this case the discrete position x_i becomes continuous and we have the following transformations:

$$\begin{aligned} \psi_i(t) &\longrightarrow \psi(x_i, t) \longrightarrow \psi(x, t) \\ m_i &\longrightarrow \Delta m(x_i) \longrightarrow dm(x) = \mu dx \\ \Delta_i &\longrightarrow \Delta x \end{aligned}$$

where μ is defined to be the local linear density ($\mu = dm/dx$) This means that the displacement ψ becomes spatially and time dependent (depends both on location x and t). In this limit the ODE system becomes a partial-differential equation:

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\psi(x,t) &= -\frac{k}{\Delta m}\left[\psi(x_{i-1},t) - 2\psi(x_i,t) + \psi(x_{i+1},t)\right] \Longrightarrow \\ \frac{\partial^2}{\partial t^2}\psi(x,t) &= -\frac{k\Delta x}{\mu}\lim_{\Delta x\to 0}\frac{\psi(x_{i-1},t) - 2\psi(x_i,t) + \psi(x_{i+1},t)}{\Delta x^2} \end{aligned}$$

where μ is defined to be the local linear density ($\mu = \lim_{\Delta x\to 0}\Delta m/\Delta_x = dm/dx$).

On the other hand, the second-order derivative of a spatially-dependent single-variable function is given by:

$$\frac{d^2}{dx^2}f(x) = \lim_{\Delta x\to 0}\frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{\Delta x^2}$$

In addition we define the tension T as,

$$T = \lim_{\Delta x\to 0}(-k\Delta x)$$

Taking all the above into account and by replacing $f(x) \rightarrow \psi(x,t)$ the Newton's law becomes:

$$\frac{\partial^2}{\partial t^2}\psi(x,t) = \frac{T}{\mu}\frac{\partial^2}{\partial x^2}\psi(x,t),$$

which after rearranging the above terms we have:

$$\boxed{\frac{\partial^2}{\partial x^2}\psi(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t) = 0, \qquad v = \sqrt{\frac{T}{\mu}}, \qquad (4.39)}$$

known as *wave equation*. In the general case, the above wave equation accepts solutions which a subset of them are known as $waves$ and generally describe the transportation of a localized initial distortion through a medium. In the present case, since the displacements were assumed along the $X-$axis it is expected that the corresponding solutions will provide *longitudinal* waves. One well known example of such travelling waves is the propagation of sound in 1-D medium.

4.7 Questions

Question 1. Assume the solution for the mass-spring system with initial conditions at $t_0 = x_0, v_0$:

$$x(t) = x_0 \sin \omega_0 t + \frac{v_0}{\omega_0} \cos \omega_0 t$$

Prove that for this SHO the total mechanical energy is a constant of motion:

$$E(t) = \frac{1}{2} k x^2(t) + \frac{1}{2} m v^2(t) = \textit{ constant.} \tag{4.40}$$

What is the value of this constant if $x_0 = 0.5$ m $v_0 = 0.5$ m/sec and $k = 1$ N/m, $m = 1$ Kg.

Question 2. [31] Confirm that the displacement of an SHO in a mass-spring system can be also expressed as,

$$x(t) = A_0 \sin(\omega_0 t + \phi_0),$$

where the amplitude A_0 and the phase ϕ are given in terms of the initial position (x_0) and velocity (v_0) as,

$$A_0 = \sqrt{x_0^2 + (\frac{v_0}{\omega_0})^2}, \qquad \phi_0 = \tan^{-1}(\frac{x_0 \omega_0}{v_0})$$

(*hint*: start from the expression $A \cos \omega_0 t + B \sin \omega_0 t$ and utilize the identities for the $\cos(a + b) = \cos a \cos b - \sin a \sin b$ or $\sin(a + b) = \sin a \cos b + \sin b \cos a$ to arrive at the desired form.)

[31] The purpose of the last two questions is to show that there are alternative expressions, though equivalent forms to represent the motion of a HO $x(t), v(t)$. These conclusions hold generally and do not apply only for the mass-spring system.

Question 3. A point-like object attached to a spring completes one oscillation every 2.4 sec. At $t = 0$ is released from rest at a distance $x_0 = 0.1$ m from it's equilibrium position.

(a) What is the eigenfrequency ω_0 of this simple harmonic oscillator system?
(b) what is the position of the object at time $t = 0.3$ sec after it's release?
(c) For the problem of question 1 what is it's acceleration at time $t = 0.3$ sec?

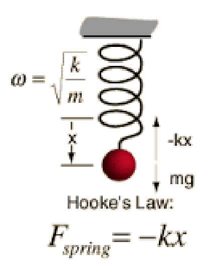


Fig 4.9. Sketch of vertical mass-spring system

Question 4. Vertical mass-spring system. Assume the vertical mass-spring system (see figure above) where a massless spring of constant k, connected with a point-like mass of mass m is placed on a gravitational field and is hanged from the free side of the spring.

Prove that this system exhibits a simple harmonic motion. What is the period of the motion if $m = 2$ Kg and $k = 4$ N/m? Would the period of this spring differ if this system was placed on Moon?

Question 5. A mass-spring system with $m = 2$ Kgr and $k = 8$ N/m is perturbeb in the following manner: At initial time $t = 0$ it is displaced by $x_0 = \sqrt{2}$ m from it's equilibrium position and it is kicked with initial velocity $v_0 = -2$ m/s. Which one from the expressions below gives the position as a function of time?

(a) What is the velocity at time $t = 1.26$ sec?

(b) What is the total mechanical energy at time $t = 0.12$ sec?

Question 6. A pendulum completes a full oscillation in 4 seconds on earth. What is the corresponding time if we instead place the pendulum on the moon?

Question 7. Spring-mass system I

Let's define as *ideal sping-mass* the idealized model where a point mass $m > 0$ is attached to a massless spring characterized by it's restoring strength through the constant $k > 0$. Assume that the general solution for position of the mass as a function of time of this harmonic oscillator is given by

$$x(t) = A_0 \sin(\omega_0 t + \phi), \tag{4.41}$$

with A_0 being the amplitude (the maximum displacement of the mass) of the motion, ω_0 its eigenfrequency and ϕ its phase angle.

(a) Assuming that at initial time $t = 0$, the displacement of the mass is x_0 and its velocity is v_0 provide the amplitude of the motion A_0, and the phase angle ϕ in terms of x_0 and v_0 as:

$$A_0 = \sqrt{x_0^2 + (v_0/\omega_0)^2}, \qquad \tan\phi = \frac{\omega_0 x_0}{v_0} \tag{4.42}$$

(b) What is the expression for the velocity of the mass as a function of time?

Question 8. Spring-mass system II

Consider the mass-spring of the above harmonic oscillator with $k = 2$ Nt/m, $m = 2$ kgr. Now consider the cases

(a) $x_0 = 0$ m, $v_0 = 2\omega_0$ m/s

(b) $x_0 = 2$ m, $u_0 = 0$.

(c) $x_0 = 2/\sqrt{2}$ m and $v_0 = \omega_0\sqrt{2}$ m/s.

For these cases write down the expressions for the position $x(t)$ and the velocity $u(t)$ and give the corresponding plots in time. How the three harmonic oscillations differ?

Question 9. Spring-mass system III

A mass (0.5 Kgr) attached to a massless spring (k= 200 N/m) is released from rest at distance 20 cm from its equilibrium position

(a) Making use of the initial conditions determine the exact form for the position of the mass as a function of time

(b) Find the maximum and minimum speed of the mass

(c) What is the maximum acceleration that the mass achieves?

(d) What is the time that the mass is halfway to the center from its original position. Having determined the time, find the velocity, acceleration, energy, kinetic and potential energy of the mass at this instant of its motion

Question 10. **Simple pendulum**. Assume a *simple pendulum* consisting of a massless, unstretchable string of length $l = 0.25$ meters. Given that pendulum is located at a place where the gravity acceleration is equal to $g = 9.81$ m/s^2:

(a) If initially the mass pulled to an angle $\theta_0 = 3.6^o$ degrees and released with no initial speed, give the angle of the pendulum as a function of time ($\theta = \theta(t)$).

(b) What is the period of oscillation of this pendulum?

(c) What the length of the simple pendulum should be, if we wanted to define the time interval of 1 sec by a full oscillation of it?

(d) What is the circular frequency in Hz in this case?

Question 11. **Coupled Oscillators I**

Assume a two-coupled-mass-spring system. The masses are equal (m) and the spring-constant known, k. The two masses are initially released from rest but displaced such that $x_1(0) = -x_2(0) = a$. Find the the subsequent motion of the masses in terms of the known quantities of the problem, m, k, a.

Question 12. **Coupled Oscillators II** Consider the simplest coupled-mass system (m, k). Find the positions $x_1(t)$ and $x_2(t)$ of the masses, by knowing that this coupled system while initially at rest, was excited by a simultaneous kick to both masses (to the right) giving to both of them speed equal to 1.25 m/s.

Question 13. **Coupled Oscillators III** Assume that a simple coupled-mass systems with ($m = 2$ Kg $k = 9$ N/m) is excited by displacing both masses at a distance equal to 0.125 cm from their respective equilibrium positions.

Calculate $x_1(t), x_2(t)$ and $v_1(t), v_2(t)$.

Question 14. **General coupled-mass HO system** The problem of the two coupled masses can be generalized to include the case where all the two masses have different masses $m_1 \neq m_2$ and the three springs have all different spring constants $k_1 \neq k_2 \neq k_3$. Apart from the fact that the relevant equations are appearing more complicated the resulting motion for this general case preserves the main characteristic of the simple coupled-mass system which is the periodic motion of the two masses with two different frequencies possible ω_+ and ω_-.

(a) Show that Newton's law for $x_1(t)$ and $x_2(t)$ ends to the following system of differential equations:

$$\ddot{x}_1 + \frac{k_1 + k_2}{m_1} x_1 = \frac{k_2}{m_1} x_2,$$
$$\ddot{x}_2 + \frac{k_3 + k_2}{m_2} x_2 = \frac{k_2}{m_2} x_1.$$

(b) Following the procedure developed for the simple couple-mass system, solve the coupled oscillator problem where $m_1 = m_2 = m$ and $k = k_1 = k_3 \neq k_2 = k'$ and show that the two eigenfrequencies ω_+ and ω_- are given by:

$$\omega_+ = \sqrt{\frac{k}{m}}, \qquad \omega_- = \sqrt{\frac{k + 2k'}{m}}$$

Chapter 5

Newtonian dynamics II - energy

5.1 Energy - Kinetic Energy- Potential Energy

Modern considerations of the dynamics of fundamental physical processes do not use the force as the central concept in their description, rather they utilize symmetry properties (time,translational,rotational, mirror, etc) to unveal quantities that remain constant during the even under question. Like manner classical mechanics, a quantity with this property is known as *energy* and in fact, the introduction of a *potential* field is what is considered to be fundamental, with the force concept being a derived. [32]

[32] Yet, there are certain processes that cannot be described in terms of potentials fields and the use of forces is the proper one. Friction forces are an example.

5.2 Equivalence of energy and forces formulation

To define the concept of energy and its main property, consider a particle of mass m moving in a 1-D space (straight line) found at time t_0 at position x_0, thus $x_0 = x(t_0)$. The particle moves under a force $F = F(x)$ (depends on the position only) and found at time t at position $x(t)$. For convenience it is assumed that the particle's mass is constant. The task for this motion is to show that there is a quantity that remains constant during this translation. This constant is what we'll be calling *energy* of the particle. To start, we do now that the system is described from the Newton' 2nd law. Then we have,

$$\begin{aligned} m\frac{dv}{dt} = F(x) \qquad &\longrightarrow \qquad mv\frac{dv}{dt} = vF(x) \qquad \longrightarrow \qquad mvdv = dxF(x) \\ &\longrightarrow \qquad \int_{v_0}^{v} mv'dv' = \int_{x_0}^{x} dx'F(x') \\ &\longrightarrow \qquad \left[\frac{1}{2}mv^2\right]_{v_0}^{v} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} dx'F(x') \quad \text{(E1)} \end{aligned}$$

Let's now define the kinetic and potential as follows. We say that the particle posseses energy due to causes; one is because of its kinetic state, namely its *velocity* alone, and is equal to,

$$T = \frac{1}{2}mv^2. \tag{5.1}$$

Additionaly, it also posseses energy because of its *position* in the force field, $F(x)$. This energy is called *potential* energy and is fully defined through its space variation properties as,

$$V(x) - V(x_0) = -\int_{x_0}^{x} dx' F(x') \tag{5.2}$$

Given the above definitions (E1) above is written as,

$$\frac{1}{2}mv_0^2 + V(x_0) = \frac{1}{2}mv^2 + V(x)$$

Noting that the times t_0, t were chosen arbitrarily the above relation holds for any values of t_0 and t and as such it is correct to say that the quantity defined as *mechanical energy*, [33]

$$\boxed{E(x,p) = \frac{1}{2}mv^2 + V(x) = \textbf{constant!}} \tag{5.3}$$

[33] If $E = mv^2/2 + V(x)$ then its time-derivative is zero:

$$\frac{dE}{dt} = mv\dot{v} + \frac{dV}{dx}v = v\left(m\dot{v} + \frac{dV}{dx}\right) = 0$$

Energy conservation law and dynamics

Conservation of energy leads to full determination of the particle's dynamics as below:

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

Since E is a constant and $dxt = dx/t$ one obtains,

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]} \qquad \longrightarrow \qquad \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}[E - V(x')]}} = \pm\int_{t_0}^{t} dt'$$

$$\boxed{t - t_0 = \pm\int \frac{dx}{\sqrt{\frac{2}{m}[E - V(x)]}}}. \tag{5.4}$$

Performing the integral leads to an expression for $x(t)$. In the general case this is not possible by analytical methods, but one can approximate the solution by numerical methods.

5.3 Potential energy

While the kinetic energy definition is a kind of 'universal' as it has the same form ($mv^2/2$) for all mechanical systems, potential energy it uniquely depends on the system. In lay language it is its identity. Different variations of the potential energy values as the system moves into the space indicates different systems and different force experienced. Within the current 1-D formulation it is easily proven that for a given potential energy field $V(x)$ we have,

$$V(x) = V(x_0) - \int_{x_0}^{x} dx' F(x') \qquad \longrightarrow \qquad F(x) = -\frac{dV}{dx} \tag{5.5}$$

So the potential energy $V(x)$ is always defined relative to a reference value $V(x_0)$. Needless to say, that suitable choices of x_0 and $V(x_)$ simplify the relevant algebra without sacrificing the correcteness of the method. [34]

[34] Usually $V(x_0) = 0$. The proper x_0 depends on the particular dependence of the potential $V(x)$ on the

Qualitative considerations of motion in a potential

One need not to rely on the complete determination of $x = x(t)$ to get understanding of the dynamics of the motion. For example, let's take a particle of mass m moving in the region of a potential energy field, $V(x)$.

Supposing the particle is released *from rest* at $x = x_0$. Therefore the initial conditions $x(t_0) = x_0$ and $v(t_0) = 0$). Then $E = V(x_0)$. Its subsequent motion depend on the initial position x_0. One can start from determining the so-called **equilibrium points**; these are points in space where the particles have the tendency to either stand still or to move around those points. From mathematical point of view the equilibrium points are the stationary points of the potential energy: [35]

[35] $V'(x) = \frac{dV(x)}{dx}$

$$V(x_0) = 0 \qquad \rightarrow \qquad m\ddot{x}(t) = -V(x_0) = 0, \qquad \text{No force present}$$

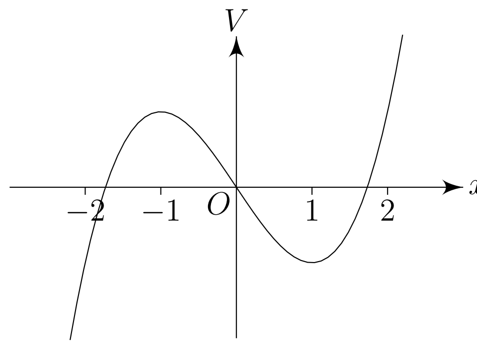


Fig 5.1: Potential $V(x)$

- $x_0 = \pm 1$: For these positions $dV(x_0/dx = 0$ and the particle stays there for all t. These points are called *equilibrium* points (Note that the particle experiences no force at these points since $F(x_0) = 0$).
- $-1 < x_0 < 2$: The particle will oscillates back and forth in the potential well. The force it experiences will always point to the origin since $F(x) = -dV/dx$.
- $x_0 < -1$: The particle eventually will fall to $x = -\infty$. The force will lead the particle to negative directions.
- $x_0 > 2$: The particle has enough energy to oveshoot the highest potential energy at $x_0 = -1$ and will escape to the negative direction $x(t = \infty) = -\infty$. [36]

[36] What about $x_0 = -2$?

5.4 Examples

In the below few examples are presented where the potential energy method is applied to solve the dynamics problem, e.g. to determine $x = x(t)$. The first step is from the given force to find the potential energy using (5.2) and then to solve (5.4).

Example 1. Free fall under constant gravitational field Let's assume a particle of mass m placed initially at $y(0) = h$ and let it to fall from rest. The latter means that its initial velocity is zero $v(0) = v_0 = 0$. By a suitable choice of the CCS ($Oxyz$) so that the weihgt force to be expressed as $F(y) = -mg\hat{y}$ and setting the reference potential value to vanish at the origin $V(0) = 0$ one may obtain for the potential energy

$$V(y) = V(0) - \int_0^y dy'(-mg) = -\int_0^y dy'(-mg) = mgy$$

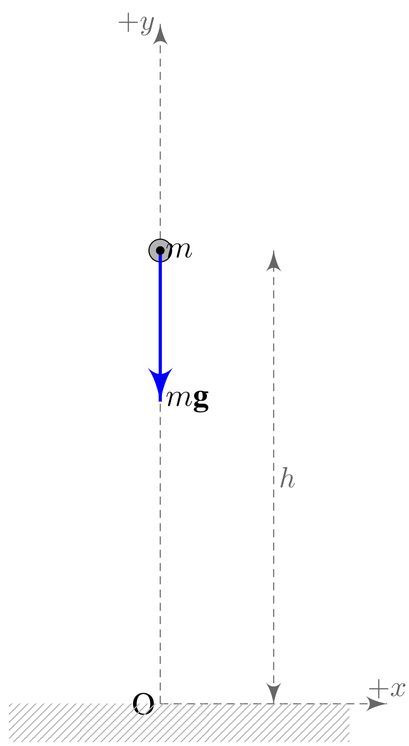


Fig 5.2: Vertical free-fall, $F(x) = mg\hat{y}$

Since the gravity field is conservative the motion constant E (mechanical energy) is given by:

$$E = \frac{1}{2}mv_0^2 + mgy_0 = 0 + mgh = mgh.$$

From relation (5.4) we have:

$$\int_{y_0}^{y} \frac{dy'}{\sqrt{mgh - mgy'}} = \sqrt{\frac{2}{m}}t \qquad \rightarrow \qquad \int_{h}^{y} \frac{dy'}{\sqrt{h-y'}} = \sqrt{2g}t$$
$$\left[-2\sqrt{h-y'}\right]_h^y = \sqrt{2g}t \qquad \rightarrow \qquad 2\sqrt{h-y} = \sqrt{2g}t,$$

from where by squaring and solving for $y(t)$ we obtain the free-fall Gallileo's equation:

$$y(t) = h - \frac{1}{2}gt^2 \tag{5.6}$$

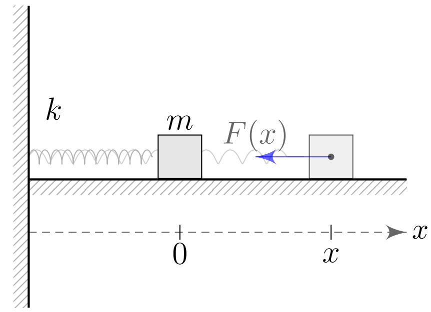


Fig 5.3: Mass-spring simple harmonic oscillator, $F(x) = -kx$

Example 2. Harmonic oscillator

Similar considerations hold for a mass-spring system, (m, k) obeying the Hooke's law $F = -kx$. For zero fiction forces ($F_T = 0$) the mass posseses a potential energy due to the stretching of the spring. Physically, no force is applied on the object when the spring has its natural length. It is convenient to choose as origin of the coordinate system this point. The object either it is displaced or it is kicked off or both. This way energy has been introduced in the system and its subsequent motion can be found either by applying the Newton's second law for $F(x) = -kx$ [37] or by relying in the conservation of energy. Energy will have the same initial value at all later times. Again applying the (5.2) we have,

$$V(x) = V(0) - \int_0^x dx'(-kx') = \frac{1}{2}kx^2. \tag{5.7}$$

One then proceeds to substitute in (5.4) and perform the integration. This is left as a problem for the reader.

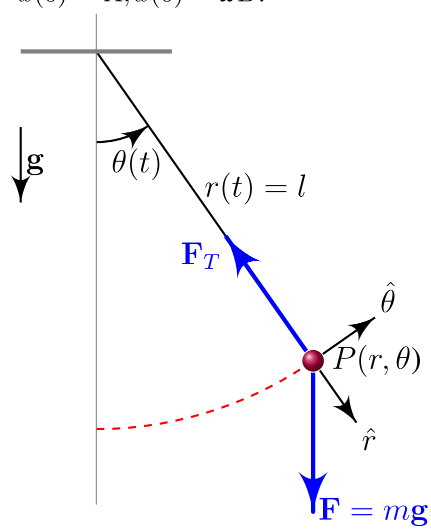
[37] Reminder: From Newton's 2nd law the equation of motion is $m\ddot{x} = -kx.$, where its general solutions is given by

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

with $\omega = \sqrt{k/m}$ and A and B are constants determined by the initial position and velocity by $x(0) = A, \dot{x}(0) = \omega B$.

Example 3. Simple pendulum in the small angle approximation, $\theta_0 \ll 1$ Consider the simple pendulum where an object of mass m is hanged by a string and moves in a Earth's gravity field (acceleration is g). The string applies tension T on the object. We raise the object initially at an angle θ_0 and leave it to evolve. The task is to find its subsequent motion, $\theta = \theta(t)$ at all later times, when $\theta_0 \ll 1$ radians.

One may use Newton's 2nd law for forces using either a CCS or a PCS system. In the present case one is benefited if a PCS system is employed given that the string's length is constant, equal to l. So effectively, the problem is reduced from a 2-D to a 1-D for the angle $\theta(t)$. Following the same lines of thinking as in the problem of the simple harmonic oscillator in the previous chapter, elimination of the r variable results to a potential energy which depends only on the θ angle.



$$V(\theta) = V(\theta_0) - \int_{\theta_0}^{\theta} d\theta'(-mgl\sin\theta') = -mgl\cos\theta, \qquad V(0) = -mg\ell. \tag{5.8}$$

$$E = T + V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta.$$

Therefore $V \propto -\cos\theta$. The stable equilibrium are at $\theta = 0$, and unstable equilibrium at $\theta = \pi$. [38]

[38] Points where $V'(\theta) = 0$. Stability depends on the sign of $V''(\theta)$.

1. If initially $E > mg\ell$, then $\dot{\theta}(t)$ never vanishes and the pendulum makes full circles.

2. If $0 < E < mg\ell$, then $\dot{\theta}(t)$ vanishes at $\theta = \pm\theta_0$ for some $0 < \theta_0 < \pi$ i.e. $E = -mg\ell \cos\theta_0$. The pendulum oscillates back and forth.

Conservation of energy allows to apply (5.4). By defining the oscillation period T as the time that takes to the object to revisit its position then it follows

$$\int_0^{\theta_0} \frac{d\theta'}{\sqrt{\frac{2E}{m\ell^2} + \frac{2g}{\ell}\cos\theta'}} = \int_0^{T_0/4} dt = \frac{T_0}{4}.$$

The pendulum's energy is given by the initial condition as $E = -mg\ell\cos\theta_0$.

$$\frac{T_0}{4} = \sqrt{\frac{\ell}{2g}} \int_0^{\theta_0} \frac{d\theta'}{\sqrt{\cos\theta' - \cos\theta_0}}.$$

In general, this integral is not trivial to evaluate in terms of familiar analytical functions, but assuming a small initial displacement $\theta_0 \ll$ [39] one obtains

$$T_0 \approx 4\sqrt{\frac{\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} = 2\pi\sqrt{\frac{\ell}{g}}\ !$$

which is the result obtained in (4.19). Note that the oscillation period is independent of the amplitude θ_0 (and of the mass m).

[39] $\cos\theta \approx 1 - \frac{1}{2}\theta^2, \quad \theta \ll 1$

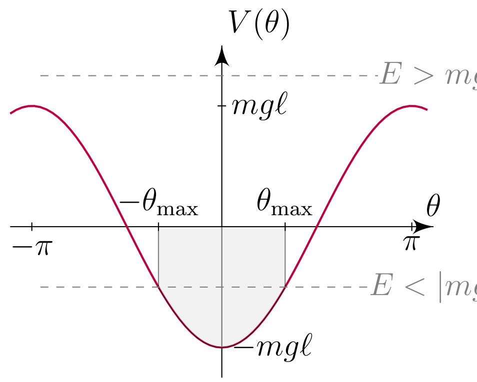


Fig 5.4: Simple pendulum. Force field sketch and potential energy.

5.5 Questions

Question 1. For a moving object moving freely use the method of (5.4) to derive a solution for the mass' position $x = x(t)$. Is it consistent with the Newton's 1st law?

Question 2. Simple mass-sping system

For a simple mass-sping system, if the *initial displacement* and the *initial velocity* of the SHO are x_0 and v_0, respectively, while the *maximum displacement* and *maximum speed* of the mass are A_0 and V_0 respectively, then show that the energy of the SHO may be writen with the following equivalent forms:

$$E = \frac{1}{2}kA_0^2 = \frac{1}{2}mV_0^2 = \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2.$$

Question 3. Spring-mass system

An object attached to a spring completes an oscillation every 2.4 seconds. At $t = 0$ is released from rest at distance $x_0 = 10$ cm from its equilibrium position.

(a) By using the initial conditions as described above, determine the exact formula for the position $x(t)$ as a function of time.

(b) What is the kinetic, potential and total energy at times 0.3, 0.6, 2.7 and 3.0 seconds?

(c) What is the instantaneous and the average power (over one period) generated by the spring's restoring force?

(d) What is the first time that the object is at the position $x = -5.0$ cm?

Question 4. For an harmonic oscillator system (say mass-sping m, k) use the method of (5.4) to derive a solution for the mass' position $x = x(t)$.

Chapter 6

Newtonian Mechanics in three dimensions

The Newton's formulation for the particle's motion is of course naturally applied to objects moving in the 3-D space. All the concepts of the kinetic, potential and mechanical energy are directly applicable to the physical three-dimensional space. In the below these necessary clarifications for this extension are discussed. It should always bear in mind that from practical point of view, depending on the physical problem under question the goal is always to reducte the dimensionality of the problem. For example the motion of an object moving on a flat plane is essentially a 2-D problem, or as we'll see later the planetary motion in the solar system can be reduced to a planar (2-D) problem. Such kind of considerations greatly simplify the complexity of the equations of motions.

6.1 Work produced by a force

The elementary work done from a force $\mathbf{F}$ on body which displaces it by $\mathbf{dr}$ is given by:

$$dW = \mathbf{dr} \cdot \mathbf{F}. \qquad \text{Elementary work} \tag{6.1}$$

In other words, the work produced is the inner product $\mathbf{F}$ times the elementary displacement $\mathbf{dr}$. The above definition implies tha the force should be considerd constant between the positions $\mathbf{r}$ and $\mathbf{r} + \mathbf{dr}$. In case the displacement is not infitesimaly small and given that the force is not constant (in contrast in the general case is function of the position), then the total work done during the displacement is given by the following *line* integral (1.28) of force:

$$W_{ab} = \int_C \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}) = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}) \qquad \text{Total Work,} \tag{6.2}$$

The rate of the generates/consumed work associated with a force is also a useful quantity and is known as the *power* generated/consumed by the force on the object:

$$P = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}, \qquad \text{Power} \tag{6.3}$$

Cartesian coordinates: If[40] the force $\mathbf{F}(\mathbf{r})$ and the elementary displacement $\mathbf{dr}$ are expressed in CCS ($\mathbf{dr} = dx\hat{x} + dy\hat{y} + dz\hat{z}$) then the elementary work is calculated as: The inner product in the integrand simplifies in the given coordinate system

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \equiv \mathbf{F}^T \cdot \mathbf{dr} = \begin{pmatrix} F_x, F_y, F_z \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dy \end{pmatrix} = F_x dx + F_y dy + F_z dz.$$

and the total work in Caresian CS,

$$W_{ab} = \int_{x_a, y_a, z_a}^{x_b, y_b, z_b} F_x dx + F_y dy + F_z dz$$

Polar coordinates: Similarly, if the force $\mathbf{F}(\mathbf{r})$ expressed as $\mathbf{F}(\mathbf{r}) = F_r \hat{r} + F_\theta \hat{\theta} + F_z dz$ and $\mathbf{dr} = dr\hat{r} + r d\theta \hat{\theta} + dz\hat{z}$ then the work in PCS is calculated as:

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \equiv \mathbf{F}^T \cdot \mathbf{dr} = \begin{pmatrix} F_r, F_\theta, F_z \end{pmatrix} \cdot \begin{pmatrix} dr \\ r d\theta \\ dz \end{pmatrix} = F_r dr + F_\theta r d\theta + F_z dz$$

$$W_{ab} = \int_{r_a, \theta_a}^{r_b, \theta_b} F_r dr + F_\theta r d\theta + F_z dz. \tag{6.4}$$

[40] For example, the work produced from the gravitational force when a mass is displaced from a position with height y_a to a position having height y_b and for $\mathbf{F} = (0, -mg, 0)$ we find:

$$\begin{aligned} W_{ab} &= \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \\ &= \int_{y_a}^{y_b} (-mg\hat{y}) \cdot (dy\hat{y}) \\ &= \int_{y_a}^{y_b} (-mg) dy \\ &= mg(y_a - y_b) \end{aligned}$$

Kinetic Energy

From purely mathematical manipulations the following relation for the work done by a force on a body of mass m displaced from the point A, $(\mathbf{r}_A)$ to B, $(\mathbf{r}_B)$:

$$W_{ab} = T_b - T_a, \tag{6.5}$$

where T

$$T \equiv \frac{1}{2} m v^2$$

is named as *kinetic energy* of the body. In other words the work produced by the force $\mathbf{F}$ is equal to the change of its kinetic energy $T_{ba} = T_b - T_a$ between the two positions A, B:

$$W_{ab} = \Delta T_{ba} = T_b - T_a \tag{6.6}$$

Potential energy. The physical quantity $V = V(\mathbf{r})$ that posseses the above properties is called body's *potential energy* and in general depends on body's position in space. The potential energy differs for each fundamental force (gravitational,electromagnetic, spring, ..). The exact form is determined through the relation (6.8). From the latter relation (6.8) the total work produced by the force during the displacement of the body from the position A and B is equal to the variation of its potential energy as below:

$$W_{ab} = \Delta V_{ab} = V(\mathbf{r}_a) - V(\mathbf{r}_b) = V_a - V_b. \tag{6.7}$$

Mechanical energy theorem

From Eqns (6.6) and (6.7) we obtain the following relations:

$$W_{ab} = \Delta T_{ba} \qquad \text{and} \qquad W_{ab} = \Delta U_{ab}$$
$$\Rightarrow T_b - T_a = U_a - U_b \quad \longrightarrow \quad T_a + U_a = T_b + U_b$$

Last relation is the *theorem of mechanical energy conservation* for a motion in a conservative field (or equivalently in a conservative field force). We then define as *mechanical energy* the following quantity:

$$E = T + U(\mathbf{r}) = \text{ constant.}$$

which has the important property that is a *constant* throught the body's motion.

Conservative force fields The forces $\mathbf{F}(\mathbf{r})$ which actually are the gradient of a scalar field $V(\mathbf{r})$ their line integral is *independent of the actual integration curve taken*. Such forces are called *conservative* [41] :

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot \mathbf{dr} = V(\mathbf{r}_a) - V(\mathbf{r}_b) \tag{6.8}$$

These forces (*conservative*) are then written as:

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) = -\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}. \tag{6.9}$$

with the nabla operator defined as a spatial derivative vector and $V(\mathbf{r})$ some scalar function of position. [42]

[41] A quick practical test to check whether a given force is conservative or not is to calculate its curl.

$\nabla \times \mathbf{F}(\mathbf{r}) = 0 \rightarrow$ conservative force

[42] In other words, for such fields, only the end points are important for the evaluation of the (line) integral and not the particular path.

Non-conservative forces field (friction) In the general case in a body there are applied forces which are not associated with any of the fundamental forces (gravity,electrical). For these forces we cannot define a potential energy. So in the presence of these forces we re-express the conservation theorem of the mechanical energy to include the work due to non-conservative forces as:

Let's assume that at a body of mass m are applied the conservative forces $\mathbf{F}_i, i = 1, 2, ..$ and the non-conservative forces $F_j, j = 1, 2, ...$ The body moves from position $A, \mathbf{r}_a$ to position B, r_b. Then we have the following relation for the mechanical energy of the body at these two points and the work produced during the motion between these two points:

$$T_a + \sum_i U_i(\mathbf{r}_a) + \sum_j W_j(\mathbf{r}_b, \mathbf{r}_a) = T_b + \sum_i U_i(\mathbf{r}_b), \tag{6.10}$$

with $W_j(\mathbf{r}_b, \mathbf{r}_b)$ the work produced by the *non-conservative* force $\mathbf{F}_j$ during the motion of the body (of mass m) from position $\mathbf{r}_a$ to position $\mathbf{r}_b$ and $U_i(\mathbf{r}_a), U_i(\mathbf{r}_b)$ body's potential energy at positions $\mathbf{r}_a$ and $\mathbf{r}_b$, respectively, *due to conservative* force $\mathbf{F}_i$.

The most characteristic example of non-conservative forces are the forces due to the touch between two bodies (friction) which transform the produced work into heat without the possibility of transfering it back. Thus, in terms of mechanical energy, the heat should be considered as 'lost' energy from the system and as an irreversible process that removes mechanical energy from the system under consideration.

6.2 Examples

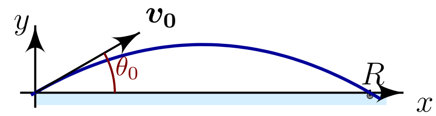


Fig 6.1: Projectile motion in the potential $U(y) = mgy$

Example 1. Projectile motion in a gravity field (low altitude approximation)

1. *Coordinates*

   Let's first consider a CCS ($Oxyz$) with the $y-$axis in parallel with gravity's force and opposite direction. We place the origin of the CS at the Earth's ground level (see figure (6.1)). In this case the position vector, $\mathbf{r}$, and the gravitational force, $\mathbf{F}$ are expressed as

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}, \qquad \mathbf{F} = -mg\hat{y}.$$

2. *Potential Energy* The potential enegy, $V(\mathbf{r})$, of the projectile in the Earth's field, can be calculated by evaluating the below (line) integral,

$$V(\mathbf{r}) = V(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}).$$

   The inner product in the integrand simplifies in the given coordinate system

$$\mathbf{dr} = dx\hat{x} + dy\hat{y} + dz\hat{z} = \begin{pmatrix} dx \\ dy \\ dy \end{pmatrix} \qquad \rightarrow$$

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \equiv \mathbf{F}^T \cdot \mathbf{dr} = \big(F_x, F_y, F_z\big) \cdot \begin{pmatrix} dx \\ dy \\ dy \end{pmatrix} = dx \cdot 0 + dy(-mg) + dz \cdot 0$$
$$= -mgdy$$

   The reference point for the potential energy is choosen the ground level ($V(\mathbf{r}_0 = 0) = 0$). Then we find:

$$V(\mathbf{r}) = V(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} = 0 - \int_{(0,0,0)}^{(x,y,z)} mgdy' = \big[\ mgy\ \big]_{(0,0,0)}^{(x,y,z)} = mgy.$$

3. *Energy conservation and EOM.*

   We can now use the energy consevation property. Initially the energy of the projectile is totally kinetic

$$E = \frac{1}{2}mv_0^2 + V(0) = \frac{1}{2}mv_0^2$$

   Since the energy is a constant of motion the following holds:

$$E = \frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 \qquad \rightarrow \qquad v^2 = v_0^2 - 2gy. \tag{6.11}$$

   The v^2 can be analyzed in its three components:

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

   From Newton's 1st law we know that when no forces are present then the velocity of an objects stays constant. Here we have force only along the $y-$axis which means that the velocities along the x, z axes will remain the same. So given the initial conditions we have,

$$v_x(t) = v_0 \cos\theta_0, \qquad\qquad v_z = 0$$

Then it results that,

$$v^2(t) = v_0^2 \cos^2\theta_0 + v_y^2(t)$$

Replacing the latter expression in (6.11) we obtain, a differential equation for the $y(t)$ component as below:

$$\begin{aligned}
v_0^2\cos^2\theta_0 + v_y^2(t) = v_0^2 - gy \quad &\longrightarrow \quad v_y^2(t) = v_0^2 \sin^2\theta_0 - 2gy(t) \\
\left[v_y(t) \equiv \frac{dy}{dt}\right] \quad &\longrightarrow \quad \frac{dy}{dt} = \pm\sqrt{v_0^2\sin^2\theta_0 - 2gy} = \sqrt{2g}\sqrt{a-y}, \\
&\longrightarrow \quad \frac{dy}{\sqrt{a-y}} = \pm\sqrt{2g}\, dt, \qquad\qquad \left[a = \frac{v_0^2\sin^2\theta_0}{2g}\right]
\end{aligned}$$

This latter equation is integrated as below:

$$\begin{aligned}
\pm t\sqrt{2g} = \sqrt{2g}\int_0^t dt' = \int_0^y \frac{dy'}{\sqrt{a-y'}} = -\int_0^y \frac{d(a-y')}{\sqrt{a-y'}} = \int_0^{a-y}\frac{du}{\sqrt{u}} = \left[2\sqrt{u}\right]_a^{a-y} \\
= 2(\sqrt{a-y} - \sqrt{a}).
\end{aligned}$$

From the latter solving for $y = y(t)$ we have,

$$\begin{aligned}
(a-y) &= \left(\sqrt{a} \pm t\sqrt{\frac{g}{2}}\right)^2 \qquad \rightarrow \\
y(t) &= \left[a - \left(\sqrt{a} \pm t\sqrt{\frac{g}{2}}\right)^2\right] = a - a \pm t\sqrt{2ga} - \frac{1}{2}gt^2,
\end{aligned}$$

from where by replacing the value of $2ga = v_0^2\sin^2\theta_0$ we end up to the same equation as in (2.43),

$$\boxed{y(t) = (v_0\sin\theta_0)t - \frac{1}{2}gt^2} \tag{6.12}$$

The negative sign (in front of the $v_0\sin\theta_0$) was ignored since for the initial velocity along the $y-$axis as $v_{0y} = v_0\sin\theta_0$.

Example 2. Projectile motion using the Newton's law for forces So we have three unknown functions, $(x(t), y(t), z(t)$ to determine and only one differential equation. We need two more equations. For these we can use the Newton's 2nd law expressed in terms of the potential energy, $\mathbf{F} = -\nabla\mathbf{V}(\mathbf{r})$:

$$\begin{aligned}
m\frac{d\mathbf{v}}{dt} = -\nabla V(\mathbf{r}) \qquad &\rightarrow \qquad m\begin{pmatrix}\dot{v}_x \\ \dot{v}_y \\ \dot{v}_z\end{pmatrix} = -\begin{pmatrix}\frac{\partial}{\partial x}(mgy) \\ \frac{\partial}{\partial y}(mgy) \\ \frac{\partial}{\partial z}(mgy)\end{pmatrix} = \begin{pmatrix}0 \\ mg \\ 0\end{pmatrix} \\
&\rightarrow \qquad \begin{cases} v_x &= v_0\cos\theta_0 \\ v_y &= v_0\sin\theta_0 - gt \\ v_z &= 0 \end{cases}
\end{aligned}$$

Now from (6.11) we get, since $v_0^2 = v_{0x}^2 + v_{0y}^2$ [43]

[43]

$$v_{0x} = v_0 \cos\theta_0,$$
$$v_{0y} = v_0 \sin\theta_0$$

Example 3. Work of a mass-sping system (Harmonic oscillator) In this case we consider a CCS ($Oxyz$) with the $x-$axis in parallel with spring's force and in opposite direction. In this case the gravitational force is expressed as $\mathbf{F} = -kx\hat{x}$ (Hooke's law). The work produced from the spring's force when the body is displaced from the position x_a to a position x_b is as follows: [44]

$$\begin{aligned} W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} &= \int_{x_a}^{x_b} (-kx\hat{x}) \cdot (dx\hat{x} + 0\hat{y} + 0\hat{z}) \\ &= \int_{x_a}^{x_b} (-kx)dy = \frac{1}{2}kx_a^2 - \frac{1}{2}kx_b^2 \end{aligned} \tag{6.13}$$

[44] Note that in accordance with (6.8) we have for $U(x)$:

$$U(x) = \frac{1}{2}kx^2 + C,$$

where again $C = const.$ is a constant that is determined as long as the reference point for the potential energy is determined. For example, if *we define* the body's position at $C, x_c = 0$ such that the potential energy is zero we find $C = 0$.

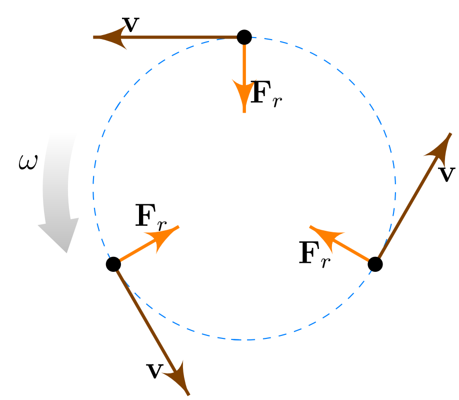
Example 4. Circular motion We have to calculate the work done by $\mathbf{F}(\mathbf{r})$ applied to a body of mass m and moves in a cycle of radius R. Assuming a CCS($Oxyz$) and the cycle lying on the plane $x - y, (z = 0)$ with its center at origin, we have:

$$r = R \qquad dr = 0,$$
$$F_r = -m\frac{v^2}{r}, \qquad F_\theta = m\frac{dv}{dt},$$

where θ the angle of the position vector with x-axis. The line integral (6.4) is written:

$$W_{ab} = m\int_{\theta_a}^{\theta_b} \frac{dv}{dt} d\theta,$$

with body's position fully determined by the angle θ.



If angular velocity is constant ($\dot{\theta} = \omega = const.$) then linear velocity is constant as well, $v_\theta = R\omega$ and thus $dv_\theta/dt = 0$. We thus conclude that:

$$W_{ab} = m\int_{\theta_a}^{\theta_b} \frac{dv}{dt} d\theta = 0,$$

as it should since the only force applied in the system is the radial one (centripetal) $F_r = -mv^2/R$ always normal to the position vector.

Chapter 7

Inverse square forces, $f(r) = -k/r^2$ Kepler's conic orbits

7.1 Central forces

An important class of force fields occuring in nature are those who posses spherical symmetry. In other words they show no any directional preference in space. The general form of such forces is the following,

$$\mathbf{F}(\mathbf{r}) = -f(r)\hat{r} \tag{7.1}$$

Accordingly such forces are associated with a potential energy field $V(r)$ expressed as,

$$f(r) = -\frac{dV}{dr} \qquad \longrightarrow \qquad V(r) = V(r_0) - \int_{r_0}^{r} dr f(r) \tag{7.2}$$

We then say that a particle posses potential energy equal to $V(r)$ which depends only on its distance from the origin of the coordinate system.

While in general a spherical coordinate system would be a suitable choice to express the dynamical Newton's laws it is also convenient to choose a polar coordinate system (r, θ, z). We then choose the central force to lie in the $r - \theta$ plain. In this case the motion along the $z-$axis it would be a uniform motion with constant velocity or if there is no initial velocity in the $z-$direction then the object will never leave the $r - \theta$ plane. So, all the object's dynamics takes place on this plane.

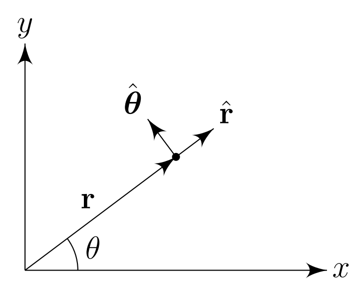


Fig 7.1

7.2 Orbits in central fields

The task in this chapter is to study the orbit of a particle in a central force,

$$m\ddot{\mathbf{r}} = -f(r)\hat{r}.$$

For central forces, in principle the motion is evolving in three dimensions however the orbit is essentially confined to a plane. As shown in class, the motion takes place in a plane passing through the origin, and perpendicular to $\mathbf{L}$ which we take to be along the $\hat{z}$ axis. It is not difficult to show that for central forces the angular momentum is also a constant of motion,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = L\hat{z} = cnst \tag{7.3}$$

To this end for $\mathbf{F} = f(r)\hat{r}$ we have for the Newton's 2nd law in polar coordinates $(\mathbf{r}, \theta, \hat{z})$:

$$m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = f(r)\hat{r}.$$

The θ component of the central force is zero, $F_\theta(r) = 0$, so the relevant equation gives,

$$m(r\ddot{\theta} - 2\dot{r}\dot{\theta}) = 0 \quad \longrightarrow \quad \frac{1}{r}\frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \longrightarrow \quad mr^2\dot{\theta} = cnst. \tag{7.4}$$

Let name $L = mr^2\dot{\theta}$. This is equal to the only z component angular momentum $\mathbf{L}$:

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = m(r\hat{\mathbf{r}}) \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = 0 + mr^2\dot{\theta}\ \hat{r} \times \hat{\theta} = mr^2\dot{\theta}\ \hat{\mathbf{z}}.$$

The above result combined with (7.4) proves that L is constant, which is the *conservation of angular momentum* principle.

Mechanical energy. Conservation of angular momentum allows the reduction of the 2D problem to an 1D problem. This is done as follows,

The radial (r) component of the equation of motion is,

$$m(\ddot{r} - r\dot{\theta}^2) = f(r).$$

Substitution of $\dot{\theta}$ using $r^2\dot{\theta} = L/m$ to obtain

$$m\ddot{r} = f(r) + \frac{L^2}{mr^3}. \tag{7.5}$$

Now by defining an effective radial potential $V_{eff}(r)$ by,

$$V_{\text{eff}}(r) = V(r) + \frac{L2}{2rm^2}, \qquad f(r) = -\frac{dV}{dr}$$

and recalling the procedure followed to determine the energy constant in the 1D case we can conclude immediately that the total energy of the particle is [45]

$$\begin{aligned} E &= \frac{1}{2}m|\dot{r}^2 + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{L}{2mr^2} + V(r) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r). \end{aligned}$$

[45] Note that $\hat{r}$ and $\hat{\theta}$ are orthogonal,

$$\dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\theta}.$$

and $\dot{\theta} = L/mr^2$

We then have arrived to the following expressions for the total energy and angular momentum of the particle,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = cnst \tag{7.6}$$

$$L = mr^2\dot{\theta} = cnst'. \tag{7.7}$$

together with the conclusion that these are constants of motion.

7.2.1 Dynamics in a central force field

In principle, one can determine the radial distance $r(t)$ by integrating the energy equation

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) \qquad \rightarrow \qquad t = \pm\sqrt{\frac{m}{2}}\int_{r_0}^{r} \frac{dr'}{\sqrt{E - \frac{L^2}{2mr'^2} - V(r')}}$$

for $t_0 = 0$ and $r(0) = r_0$. Unfortunately, for the potential fields of interest is not so easy to evaluate the integral. An alternative approach is insteat to find the time evolution of $r(t), \theta(t)$ is to calculate the shape $r(\theta)$ of the orbit.

Binet's equation Our starting point will be the radial equation for the force The radial equation of motion

$$m\ddot{r} = \frac{L^2}{mr^3} + f(r)$$

We can arrive to a simple differential equation by setting,

$$u = \frac{1}{r}, \qquad h = \frac{L}{m}$$

Then the following relation for the derivatives over angle are true:

$$\begin{aligned}
\dot{r} &= \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta}, \\
\ddot{r} &= \frac{d}{dt}\left[-h\frac{du}{d\theta}\right] = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h\frac{d^2u}{d\theta^2}\frac{h}{r^2} = -h^2u^2\frac{d^2u}{d\theta^2}.
\end{aligned}$$

By replacing the above relations to the radial equation of motion above we find,

$$-mh^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = f\left(\frac{1}{u}\right).$$

Finally one may substitute the value of h to arrive at:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{f(1/u)}{u^2}, \qquad \text{Binnet's equation} \tag{7.8}$$

[46]

Finally, provided that $u = u(\theta)$ is known one can work out the time-dependence via

$$\dot{\theta}(t) = \frac{L}{m}u^2. \tag{7.9}$$

[46] The Binnet equation for a inverse-square law force $f(r) = -kr^2 = -ku^2$ reduces to an easily solvable equation, namely to that of an harmonic oscillator!

7.3 The Kepler orbits

The Kepler problem is the study of the orbits of two objects interacting via an inverse square law central force. The target is to classify the possible orbits and study their properties. For the classical Kepler problem is the planetary motion under the Sun's gravitational field, or more generally the orbit of celestial objects.

Shapes of orbits

For a planet orbiting the sun, the potential and force are given by

$$F(r) = -\frac{GMm}{r^2},$$

Defining $k = GMm$ and replacing $u = 1/r$ Binet's equation becomes linear to, [47]

$$\frac{d^2u}{d\theta^2} + u = \frac{km}{L^2}. \tag{7.10}$$

[47] This differential equation is equivalent to the driven harmonic oscillator problem with constant force. For example, mass-spring system under a constant force.

This is a differential equation which is easily solved with the general solution,

$$u(\theta) = \frac{mk}{L^2} + A\cos(\theta - \theta_0)$$

where $A \geq 0$ and θ_0 are arbitrary constants.

1. $A = 0$ Then $u = mk/L^2$ is constant, and the orbit is circular,

$$r(\theta) = \frac{L^2}{mk}$$

2. $A \neq 0$ The polar coordinate system can always be re-oriented such that $\theta_0 = 0$. It is then concluded that the orbit of a planet around the sun is given by

$$r = \frac{p}{1 + e\cos\theta}, \qquad p = \frac{L^2}{mk}, \qquad e = \frac{AL^2}{mk} \tag{7.11}$$

The constant A can be related with the total energy of E as, [48]

$$e = \sqrt{1 + \frac{2EL^2}{mk^2}} \tag{7.12}$$

This is the polar equation of a conical orbit, with a focus (the sun) at the origin.

[48] For this one should note that at the closest approach $E = V(r_{min})$. See below for an alternative route.

[49] The dimensionless parameter in the equation of orbit is the *eccentricity* and essentially determines the shape of the orbit. [50] There are three different possibilities:

[49] Eccentricity $e = \frac{AL^2}{m^2k} > 0$

[50] The conic orbit is expressed in the Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$ as,

$$(1-e^2)x^2 + 2epx + y^2 = p^2.$$

- **Ellipse:** ($0 \leq e < 1$) Then r is restricted by

$$\frac{p}{1+e} \leq r \leq \frac{p}{1-e}.$$

and can be put into the equation of an ellipse centered on $(-ea, 0)$,

$$\boxed{\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a = \frac{p}{1-e^2}, \qquad b = \frac{p}{\sqrt{1-e^2}} \leq a. \quad (7.13)}$$

Parameters a and b are the semi-major and semi-minor axis. p is the *semi-latus rectum*. One focus of the ellipse is at the origin. If $e = 0$, then $a = b = p$ and the ellipse is a circle.

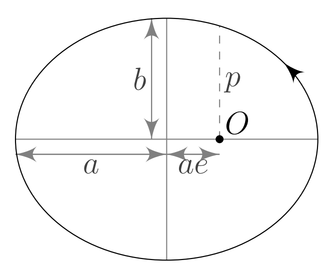


Fig 7.2: Elliptical orbit when for $1 < e < 0$.

- **Hyperbola:** ($e > 1$). For $e > 1$, $r \to \infty$ as $\theta \to \pm\alpha$, where $\alpha = \cos^{-1}(1/e) \in (\pi/2, \pi)$. Then (†) can be put into the equation of a hyperbola centered on $(ea, 0)$,

$$\boxed{\begin{aligned} \frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} &= 1, & (7.14)\\ a = \frac{p}{e^2-1}, \qquad b &= \frac{p}{\sqrt{e^2-1}}. & (7.15)\end{aligned}}$$

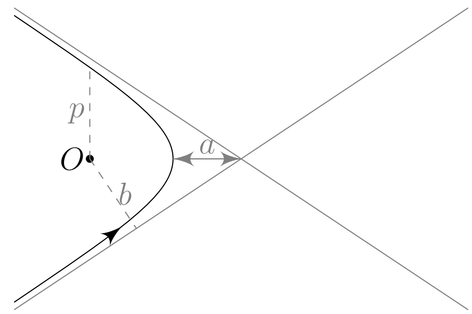


Fig 7.3: Hyperbolic orbit when for $e > 1$.

This corresponds to an unbound orbit that is deflected (scattered) by an attractive force.

Parameter b is both the semi-minor axis and the *impact parameter*. It is the distance by which the planet would miss the object if there were no attractive force.

The asymptote is $y = \frac{b}{a}(x - ea)$, or

$$x\sqrt{e^2-1} - y = eb.$$

- **Parabola:** ($e = 1$). Then conical orbit,

$$r = \frac{p}{1+\cos\theta}.$$

becomes the equation of a parabola, $y^2 = \ell(\ell - 2x)$. We see that $r \to \infty$ as $\theta \to \pm\pi$. The trajectory is similar to that of a hyperbola.

Energy and eccentricity

The path of a planet is determined by its angular momentum L and its energy E. For this we need to relate these two constants with the eccentricity of the orbit, e. This may done as below

$$\begin{aligned} E &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}\\ &= \frac{1}{2}mh^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - ku \end{aligned}$$

Substitute $u = \dfrac{1}{p}(1 + e\cos\theta)$ and $p = \dfrac{L^2}{mk}$, to arrive at,

$$E = \frac{mk^2}{2L^2}(e^2-1), \qquad e = \sqrt{1 + \frac{2EL^2}{mk^2}} \tag{7.16}$$

which is independent of θ, as it must be (since it is a constant).

The range of the values of the eccentricity, e, is related to the orbit's shape: We have the following cases,

$e = 0$ This results to $E = -1$ and the orbit is bounded and circular.

$e < 1$ This results to $E < 0$ and the orbit is bounded.

$e = 1$ In this case, $E = 0$ and the orbit is parabolic orbit, in fact it is *'marginally bound'*.

$e > 1$ Thus $E > 0$ and the orbit is unbounded.

Note that the condition $E > 0$ is equivalent to

$$v_{\text{esc}} > \sqrt{\frac{2GM}{r}}$$

, which means that there is enough kinetic energy to escape orbit.

Kepler's laws of planetary motion

[51] Let's see now whether the conical orbits could have predict the Kepler's laws:

1 [Kepler's 1st law]

The orbit of each planet is an ellipse with the Sun at one focus.

2 [Kepler's 2nd law]

The line between the planet and the sun sweeps out equal areas in equal times.

3 [Kepler's 3rd law]

The square of the orbital period is proportional to the cube of the semi-major axis,

$$T^2 \propto a^3.$$

[51] Kepler's law 1 has already shown as the solution of the motion of the inverse-square law of gravity.
Kepler's law 2 is essentially the conservation of angular momentum: The area swept out by moving $d\theta$ is $dA = \frac{1}{2}r^2\, d\theta$ (area of sector of circle). So,

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{const.}$$

This is true for *any* central force.

Generally the planets have very low eccentricity (i.e. almost circular orbit), but asteroids/comets are ellipses, i.e. orbits of high eccentricity. In other solar systems, even planets have have highly eccentric orbits. As we've previously shown, it is also possible for the object to have a parabolic or hyperbolic orbit. However, the trend is not to call these "planets".

Qualitative study of the radial potential energy We have the following effective potential

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{L^2}{2mr^2}.$$

Therefore there are two terms of opposite signs and different dependencies on $1/r$, $1/r^2$. For small r, the $\sim 1/r^2$ dominates and $V_{\text{eff}}(r)$ is large. For large r, the term $\sim 1/r$ dominates. Then $V_{\text{eff}}(r)$ asymptotically approaches 0 from below.

The minimum of V_{eff} is at

$$R = \frac{L^2}{2mk}, \qquad E_{\min} = -\frac{mk^2}{2L^2}, \qquad e = 1$$

We have a few possible types of orbits:

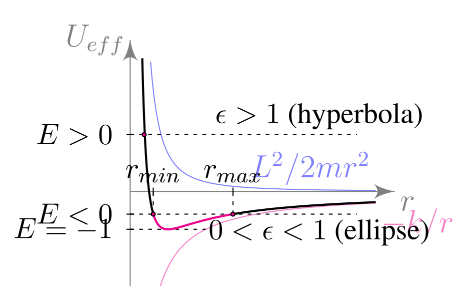


Fig 7.4: Effective potential $V_{eff}(r) = -1/r + 1/r^2$, $(L^2 = 2m,\ k = 2)$.

- If $E = E_{\min}$, then r remains at r_* and $\dot{\theta} = L/mr^2$ is constant. So we have a uniform motion in a circle.

- If $E_{\min} < E < 0$, then r oscillates and $\dot{r} = L/mr^2$ does also. This is a non-circular, bounded orbit.

- If $E \geq 0$, then r comes in from ∞, reaches a minimum, and returns to infinity. This is an unbounded orbit.

We will later show that in the case of motion in an inverse square force, the trajectories are conic sections (circles, ellipses, parabolae and hyperbolae).

7.4 Questions

Question 1. Central force Show that if $\mathbf{F}(\mathbf{r}) = -f(r)\hat{r}$ is a central force, then

$$\mathbf{F} = -\nabla V = -\frac{dV}{dr}\hat{r}.$$

Question 2.

The spherical symmetry property of central forces give rise to an additional conserved quantity known as *angular momentum* defined by,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}, \tag{7.17}$$

Show that the angular momentum of a particle moving in the region of a central force remains constant.

Question 3. Kepler's 3rd law

Assume an object of mass m moving on an ellipse in the potential field $V(r) = -k/r$. Its angular momentum is L and its energy E.

(a) Calculate the total area of the ellipse as by integrating for a full period of the second law and show that

$$\pi ab = T\frac{L}{2m}.$$

where a, b are the semi-major and minor-axes of the ellipse and T the orbital period.

(b) Then use the relations

$$b = \frac{\lambda}{\sqrt{1-e^2}}, \qquad a = \frac{\lambda}{1-e^2}, \qquad \lambda = \frac{L^2}{mk}$$

where e is the ellipse's eccentricity, to show that,

$$T^2 = \frac{(2\pi)^2 a^3}{k}.$$

(c) In the case of the solar system, specialize k and estimate the Earth's orbital period around the Sun. For the numerical values of the required quantities use published values in the literature or on the internet.

Question 4. Circular orbits

Prove the Kepler's 3r law for circular orbits starting from the radial equation of motion.

Question 5. Halley's comet

Given that the Halley's comet has an elliptical orbit. Its last perihelion was 9/02/1986 while the next one is around 28/07/2061). Its perihelion has a distance 0.586 AU from the Sun. Calculate its eccentricity.

Chapter 8

Non-conservative forces, $F = F(v)$

In this chapter is discussed the case of forces where changes in the energy of the mechanical system from one point to another depends on its full path in the space and not simply on the end points. A typical class of such forces are those that are velocity dependent, namely, $F = F(v)$. As a result the concept of potential energy becomes questionable in general and certainly cannot be regarded as an alternative descrption of the particle's dynamics. Such examples are the friction force (e.g. motion in a rough surface, fluid motion) and the magnetic force.

8.1 Motion in a uniform magnetic field

The most notable example of such force is the magnetic force exerted on a moving charge. Assuming that a particle of mass, m and charge, q, moving with velocity $\mathbf{v}$ is placed in a region of space where electric, $\mathbf{E}$, and magnetic fields, $\mathbf{E}$ are present then the force governing its motion is,

$$F_L = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \tag{8.1}$$

In this case the Newton's 2nd law dictates, that,

$$m\ddot{\mathbf{r}} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \tag{8.2}$$

It is obvious that the magnetic force is vanishing when the particles is motionless, $v = 0$. In a region where only magnetic fields are present (e.g. the interior of an inductor) then it is reduced to,

$$m\ddot{\mathbf{r}} = q\mathbf{v} \times \mathbf{B}, \tag{8.3}$$

for $\mathbf{E} = 0$. So we assume $\mathbf{v}_0 = (v_{0x}, v_{0y}, v_{0y})$ and choose a CCS system with the magnetic field along the $z-$axis, $\mathbf{B} = (0, 0, B)$. Then (8.3) expands to a coupled system of 1st-order differential equations for the three componets of the position vector, $\mathbf{r} = (x, y, z)$,

$$m\ddot{x} = qB\dot{y} \tag{8.4}$$
$$m\ddot{y} = -qB\dot{x} \tag{8.5}$$
$$m\ddot{z} = 0 \tag{8.6}$$

From the last equation it is easily concluded a uniform motion along the z-axi, $\mathbf{v} = v_{0z}\hat{z}$, where v_{0z} is the initial velocity along the $z-$axis. The interesting motion is on the $x - y$ plane as the equation there are coupled. There are few ways that one may solve this system of equation. An elegant approach is to use the powerful complex arithmetics, but for now will try to skip it. Let's set, $\dot{x} = v$ and $\dot{y} = u$. Then we have,

$$\dot{x} = v \quad (8.4) \Longrightarrow \quad \dot{u} = +\Omega_0 v, \qquad \Omega_0 = qB/m \tag{8.7}$$
$$\dot{y} = u \quad (8.5) \Longrightarrow \quad \dot{v} = -\Omega_0 u \tag{8.8}$$

In this case if one takes the time derivative of $u(t)$ one obtains:

$$\ddot{u} = +\Omega_0 B\dot{v} = \Omega_0 B(-\Omega_0)u \qquad \Longrightarrow \qquad \ddot{u}(t) + \Omega_0^2 \ddot{u}(t) = 0.$$

A similar equation is obtained for the v component. It will be shown that this represents a simple harmonic oscillator system, known to oscillate with constant period $T_0 = 2\pi/\Omega_0$ around its equilibrium point. At this stage will not go further in the analysis and let this task for the relevant chapter where harmonic oscillator systems are treated.

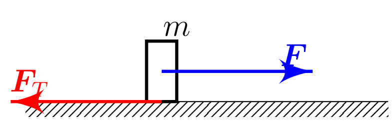
8.2 Friction

Other mechanical systems that evolve under the influence of forces which depend on their kinetic state arise as the statistical sum of numerous individual fundamental forces (gravity/electromagnetic), randomly exerted on the body. Their velocity dependence is then modelled empirically. The net physical result is to oppose the motion and as such they normally exhibit as *resistance forces* (friction). Again we start by choosing a CCS($Oxyz$) with the $y-$axis opposite to the gravitational force, $\mathbf{F} = -mg\hat{y}$ and assuming that the initial time t_0 we have $\mathbf{r}(t_0) = \mathbf{r}_0 = (x_0, y_0, z_0)$ and $u(t_0) = \mathbf{u}_0 = (u_{x0}, u_{y0}, u_{z0})$ for body's position and velocity. The equation of motions are given:

The equation of motion from Newton's 2nd law can be determined as:

$$m\frac{d}{dt}\mathbf{v}(t) = \mathbf{F} + \mathbf{F}_T, \tag{8.9}$$

with $\mathbf{F}_T$ resistance force and $\mathbf{F} = m\mathbf{g}$ the gravity force. In normal physical conditions the resistance forces have opposite direction with the body's velocity and magnitude that depend's linearly or the square of body's speed:



$$\mathbf{F}_T = -kv\hat{v} \qquad \text{or} \qquad \mathbf{F}_T = -kv^2\hat{v},$$

where k is a constant which depends on the particular problem. In the case of friction force is known only empirically. Drag forces which show dependance on the square of the speed appear to large objects moving in low viscosity media (e.g. tennis ball through air).
[52]

8.2.1 Free fall in a fluid: $\mathbf{F}_T = -kv\hat{v}$

Assume free fall with in the gravitational field with constant $\mathbf{g} = -g\hat{y}$ and initial velocity $\mathbf{v}_0 = v_0\hat{y}$. From Newton's 2nd law it is derived that the object moves only

[52] For example, for a solid sphere of radius R moving in a fluid the drag force is proportional $-kv$ is taken to be

$$k = 6\pi\mu R, \qquad \text{Stoke's law}$$

where μ is the viscosity of

along the yy axis. In this case from (8.9) one obtains:

$$m\frac{d^2}{dt^2}y(t) = mg - kv(t). \tag{8.10}$$

By solving the above 2nd-order differential order with given initial conditions (let's choose here $y(0) = 0, v(0) = 0$) will give for the velocity:

$$m\frac{dv}{dt} = mg - kv(t) \qquad \Longrightarrow \qquad \frac{dv}{g-(k/m)v} = dt$$
$$\Longrightarrow \qquad \int_{v_0}^{v(t)} \frac{dv'}{g-(k/m)v'} = \int_0^t dt' = t - 0.$$

Finally by performing the integral for the velocity, v we may obtain: [53]

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \tag{8.11}$$

and

$$y(t) = \frac{mg}{k}\left[t - \frac{m}{k}(1 - e^{-\frac{k}{m}t})\right]. \tag{8.12}$$

[53]
$$\int_0^v \frac{dv'}{g-\frac{kv'}{m}} = -\frac{m}{k}\int_0^v \frac{d(g-\frac{kv'}{m})}{g-\frac{kv'}{m}} = -\frac{m}{k}\int_g^{g-\frac{kv}{m}} \frac{ds}{s} = -\frac{m}{k}[\ln s]_g^{g-\frac{kv}{m}} = -\frac{m}{k}\ln\frac{g-\frac{kv}{m}}{g}$$

8.2.2 Motion on a rough surface: damped harmonic oscillator

We say that a physical system represents a damped harmonic oscillator (DHO) when we insert a damping force in a simple harmonic oscillator system. This damping force represents the influence of the environment on the motion of the simple harmonic oscillator. In general the overall result of the damping force is to extract energy from the SHO. The SHO cannot oscillate for ever and eventually it will stop oscillating. In most cases, the total energy of the SHO dissipates to the environment as a heat.

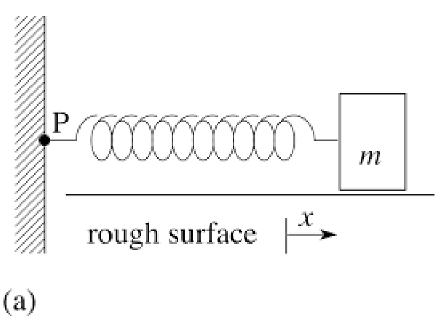


Fig 8.1. Sketch of mass-spring system with damping, namely the damped harmonic oscillator (DH0)

Spring-mass system Assuming an ideal mass-spring system, and as usual if x represents the displacement of the mass from it's equilibrium position we can discuss the basic properties of a DMO. The forces acting on the mass are the force from the spring (Hooke's law), $F = -kx$ and the friction force due to the contact of the mass with the ground. This latter force is opposed the motion of the mass and model it as $-bv$. Taking into the account the above expressions for the forces we write for the

2nd Newton's law:

$$\begin{aligned}
\sum_i F_i = m\frac{d^2}{dt^2}x(t) &\implies -kx(t) - bv(t) = ma(t) \\
&\implies -kx(t) - b\dot{x}(t) = m\ddot{x}(t) \\
&\implies m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0 \\
&\implies \ddot{x}(t) + \frac{b}{m}\dot{x}(t) + \frac{k}{m}x(t) = 0
\end{aligned}$$

Therefore, we obtain the following differential equation (ODE) for a damped harmonic oscillator system:

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = 0, \qquad \omega_0 = \sqrt{\frac{k}{m}}, \qquad \gamma = \frac{b}{m}. \tag{8.13}$$

Solutions for the DHO: The solution of the above differential equation for the DHO depends on the relation between the eigenfrequency ω_0 and the damping parameter γ. We can distinguish among the three different solutions as follows:

1. **Light damping**: $\omega_0 > \gamma/2$. In this case the motion is again oscillatory, however when compared with the undamped harmonic oscillator ($\gamma = 0$) two important points should be emphasized: (a) the amplitude $A(t)$ of the oscillations decays exponentially and (b) the frequency ω of the oscillation is decreased by an amount that depends on the value of the damping parameter γ. In this case the oscillation of the mass is expressed as below:

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1 \cos\omega t + A_2 \sin\omega t), \qquad \omega_0 > \frac{\gamma}{2} \tag{8.14}$$

   - *Angular eigenfrequency* ω (rad/sec): It's value is characteristic of the system:

$$\omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}, \qquad \omega = \frac{2\pi}{T} = 2\pi f, \tag{8.15}$$

     where T is the oscillation's period and f is the frequency measured in sec^{-1} (Hz).

2. **Heavy damping**: $\omega_0 < \gamma/2$. In this case there are *no oscillations* at all and the mass returns to it's equilibrium position at large times.

$$x(t) = e^{-\frac{\gamma}{2}t}\left(A_1 e^{\alpha t} + A_2 e^{-\alpha t}\right), \qquad \omega_0 < \frac{\gamma}{2} \tag{8.16}$$

   - α (Hz): It's value is characteristic of the system and the damping environment:

$$\alpha = \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2},$$

3. **Critical damping**: $\omega_0 = \gamma/2$, Again, there are *no oscillations* and the mass returns to it's equilibrium position without passing it. The exact motion is given by the following expression:

$$x(t) = e^{-\frac{\gamma}{2}t}\left(A_1 + A_2 t\right), \qquad \omega_0 = \frac{\gamma}{2} \tag{8.17}$$

In all the above cases A_1, A_2 are constants (independent on time) that are determined through the initial conditions. As initial conditions should be understood the following equations, which provide the position ($x(0)$) and the velocity $\dot{x}(t)$) at the initial time $t_0 = 0$:

$$x(0) = x_0 \qquad\qquad v(0) = \dot{x}(0) = v_0, \tag{8.18}$$

Mechanical Energy of a DHO and Quality factor The energy for a DHM is *not a constant* of the motion as it decreases as a function of the time. Eventually all the energy of the DHO will be dissipated into the surrounding environment. A measure of the rate that energy is dissipating is expected to be the γ parameter which describes the strengh of the system-environment interaction.

In the present case, we'll examine the case of light damping of a good harmonic oscillator by assumming $\omega_0 >> \gamma$. The mechanical energy of the mass-sping system has be defined to be the sum of the kinetic and the potential energy:

$$E(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2, \tag{8.19}$$

For the case of light damping we have for the position $x(t)$:

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1\cos\omega t + A_2\sin\omega t),$$

Taking the derivative of the position we find for the velocity $v(t) = \dot{x}$:

$$\begin{aligned}
v(t) &= e^{-\frac{\gamma}{2}t}(\omega A_2 - \frac{\gamma}{2}A_1)\cos\omega t - (A_2\frac{\gamma}{2} + \omega A_1)\sin\omega t) \\
&\simeq e^{-\frac{\gamma}{2}t}(\omega A_2 + 0)\cos\omega t - (0 + \omega A_1)\sin\omega t) \\
&= e^{-\frac{\gamma}{2}t}\omega(A_2\cos\omega t - A_1\sin\omega t),
\end{aligned}$$

where from going from the first to the second line we used that $\omega_0 >> \gamma$ and ignored the terms containing γ by setting it zero. Substituting the above expressions into the expression for the energy [(Eq.(8.19))] we get:

$$\begin{aligned}
E &= \frac{1}{2}e^{-\gamma t}\left[k(A_1\cos\omega t + A_2\sin\omega t)^2 + m\omega^2(A_2\cos\omega t - A_1\sin\omega t)^2\right] \qquad (\omega \to \omega_0) \\
&\simeq \frac{1}{2}e^{-\gamma t}\left[k(A_1\cos\omega_0 t + A_2\sin\omega_0 t)^2 + m\omega_0^2(A_2\cos\omega_0 t - A_1\sin\omega_0 t)^2\right] \qquad (k = m\omega_0^2) \\
&= \frac{1}{2}e^{-\gamma t}k\left[(A_1\cos\omega_0 t + A_2\sin\omega_0 t)^2 + (A_2\cos\omega_0 t - A_1\sin\omega_0 t)^2\right] \\
&= \frac{1}{2}e^{-\gamma t}k\left[(A_1^2 + A_2^2)\cos^2\omega_0 t + (A_1^2 + A_2^2)\sin^2\omega_0 t\right. \\
&\qquad\qquad \left. +2A_1A_2\cos\omega_0 t\sin\omega_0 t - 2A_1A_2\cos\omega_0 t\sin\omega_0 t\right] \\
&= \frac{1}{2}e^{-\gamma t}k(A_1^2 + A_2^2)(\cos^2\omega_0 t + \sin^2\omega_0 t) = \frac{1}{2}e^{-\gamma t}kA_0^2.
\end{aligned}$$

where A_0 is the amplitude of the motion *at initial time*. In going from the first line to the second we assumed that $\omega \simeq \omega_0$ which constitutes a good approximation if we recall that $\omega = \sqrt{(\omega_0 - (\gamma/2)^2)}$ and $\gamma << \omega_0$. In addition from the second line to the third we used the relation $\omega_0^2 = \sqrt{(k/m)} \to k = m\omega_0^2$. We may summarize the result by setting $E_0 = kA_0^2/2$ and write for the mechanical energy of this underdamped DHO:

$$\boxed{E(t) \simeq E_0 e^{-\gamma t}, \qquad E_0 = \frac{1}{2}kA_0^2 = \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2. \qquad (8.20)}$$

Therefore we see that eventually the energy of the DHO will vanish ($E(t \to \infty) \to 0$). The decay rate of the energy (energy dissipation) is given by the γ.

Quality factor A quantity useful to characterize the DHO is the *quality factor*, which is a measuere of the relative loss of energy of the DHO within a oscillation period:

$$Q = \frac{\omega_0}{\gamma}.$$

The way that this definition is derived goes along the following lines: first assume the energy of the DHO at a particular time t:

$$E(t) = E_0 e^{-\gamma t}.$$

Then let's ask for the energy at a time after a period of oscillation $t = t + T_0$, where $T_0 = 2\pi/\omega_0$:

$$E(t + T_0) = E_0 e^{-\gamma(t+T_0)} = E_0 e^{-\gamma t} e^{-\gamma T_0} = E(t) e^{-\gamma T_0}$$

Now let's form the loss of energy within this period and divide by $E(t)$. The absolute value of it is

$$\begin{aligned}
\left|\frac{\Delta E(t)}{E(t)}\right| \equiv \frac{E(t) - E(t+T_0)}{E(t)} &= 1 - e^{-\gamma T_0} = 1 - e^{-\gamma(\frac{2\pi}{\omega_0})} \\
&= 1 - [1 - (2\pi\frac{\gamma}{\omega_0}) + \frac{1}{2!}(2\pi\frac{\gamma}{\omega_0})^2 - ...] \\
&\simeq 2\pi(\frac{\gamma}{\omega_0}) = \frac{2\pi}{Q}
\end{aligned}$$

From the above relation we can see that a good oscillator ($|\Delta E/E| << 1$) requires high-Q! Therefore, *the higher the quality factor the better the DHO*. Using other words, we can also say that high quality factor of an harmonic oscillator suggests for weak interaction with it's environment.

8.3 Examples

Example 1. Underdamped solutions of DHO in terms of x_0 and v_0 For the case of light dampong ($\omega_0 > \gamma$) calculate the constants A_1, A_2 of Eq. (8.14) in terms of the initial conditions $x(0) = x_0$ and $v(0) = v_0$.

(a) Give the solutions in terms of v_0 if $x_0 = 0$.

(b) Give the solutions in terms of x_0 and if $v_0 = 0$.

(c) Take the case that both x_0 and v_0 are non-zero. However set $\gamma = 0$. Does the result coincide with the simple harmonic solution?

For the case of light damping we have for the position $x(t)$:

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1 \cos \omega t + A_2 \sin \omega t),$$

Taking the derivative of the position we find for the velocity $v(t) = \dot{x}$:

$$v(t) = e^{-\frac{\gamma}{2}t}\left[(\omega A_2 - \frac{\gamma}{2}A_1)\cos\omega t - (A_2\frac{\gamma}{2} + \omega A_1)\sin\omega t\right]$$

Now we use the initial conditions and we find:

$$\begin{aligned} x(0) = x_0 \Longrightarrow 1 \cdot (A_1 \cdot 1 + A_2 \cdot 0) = x_0 \Longrightarrow \qquad & A_1 = x_0, \\ v(0) = v_0 \Longrightarrow 1 \cdot ((\omega A_2 - \frac{\gamma}{2}A_1) + 0) = v_0 \Longrightarrow \qquad & (\omega A_2 - \frac{\gamma}{2}A_1) = v_0, \end{aligned}$$

Solving for the above system in terms of A_1 and A_2 we obtain:

$$A_1 = x_0, \qquad A_2 = \frac{v_0 + x_0\gamma\omega/2}{\omega}$$

That results to the following exressions for $x(t)$ and $v(t)$:

$$\boxed{\begin{aligned} x(t) &= e^{-\frac{\gamma}{2}t}(x_0 \cos\omega t + \frac{v_0 + x_0\gamma\omega/2}{\omega}\sin\omega t), \\ v(t) &= e^{-\frac{\gamma}{2}t}(v_0 \cos\omega t - \frac{x_0\omega_0^2 + v_0\gamma\omega/2}{\omega}\sin\omega t), \end{aligned}}$$

where the relation $\omega^2 = \omega_0^2 - (\gamma/2)^2$ was used to introduce ω_0 into the expressions. Having the general solution for the light-damping case we can consider the special cases:

(a) If $x_0 = 0$ we have $A_1 = 0$ and $A_2 = v_0/\omega$. Then $x(t)$ and $v(t)$ are written as:

$$\begin{aligned} x(t) &= \frac{v_0}{\omega}e^{-\frac{\gamma}{2}t}\sin\omega t \\ v(t) &= e^{-\frac{\gamma}{2}t}\left(v_0 \cos\omega t - \frac{v_0\gamma}{2\omega}\sin\omega t\right) \end{aligned}$$

(b) if $v_0 = 0$ we have $A_1 = x_0$ and $A_2 = (\gamma/2\omega)x_0$. Then $x(t)$ and $v(t)$ are written as:

$$\begin{aligned} x(t) &= x_0 e^{-\frac{\gamma}{2}t}\left(\cos\omega t + \frac{\gamma}{2\omega}\sin\omega t\right) \\ v(t) &= -\frac{x_0\omega_0^2}{\omega}e^{-\frac{\gamma}{2}t}\sin\omega t \end{aligned}$$

(c) Setting $\gamma = 0$, we end up with $A_1 = x_0$ and $A_2 = v_0/\omega$ and $\omega = \omega_0$. The decaying factor $e^{-\gamma t/2} = 1$ is becoming unity and we obtain the oscillatory solutions for the SHO:

$$\begin{aligned} x(t) &= x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t, \\ v(t) &= v_0 \cos\omega_0 t - x_0\omega_0 \sin\omega_0 t, \end{aligned}$$

We left to the reader to confirm that these relations are indeed the solutions for the SHO when both initial conditions $x(0) = x_0$ and $v(0) = v_0$ are both different than zero.

Alternative forms for the light-damping DHO Show that the position $x(t)$ of a DHO can also be expressed as below:

$$x(t) = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega t - \phi), \tag{8.21}$$

where A_0 is the amplitude of the motion and ϕ is known as the initial phase. Find A_0 and ϕ in terms of the constants A_1, A_2 of Eq. (8.14).

Eq. (8.14) gives for $x(t)$:

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1 \cos \omega t + A_2 \sin \omega t)$$

If we set,

$$A_1 = A_0 \cos \phi, \qquad A_2 = A_0 \sin \phi \tag{8.22}$$

we may rewrite the equation for $x(t)$:

$$x(t) = e^{-\frac{\gamma}{2}t} A_0(\cos \phi \cos \omega t + \sin \phi \sin \omega t) = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega t - \phi)$$

A_0 and ϕ can be found from Eqns (8.22) by first adding A_1^2 and A_2^2:

$$A_1^2 + A_2^2 = A_0^2(\cos \phi + \sin^2 \phi) = A_0^2, \quad \Longrightarrow \boxed{A_0 = \sqrt{A_1^2 + A_2^2}}. \tag{8.23}$$

while by dividing A_2 and A_1 we obtain:

$$\boxed{\tan \phi = \frac{A_2}{A_1}} \tag{8.24}$$

Example 2. Damped mass-spring system For a underdamped mass-spring system with $m = 1$ Kgr and $k = 2$ N/m we assume that initially the mass is found at the equilibrium position ($x = 1$ m) and is released from rest. We assume that the friction of the surface (that the spring-mass system is located) is opposite to the direction of the velocity of the mass and proportional to it's velocity. We express this force as $F_T = -bv(t)$, with $b = 1$ kgr/sec.

(a) Determine the motion of the mass at any later time by considering the following form for the position $x(t)$:

$$x(t) = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi),$$

(b) What is the energy as a function of time $E = E(t)$?

(c) Plot the $x = x(t), v = v(t)$ and $E = E(t)$.

First we determine the eigenfrequency of the system ω_0 and the damping parameter γ:

$$\begin{aligned}\omega_0 &= \sqrt{\frac{k}{m}} = \sqrt{\frac{2\,N/m}{1\,kgr}} = \sqrt{2}\,\text{Hz} \\ \gamma &= \frac{b}{m} = \frac{1\,Kgr/sec}{1\,kgr} = 1\ \text{sec}^{-1} = 1\ \text{Hz}\end{aligned} \tag{8.25}$$

Since $\omega_0 > \gamma$ then we we have the case of *light damping* oscillators. We can use the following form for the position $x(t)$:

$$x(t) = A_0 e^{-\frac{\gamma}{2}t}\cos(\omega t + \phi), \tag{8.26}$$

The frequency of the oscillatory (but decaying) motion is equal to:

$$\omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2} = \sqrt{2 - 1/4} = \sqrt{7/4} = 1.3229\,\text{Hz}$$

At this stage it remains to calculate A_0 and the phase angle ϕ. To this end we are going to use the initial conditions. We always need to calculate the velocity of the mass, which is given as:

$$v(t) = \frac{dx}{dt} = \dot{x}(t) = -A_0 e^{-\frac{\gamma}{2}t}(\omega\sin(\omega t + \phi) + \frac{\gamma}{2}\cos(\omega t + \phi) \tag{8.27}$$

Applying the initial conditions to Eqns. (8.26) and (8.27) we have:

$$\begin{aligned}x(0) &= 1 \\ &\Longrightarrow A_0 e^{-\frac{\gamma}{2}\cdot 0}\cos(\omega\cdot 0 + \phi) = 1 \\ &\Longrightarrow A_0\cos\phi = 1 \Longrightarrow A_0 = \frac{1}{\cos\phi} \\ v(0) &= 0 \\ &\Longrightarrow -A_0 e^{-\frac{\gamma}{2}\cdot 0}\left[\omega\sin(\omega\cdot 0 + \phi) + \frac{\gamma}{2}\cos(\omega\cdot 0 + \phi)\right] = 0 \\ &\Longrightarrow \omega\sin\phi = -\frac{\gamma}{2}\cos\phi, \\ &\Longrightarrow \tan\phi = -\frac{\gamma}{2\omega}\end{aligned}$$

Finally we end up to the following two equation for the amplitude and the phase:

$$\begin{aligned}\tan\phi &= -\frac{\gamma}{2\omega} = -\frac{1}{2\cdot 1.3229} \\ &\Longrightarrow \phi = \tan^{-1}(-0.377957) = -20.696^o = -0.3612\ \text{rad} \\ A_0 &= \frac{1}{\cos\phi} \\ &\Longrightarrow A_0 = 1/\cos(-0.3612) = 1/0.9354 = 1.067\,\text{m}\end{aligned}$$

Therefore the position and the velocity of the mass are given as:

$$\begin{aligned}x(t) &= 1.067e^{-0.5t}\cos(1.3229t - 0.3612) \\ v(t) &= -1.067e^{-0.5t}(1.3229\sin(1.3229\,t - 0.3612) + 0.5\cos(1.3229\,t - 0.3612)) \\ E(t) &= \frac{1}{2}kx^2(t) + \frac{1}{2}mv^2(t) = x^2(t) + 0.5v^2(t)\end{aligned}$$

In Figure 1, the relevant plots are shown.

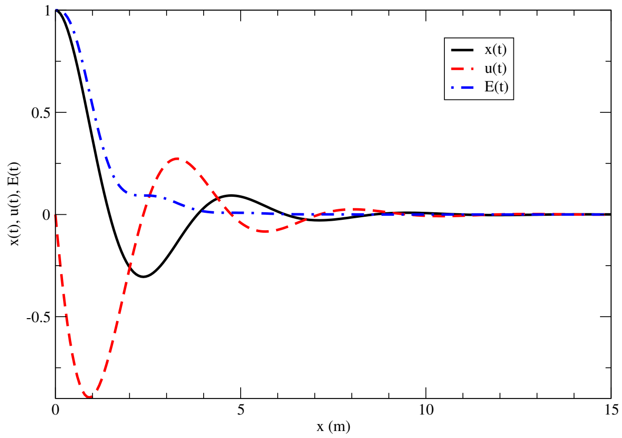


Fig 8.2. Plot of position, velocity and energy as a function of time for the spring mass-system of the application of the theory. Parameters of the system are $m = 1$ Kgr, $k = 2$ N/m, $b =$ 1 kgr/m $x(0) = 1$ m and $u(0) = 0$ m/s. The period of the oscillations are $T = 2\pi/\omega =$ 4.71 sec.

8.4 Appendix: Solving the damped harmonic equation

Assume[1] the DHO equation supplemented with the initial conditions at time $t_0 = 0$:

$$\boxed{\begin{array}{ll} \ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = 0, \qquad & \omega_0 = \sqrt{\dfrac{k}{m}}, \qquad \gamma = \dfrac{b}{m} \\ x(0) = x_0, \qquad v(0) = v_0. & \end{array}}$$

Eq. (8.13) represents a 2nd-order ordinary differential equation (ODE) with constant coefficients. Standard theory of ODEs proves that the general solution is written as a linear combination of two (independent) solutions as:

$$x(t) = A_1 x_1(t) + A_2 x_2(t) \tag{8.28}$$

with the constants A_1, A_2 determined through the values of the $x(t_0)$ and it's derivative $dx(t_0)/dt$ at a specified time t_0. Usually, and inn the present case we take them as the initial conditions of the position $x(0) = x_0$ and the velocity $v(0) = v_0$ at time $t_0 = 0$ (for simplicity).

A standard procedure to calculate $x_1(t), x_2(t)$ is to seek for solutions of the form

$$x_i(t) = A_i e^{\sigma_i t}.$$

Substituting the above expression into Eq (8.13) we end up to the following

$$(\sigma_i^2 + \gamma\sigma_i + \omega_0^2)A_i e^{\sigma_i t} = 0 \quad \Longrightarrow \quad \sigma_i^2 + \gamma\sigma_i + \omega_0^2 = 0,$$

[1] Complex arithmetics is used in this section.

Heavy damping (overdamped): $\gamma/2 < \omega_0$**:** Since neither A_i nor the exponent can be zero (for finite values of time) we end up to the second-order algebraic equations, which in general provides two solution for the possible σ_i:

$$\sigma_i^2 + \gamma\sigma_i + \omega_0^2 = 0 \quad \Longrightarrow \quad \sigma_{1,2} = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right)$$
$$= -\frac{\gamma}{2} \pm \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2}, \qquad \frac{\gamma}{2} > \omega_0.$$

The above values of $\sigma_{1,2}$ are acceptable only in the case of $\gamma/2 > \omega_0$. In this case, we arrive to two solutions which are independent each other and the general solution Eq.(8.28) can now be re-written as,

$$\boxed{x(t) = e^{-\frac{\gamma}{2}t}(A_1 e^{at} + A_2 e^{-at}), \qquad a = \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2}, \qquad \frac{\gamma}{2} > \omega_0.} \quad (8.29)$$

representing the *overdamped solution*.

Light damping (underdamped): $\gamma/2 < \omega_0$**:** In that case we work as follows: we start from the roots of the algebraic equation for the σ's, and rewrite them as:

$$\sigma_{1,2} = -\frac{\gamma}{2} \pm \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2} = -\frac{\gamma}{2} \pm \sqrt{-[\omega_0^2 - (\frac{\gamma}{2})^2]}$$
$$= -\frac{\gamma}{2} \pm \sqrt{\imath^2[\omega_0^2 - (\frac{\gamma}{2})^2]} = -\frac{\gamma}{2} \pm \imath\sqrt{\omega_0^2 - (\frac{\gamma}{2})^2},$$

where going from the first line to the second we used the well known property of the imaginary unit $\imath^2 = -1$ and from the second to the third line we used $\sqrt{\imath^2} = \imath$. In this case we obtain the general solution of Eq. (8.28) as a linear combination of two complex exponentials $(e^{\pm\imath\omega t})$

$$x(t) = e^{-\frac{\gamma}{2}t}(A_1' e^{\imath\omega t} + A_2' e^{-\imath\omega t}), \qquad \omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}, \qquad \frac{\gamma}{2} < \omega_0. \quad (8.30)$$

We can reexpress the above solution in terms of the real function $\sin\omega t$ and $\cos\omega t$ to obtain:

$$\boxed{x(t) = e^{-\frac{\gamma}{2}t}(A_1 \cos\omega t + A_2 \sin\omega t), \qquad \omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}, \qquad \frac{\gamma}{2} < \omega_0.} \quad (8.31)$$

To the above end one needs to utilize the other very well known property of a complex exponential $e^{\imath x} = \cos x + \imath \sin x$. This last step is left as an exercise to the reader.

Critical damping (underdamped): $\gamma/2 = \omega_0$: In this case we obtain only one root for the σ, namely $\sigma = -\gamma/2$ and thus with this method we can obtain only one from the required two (independent) solutions:

$$x(t) = A_1 e^{-\frac{\gamma t}{2}} + x_2(t)$$

The second solution $x_2(t)$ can be found using a different method (is beyond the scope of the present lecture). For completeness reason it is found that $x_2(t) = A_2 t e^{-\frac{\gamma t}{2}}$ an results to the following general solution in the case of critical damping:

$$\boxed{x(t) = e^{-\frac{\gamma}{2}t}(A_1 + A_2 t), \qquad \omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}, \qquad \frac{\gamma}{2} = \omega_0. \qquad (8.32)}$$

8.5 Questions

1. **Friction force, $F_T = -kv$.** How the results for the velocity (8.11) and position (8.12) whould change if $v(0) = v_0 \neq 0$?

2. **Friction force, $F_T = -kv$.** How the results for the velocity (8.11) and position (8.12) whould change if $F_T = -kv^2$?

   (hint: You may also assume $v_0 \neq 0$ and at the end set $v_0 = 0$).

3. **Projectile motion and fluid drag** When a solid object moves through a fluid (i.e. liquid or gas), it experiences a *drag force*. Consider a projectile moving in a viscus liquid under a uniform gravitational field. At $t = 0$, the projectile is thrown with velocity $\mathbf{v}$. Model the drag force as $F(v) = -kv$. Determine its subsequent motion.

4. **Alternative forms for DHO I** Starting from the standard solution for the position $x(t) = A_1 \cos \omega t + A_2 \sin \omega t$ show that it can be rewritten as:

   $$x(t) = A_0 e^{-\gamma t/2} \cos(\omega t + \phi)$$

   Give A_0 and ϕ in terms of A_1 and A_2. (hint: express A_1 and A_2 in a terms of A_0 ϕ in a similar way as shown in the application section (where a similar problem is solved).

5. **Power for a mass-spring DHO.**

   Starting from the expression $E = mv^2/2 + kx^2/2$ for the damped mass-spring system (damping parameter γ) show that the energy decreases as:

   $$\frac{d}{dt}E(t) = -\gamma m v^2. \tag{8.33}$$

   The fact that the above expression can only be negative allows to conclude that for a DHO there is net energy loss from the DHO system to its environment. In the mass-spring system the energy is transformed to heat due to the friction (contact) forces.

   (Hint: Take the first derivative in time of the mechanical energy given above, $E(t)$, and then replace the derived term that includes $-kx$ using the DHO equation ($m\ddot{v} = -kx(t) - bv(t)$).)

6. **Tunning fork**. A tunning fork vibrates with frequency $f = 440$ Hz. It is measured that the generated sound decreases by a factor 5 within 4 seconds. Provided that the intensity of the sound is proportional to the energy of the tunning fork find (a) it's eigenfrequency f_0 and (b) it's quality factor $Q \equiv \omega_0/\gamma$.

   (hint: Treat the tunning fork as a light-damping DHO system where $\omega_0 << \gamma/2$ use the corresponding approximate expression for it's energy, [Eq. (8.19)] as a function of time).

7. **Spring-mass system 1** For the spring-mass system of application 1, determine the motion and derive the relevant plots for $x(t)$ and $v(t)$ when the damping constant $b = 4$ Kgr/sec.

8. **Spring-mass system 2** For the spring-mass system of application 1, determine the motion and derive the relevant plots for $x(t)$ and $v(t)$ when the damping constant $b = 2\sqrt{2}$ Kgr/sec.

9. **Spring-mass system 3** An object of mass (0.5 Kg) attached to a masless spring (k= 200 N/m) with a damping parameter $b = 10$ at initial time $t_0 = 0$ is found at its equilibrium position with velocity $v(0) = 2$ m/sec.

   (a) Making use of the initial conditions calculate the functional dependance of the position of the mass on time.

   (b) Find the maximum and minimum speed of the mass.

   (c) What is the maximum acceleration achieved by the mass?

Chapter 9

Lagrangian Mechanics

In this section it is introduced an alternative description of the dynamics of mechanical systems, due to a number of people, including the names of Leibnitz, Bernoulli, Euler, Lagrange etc.. to mention a few. This method it is has been proven practically considerably more useful than Newton's method (especially for more complicated mechanical systems). Importantly, the conceptual difference are directly extensible to other domain of physics and its viewpoint is not restricted strictly to mechanical systems. A very brief exposition, without digging the deeper principles is presented below.

9.1 Recap of the Newton's approach

According to Newton's method the spatial motion of particle is governed by (2nd law):

$$F(x) = m\frac{d^2x}{dt^2} \tag{9.1}$$

Its **position** (here denoted x) depends on time, thus giving rise to its associated **velocity**, $v(t)$ and acceleration $a(t)$:

$$v = \dot{x} = \frac{dq}{dt}, \qquad a = \ddot{x} = \frac{d^2x}{dt^2}$$

In the Newton's 2nd-law, (9.1), above $m > 0$ is the **mass** of the particle, and F is the so-called **force**. Newton claimed that the particle's motion satisfies (9.1) which, mathematically, is a 2nd-order differential equation with a unique solution given some $x(t_0)$ and $v(t_0)$, provided the force takes finite values. So, given the mass of the particle the task is to find all the forces acting on the particle and apply the law to find its motion; Compactly:

$$F(x) = m\frac{d^2x}{dt^2} \quad \longrightarrow \quad x = x(t) \tag{9.2}$$

At this point, no further consideration it would be needed if the Newton's law wouldn't lead to the acknowledgment that there are few quantities that remain constants in the entire 'life' of particle; These are customarily called energy, momentum and (in particular cases) angular momentum, etc... The existence of such constants allows for alternative treatments of the particle's dynamics and more importantly lead to more fundamental physical concepts than the force concept.

So, as has discussed at the early chapters the quantity:

$$T(t) = \frac{1}{2} m\, v^2(t), \qquad \text{Kinetic energy} \tag{9.3}$$

has some interesting properties. For example, it changes in time as:

$$\dot{T} = \frac{d}{dt} T(t) = mv(t)a(t) \quad \longrightarrow \quad dT(t) = F(x(t))v(t)\, dt,$$

where $dT(t) = T(t+dt) - T(t)$. So, $T(t)$ goes up when you push an object in the direction of its velocity, and goes down when you push it in the opposite direction. In the former case some **work** is generated from the force while in the latter case some **work** is wasted. By adding all the elementary changes in the kinetic energy between t_1 and t_2 we see, [54]

$$\begin{aligned} T(t_2) - T(t_1) &= \int_{t_1}^{t_2} dt\, F(t)\, v(t) \\ &= \int_{x_1}^{x_2} F(x) dx \end{aligned}$$

So, the change of kinetic energy is equal to the **work** done by the force, that is, the integral of F along the curve $x(t)\colon t_0 < t < t_1$. From the latter expression of the above equation we can see that we can relate the change of the kinetic energy between two times (t_0, t_1) with the *positions* taken by the particle in between, *with no reference to the time elapsed*. In the special case where [55] ,

$$F(x)dx = -dV(x) \quad \longrightarrow \quad F(x) = -\frac{dV}{dx},$$

we end up to the following equation,

$$\begin{aligned} \Delta T_{21} = \mathbf{T(t_2)} - \mathbf{T(t_1)} &= \int_{t_1}^{t_2} dt\, F(t)\, v(t) \\ &= \int_{x_1}^{x_2} F(x) dx = \mathbf{V(x_1)} - \mathbf{V(x_2)} = -\Delta V_{21}, \end{aligned}$$

which in turn entails to, [56]

$$\Delta T_{21} = -\Delta V_{21} \quad \longrightarrow \quad T(t_1) + V(x_1) = T(t_2) + V(x_2)$$

From the above definition we see that $V(x)$ is unique up to an additive constant; This means that makes no difference whether we use $V(x)$ or $V(x) + c$ to work out the particle's mechanical properties. Given the above development we conclude that there is a quantity which is *constant of motion* of the particle (named as **total energy**):

$$\boxed{E = T(v(t)) + V(x(t)) = T(v) + V(x) = const. \tag{9.4}}$$

where $V(t) = V(x(t))$ is called the **potential energy** of the particle [57] We also see that *conservative* forces cause the particle to move in such a way that changes in the

[54] Note that $F(t) = F(x(t))$

[55] A force with this property is called **conservative**.

[56] In 3 dimensions, *Stokes'* theorem which relates line integrals to surface integrals of the curl implies that the change in kinetic energy $T(t_2) - T(t_1)$ is independent of the curve going from $\mathbf{r}(t_1) = a$ to $\mathbf{r}(t_2) = b$ iff

$$\nabla \times F(\mathbf{r}) = 0 \Leftrightarrow \mathbf{F} = -\nabla V(\mathbf{r})$$

In fact the above condition hold for multidimensional systems as well due to a generalized Stokes' theorem.

[57] Walking backwards note that $F = ma = m\dot{v}$ implies

$$\frac{d}{dt}\left[T(t) + V(x(t))\right]$$
$$= mv\dot{v} + \frac{dV(x)}{dx}\frac{dx}{dt}$$

potential energy depends only on the end positions and *not on the particular path followed*.

These invariance properties of the particle's total, kinetic and potential energy are originating from a more general physical principle which governs nature and have led to an alternative consideration of the physical events having with central concept the particle's potential energy as it moves in the space, rather than the force exerted in the particle. The subject of the following section is exactly this alternative method first pushed forward by Langrange and Euler.

9.2 Lagrangian method

Before we start, it is convenient to represent the space variables not by the convential x variable but by q as a reminder that the dependence variable can be of any nature (one-dimensional space, angle, radial distance, r, cylindrical distance, or some combination of them). In this case we define the *generalized coordinate* q and its corresponding *generalized velocity*, *generalized momentum* by, [58]:

$$q, \qquad \dot{q} = \frac{dq}{dt}. \tag{9.5}$$

Having defined the proper (generalized) coordinates to describe the particle's motion a quantity called **Langrangian** is defined by,

$$\mathcal{L}(q,\dot{q},t) = T(\dot{q}) - V(q) = \frac{1}{2}m\dot{q}^2 - V(q), \quad \textbf{Lagrangian} \tag{9.6}$$

called the **Lagrangian**. Then within the Langrance method the equation(s) of motion (EOM) for the particle are derived by,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}, \qquad \textbf{Euler–Lagrange equations} \tag{9.7}$$

Before to proceed to specific examples demonstrating this method let's see whether the E-L equations are consistent with the Newton's 2nd law.

First note that,

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial}{\partial q}\left(T(\dot{q}) - V(q)\right) = 0 - \frac{\partial V}{\partial q} = F(q).$$

The latter component is called the *generalized force* 'acting' on the particle [59]. Accordingly,

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left(T(\dot{q}) - V(q)\right) = m\dot{q} - 0 = p$$

is equal to mass times its (generalized) velocity, also known as its **generalized momentum**. With these definitions the Euler–Lagrange equations say simply that,

$$\frac{dp}{dt} = F.$$

The time derivative of momentum is force, which is the Newton's 2nd law!

[58] Again, for example:

$$x \to q, \qquad \dot{x} \to \dot{q}$$
$$\theta \to q \qquad \dot{\theta} \to \dot{q}$$

[59] It coincides with the usual force only when $q = x$. When, for example, $q = \theta$ then the generalized force corresponds to the torque.

Lagrange	Newton	Generalized
$\dfrac{\partial L}{\partial \dot{q}}$	$m\dot{q} = mv = p$	Momentum
$\dfrac{\partial L}{\partial q}$	$-\dfrac{\partial V}{\partial q} = F$	Force

Table 9.1. *A comparison of the central quantities in the Euler-Langrange and Newton's theories for a mechanical system*

Energy conservation It is instructive to take the Lagranian's derivative. Then it can be proven that,

$$\frac{d}{dt}\mathcal{L} = \frac{d}{dt}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \qquad \rightarrow \qquad \frac{d}{dt}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}\right) = 0 \tag{9.8}$$

This says that the quantity defined by,

$$E = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \tag{9.9}$$

is a constant of motion. In the case where,

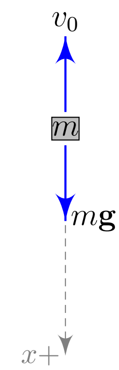
$$\mathcal{L} = T(\dot{q}) - V(q) = \frac{1}{2}m\dot{q}^2 - V(q) \qquad \rightarrow \qquad \dot{q}\frac{\partial \mathcal{L}}{\partial q} = \dot{q}m\dot{q} = m\dot{q}^2 = 2T.$$

So eventually we have,

$$E = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} = 2T - (T - V) = T(\dot{q}) + V(q) = \text{const!}$$

The latter relation is nothing else than the expression for the conserved quantity, known as energy, of a mechanical system according to Lagrangian method. This coincides with the one derived using the Newtonian method.

9.3 Examples



Example 1. Free-fall body

Our purpose here to derive the EOMs using the E-L equations. We need to construct the Langrangian of the system, which requires an expression for its kinetic and potential energy.

1. First a suitable coordinate system to describe the motion needs to be set. For this, a vertical 1-dimensional axis (x) is defined with its positive axis pointing downwards.

2. In this case, the particle's position is $x(t)$, its velocity is $v(t) = \dot{x}$. Then its kinetic energy is $T = mv^2/2$. The potential energy is, when its position is $x(t)$ above Earth's ground,

$$V(x) = V(0) - \int_0^x dx(mgx) = -mgx, \qquad V(0) = 0.$$

3. The particle's Langrangian is,

$$\mathcal{L}(x, \dot{x}, t) = T(\dot{x}) - V(x) = \frac{1}{2}m\dot{x}^2 + mgx$$

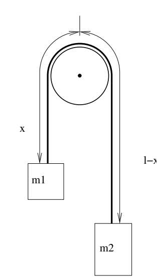
4. We need the partial derivatives of $\mathcal{L}$. For, $q = x$ we have,

$$\frac{\partial}{\partial \dot{x}}\mathcal{L} = m\dot{x} + 0 = m\dot{x}, \qquad \frac{\partial}{\partial x}\mathcal{L} = 0 + mg = mg$$

5. Then applying the E-L equations we find:

$$\frac{d}{dt}(m\dot{x}) = mg \qquad \rightarrow \qquad \ddot{x} = g$$

Example 2. The Atwood Machine



Consider a frictionless cylindrical pulley (of radius R) with two masses, m_1 and m_2, hanging with a string of length ℓ from it:

The task her is to quantify the motion of the masses. To this end, we assume a coordinate system placed horizontally at the center of the pulley, with its $y-$axis pointing downwards. Since the pulley is a cylinder of radius R we may represent the masses position by:

$$m_1 : \qquad \mathbf{r}_1(t) = (x_1, y_1, z_1) = (-R, y_1(t), 0)$$

and

$$m_2 : \qquad \mathbf{r}_2(t) = (x_2, y_2, z_1) = (-R, y_2(t), 0)$$

The two masses are moving under the constant gravity field of acceleration, $\mathbf{g} = g\hat{y}$, so the kinetic energy is

$$T(\dot{y}_1, \dot{y}_2) = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2,$$

whereas the potential energy is given by,

$$V(y_1, y_2) = -m_1 g y_1 - m_2 g y_2$$

In addition, note that the following relation between x_1 and x_2 holds:

$$y_1(t) + y_2(t) = \ell. \tag{9.10}$$

At this point we have two different ways to proceed. Either, we express everything in terms of one of the coordinates (say $y_1(t)$), determine the EOMs and solve for $y_1(t)$ and then use the latter equation to find $y_2(t)$, or we continue like the two coordinates are independent each other and have a two-coordinates problem to find the EOMs and solve for $y_1(t), y_2(t)$.

The former approach is certainly simpler as it is 1-dimensional problem, while the second one is slightly more complicated (but not that much because the constraint $y_1(t) + y_2(t) = \ell$ can always be employed at any stage of the solution.

Set: $y_2(t) = \ell - y_1(t)$

1. *Coordinates* All the involved quantities below will be written in terms of the mass '1' position and speed $y_1(t), \dot{y}_1(t)$:

2. *Lagrangian:*

   Noting that,

   $$\dot{y}_2 = 0 - \dot{y}_1 \quad \rightarrow \quad \dot{y}_2^2 = \dot{y}_1^2$$

   The kinetic and potential energy for the system become:

   $$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_1^2\frac{1}{2}(m_1 + m_2)\dot{y}_1^2$$
   $$V = -m_1 g y_1 - m_2 g(\ell - y_1) = -(m_1 - m_2)g y_1 - m_2 g\ell$$

   Then the Langrangian is given by:

   $$\mathcal{L}(y_1, \dot{y}_1) = \frac{1}{2}M\dot{y}_1^2 + Mgy_1 + m_2 g\ell \tag{9.11}$$

3. E-L equations

   To employ the E-L equations we calculate the partial derivatives of $\mathcal{L}$:

   $$\frac{\partial}{\partial \dot{y}_1}\mathcal{L} = (m_1 + m_2)\dot{y}_1, \qquad \frac{\partial}{\partial y_1}\mathcal{L} = (m_1 - m_2)g$$

   Then the E-L equations give:

   $$\frac{d}{dt}((m_1 + m_2)\dot{y}_1) = (m_1 - m_2)g \quad \rightarrow \quad \ddot{y}_1(t) = \frac{m_1 - m_2}{m_1 + m_2}g$$

   So this describes a falling object in a downwards gravitational acceleration $a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$

   Finally by using $x_1(t) + x_2(l) = \ell$ we end up to have determined the acceleration for both the masses [60] :

[60] Note the following special cases:

$$m_1 = m_2 \quad \rightarrow \quad \ddot{y}_1 = 0$$

and

$$\ddot{y}_1(t) = \frac{m_1 - m_2}{m_1 + m_2} g, \qquad \ddot{y}_1(t) = \frac{m_2 - m_1}{m_1 + m_2} g. \tag{9.12}$$

It is straightforward to integrate the expression for $\ddot{y}_1(t)$ twice (with the proper initial conditions) to obtain the complete solution to the motion $y_1(t)$ and $\dot{y}_2(t)$ [61].

[61] You can do it yourself for example when,

$$y_1(0) = 0, \qquad \dot{y}_1(0) = 0$$

Example 3. Simple horizontal mass-spring system 1-D

Assume an horizontal mass-spring system, with no-friction, characterized by the mass m and the spring's constant, k. We want to derive the EOM for this system. We expect this to be that of the harmonic oscillator.

We need to construct the Langrangian of the system, which requires an expression for its kinetic and potential energy.

1. First a suitable coordinate system to describe the motion needs to be set. For this, an horizontal 1-dimensional axis (x) with its origin placed at the position where the spring has its natural length.

2. As usual, the particle's position is $x(t)$, its velocity is $v(t) = \dot{x}$. Then its kinetic energy is $T = mv^2/2$. The potential energy is, when its position is $x(t)$ above Earth's ground,

$$V(x) = \frac{1}{2} k x^2$$

3. The particle's Langrangian is,

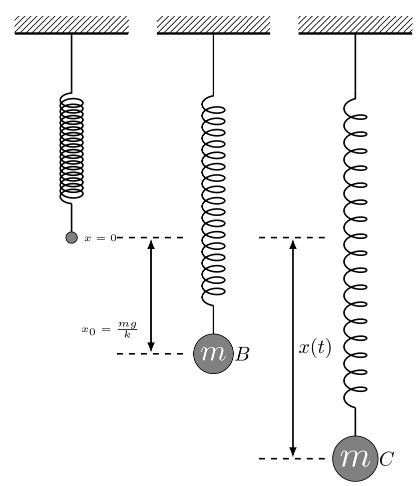
$$\mathcal{L}(x, \dot{x}, t) = T(\dot{x}) - V(x) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m x^2$$

4. We need the partial derivatives of $\mathcal{L}$. For, $q = x$ we have,

$$\frac{\partial}{\partial \dot{x}} \mathcal{L} = m\dot{x} + 0 = m\dot{x}, \qquad \frac{\partial}{\partial x} \mathcal{L} = 0 - kx = -kx$$

5. Then applying the E-L equations we find:

$$\frac{d}{dt}(m\dot{x}) = -kx \qquad \rightarrow \qquad \ddot{x} + \frac{k}{m} x = 0.$$



Example 4. Simple vertical mass-spring system 1-D

Assume a vertical mass-spring system, in the Earth's gravity field, characterized by the mass m, the spring's constant, k and the acceleration g. At time $t = 0$ the mass is at rest and is kicked-off downwards. We want to find the EOM.

The new element in this problem, in reference to the corresponding horizontal mass-spring problem is that the mass posses additional potential energy due to the Earth's gravitational field.

1. First a suitable coordinate system to describe the motion needs to be set. For this, a vertical 1-dimensional axis (x) is defined with its positive axis pointing downwards.

2. As usual, the particle's position is $x(t)$, its velocity is $v(t) = \dot{x}$. Then its kinetic energy is $T = mv^2/2$.

   It is convenient to set a common reference point for the spring's and Earth's potential energy. We'll set it to be the point where the spring has its natural length.

   Then the potential energy is, when its position is $x(t)$ away from the origin we have [62] ,
$$V(x) = \frac{1}{2}kx^2 - mgx$$

3. The particle's Langrangian is,
$$\mathcal{L}(x, \dot{x}, t) = T(\dot{x}) - V(x) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}mx^2 + mgx$$

4. We need the partial derivatives of $\mathcal{L}$. For, $q = x$ we have,
$$\frac{\partial}{\partial \dot{x}}\mathcal{L} = m\dot{x} + 0 = m\dot{x}, \qquad \frac{\partial}{\partial x}\mathcal{L} = 0 - kx + mg = -kx + mg$$

5. Then applying the E-L equations we find:
$$\frac{d}{dt}(m\dot{x}) = -kx + mg \qquad \rightarrow \qquad m\ddot{x} + kx = mg.$$

The solution is periodic and is easily found to be (it can be even guessed)
$$x(t) = \frac{mg}{k} + A\cos\omega_0 t + B\sin\omega_0 t, \qquad \omega_0 = \sqrt{\frac{k}{m}}. \tag{9.13}$$

The constants, A and B may be found from the initial conditions. Since the mass is at rest initially, we then have,
$$\ddot{x}(0) = 0 \qquad \rightarrow \qquad 0 + kx(0) = mg \qquad \rightarrow \qquad x(0) = \frac{mg}{k}.$$

Substituting this value to (9.13) we obtain,
$$\frac{mg}{k} = \frac{mg}{k} + A \qquad \rightarrow \qquad A = 0.$$

The B constant is calculated by considering the initial velocity $v(0) = v_0$. From the general solution (9.13) we have,
$$v(t) = B\omega_0 \cos\omega_0 t.$$

Therefore we havem
$$v_0 = B\omega \qquad \rightarrow \qquad B = \frac{v_0}{\omega}.$$

Summarizing the solution for the given initial conditions is found to be

[62] Note that,
$$F(x) = -\frac{dV}{dx} = -kx + mg$$

$$x(t) = \frac{mg}{k} + \frac{v_0}{\omega_0}\sin\omega_0 t$$
$$v(t) = v_0\cos\omega_0 t$$

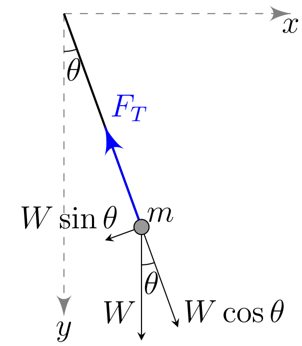


Fig 9.1: Idealized simple pendulum sketch. The point mass moves under the weight force, $W = mg$. At the bottom figure the coordinates axes are shown explicitly

Example 5. Simple pendulum The task of this example is to derive the pendulum equation using the Euler-Lagrangian's methodology. Again, the most important part of the solution consists of constructing a suitable Lagrangian for the system. As usual, we choose first (a) a convenient coordinate system (b) we write down the kinetic and potential energy in terms of these coordinates and finally (c) we take the E-L equations in order to find the EOMs:

1. *Coordinates*. Looks like the polar coordinate system is the proper one (the motion takes place in a plane normal to the ground). Then we choose the plane such that the $z-$axis is normal to the pendulum's plane. In this case in the below we may ignore the $z-$axis since $z = 0$. Therefore we have,
$$x = L\sin\theta, \qquad y = L\cos\theta.$$
An important simplification it has already taken place here since it is assumed that the distance of the mass is equal to $r = L$ (or equivalently $\dot{r} = 0$). This would mean,
$$\dot{x} = L\dot{\theta}\cos\theta, \qquad \dot{y} = -L\dot{\theta}\sin\theta$$

2. *Kinetic and potential energy*,
$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mL^2\dot{\theta}^2$$
For the potential energy we have,
$$V(y) = mgy = mgL(1 - \cos\theta),$$
where the zero-potential energy level is at the mass's lowest point ($\theta = 0$).

3. *Lagrangian* The system's Lagrangian now is given by,
$$\mathcal{L}(\theta, \dot{\theta}, t) = T - V = \frac{1}{2}mL^2\dot{\theta}^2 - mgL(1 - \cos\theta)$$

4. Partial derivatives and E-L equations:
$$\frac{\partial}{\partial\dot{\theta}}\mathcal{L} = mL^2\dot{\theta} - 0 = mL^2\dot{\theta} \qquad \frac{\partial}{\partial\theta}\mathcal{L} = 0 - mgL(0 + \dot{\theta}\sin\theta) = -mgL\dot{\theta}\sin\theta$$
By applying the E-L equations we find:
$$\frac{d}{dt}(mL^2\dot{\theta}) = -mgL\dot{\theta}\sin\theta \qquad \rightarrow \qquad \boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = 0, \qquad \text{pendulum equation!}}$$
The last equation is the same as the one we found using the Newtonian method ((4.13)). From this point on the effort is on the solution of this differential equation. Everything that was discussed at the relevant chapter apply fully (e.g. the small-angle approximation).

9.4 Questions

Question 1. Time derivative of Lagrangian

Assume a Lagrangian which is not dependend explicitly in time, $\mathcal{L} = \mathcal{L}(q, \dot{q})$. By assuming the total differential of $\mathcal{L}$ and the E-L equations show that:

$$\frac{d}{dt}\mathcal{L} = \frac{d}{dt}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

Question 2. Bead on a swinging rod (pendulum)

Take the case of the simple pendulum but allow the mass to slide freely on the rod. That would mean that its distance from the rotating center is not constant of motion (So the constraint $r = L$ is no longer true).

(a) Show that an expression for its Lagrangian in a polar coordinate system can be,

$$\mathcal{L}(\theta, \dot{\theta}, t) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mg(L - r\cos\theta)$$

(b) Provide the E-L equations for this system.

(c) Now apply the case where the mass is actually fixed at the edge of the rod and show that the EOM lead to the *pendulum equation*.

Question 3. Projectile motion

Using the Lagrangian approach derive the EOMs for the problem of projectile motion, where a mass is moving under the Earth's gravitational field (See figure (6.1)).

Question 4. Equation of motion for central fields

Assume an object of mass m having a potential energy which is spherically symmetric in polar coordinate system, namely $V = V(r)$.

(a) Applythe Lagrangian method in a polar coordinate system (ignore the $z-$coordinate) and derive the EOMs for the r, θ position variables and show that

$$m\ddot{r} = -\frac{dV}{dr} + \frac{L^2}{mr^3}, \qquad mr^2\dot{\theta} = \text{constant},$$

(b) Starting from the definition of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ show that the constant in the latter equation is the magnitude of $\mathbf{L}$.

Question 5. Isotropic harmonic oscillator

Assume a point-like object of mass m attached to a spring constant k, free to move on a plane (2-dimensional). Assuming a coordinate system where the $x - y$ plane resides on the plane and ignoring the $z-$axis the system is isotropical on the plane in the sense that its potential energy is given by

$$V(r) = \frac{1}{2}k(x^2 + y^2)) = \frac{1}{2}kr^2.$$

where $r^2 = x^2 + y^2$.

(a) Using the Lagrangian method provide the EOM for the mass. You may use either the polar or the Cartesian CS to solve the problem.

(b) Calculate the force exerted on the mass from the spring. Apply the Newton's 2nd law and show that you arrive at the same EOM coming from the Lagrangian method

Question 6. Harmonic Oscillator energy- mass-sping

Use the definition (9.9) to give the energy of an harmonic oscillator mass-sping (m, k) system.

Question 7. Harmonic Oscillator energy- Pendulum

Use the definition (9.9) to give the energy of an harmonic oscillator pendulum (m, L) system in the small-angle approximation.