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Classical Electrodynamics

2020-21 class

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Classical Electrodynamics, 2020-21 class

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To Coulomb-Faraday-Maxwell-Hertz

– LAAN

These notes are not yet fully endorsed by myself and should be used with some caution. Note that lots of typos should be present. Also, please, do not hesitate to communicate such suspected typos or possible inaccuracies (Nov 2019)

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1 Maxwell equations in vacuum

1.1 Differential microscopic form of MEs

The electric and magnetic fields in vacuum satisfy the so-called (microscopic) *Maxwell equations* (MEs). There are two different formulations of the MEs, namely the *differential* and the *integral* form.

Differential form of MEs The MEs in this formulation are expressed by providing the divergence and the rotation of the electric, $\mathbf{E}(\mathbf{r}, t)$, and the magnetic, $\mathbf{B}(\mathbf{r}, t)$ fields in terms of the electric, ε_0, and magnetic, μ_0, constants of the vacuum the presence of charge and current density, $\rho(\mathbf{r}, t)$, $\mathbf{j}(\mathbf{r}, t)$ as well as their partial derivatives $\partial_t \mathbf{E}(\mathbf{r}, t)$ and $\partial_t \mathbf{B}(\mathbf{r}, t)$:

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\varepsilon_0} \qquad \textbf{(ME1)} \qquad (1.1)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \qquad \textbf{(ME2)} \qquad (1.2)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \qquad \textbf{(ME3)} \qquad (1.3)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{j}(\mathbf{r}) + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r}, t), \qquad \textbf{(ME4)} \qquad (1.4)$$

All the involved physical quantities ($\mathbf{E}, \mathbf{B}, \rho, \mathbf{j}$) generally are considered time-dependent. From the above MEs it is immediately concluded that EM fields are generated due to the existense of:

(i) *Matter:* charged matter [charge density, $\rho(\mathbf{r}, t)$, in Eq. (ME1)] and/or moving charges [electric current density, $\mathbf{j}(\mathbf{r}, t)$, in ME4)]
(ii) *Time-dependent fields:* $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ in Eqs (ME3) and (ME4).

Integral form of MEs And alternative form of the MEs is their expression in terms of their electric and magnetic fluxes $\Phi_E(t)$, $\Phi_B(t)$ and the presence of charge(s) $q(t)$ and current(s) $i(t)$. More specifically,

$$\oint_A \mathbf{da} \cdot \mathbf{E}(\mathbf{r}, t) = \frac{q(t)}{\varepsilon_0}, \qquad \text{Gauss law,} \quad \textbf{(ME1')} \qquad (1.5)$$

$$\oint_A \mathbf{da} \cdot \mathbf{B}(\mathbf{r}, t) = 0 \qquad \textbf{(ME2')} \qquad (1.6)$$

$$\oint_C \mathbf{dr} \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{d}{dt}\Phi_B(t), \qquad \text{Faraday's law,} \quad \textbf{(ME3')} \qquad (1.7)$$

$$\oint_C \mathbf{dr} \cdot \mathbf{B}(\mathbf{r}, t) = \mu_0 i + \mu_0 \varepsilon_0 \frac{d}{dt}\Phi_E(t), \qquad \text{Ampere's law} \quad \textbf{(ME4')} \qquad (1.8)$$

where A is defined as the surface that enclosed volume V and C defined to be the line curve that enclosed surface A. In the above integral formulation of the MEs the following physical quantities were defined:

$$q(t) \equiv \int_V d^3\mathbf{r}\rho(\mathbf{r},t), \qquad \text{total charge enclosed in volume V} \tag{1.9}$$

$$i(t) \equiv \int_A \mathbf{da}\cdot\mathbf{j}(\mathbf{r},t), \qquad \text{flux of electric current density flux} \tag{1.10}$$

$$\Phi_B(t) \equiv \int_A \mathbf{da}\cdot\mathbf{B}(\mathbf{r},t) \qquad \text{magnetic field flux} \tag{1.11}$$

$$\Phi_E(t) \equiv \int_A \mathbf{da}\cdot\mathbf{E}(\mathbf{r},t) \qquad \text{electric field flux} \tag{1.12}$$

• **Homework**. Utilizing Gauss and Stokes mathematical identities derive the integral MEs, [ME1′-4′] starting from the differential MEs [ME1-4].

Some points to emphasize here:

- The above MEs are expressed in the SI unit system where the independent base units are the unit mass (kg), time (s), length (m) and current (A). That means the charge unit is the Coulomb (Cb = A· s) and that the Coulomb force $F_e(r_{12})$ between two charges q_1 and q_2 at fixed distance r_{12}, is given by,

$$\mathbf{F}_e(r_{12}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{12}}\hat{r}_{12}, \qquad [\text{Nt}] \tag{1.13}$$

  while the magnitude of the magnetic force between two parallel wires in distance r_{12} carrying static currents i_1 and i_2 is given by,

$$F_b(r_{12}) = \frac{\mu_0}{4\pi}\frac{i_1i_2}{r_{12}} \qquad [\text{Nt}] \tag{1.14}$$

  In this system,

$$\mu_0 = 4\pi\times10^{-7}\,Nt/A^2, \qquad \varepsilon_0 = 8.854187817\times10^{12}\,\text{A}^2\,\text{s}^{\,2}\,/\text{Kg m}^3.$$

- μ_0, ε_0 are constants related with the magnetic and electric properties of the vacuumn, the value of which determines the light's speed *in vacuumn*:

$$c \equiv \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 299792458\,m/s \tag{1.15}$$

- The important remark here is that in the integral form of the MEs1′-2′ is valid for any shape of the surface A that encloses the charge q. Similarly, Eqs ME3′-4′ hold for any line C that surrounds the surface A that the currents i pass through.
- Note that when all the quantities are all time-dependent Eqs (ME1-4) are decoupled, since $\partial_t\mathbf{B}(\mathbf{r}) = 0$ and $\partial_t\mathbf{E}(\mathbf{r}) = 0$. In this case the properties of the electric and magnetic field can be studied separately, giving rise to *electrostatics* (ME1-2) and *magnetostatic* (ME3-4) EM theory, namely,

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})/\epsilon_0, \qquad \nabla\times\mathbf{E}(\mathbf{r}) = 0, \tag{1.16}$$

  and

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0, \qquad \nabla\times\mathbf{E}(\mathbf{r}) = \mu_0\mathbf{j}(\mathbf{r}). \tag{1.17}$$

All the above laws are expressed in vectorial form which makes them independent on the choice of the coordinate system. As such, any coordinate system can be applied for the solution of the MEs. Most often, space symmetry of the charge distributions and currents provide strong indications for the most convenient choice of the coordinate system. Needless to say, that, no matter the particular coordinate system chosen to work, the solutions of the ME1-4 equations should be unique. The only difference among the coordinate systems is the workload to obtain the final results.

1.2 Mechanical effect of EMs on charges

The mechanical effects of EM fields, $\mathbf{E}(\mathbf{r},t), \mathbf{B}(\mathbf{r},t)$ on a charge q', moving with velocity $\mathbf{v}'$ and distributed over a volume V, ($q' = \int \rho'(\mathbf{r},t)dV$, where ρ' is the charge density) are determined through the Lorentz-Coulomb force density, $\mathbf{f}(\mathbf{r},t)$:

$$\mathbf{f}(\mathbf{r},t) = \frac{d\mathbf{F}}{dV} = \rho'\mathbf{E}(\mathbf{r},t) + \mathbf{j}'(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t). \tag{1.18}$$

In the above, $\mathbf{j}'(\mathbf{r},t) = \rho'(\mathbf{r},t)\mathbf{v}$ is the charge's current density.

Integration over the the whole volume of the above force ($\mathbf{F}(t) = \int dV \mathbf{f}(\mathbf{r},t)$) density will give us the total force on the volume due to the presence of EM fields. This total force may be expressed as,

$$\mathbf{F}(t) = q'(t)[\mathbf{E}(t) + \mathbf{v}' \times \mathbf{B}(t)]. \tag{1.19}$$

Eq. (1.18) is the differential form of the Lorentz-Coulomb force Eq. (1.19).

Charge in an EM field In the special case of a point-like particle of mass m' and charge q' in the presence of an EM field, $\mathbf{E}, \mathbf{B}$ we can find its motion by use of the Newton's second law. Then the expression (1.18) is used for the force exerted on the charge:

$$m' \frac{d^2}{dt^2}\mathbf{r}(t) = q'\mathbf{E}(\mathbf{r},t) + q'\mathbf{v}' \times \mathbf{B}(\mathbf{r},t), \qquad \mathbf{r}(0) = \mathbf{r}_0, \quad \mathbf{v}(0) = \dot{\mathbf{r}}(0) = \mathbf{v}_0.$$

Note that the electric component of the force ($f_e = q'\mathbf{E}$) exceeds the magnetic component of the force ($f_b = q'\mathbf{v}' \times \mathbf{B}$) by a factor equal to c/v:

$$\frac{f_e}{f_b} = \frac{|\mathbf{f}_e|}{|\mathbf{f}_b|} = \frac{qE_0}{qvB_0} = \frac{q(cB_0)}{qvB_0} = \frac{c}{v} >> 1 \tag{1.20}$$

It is only at relativistic energies that the magnetic force, f_b, is comparable to the electric one, f_e. Therefore for a moving charge moving with a non-relativistic speed ($v << c$) we often ignore the magnetic components in analyzing it's motion.

1.3 Charge-density current continuity equation

Use of the MEs can lead to the *charge-conservation law* expressed as:

$$\boxed{\nabla \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial}{\partial t}\rho(\mathbf{r}, t) = 0 \Longleftrightarrow i(t) = -\frac{d}{dt}q(t)} \tag{1.21}$$

• **Homework**. Derive the above continuity laws for a distribution charge, $\rho(\mathbf{r}, t)$ and its current density, $\mathbf{j}(\mathbf{r}, t)$.

Hint: Take the partial derivative in time of the ME1 and then make use of ME2. To derive the equation for the current, $i(t)$, integrate over volume and make use of the Gauss identity for volume integrals.

1.4 Electromagnetic energy and Poynting vector

EM field energy, U_{EM}. Assume a region with present some charge distribution ρ, current density current, $\mathbf{j}$, and EM fields $\mathbf{E}, \mathbf{B}$. This system, being isolated, has its total energy constant. The energy of the system is the sum of its mechanical energy (energy of the matter) and its EM field energy. Generally, the EM fields exerts a force $\mathbf{f}$ on this distribution (Lorentz force). Thus the charge distribution is generally accelerated (changes its kinetic energy) and as a result we have the production of *mechanical* power:

$$P_M = \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t). \tag{1.22}$$

This mechanical power is offered by the EM field and as such there should a relation which balances the total energy of the charge-distribution and the EM field energy. In other words since the total energy should be constant then any change of the mechanical energy of the charge/density distribution should be balanced by equal and opposite change of an EM energy, $U_M = U_{EM}$. However any expression for the EM field energy,U_{EM}, is meaningfull only when it is expressed only on purely EM field quantities. To this end an elimination of the current density, j from Eq (1.22) should be performed. For this we make use of ME4 and after some manipulations involving standard vector field mathematical identities we arrive to the following relation:

$$\boxed{\nabla \cdot \mathbf{S} + \frac{\partial}{\partial t} U_{EM} = -\mathbf{j} \cdot \mathbf{E}} \tag{1.23}$$

where the EM field density energy was defined to be:

$$\boxed{U_{EM}(\mathbf{r}, t) = \frac{\varepsilon_0}{2} \mathbf{E}^2(\mathbf{r}, t) + \frac{1}{2\mu_0} \mathbf{B}^2(\mathbf{r}, t).} \tag{1.24}$$

The physical meaning of the above quantity is that it represents the amount of the EM energy contained in an elementary volume dV at position $\mathbf{r}$ at at time t. The above energy-balance relation represents the *energy conservation theorem* for EM fields in matter.

• **Homework**. Derive the above differential energy conservation law for a distribution charge, $\rho(\mathbf{r}, t)$ and its current density, $\mathbf{j}(\mathbf{r}, t)$.

Hint: Eliminate $\mathbf{j}$ by use of ME4. Then proceed by utilizing properly the remaining ME (ME1-3) as well as standard identities for cross products of vector fields.

Poynting vector, S. Moreover we may obtain the integral expression for the energy conservation theorem, by volume integration of Eq. (1.23), to obtain:

$$\frac{d}{dt}(W_M + W_{EM}) = -\oint_A \mathbf{S} \cdot \mathbf{da}, \tag{1.25}$$

where W_M is the mechanical work of the charge/current distributions contained in a volume V (due to forces exerted by the EM field). W_{EM} is the total EM energy contained in the *same* volume V. These are defined as:

$$W_M(t) = \int_V dV \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t), \qquad W_{EM}(t) = \int_V dV U_{EM}(\mathbf{r}, t) \tag{1.26}$$

What this integral energy balance relation tell us is that the rate of change of mechanical energy (Mechanical power) of the matter included in the volume equals with the (a) the EM-field energy change rate (EM power) plus (b) the flux of a vector field, **S** through a surface, A that encloses the volume, V. This vector field, **S** is known as *Poynting vector*, and generally is associated with the radiation energy flow. More specifically, The amount and the direction of the EM power transferred per unit area is provided by the flux of the Poynting vector **S**,

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{\mu_0}(\mathbf{E}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t)). \tag{1.27}$$

• **Homework**. Derive the integral energy conservation law [Eq. (1.25)] for matter in EM fields.

At this point we may relate the well known *intensity (irradiance)* of the EM field with the above Poynting vector. The irradiance (or intensity) is simply the time-average of the magnitude of the Poynting vector over a chosen interval time, τ:

$$I \equiv \langle S \rangle_\tau \equiv \frac{1}{\tau}\int_0^\tau dt' S(\mathbf{r},t'). \tag{1.28}$$

As final comment of this section we note that the expressions of the EM field energy density U_{EM} and the associated Poynting vector, **S**, are the result of the more general requirement of the conservation of energy of an isolated system (as is the present EM field/matter system).

1.5 Tutorial problems

(i) **Integral form of the MEs.** Starting from the differential form of the MEs and using the Gauss and Stokes theorems obtain the corresponding integral form.

(ii) **Charge conservation law.** Starting from the differential form of the MEs prove the charge conservation law (1.21) both the differential expression and the integral one.

(iii) **Conservation of mechanical-EM energy law.** Starting from the integral form of the MEs derive the mechanical-EM energy conservation law (1.23). Explain the physical meaning of the various terms in this equation.

(iv) **Poynting vector** In vacuum, for the following EM field

$$\mathbf{E}(z,t) = -cE_0 \sin(kz - \omega t)\hat{y}$$
$$\mathbf{B}(z,t) = +E_0 \sin(kz - \omega t)\hat{x}$$

with $E_0 = 10^6$ V/m and c the light speed in vacuum.
Find the Poynting vector **S** and the average power (over a field cycle $T = 2\pi/\omega$) crossing a circular area of radius 2 m lying in the xy-plane (plane $z = 0$). Note that not any knowledge about the frequency of the field is required to get the result.
(b) Calculate and show that the Poynting vector $\mathbf{S}(\mathbf{r}, t)$ of this field is a constant vector field. Calculate it's irradiance (intensity) in W/cm^2.

(v) **Poynting vector of *circularly polarized* EM field** In vacuum, an EM field varies only along the z-axis as

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t)\hat{x} + E_0 \sin(kz - \omega t)\hat{y}$$
$$\mathbf{B}(z,t) = -B_0 \sin(kz - \omega t)\hat{x} + B_0 \cos(kz - \omega t)\hat{y}$$

with $\omega = ck = 2\pi/\lambda$ (wavelength $\lambda = 800$ nm), amplitude $E_0 = 10^6$ V/m and $E_0 = cB_0$ and c the light speed in vacuum.
(a) Prove that **E** and **B** are perpendicular each other at all points in space and at all times.
(b) Calculate and show that the Poynting vector $\mathbf{S}(\mathbf{r}, t)$ of this field is a constant vector field. Calculate it's irradiance (intensity) in W/cm^2.
(c) Calculate the values of **E** and **B** and **S** for at $\mathbf{r}_1 = (0, 0, 0)$ at $t_1 = 0$. Make a figure that shows these three vectors fields.

(vi) ⋆ **Potential formulation of the MEs in the *Lorentz* gauge.**[*] The set of the MEs can be substituted by a set of partial differential equations (PDE) for the electromagnetic potentials $\mathbf{A}(\mathbf{r}, t)$, $\phi(\mathbf{r}, t)$ related with the $\mathbf{E}, \mathbf{B}$ fields as:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\phi(\mathbf{r},t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) \tag{1.29}$$
$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t). \tag{1.30}$$

From the above it results that if $\mathbf{A}, \phi$ are known functions of **r** and **t**then $\mathbf{E}, \mathbf{B}$ can be evaluated.
(a) Assuming the fields in the vacuum ($\rho = \mathbf{j} = 0$) and the so-called

[*] Questions with stars represent a bit more advanced subject not necessarily for consideration.

Lorentz gauge for the potential fields **A** and ϕ:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi = 0 \qquad \text{Lorenz gauge} \tag{1.31}$$

show that the fields $\mathbf{A}, \phi$ satisfy the wave equation:

$$\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = 0$$
$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = 0$$

(b) Show that the electric **E** and magnetic **B** field as defined above satisfy the four MEs [2.1a-2.1d].

(vii) ⋆ **Potential formulation of the MEs in the *Coulomb* gauge.**
Derive the partial differential waves for the potential fields $\mathbf{A}(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$ if the following condition is true (in place of the *Lorentz* gauge):

$$\nabla \cdot \mathbf{A} = 0 \qquad \text{Coulomb gauge}$$

2 Monochromatic Plane waves in vacuum

2.1 General plane waves as solutions of vacuum MEs

We assume the vacuumn, free of charges and current densities, such as $\rho = \mathbf{j} = 0$. In this case the MEs may be expressed as ($c = 1/\sqrt{\mu_0\varepsilon_0} = 2.889 \times 10^8$ m/s):

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0 \qquad \textbf{(MEV1)} \qquad (2.1a)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \qquad \textbf{(MEV2)} \qquad (2.1b)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \qquad \textbf{(MEV3)} \qquad (2.1c)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r}, t), \qquad \textbf{(MEV4)} \qquad (2.1d)$$

The form of the MEs in the vacuumn suggests that the generator of electric field $\mathbf{E}(\mathbf{r}, t)$ in the vacuumn is the time-dependent magnetic field $\mathbf{B}(\mathbf{r}, t)$ and vice-versa. Using the above MEs we find that both $\mathbf{E}, \mathbf{B}$ satisfy a *wave equation* as:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left\{\begin{array}{c} \mathbf{E}(\mathbf{r}, t) \\ \mathbf{B}(\mathbf{r}, t) \end{array}\right\} = 0. \qquad (2.2)$$

• **Homework**. Using the MEV1-4 equations prove that the electric and magnetic components of an EM field in vacuum satisfies the wave equation. Hint: For the electric field take the curl of MEV3 and then use MEV4. For the magnetic field $\mathbf{B}(\mathbf{r}, t)$ take the curl of MEV4 and and then use of MEV3.

From standard mathematical analysis of the *wave equation* results that the electric and the magnetic fields in the vacuumn must have the following functional dependence on space and time*:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{k} \cdot \mathbf{r} \pm \omega t + \phi), \qquad \mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\mathbf{k} \cdot \mathbf{r} \pm \omega t + \phi), \qquad (2.3)$$

where conventionally $\mathbf{k}$ is called wavevector, denoting the direction of the wave propagation and ω the angular frequency of the wave. The magnitude of the wavevector (wavenumber) is related with the periodic variation in space while the angular frequency with the periodic variation of the wave in time. Therefore $\mathbf{E}, \mathbf{B}$ in vacuumn without charges/currents are waves that are propagating along a direction $\mathbf{k}$, with velocity equal to the speed of light c, while the frequency is related with the magnitude of the wave vector as $\omega = ck$. Furthemore, the wavenumber and the angular frequency can also be defined in terms of the wave's wavelength (λ) and the period (T) as:

$$k = 2\pi/\lambda, \qquad \omega = 2\pi/T \qquad (2.4)$$

Further analysis of the MEs in vacuumn [Eq.(4.5)] leads to the general conclusion that for a plane EM wave of frequency $\omega = ck$ the vectors $\mathbf{E}, \mathbf{B}, \mathbf{k}$ form

* The opposite is not true. There are fields $\mathbf{F}(\mathbf{r}, t)$ which satisfy the wave equation but there *are not waves*. Standard example is the *standing wave*, made up by two identical waves travelling along opposite directions, $\mathbf{E}_S(\mathbf{r}, t) = \mathbf{E}^{(+)}(\mathbf{k} \cdot \mathbf{r} - \omega t) + \mathbf{E}^{(-)}(\mathbf{k} \cdot r + \omega t)$. Both fields, $\mathbf{E}^{(+)}$ and $\mathbf{E}^{(-)}$ represent waves but their sum not, in spite of that is a solution of the WE (as sum of solutions of the WE)

an (right-handed) orthogonal triad, while the ratio of the electric and the magnetic field is equal to the speed of light:

$$\boxed{\mathbf{E}(\mathbf{r},t)\perp\mathbf{B}(\mathbf{r},t)\perp\mathbf{k}, \quad \omega = ck \quad \mathbf{E}(\mathbf{r},t) = c\mathbf{B}(\mathbf{r},t), \quad \mathbf{S}(\mathbf{r},t) = c\varepsilon_0\mathbf{E}^2(\mathbf{r},t)\hat{k},}$$

2.2 Monochromatic plane waves and MEs in k-space

Monochromatic plane waves (MPW) of frequency ω are travelling waves of the form:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} \pm \omega t + \phi) = \mathrm{Re}(\mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r} \pm \omega t + \phi)}), \tag{2.5}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} \pm \omega t + \phi) = \mathrm{Re}(\mathbf{B}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r} \pm \omega t + \phi)}), \tag{2.6}$$

with $\mathbf{E}_0, \mathbf{B}_0$ the amplitudes being constant vectors and $\omega = ck$. The phase ϕ in the above is a constant, real number related with the definition of the time origin. As long as we examine the properties of a single MPW, we can set $\phi = 0$ without any loss of the generality*.

At this stage it is worth to introduce the complex representation of the electric and the magnetic fields e.g.

$$\mathbf{E} = \mathrm{Re}(\mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r} \pm \omega t)}) \longrightarrow \mathbf{E} = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

a representation that offers great practical convenience. The corresponding complex-number algebra (c-algebra) facilites enormously algebraic manipulations. Nevertheless it should be very clear that the $\mathbf{E}$ and $\mathbf{B}$, being observables, are real numbers. Following this rule, one performs the calculation in complex arithmetic and at the end of the calculations *we are restricted only on to the real part of the complex representation of the fields* $\boldsymbol{E}, \boldsymbol{B}$.

The MEs in the case of MPW traveling in an unbounded domain, are simplified since they can be expressed as mere algebraic equations for the amplitudes $\mathbf{E}_0$ and $\mathbf{B}_0$. The underlying reason for this is that for PMW the spatial-temporal dependence is fixed, namely, $\sim \cos(\mathbf{k} \cdot \pm \omega t)$. Then we have the following k-MEVs:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \qquad \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{2.7}$$

$$\mathbf{k} \cdot \mathbf{E}_0 = 0, \qquad \textit{(k-MEV1)} \tag{2.8a}$$

$$\mathbf{k} \cdot \mathbf{B}_0 = 0, \qquad \textit{(k-MEV2)} \tag{2.8b}$$

$$\mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0, \qquad \textit{(k-MEV3)} \tag{2.8c}$$

$$\mathbf{k} \times \mathbf{B}_0 = -\omega \mu_0 \varepsilon_0 \mathbf{E}_0, \qquad \textit{(k-MEV4)} \tag{2.8d}$$

The above form of the MEs are also known as *inverse space MEs* or *k-space MEs*. The proof of the above algebraic equations is straightforward if one adopts the complex representation of the field and recalls the following actions of the Laplace and the time-derivative operators on monochromatic fields:

$$\nabla(\mathbf{F}_\pm) = \imath \mathbf{k} \cdot \mathbf{F}_\pm, \qquad \frac{\partial}{\partial t}(\mathbf{F}_\pm) = \pm \imath \omega \mathbf{F}_\pm, \qquad \mathbf{F}_\pm = \mathbf{F}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r} \pm \omega t))} \tag{2.9}$$

* The situation is different when we examine the properties of many MPWs. For example, for two different MPWs the relative phase $\phi = \phi_1 - \phi_2$ has a decisive role in the composite's wave polarization properties (see relevant section in later chapter (Composite fields)).

Utilizing the algebraic form of the MEs [Eq.(8.11)] we arrive to the following formulas:

$$\mathbf{E}_0 \perp \mathbf{B}_0 \perp \mathbf{k}, \qquad \mathbf{E}_0 = c\mathbf{B}_0, \qquad \omega = c|\mathbf{k}| \tag{2.10a}$$

$$\mathbf{S}(\mathbf{r}, t) = c\varepsilon_0 \mathbf{E}^2(\mathbf{r}, t)\hat{k}, \qquad I = \langle S \rangle = c\varepsilon_0 \frac{E_0^2}{2} \tag{2.10b}$$

• **Homework**. Prove relations (Eq. 2.9) for $\mathbf{F}_-$ in a Cartesian coordinate system assuming $\mathbf{k} = (k_x, k_y, k_z)$, $\mathbf{r} = (x, y, z)$ and $\mathbf{F}_0 = (F_{0x}, F_{0y}, F_{0y})$.

• **Homework**. Using the mathematical properties of a PMW (Eq. 2.9) to derive the k-space MEs (k-MEV1-4), starting from the MEV1-4s.

• **Homework**. Using the k-space ME equations (k-MEV1-4) prove Eqs. (2.10a) (2.10b).

2.3 Tutorial problems

(i) Electromagnetic wave equation. Starting from the MEs in vacuumn without charges and currents prove that $\mathbf{E}, \mathbf{B}$ satisfy a wave equation (5.7).

(ii) **Integral form of the MEs in vacuumn** Starting from the differential form of the MEs and using the Gauss and Stokes theorems obtain the corresponding integral form.

$$\oint_A \mathbf{da} \cdot \mathbf{E}(\mathbf{r}, t) = 0, \qquad \oint_A \mathbf{da} \cdot \mathbf{B}(\mathbf{r}, t) = 0$$
$$\oint_C \mathbf{dr} \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{d}{dt}\Phi_B(t), \qquad \oint_C \mathbf{dr} \cdot \mathbf{B}(\mathbf{r}, t) = \mu_0 \varepsilon_0 \frac{d}{dt}\Phi_E(t).$$

How Φ_E, Φ_B in the above are defined?

(iii) Algebraic form of the MEs. Derive the algebraic form of the MEs for the plane monochromatic wave with electric and magnetic field expressed as:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}+\omega t)}, \qquad \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}+\omega t)}$$

(iv) **Plane waves**. Prove that the MEs in vacuumn with no charges and currents present, predict that the electric ($\mathbf{E}$), magnetic ($\mathbf{B}$) and the wavevector $\mathbf{k}$ form a right-handed orthogonal triangle such that $\mathbf{E}\perp\mathbf{B}\perp\mathbf{k}$.

(v) **Poynting vector of a plane monochromatic wave**. Assume a monochromatic plane wave of frequency ω and amplitude $\mathbf{E}_0$.
(a) If the plane wave is linearly polarized along the x-axis ($\mathbf{E}_0 = E_0\hat{x}$) and propagates along the z-axis, express it's Poynting vector in terms of E_0 and the wavevector $\mathbf{k}$.
(b) Moreover, show that if one choses to use the complex representation of the fields e.g. $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}\pm\omega t)}$, $\quad \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}\pm\omega t)}$, then the irradiance I can be written as:

$$I = \langle S \rangle = \frac{\mathbf{E} \times \mathbf{B}^\star}{2\mu_0} = \langle \frac{\mathrm{Re}(\mathbf{E}) \times \mathrm{Re}(\mathbf{B})}{\mu_0} \rangle = \frac{1}{T}\int_0^T dt(\frac{\mathrm{Re}(\mathbf{E}) \times \mathrm{Re}(\mathbf{B})}{\mu_0}),$$

with Re($\mathbf{E}$) and Re(B) denoting the corresponding real parts and $T = 2\pi/\omega$.

(vi) Plane monochromatic waves in vacuumn. The magnetic field associated with a plane monochromatic wave propagating along the z-axis is given by

$$\mathbf{B}(z,t) = B_0 \sin(kz - \omega t)\hat{x} + B_0 \cos(kz - \omega t)\hat{y}$$

The amplitude of the corresponding electric field is $E_0 = 10^8$V/cm and the wavelength is $\lambda = 800$ nm. This wavelength is the operating wavelength for the Ti:Sapphire laser, one of the most powerful lasers.
(a) Provide the algebraic form of the MEs for plane monochromatic waves in vacuumn ($\rho = 0, \mathbf{j} = 0$).
(b) Determine the associated electric field $\mathbf{E}(z,t)$. Draw the electric and magnetic field at $z = 0$ and $t = 0$.
(c) Calculate the Poynting vector **S** and show that it is a constant vector, in the direction of wave propagation. Calculate the EM energy density and the linear momentum density carried by the monochromatic wave.
(d) Calculate the force exerted on a square surface with side $d = 5$ mm in the case of full reflection of the radiation.

(vii) Linearly polarized light. The electric field associated with a plane monochromatic wave propagating along the z-axis is given by

$$\mathbf{E}(z,t) = E_0 e^{(kz-\omega t)}\hat{x}$$

The amplitude of the corresponding electric field is $E_0 = 10^8$V/cm and the wavelength is $\lambda = 800$ nm.
(a) Calculate the associated magnetic field $\mathbf{B}(z,t)$ and show that the Poynting vector **S** is a constant vector, in the direction of wave propagation. Calculate the EM energy density.
(c) What is the EM energy passed through a surface (of area $A = 10$ mm^2) vertical to the propagation direction.

(viii) Standing wave. Assume two EM waves which propagate in opposite directions as:

$$\mathbf{E}_1(z,t) = E_0 \sin(kz - \omega t)\hat{x}, \qquad \mathbf{E}_2(z,t) = E_0 \sin(kz + \omega t)\hat{x}$$

Calculate the Poynting vector for the total EM field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$. Does the field **E** represent a wave?

3 Wavepackets in vacuum

Although the monochromatic plance waves of Eqns (2.5,2.6) are solutions of the MEs (or equivalently of the corresponding wave equations Eqns (5.7)) they can't be accepted as realizable physical quantities, as they extend spatially at infinite distances. The proof is relatively easy, given that the amplitudes E_0, B_0 are constants (and therefore independent on the position), since integration over the full (infinite) space results to an infinite value for both the total EM energy and momentum carried by monochromatic plane waves.

3.1 Energy carried by a monochromatic wave.

In the below we'll see that, although the MPWs [see Eqs (2.5),(2.6)] are solutions of the MEs (or equivalently of the associated WE (5.7)) they can't be accepted as realizable physical observables, as their form implies that extend, spatially, at infinite distances.

To see the ramifications of this infinite extension of the MPWs, Let's consider the complex representation of a MPW of frequency ω in vacuumn which propagates along the z direction:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}(\mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)}) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t) = E_0\cos(k_z z-\omega t)$$

since $\mathbf{k}\cdot\mathbf{r} = k_z z$ For the above EM field the EM energy density is given by:

$$\begin{aligned} U_{EM}(\mathbf{r},t) &= \frac{1}{2}\epsilon_0(|\,\mathrm{Re}(E(\mathbf{r},t))|^2 + |c\,\mathrm{Re}(B(\mathbf{r},t))|^2) = \epsilon_0(\mathrm{Re}(E(\mathbf{r},t))^2 \\ &= \epsilon_0 E_0^2\cos^2((k_z z-\omega t) \end{aligned}$$

since $\omega = ck$. The total EM energy is obtained if we integrate U_{EM} over the whole space, as

$$\begin{aligned} W_{EM} &= \int dV\, U_{EM}(\mathbf{r},t) = \int dxdydz\, U_{EM}(\mathbf{r},t) = \int dxdydz\, E_0^2\cos^2((kz-\omega t) \\ &= E_0^2\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz\cos^2(kz-\omega t) \\ &= \varepsilon_0 E_0^2\,[x]_{-\infty}^{+\infty}\times[y]_{-\infty}^{+\infty}\times\left[\frac{kz+\cos(kz-\omega t)\sin(kz-\omega t)}{2k}\right]_{-\infty}^{+\infty} \\ &= \frac{\varepsilon_0 E_0^2}{2}\,[x]_{-\infty}^{+\infty}\times[y]_{-\infty}^{+\infty}\times\left\{[z]_{-\infty}^{+\infty} + \cancelto{0}{\left[\frac{\cos(kz-\omega t)\sin(kz-\omega t)}{2k}\right]_{-\infty}^{+\infty}}\right\} \\ &= \frac{\varepsilon_0 E_0^2}{2}\,[x]_{-\infty}^{+\infty}\times[y]_{-\infty}^{+\infty}\times[z]_{-\infty}^{+\infty} = \infty! \end{aligned}$$

The fact that the amplitude E_0 is constant everywhere in space (in all three directions) brings E_0 outside the respective integrals making them infinite. It is then appears that the initial assumption of constant $\mathbf{E}_0$ is not physically realizable and that physical EM fields have spatially and/or time-dependent amplitude $E_0 = E_0(\mathbf{r},t)$ such that $E(\mathbf{r},t)\to 0$ when $r=\sqrt{x^2+y^2+z^2}\to\infty$. In other words, real EM fields are actually 'localized' fields. Nevertheless the exact form of localization is not arbitrary but rather posses certain properties, derived from the MEs. This is the subject of the next section.

3.2 Wavepackets

The usefulness of the idealized concept of monochromatic plane waves lies on the fact that any EM field can be constructed as a linear combination of MPWs*:

$$\mathbf{E}(\mathbf{r}, t) \sim \sum_{n=1,2,..} c_n \mathbf{E}_n(\mathbf{r}, \omega_n), \qquad \omega_n = ck_n.$$

Therefore, the properties of the individual components of the general EM field, $\mathbf{E}_n$, will determine the properties of the composite general EM field, $\mathbf{E}$. The price to pay is that we can't any more speak about strictly monochromatic waves but for wavepacket(s) containing different frequencies $\omega_{\mathbf{k}}$. Needles to say that the above conclusion holds for any monochromatic plane wave travelling to any arbitrary direction $\mathbf{k}$ (since in vacuum where no predefined axis exists we can always define a coordinate system $\mathbf{k}$ with the $z-$axis along the propagation vector).

In contrast, linear combination of monochromatic plane waves can lead localized solutions (that do not extend infinitely in space in any direction), known as *wavepackets*. This lead us to the following section where the properties of these physically realizable solutions of the source-free MEs.

One-dimensional wavepackets. To keep the discussion as simple as possible we'll discuss the case of a wavepacket with the electric field linearly polarized along the x-axis, propagating along the positive z-axis. We'll also assume an one-dimensional treatment in the sense that the field is spatially dependent *only along the* $z - axis$ while along $xy-$plane it has constant value. Moreover we assume that the wavepacket has finite extension beyond a distance a along the $x-$ and $y-$axes. Namely this means that,

$$\mathbf{E}(\mathbf{r}, t) = \hat{x}E(x, y, z, t) = \begin{cases} \hat{x}E(z, t), & -a \leq x, y \leq a \text{ and } -\infty \leq z \leq \infty \\ 0, & otherwise \end{cases}$$

***Important note**: the above simplification is used only for convenience reasons and has loose physical meaning when compared to realistic situations. A rigorous treatment requires a 3D analysis, given the finite extension of the wavepacket along the x− and y− axes.*

Provided the above clarification, we define as *wavepacket* the following linear combination of monochromatic plane waves:

$$E(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk_z \mathscr{E}(k_z) \cos(k_z z - \omega_{k_z} t), \tag{3.1}$$

$$\mathscr{E}(k_z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \left[E(z, 0) + \frac{\imath}{\omega_{k_z}} \frac{\partial}{\partial t} E(z, 0)\right] e^{-ik_z z}, \tag{3.2}$$

The above relation is called the Fourier decomposition (or spectral decomposition) in monochromatic plane waves of the EM field. The above relations are of very fundamental nature and of major importance in the EM field theory†.

* Mathematically this is nothing else than the Fourier expansion (discrete or continuous).
† Actually, this is true for their 3-D generalization.

Electric field. According to the above pair of equations the complete determination of the electric component of the EM field requires,

the initial value both for the field, $E(z,0)$ and its partial time-derivative, $\partial_t E(z,0)$. This is a requirement coming from the fact that the wave equation includes second-order derivatives both in space and time.
The relation, $\omega_{k_z} = \omega(k_z)$ between the frequency and the wavenumber in the propagation medium. This latter relation is known as *dispersion relation* and is a characteristic property of the material as regards its interaction with the EM fields.

From a practical point of view it is often the case that the temporal profile of an EM pulse is not available (especially in the case EM fields of ultra-short duration at the fs scale, such as e.g. modern lasers) however it's frequency content can be routinely measured (note that if $\omega_{k_z} = \omega(k_z)$ then if $\mathcal{E}(\omega_{k_z})$ is known then also $\mathcal{E}(k_z)$ is necessarily known).

Magnetic field. Having calculated the electric field, $E(z,t)$ we can calculate the associated magnetic field $B(z,t)$ by direct integration of the ME (8.11b), $\partial_t \mathbf{E}(z,t) = \nabla \times \mathbf{E}(z,t)$. Doing this we find,

$$c\mathbf{B}(z,t) = \hat{k}_z \times \mathbf{E}(z,t) \tag{3.3}$$

3.3 Wavepacket propagation in vacuum

In vacuum we have $\omega_k = ck_z$, $\hat{k} = \hat{k}_z$ with the propagation speed equal to c for all frequencies (non-dispersive case). The EM energy and momentum density carried by the wavepacket and it's Poynting vecor are given by,

$$\mathbf{S}(z,t) = c\epsilon_0|\mathbf{E}(z,t)|^2\hat{k} \tag{3.4}$$

$$U_{EM}(z,t) = \frac{1}{2}\epsilon_0(|E(z,t)|^2 + |cB(z,t)|^2) = \epsilon_0|\mathbf{E}(z,t)|^2, \tag{3.5}$$

$$\mathbf{g}(z,t) = \frac{\mathbf{S}(z,t)}{c^2} = \frac{\epsilon_0}{c}|\mathbf{E}(z,t)|^2\hat{k} \tag{3.6}$$

while the total EM energy and momentum carried by a general EM wave, are given by,

$$W_{EM} = \int dV\ U(\mathbf{r},t), \tag{3.7}$$

$$\mathbf{P}_{EM} = \int dV\ \mathbf{g}(\mathbf{r},t). \tag{3.8}$$

Finally the definition of the intensity is given as usual from the expression $I = \langle S \rangle$ with the time-average now taken over an appropriate time interval (which in the general case should cover several periods $2\pi/\omega$).

Quasi-monochromatic wavepackets. These are localized wavepackets of the type presented in Eqns (3.1, 3.2), however the $\mathcal{E}(k)$ is strongly peaked around a single wavenumber k_0. For example we can consider a Gaussian functional form for the wavenumber distribution such as,

$$\mathcal{E}(k_z) = \frac{\mathcal{E}_0}{\Delta k_z} e^{-\frac{(k_z-k_0)^2}{2(\Delta k_z)^2}}, \qquad \Delta k_z << k_0. \tag{3.9}$$

The relation $\Delta k_z << k_0$ ensures that $\mathcal{E}(k_z)$ is *strongly-peaked* around k_0.

In this case, we can find for $E(z,t)$ from Eq. (3.1):

$$\begin{aligned}
E(z,t) &= Re\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk_z \frac{\mathcal{E}_0}{\Delta k_z} e^{-\frac{(k_z-k_0)^2}{2(\Delta k_z)^2}} e^{i(k_z z - ck_z t)}, \qquad (\omega_k = ck_z)\\
&= Re\frac{1}{\sqrt{2\pi}}\int d(k_z-k_0)\frac{\mathcal{E}_0}{\Delta k_z} e^{-\frac{(k_z-k_0)^2}{2(\Delta k_z)^2}} e^{ik_z(z-ct)} \qquad (dk_z = d(k_z-k_0))\\
&= Re\frac{\mathcal{E}_0}{\sqrt{2\pi}\Delta k_z} e^{ik_0(z-ct)} \int d(k_z-k_0) e^{-\frac{(k_z-k_0)^2}{2(\Delta k_z)^2}} e^{i(k_z-k_0)(z-ct)} \qquad (s \equiv k_z - k_0)\\
&= Re\frac{\mathcal{E}_0}{\sqrt{2\pi}\Delta k_z} e^{ik_0(z-ct)} \int ds e^{-\frac{s^2}{2(\Delta k_z)^2}+is(z-ct)} \qquad \left(\int_{-\infty}^{\infty} ds\ e^{-\beta s^2+\alpha s+\gamma} = \sqrt{\frac{\pi}{\beta}} e^{\frac{\alpha^2}{4\beta}+\gamma}\right)\\
&= Re\frac{\mathcal{E}_0}{\sqrt{2\pi}\Delta k_z} e^{ik_0(z-ct)} \sqrt{\frac{\pi}{1/2(\Delta k_z)^2}} e^{\frac{(i(z-ct))^2}{4/2(\Delta k_z)^2)}}, \qquad (\Delta z \equiv 1/\Delta k_z)\\
&= Re\frac{\mathcal{E}_0}{\sqrt{2\pi}\Delta k_z}\sqrt{2\pi}\Delta k_z e^{-\frac{(z-ct)^2}{2(\Delta z)^2}} e^{i(k_0 z - ck_0 t)} = \mathcal{E}_0 e^{-\frac{(z-ct)^2}{2(\Delta z)^2}} \cos(k_0 z - \omega_0 t)
\end{aligned}$$

The electric component of the EM field then is taken as the real part of it's complex representation:

$$E(z,t) = \mathcal{E}(z,t)\cos(k_0 z - \omega_0 t), \qquad \mathcal{E}(z,t) = \mathcal{E}_0 e^{-\frac{(z-ct)^2}{2(\Delta z)^2}} \tag{3.10}$$

The magnetic component is then given by:

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{k} \times \mathbf{E}(z,t) = \hat{y}\frac{\mathcal{E}_0(z,t)}{c}\cos(k_0 z - \omega_0 t)$$

3.4 Tutorial problems

$f(z,t) = f_1(z - vt) + f_2(z + vt)$ is a solution of the following wave equation:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) f(z,t) = 0$$

Pulse harmonic decomposition. Show that fields $\mathbf{F}(\mathbf{r}, t)$ formed as a linear combination of plane waves,

$$\mathbf{F}(\mathbf{r}, t) = \sum_n c_n e^{\imath(\mathbf{k}_n \cdot \mathbf{r} - \omega_n t)}, \qquad \omega_n = ck_n.$$

satisfies the wave equation as well.

Standing wave. Assume two EM waves which propagate in opposite directions as:

$$\mathbf{E}_1(z,t) = E_0 \sin(kz - \omega t)\hat{x}, \qquad \mathbf{E}_2(z,t) = E_0 \sin(kz + \omega t)\hat{x}$$

Calculate the Poynting vector for the total EM field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$. Does the field $\mathbf{E}$ represent a wave?

Momentum of a monochromatic plane wave. Show that the total momentum of a monochromatic plane wave is infinite (and as such a monochromatic plane wave cannot be an acceptable physical field).

Momentum of a quasi-monochromatic wavepacket. Calculate the total momentum of a quasi-monochromatic wavepacket wave with electric field given by,

$$\mathbf{E}(z,t) = \hat{x}\mathcal{E}_0 e^{-\frac{(z-ct)^2}{2(\Delta z)^2}} \cos(k_0 z - \omega_0 t),$$

where $k_0 \Delta z >> 1$.

Plane waves in simple matter: dielectrics

4

4.1 Electromagnetic waves in dielectrics

Propagation of EM waves in a medium results to the *polarization* **P** and *magnetization* **M** of the medium, which in turn adds to the appearance of effective electric (**D**, named here as *electric displacement*) and magnetic fields (**H**, named here as *magnetic* intensity)*. In the present chapter the propagation of the EM plane waves in *simple* matter with certain properties will be discussed. In particular we'll be discussing the case of materials where the only charged particles present are those that are bound to the atoms (or molecules) that constitute the material. Briefly, and within a loose description, the **E**, **B** components of a EM field, propagating through a dielectric medium, exert forces to the charges (e.g. electrons) that are bound to atoms (bound charges). Due to these EM forces the microscopic (atomic dimension) charge distribution is rearranged so that effectively are displaced relative to their initial distribution (but still bound). The net effect of this 'coordinated' electronic displacement, mainly due to the electronic component of the EM field **E**, is the appearance of bound charges and currents ($\rho_b, \mathbf{j}_b$) which eventually contribute to the creation of a macroscopic (several atomic dimensions) polarization and magnetization of the material.

Without going to the mathematical details of how to describe EM fields in dielectrics (or materials in general) we only give here the final conclusions of EM theory in materials which in summary are as below:

– First all the relevant fields that describe the interaction of EM fields and matter are described in an spatial average fashion rather than their microscopic version:

$$\mathbf{F}(\mathbf{r},t) \equiv \lim_{dV'\to 0} \frac{\int dV'\mathbf{F}(\mathbf{r}',t)}{dV'}, \qquad \mathbf{F} = \mathbf{E}, \mathbf{B}, \mathbf{P}, \mathbf{M}, \mathbf{D}, \mathbf{M}$$

where now **r** represents the position of an elementary volume dV' which contains several atoms inside so that a converged averaged quantity can be defined. It is in this context that the subsequent relations among the various fields have a physical meaning and are related with the experimental findings.

– Dielectric materials are now described by their magnetic and electric properties characterized by the magnetic permeability (μ) and electric permitivity (ε).

– The effective electric and magnetic fields inside the dielectric are proportional to the external **E**, **B** fields, given below by the following *constitutive relations*†:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \equiv \varepsilon \mathbf{E}, \qquad \varepsilon = \text{const.} \tag{4.1}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M} \equiv \mu^{-1}\mathbf{B}, \qquad \mu = \text{const.}, \tag{4.2}$$

where the electric permitivity ε and magnetic permeability μ are constants‡.

* **D**, **H** are also known as *auxilliary* electric and magnetic field, respectively

† Note that these relations imply a linear relation of the induced polarization and magnetization on the applied (external) EM fields **E**, **B**. Namely, in an homogeneous and isotropic dielectric medium these relations are as $\mathbf{P} = \chi\varepsilon_0\mathbf{E}$ and $\mathbf{M} = \chi_m\mathbf{H}$ where χ, χ_m are known as electric and magneric susceptibilitie, respectively.

‡ This in contrast with real materials where in general are *dispersive*, meaning that $\varepsilon = \varepsilon(\omega)$,

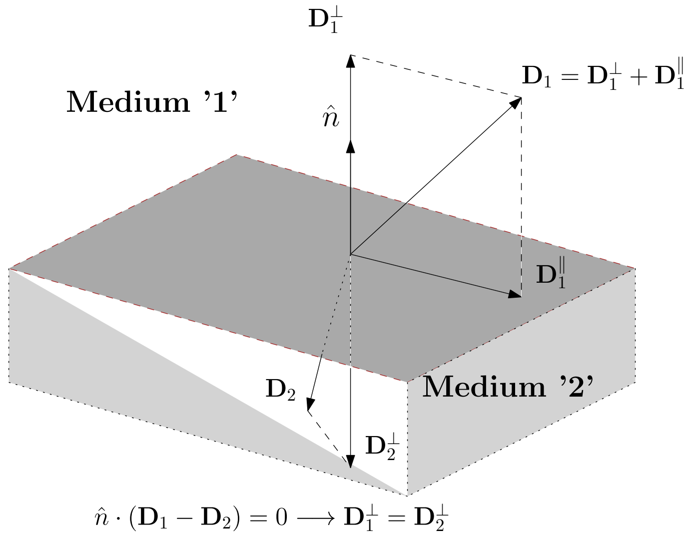

Figure 4.1: Matching conditions for the effective electrive field on the interface boundary between two dielectric media. The normal components of $\mathbf{D}_1, \mathbf{D}_2$ to the interface plane should be equal in magnitude.

– Starting from the known MEs in vacuumn, after taking into account the constitutive relations (5.2) the *macroscopic MEs* are as below:

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0, \qquad \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{4.3a}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \qquad \nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t). \tag{4.3b}$$

and/or spatially dependent and/or anisotropic.

4.2 Boundary conditions of MEs at interfacing dielectric media

An extra feature with comparison with the vacuumn is the possibility of having propagation through different dielectrics, characterized by ε, μ. Physically, and mathematically, some extra conditions should be considered for the EM fields at the interface boundary which relate the electromagnetic fields $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ between the two sides of the interface boundary*.

In the particular case of two dielectrics '1' and '2' with magnetic and electric constants $\varepsilon_i, i = 1, 2$ and $\mu_i, i = 1, 2$, respectively, direct application of the *integral form* of the MEs and results to the following relations:

$$D_1^{\perp} - D_2^{\perp} = 0 \qquad \Longleftrightarrow \qquad \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0 \tag{4.4a}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \Longleftrightarrow \qquad \mu_1 H_1^{\perp} - \mu_2 H_2^{\perp} = 0 \tag{4.4b}$$

$$E_1^{\parallel} - E_2^{\parallel} = 0 \qquad \Longleftrightarrow \qquad \varepsilon_1^{-1} D_1^{\parallel} - \varepsilon_2^{-1} D_2^{\parallel} = 0 \tag{4.4c}$$

$$H_1^{\parallel} - H_2^{\parallel} = 0 \qquad \Longleftrightarrow \qquad \mu_1^{-1} B_1^{\parallel} - \mu_2^{-1} B_2^{\parallel} = 0. \tag{4.4d}$$

* In the case of vacuumn propagation where only one medium was treated (the vacuumn) boundary conditions were imposed at infinite distances with the requirement that the EM fields should vanish, $\mathbf{E}(\mathbf{r} \to \infty, t) \to 0$ (similarly for $\mathbf{B}$). As explained in the class, this requirement excludes purely monochromatic waves as realistic EM waves, since in this case certain physical properties of the EM field (total energy, total momentum) diverge. MEs solutions predict that only (localized) superpositions of monochromatic waves are realizable.

4.3 Dielectrics (insulators)

Dielectrics are non-conducting materials where no any other charges and currents are present, apart from the effective bound charges and currents. The MEs for a dielectric medium are now expressed as:

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0, \qquad \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{4.5a}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \qquad \nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t). \tag{4.5b}$$

Wave equation in dielectrics. Using the above MEs we find that both $\mathbf{E}, \mathbf{B}$ satisfy a *wave equation* as:

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{F}(\mathbf{r}, t) = 0, \qquad \mathbf{F} = \mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}, \qquad v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4.6}$$

If one compares the above wave equation with the form of the 'vacuumn' wave equation, then it can be concluded that electric and magnetic fields in dielectric media are also waves propagating with at a reduced speed equal to v. As such any conclusion for the propagation of waves in a vacuumn is still valid provided the following substitutions are made:

$$\boxed{\varepsilon_0 \longrightarrow \varepsilon, \qquad \mu_0 \longrightarrow \mu, \qquad c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \longrightarrow v = \frac{1}{\sqrt{\mu\varepsilon}}}$$

Therefore $\mathbf{E}, \mathbf{D}$ and $\mathbf{B}, \mathbf{H}$ in a dielectric medium are waves that are propagating along a direction $\mathbf{k}$, with speed equal to v, while the frequency is related with the magnitude of the wave vector as $\omega = vk$. For dielectrics two widely used quantities (as they allow an even compact description of the medium) have also be defined, namely the *index of refraction* (n) and the *intrisinc impedance* (Z) of the medium:

$$\boxed{n \equiv \frac{c}{v} = c\sqrt{\mu\varepsilon}, \qquad \text{index or refraction}, \qquad Z \equiv \sqrt{\frac{\mu}{\varepsilon}}, \qquad \text{impedance.}} \tag{4.7}$$

By definition, the index of refraction is dimensioless and takes the value of 1 in vacuumn. The intrinsic impedance has the dimensions of resistance and takes the value of $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 377$ Ohm in vacuumn. Furthemore, the wavenumber and the angular frequency are still defined in terms of the wave's wavelength (λ) and the period (T) as:

$$k = \frac{2\pi}{\lambda} = \omega\sqrt{\mu\varepsilon} = n\frac{\omega}{c}, \qquad \omega = \frac{2\pi}{T} = vk. \tag{4.8}$$

Following further the analogy with the analysis of the MEs in vacuumn [Eq.(4.5)] we reach to the general conclusion that for a plane EM wave of frequency $\omega = vk$ the vectors $(\mathbf{E}, \mathbf{D})$,$(\mathbf{B}, \mathbf{H})$ and $\mathbf{k}$ form an (right-handed) orthogonal triad, while the ratio of the electric and the magnetic field is equal to the velocity:

$$\boxed{(\mathbf{E},\mathbf{D})\perp(\mathbf{B},\mathbf{H})\perp\mathbf{k}, \qquad \mathbf{E} = v\mathbf{B} \qquad \text{or} \qquad \mathbf{E} = Z\mathbf{H}}, \tag{4.9}$$

and the energy-related quantities are given as,

$$\boxed{\mathbf{S} = \mathbf{E}\times\mathbf{H} = vU_{EM}\hat{k}, \qquad U_{EM} = \frac{1}{2}\varepsilon\mathbf{E}^2 + \frac{1}{2}\mu\mathbf{H}^2} \tag{4.10}$$

where U_{EM} is the energy density of the EM wave and **S** is the Poynting vector.

4.4 Algebraic form of the MEs for plane monochromatic waves

As known, the plane monochromatic waves (PWs) of frequency ω are waves of the form*:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} \pm \omega t) = \mathrm{Re}(\mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}\pm\omega t)}), \tag{4.11}$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 \cos(\mathbf{k} \cdot \mathbf{r} \pm \omega t) = \mathrm{Re}(\mathbf{H}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}\pm\omega t)}), \tag{4.12}$$

$$\omega = vk \tag{4.13}$$

with $\mathbf{E}_0, \mathbf{H}_0$ the amplitudes, being constant vectors, and $\omega = vk$.

Similarly, as in the case of vacuumn propagation, for plane waves the MEs can be trasformed to algebraic equations for $\mathbf{E}_0, \mathbf{D}_0$ and $\mathbf{B}_0, \mathbf{H}_0$. For example for the $\mathbf{E}$ and $\mathbf{H}$ the algebraic form of the MEs are as follows:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \qquad \mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

$$\mathbf{k} \cdot \mathbf{E} = 0, \qquad \mathbf{k} \cdot \mathbf{H} = 0, \tag{4.14a}$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}, \qquad \mathbf{k} \times \mathbf{H} = -\omega\varepsilon\mathbf{E}, \tag{4.14b}$$

where the substitution rule $\nabla \to \imath\mathbf{k}$ and $\partial/\partial t \to -\imath\omega$ was applied. The proof of the above algebraic equations proceeds identically as in the case of PWs in vacuumn, with the standard replacement of $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$. Similar expressions that relate the other two EM fields $\mathbf{D}, \mathbf{B}$ can easily be found by considering that $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$.

Question: Provide the above algebraic form of the MEs if

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}+\omega t)}, \qquad \mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}+\omega t)},$$

* It is convenient to study $\mathbf{E}$ and $\mathbf{H}$ as the other two fields, $\mathbf{D}$ and $\mathbf{B}$ differ only by a multiplication factor $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, respectively.

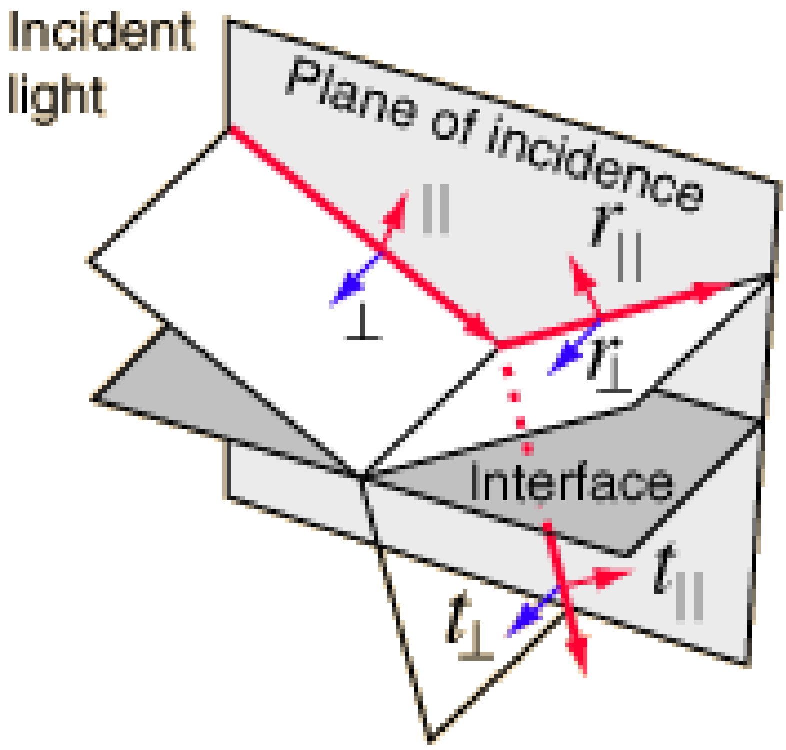

Figure 4.2: Fresnel scheme for reflection and refraction of plane monochromatic waves through an infinite *interface plane*. The $\perp$ and $\|$ components of the relevant fields $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ are defined with respect to the *plane of incidence*. The medium above the interface plane is characterized by ε_1, μ_1 while the one below the interface plane by ε_2, μ_2. Angle of incidence (θ_i), reflection (θ_r) and refraction (θ_t) are defined with respect to a vertical axis to the interface plane.

4.5 Fresnel Equations: reflection and refraction laws at interfaces

Assume two dielectric media $\varepsilon_i, \mu_i, i = 1, 2$ and PWs of the form (7.24). Focusing at the properties of the electric components of the EM fields*. In this case, let's define the *incident*, *reflected* and *transmitted* electric fields as:

$$\mathbf{E}_i(\mathbf{r}, t) = \mathbf{E}_{0i} \cos(\mathbf{k}_i \cdot \mathbf{r} - \omega_i t), \qquad \text{incident wave} \tag{4.15}$$

$$\mathbf{E}_r(\mathbf{r}, t) = \mathbf{E}_{0r} \cos(\mathbf{k}_r \cdot \mathbf{r} - \omega_r t) \qquad \text{reflected wave} \tag{4.16}$$

$$\mathbf{E}_t(\mathbf{r}, t) = \mathbf{E}_{0t} \cos(\mathbf{k}_t \cdot \mathbf{r} - \omega_t t) \qquad \text{transmitted wave} \tag{4.17}$$

with $\mathbf{E}_{0i}, \mathbf{E}_{0r}$ and $\mathbf{E}_{0t}$ being constant vectors, while $\omega_i = v_i k_i$, $\omega_r = v_r k_r$ and $\omega_t = v_t k_t$ (see caption in Fig (4.2)).

Our task is to calculate the EM fields everywhere in space, provided that we know only the incident field and the electric/magnetic properties of the two media. In other words, knowing $\mathbf{E}_{0i}, \mathbf{k}_i, \omega_i$ and $\mu_i, \varepsilon_i, i = 1, 2$, what are the $\mathbf{E}_r, \mathbf{E}_r$?

At this point we should make a choice about the direction of the electric field polarization $\mathbf{E}_{0i}$ and the plane of incidence (PofI). There are two possible configurations, named $\perp$ (or 's', TE) polarization ($\mathbf{E}_i \perp PoI$) and $\|$ (or 'p' or TM) polarization ($\mathbf{E}_i \parallel PoI$).

* Bear in mind that knowledge of the electric field suffices to determine the remaining $\mathbf{D}, \mathbf{B}, \mathbf{H}$ components.

Perpendicular polarization ($\mathbf{E}_i \perp$ PoI)

More specifically, we aim to calculate $\mathbf{k}_r, \omega_r, \mathbf{E}_{r0}$, for the reflected electric field and $\mathbf{k}_t, \omega_t, \mathbf{E}_{t0}$ for the transmitted electric field in terms of the $\mathbf{k}_i, \omega_i, \mathbf{E}_{i0}$ of the incident field. Having that calculated, the associated magnetic fields are calculated by using the following algebraic ME relation:

$$\mathbf{H}(\mathbf{r}, t) = \frac{1}{\omega\mu}\mathbf{k} \times \mathbf{E}(\mathbf{r}, t)$$

Cartesian representation for the k and E vectors First we choose a Cartesian coordinate system conveniently placed for our purposes. By taking the interface to lie in the $xz-$plane, then the $y-$axis is perpendicular to the interface plane. We define as $0 \leq \theta_i, \theta_r, \theta_t \leq \pi/2$ the angles of the wavevectors $\mathbf{k}_i, \mathbf{k}_r, \mathbf{k}_t$, respectively, with the $y-$axis.

In this case, the wave vectors for the electric fields in this coordinate system are expressed as:

$$\begin{aligned}
\mathbf{k}_i &= (k_{ix}, k_{iy}, k_{iz}) = (k_i \sin\theta_i, -k_i \cos\theta_i, 0), & \longrightarrow \quad & k_{ix} = k_i \sin\theta_i \\
\mathbf{k}_r &= (k_{rx}, k_{ry}, k_{rz}) = (k_r \sin\theta_r, k_r \cos\theta_r, 0) & \longrightarrow \quad & k_{rx} = k_r \sin\theta_r \\
\mathbf{k}_t &= (k_{tx}, k_{ty}, k_{tz}) = (k_t \sin\theta_t, -k_t \cos\theta_t, 0), & \longrightarrow \quad & k_{tx} = k_t \sin\theta_t.
\end{aligned}$$

For the components of the electric field :

$$\begin{aligned}
\mathbf{E}_i &\equiv (E_{ix}, E_{iy}, E_{iz}) = (0, 0, E_{iz}) \\
\mathbf{E}_r &\equiv (E_{rx}, E_{ry}, E_{rz}) = (0, 0, E_{rz}) \\
\mathbf{E}_t &\equiv (E_{tx}, E_{ty}, E_{tz}) = (0, 0, E_{tz})
\end{aligned}$$

while for the magnetic field components $\mathbf{B}_i, \mathbf{B}_r, \mathbf{B}_t$, (by using trigronometry) in terms of the incident, reflected and transmission angles, we have:

$$\begin{aligned}
\mathbf{B}_i &\equiv (B_{ix}, B_{iy}, B_{iz}) = (-B_{0i} \cos\theta_i, B_{iy}, 0) & \Longrightarrow B_{ix} &= -B_{0i} \cos\theta_i \\
\mathbf{B}_r &\equiv (B_{rx}, B_{ry}, B_{rz}) = (B_{0r} \cos\theta_r, B_{ry}, 0) & \Longrightarrow B_{ix} &= B_{0r} \cos\theta_r \\
\mathbf{B}_t &\equiv (B_{tx}, B_{ty}, B_{tz}) = (-B_{0t} \cos\theta_t, B_{ty}, 0) & \Longrightarrow B_{tx} &= -B_{0t} \cos\theta_t
\end{aligned}$$

We also have:

$$\begin{aligned}
\mathbf{E}_1(\mathbf{r}, t) &= \mathbf{E}_i(\mathbf{r}, t) + \mathbf{E}_r(\mathbf{r}, t), & &\text{region '1'}, \\
\mathbf{E}_2(\mathbf{r}, t) &= \mathbf{E}_t(\mathbf{r}, t), & &\text{region '2'}
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}_1(\mathbf{r}, t) &= \mathbf{H}_i(\mathbf{r}, t) + \mathbf{H}_r(\mathbf{r}, t), & &\text{region '1'}, \\
\mathbf{H}_2(\mathbf{r}, t) &= \mathbf{H}_t(\mathbf{r}, t) & &\text{region '2'}
\end{aligned}$$

Derivation of the Kinematic laws. The MEs boundary conditions for the parallel components of the electric fields [Eq. (5.15c)] at the *interface plane*($y =$

0) of two media '1' and '2' at all instants, are as below:

$$
\begin{aligned}
&E_2^{\|}|_{y=0} = E_1^{\|}|_{y=0} \Longrightarrow \\
&\qquad \Longrightarrow E_t^{\|}(\mathbf{r},t)|_{y=0} = E_i^{\|}(\mathbf{r},t)|_{y=0} + E_r^{\|}(\mathbf{r},t)|_{y=0} \\
&\qquad \Longrightarrow E_{t0}^{\|}\cos(\mathbf{k}_t\cdot\mathbf{r}-\omega_t t)|_{y=0} = E_{i0}^{\|}\cos(\mathbf{k}_i\cdot\mathbf{r}-\omega_i t)|_{y=0} + E_{r0}^{\|}\cos(\mathbf{k}_r\cdot\mathbf{r}-\omega_r t)|_{y=0} \\
&\qquad \Longrightarrow (\mathbf{k}_t\cdot\mathbf{r}-\omega_t t)|_{y=0} = (\mathbf{k}_i\cdot\mathbf{r}-\omega_i t)|_{y=0} = (\mathbf{k}_r\cdot\mathbf{r}-\omega_r t)|_{y=0}
\end{aligned}
$$

From the last equalities we conclude the separate conditions by applying it twice for $\mathbf{r} = 0$ and for $t = 0$:

- $\mathbf{r} = (x, y, z) = 0$:

$$\omega_i t = \omega_r t = \omega_t t \qquad \Longrightarrow \boxed{\omega_r = \omega_i = \omega_t} \qquad \textit{frequency law}$$

- $t = 0$:

$$
\begin{aligned}
(\mathbf{k}_t\cdot\mathbf{r})|_{y=0} = (\mathbf{k}_i\cdot\mathbf{r})|_{y=0} = (\mathbf{k}_r\cdot\mathbf{r})|_{y=0} &\Longrightarrow x k_{tx} = x k_{ix} = x k_{rx} \\
&\Longrightarrow k_t\sin\theta_t = k_i\sin\theta_i = k_r\sin\theta_r
\end{aligned}
$$

Finally we have from the above equations, since $k_i = k_r$:

$$
\begin{aligned}
k_i\sin\theta_i = k_r\sin\theta_r &\qquad\Longrightarrow \boxed{\theta_i = \theta_r} &&\qquad \textit{Refraction law} \\
k_i\sin\theta_i = k_t\sin\theta_t &\qquad\Longrightarrow \boxed{n_i\sin\theta_i = n_t\sin\theta_t} &&\qquad \textit{Snell's law,}
\end{aligned}
$$

where n_i, n_t are the refraction indices of the two media,

$$n_i = \frac{ck_i}{\omega} = \frac{c}{v_i}, \qquad n_t = \frac{ck_t}{\omega} = \frac{c}{v_t}.$$

and ω defined as $\omega = \omega_t = \omega_i$.

Derivation of the EM field amplitudes. Next step is to calculate the amplitudes of the reflected (E_{r0}) and transmitted (E_{t0}) electric field in terms of the amplitude of the incident field E_{i0}. Given that the uknown amplitudes are two (E_{r0}, E_{t0}) we need to formulate two equations that include these amplitudes. To this end, again the boundary conditions for the electric component will be recalled:

$$
\begin{aligned}
E_1^{\|}(0,t) = E_2^{\|}(0,t) &\Longrightarrow E_{i0}\cos(-\omega_i t) + E_{r0}\cos(-\omega_r t) = E_{t0}\cos(-\omega_t t) \\
&\Longrightarrow \boxed{E_{i0} + E_{r0} = E_{t0}}
\end{aligned}
\tag{4.18}
$$

where the *frequency law* ($\omega_i = \omega_r = \omega_t$) was used. In addition, for the magnetic components we have:

$$
\begin{aligned}
H_1^{\|}(0,t) = H_2^{\|}(0,t) &\Longrightarrow \frac{B_1^{\|}(0,t)}{\mu_1} = \frac{B_2^{\|}(0,t)}{\mu_2} \\
&\Longrightarrow \boxed{\frac{B_{ix}}{\mu_1} + \frac{B_{rx}}{\mu_1} = -\frac{B_{tx}}{\mu_2}}.
\end{aligned}
$$

In the above we used the fact that the parallel component of $\mathbf{B}$ to the interface plane is the component along the $x-$axis. In other words $\mathbf{B}_i^{\|} \equiv B_{ix}$,$\mathbf{B}_r^{\|} \equiv B_{rx}$,$\mathbf{B}_t^{\|} \equiv B_{tx}$. The latter expression, by substituting the expressions for

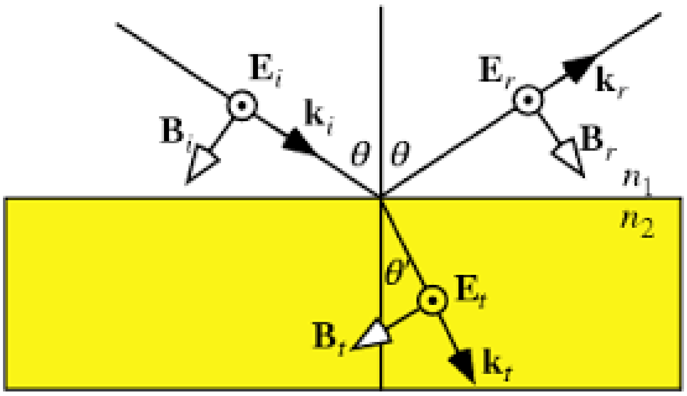

Figure 4.3: Sketch for the s-polarized ($\mathbf{E}_i \perp$ PoI) incidence of an EM wave on a infinitely flat interface boundary between two dielectrics. In the text the refraction and Snell laws that relate the incidence and reflections angles as $\theta_i = \theta_r = \theta$ and $n_1 \sin\theta = n_2 \sin\theta'$, where $\theta_t = \theta'$.

$\mathbf{B}_{ix}, \mathbf{B}_{rx}, \mathbf{B}_{tx}$ with those we derived earlier (in terms of the incident, transmission and reflection angles) is rewritten as:

$$-\frac{B_{i0}\cos\theta_1}{\mu_1} + \frac{B_{r0}\cos\theta_1}{\mu_1} = -\frac{B_{t0}\cos\theta_2}{\mu_2}. \tag{4.19}$$

From the above relations and considering that $B_{j0} = E_{j0}/v_j$ for $j = i, r, t$ we substitute to the boundary conditions (4.18) and (4.19) and we get:

$$E_{i0} + E_{r0} = E_{t0}, \qquad -\frac{E_{i0}\cos\theta_1}{v_1\mu_1} + \frac{E_{r0}\cos\theta_1}{v_1\mu_1} = -\frac{E_{t0}\cos\theta_2}{v_2\mu_2}$$

which after rearranging and by noticing that $Z_i = \mu_i v_i, i = 1, 2$the terms we obtain:

$$\boxed{\begin{aligned} E_{r0} - E_{t0} &= E_{i0} \\ Z_2\cos\theta_1 E_{r0} + Z_1\cos\theta_2 E_{t0} &= Z_2\cos\theta_1 E_{i0} \end{aligned}}$$

The above system of equations is a 2×2 algebraic system for the unknowns E_{r0} and E_{t0} and is solved to provide:

$$E_{0r} = \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_1\cos\theta_2}E_{0i}, \qquad E_{0t} = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2}E_{0i}. \tag{4.20}$$

where $Z_i = Z_r = \sqrt{\mu_1/\varepsilon_1}$ and $Z_r = \sqrt{\mu_2/\varepsilon_2}$. Also note that $v_i = v_r = v_1$ and $v_t = v_2$.

With the above relation we are able to calculate the EM field in both media '1' and '2'.

Parallel polarization ($\mathbf{E}_i \parallel$ PoI)

Following similar considerations we can derive the electric and magnetic fields for the alternative configuration of 'parallel' (or 'p') configuration. This is left for the student. The results should be as below:

$$E_{0r} = \frac{Z_1\cos\theta_1 - Z_2\cos\theta_2}{Z_1\cos\theta_1 + Z_2\cos\theta_2}E_{0i}, \qquad E_{0t} = \frac{2Z_2\cos\theta_1}{Z_1\cos\theta_1 + Z_2\cos\theta_2}E_{0i}. \tag{4.21}$$

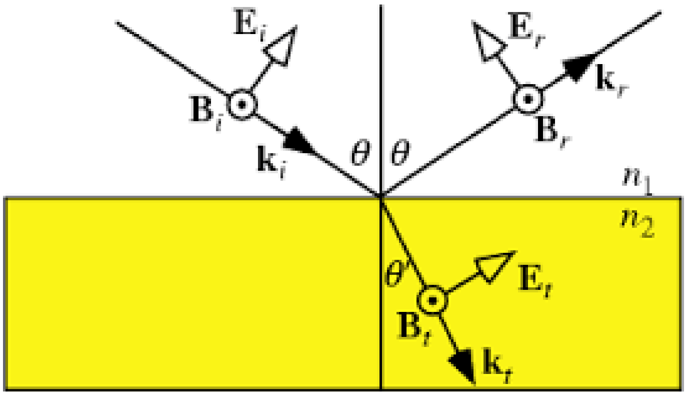

Figure 4.4: Sketch for the p-polarized ($\mathbf{E}_i \parallel$ PoI) incidence of an EM wave on a infinitely flat interface boundary between two dielectrics. The incidence and reflection angles are denoted by θ while the refraction angle by θ'.

Question: Following a similar procedure derive similar relations in the case that the electric field is parallel to the PoI ($\mathbf{E}_i \parallel PoI$) (or 'p' or parallel polarization).

Summary of kinematic and Fresnel laws

To summarize, the kinematic laws are as follows:

(i) Kinematic laws:

$$\omega_i = \omega_r = \omega_t \qquad \textit{frequency law} \qquad (4.22a)$$

$$\theta_i = \theta_r \qquad \textit{Refraction law} \qquad (4.22b)$$

$$n_i \sin\theta_i = n_t \sin\theta_t \qquad \textit{Snell's law} \qquad (4.22c)$$

The above laws are general and hold both for the *parallel* and *perpendicular* incidence to PoI.
Finally, we can define the transmission ($t_j, j = \perp, \parallel$) and reflection ($r_j, j = \perp, \parallel$) *amplitude coefficients* as follows:

(ii) Fresnel relations

$\mathbf{E}_0 \parallel$ to incident plane

$$r_{\parallel} \equiv \left[\frac{E_{0r}}{E_{0i}}\right]_{\parallel} = \frac{Z_1\cos\theta_1 - Z_2\cos\theta_2}{Z_1\cos\theta_1 + Z_2\cos\theta_2}, \qquad t_{\perp} \equiv \left[\frac{E_{0t}}{E_{0i}}\right]_{\parallel} = \frac{2Z_2\cos\theta_1}{Z_1\cos\theta_1 + Z_2\cos\theta_2},$$

$\mathbf{E}_0 \perp$ to incident plane

$$r_{\perp} \equiv \left[\frac{E_{0r}}{E_{0i}}\right]_{\perp} = \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_1\cos\theta_2}, \qquad t_{\perp} \equiv \left[\frac{E_{0t}}{E_{0i}}\right]_{\perp} = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2},$$

Power transmission

Based on the amplitude reflection and transmission coefficients one can calculate the power transmission coefficients as*:

$$R_j \equiv \left[\frac{\hat{n}\cdot\langle \mathbf{S}_r\rangle}{\hat{n}\cdot\langle \mathbf{S}_i\rangle}\right] = |r_j|^2, \qquad j = \perp, \| \tag{4.23}$$

$$T_j \equiv \left[\frac{\hat{n}\cdot\langle \mathbf{S}_t\rangle}{\hat{n}\cdot\langle \mathbf{S}_i\rangle}\right] = \frac{Z_1\cos\theta_2}{Z_1\cos\theta_1}|t_j|^2, \qquad j = \perp \; \| \tag{4.24}$$

$\langle S_i\rangle$, $\langle S_r\rangle$ and $\langle S_t\rangle$ are the time-average of the corresponding Poynting vector and $\hat{n}$ is a unit vector normal to the interface plane and pointing upwards (from medium '2' to medium '1'). Note that due to energy conservation law (since no dissipation is present)

$$R_j + T_j = 1, \qquad j = \perp \; \|$$

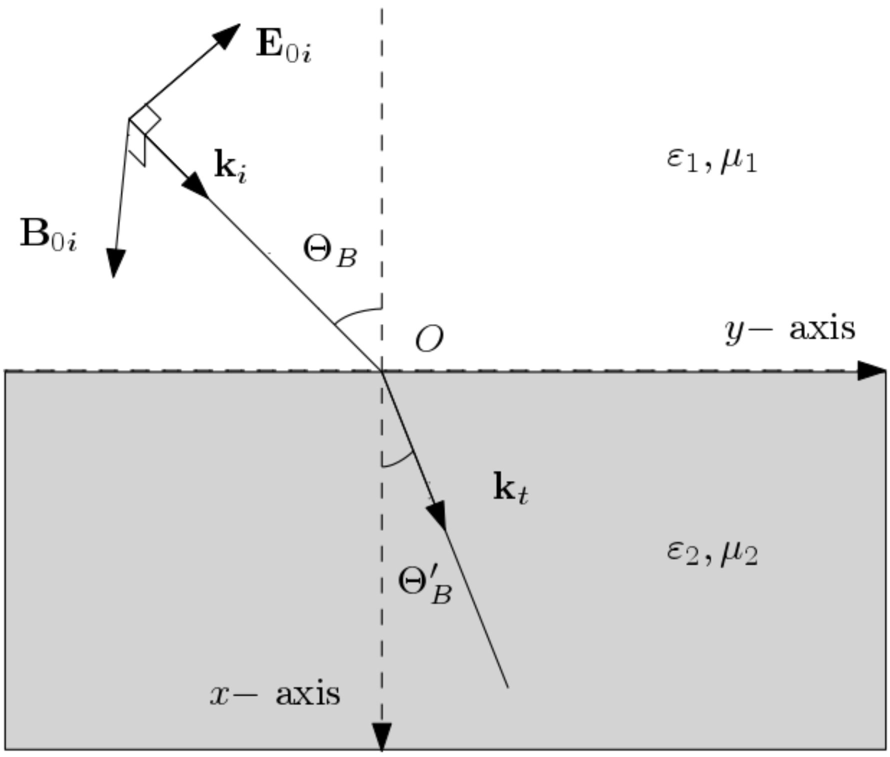

Figure 4.5: Fresnel scheme for p-incidence of a MPW at a Brewster angle.

Brewster angle

Brewster angle is this particular incident angle of an EM wave with its electric component parallel to the plane of incidence **E**|| PoI (or $p-$incidence) with no wave reflected. Equivalently the reflection coefficient at this angle vanishes. It is also known that such angle doesn't exist for an incident EM wave having its electric component perpedicular to the plane of incidence, **E**⊥PoI (or $s-$incidence).

* The derivation of the following relations are left as tutorial problem.

For example for non-magnetic media ($\mu_1 = \mu_2 = \mu_0$), starting from the condition that the reflection coefficient should vanish (at the Brewster angle) and the Snell's law, we end up to, the following expression:

$$\boxed{\tan^2\Theta_B = \frac{n_2}{n_1}, \qquad \textbf{Brewster angle for } p\textbf{-incidence } (E\|PoI)} \tag{4.25}$$

where $n_i, i = 1, 2$ are the refraction indices of the two dielectric media, '1' and '2'.

In this case the transmitted angle, Θ_B', it can be shown that is given by,

$$\boxed{\Theta_B' = \frac{\pi}{2} - \Theta_B, \qquad \textbf{Transmitted angle at Brewster angle}} \tag{4.26}$$

• **Homework**. Show that for s−wave incidence the corresponding 'vanishing' condition for the reflected wave requires that,

$$\tan^2\Theta_B = -1, \qquad s\textbf{-incidence, } (E\perp PoI)$$

which leads to the conclusion that there is no Brewster angle in this case.

4.6 Tutorial problems

(i) **Plane monochromatic waves in vacuum**
Consider a monochromatic plane wave, expressed in a Cartesian coordinate system as $E(z,t) = E_0 \sin(kz - \omega t)$, with $\omega = 2.356 \times 10^{15}$ Hz its angular frequency and k its wavenumber. The amplitude of its magnetic component is equal to $B_0 = 1$ mT.

(a) Using the MEs show that $E_0 = cB_0$.
(b) Provide the expressions for the electric, **E**, and magnetic, **B**, components of the field.
(c) Calculate the Poynting vector **S** and make its plot at $r = (1, 1, 0)$ at time $t = 10^{-14}$ sec.
(d) Starting from the definition of the Poynting vector show that the irradiance of the field $I \equiv \langle S \rangle$ (the brackets denote time-averaging) is given by, $I = c\varepsilon_0 E_0^2/2$ where ε_0 and μ_0 are vacuum's electric permitivity and magnetic permeability, respectively.
(e) Calculate the power incident to a surface, at an angle with the normal equal to 30^o. The area has 12 cm^2. If the radiation consists of photons, with each one carrying energy $E_\gamma = \hbar\omega$, calculate the number of photons incident in this surface per second. ($\hbar = 1.0545718 \times 10^{-34}$Js).
(f) If the momentum of the wave is defined by $\mathbf{g} = \mathbf{S}/c^2$ then calculate the radiation pressure on the surface of the case (e), when *all the photons are absorbed*.

(ii) Parallel to incidence plane. Prove the the kinematic laws and the Fresnel relations when the electric field is parallel to the incidence plane (see Fig. 4.2) (hint: use the appropriate ME boundary conditions and show that:

$$E_{0i} \cos\theta_1 - E_{0r} \cos\theta_1 = E_{0t} \cos\theta_2, \qquad H_{0i} + H_{0r} = H_{0t},$$

where θ_1 is the angle of incidence and θ_2 is the angle of the transmitted wave. Using the above relations find the corresponding reflection $r_\parallel$ and transmission $t_\parallel$ coefficients.

(iii) Power coefficients. Prove the relations (4.23) and (4.24) for both cases ('p' and 's' polarization).

(iv) Normal incidence. Provide the reflection and transmission coefficients when the incident wave hits vertically the interface plane and show that

$$r = r_\perp = r_\parallel = \frac{n_1 - n_2}{n_1 + n_2}, \qquad t = t_\perp = t_\parallel = \frac{2n_1}{n_1 + n_2},$$

(v) Time-averaged Poynting vector of an MPW.
Assuming the definition of a time-averaged quantity as $\langle f(\mathbf{r}) \rangle \equiv \int_0^T dt f(\mathbf{r}, t)/T)$, show that:

$$\langle \mathbf{S} \rangle = \frac{1}{2Z} |\mathbf{E}_0|^2 \hat{k}.$$

where Z is the impedance of the medium, E_0 the amplitude of the EM field and $\hat{k} = \mathbf{k}/k$, with k its wavenumber.

(vi) Power transport at normal incidence. Starting from the general expressions for the power coefficients, provide the corresponding expressions

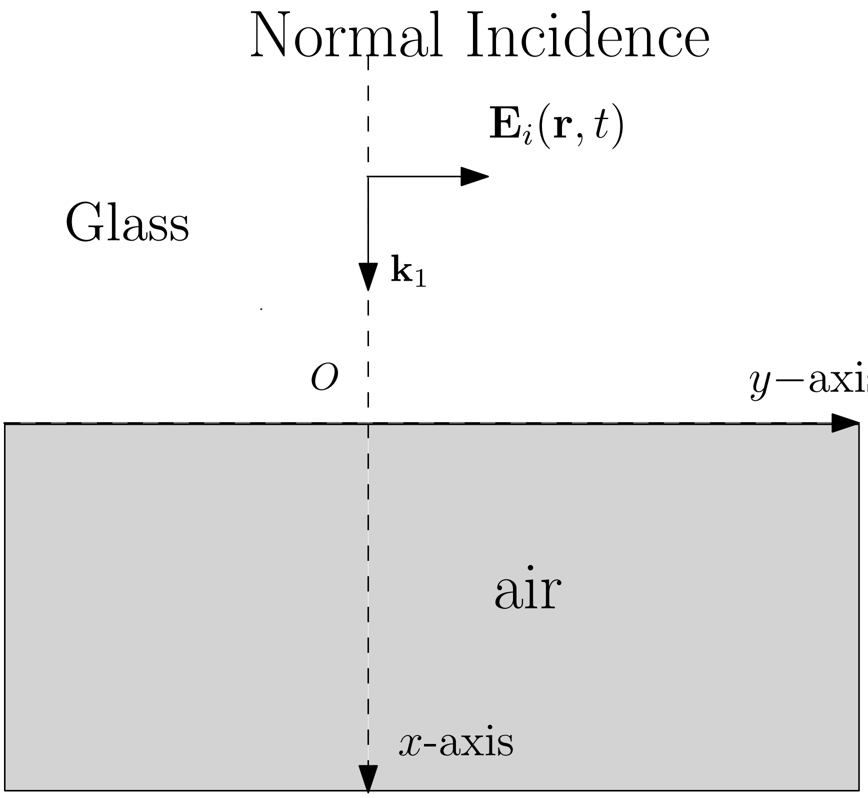

Figure 4.6: Question: Normal incidence Fresnel Equations (FEs)

in the case of normal incidence and show that:

$$R = R_{\parallel} = R_{\perp} = r^2 = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2, \qquad T = T_{\parallel} = T_{\perp} = \frac{4Z_1Z_2}{(Z_2 + Z_1)^2}$$

(a) Show that the above relations for normal incidence and non-magnetic materials $\mu_1 = \mu_2 = \mu_0$ reduces to:

$$R = R_{\parallel} = R_{\perp} = r^2 = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2, \qquad T = T_{\parallel} = T_{\perp} = \frac{4n_1n_2}{(n_2 + n_1)^2}$$

(b) For the last case prove $R + T = 1$ (due to energy conservation). Of course the relation $R + T = 1$ holds for all the above cases.

(vii) Normal incidence Fresnel Equations. Assume an electromagnetic plane wave of irradiance $I_1 = 10^6$ W/cm^2 propagating in glass (medium '1') surrounded by air (medium '2') (see relevant figure). Suppose the electromagnetic wave hits the interface boundary between glass and water with its propagation wavevector $\mathbf{k}_1$ normal to the interface plane. For a wavelength of the wave equal to $\lambda_1 = 800$ nm and at around room temperature 20^o the propagation speed of the light in the air is circa that of the light speed in the vacuum, c. In glass the light propagates with speed $v_1 = 2c/3$. Both the glass and air are non-magnetic materials (meaning that we can take $\mu_w = \mu_a = \mu_0$), where μ_w and μ_a the magnetic permeabilities of the water and air respectively. Assume a Cartesian coordinate system $Oxyz$, with the air occupying the $x > 0$ region, yz the interface plane between the glass and the air and the electric component of the incident plane wave in the water expressed as, $\mathbf{E}_i(\mathbf{r}, t) = \hat{y}E_{0i} \cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t)$,

(a) Provide the corresponding expressions for the reflected ($\mathbf{E}_r$) and transmitted waves ($\mathbf{E}_t$).
(b) Calculate the reflected ($R_\perp$) and the transmitted ($T_\perp$) power as the wave passes from water to the air.

(viii) Laser beam I. Laser beam transfering power of 20 Watts, focused on a circular surface of diameter $d = 2$ mm propagates through a glass ($n_{glass} \sim 1.6$) with incident angle $\theta_i = 0^o$.
(a) What are the electric and auxiliary magnetic field amplitudes $\mathbf{E}_0$ and $\mathbf{H}_0$, relative to the corresponding values in air? (Use that $\langle S_{glass}\rangle = \langle S_{air}\rangle$ (conservation of energy).
Also, note that $\mu_{glass} = \mu_{air} = \mu_0$ and $\varepsilon_{air} = \varepsilon_0$.
(b) Find the numerical values of the electric fields E_0, H_0 in air and glass.

(ix) Laser beam II. A beam of light in air strikes the surface of a smooth piece of plastic having an index of refraction of 1.55 at an angle with the normal of 20^o. The incident light has component E-field amplitudes parallel and perpedicular to the plane-of-incidence of 10 V/cm and 20 V/m, respectively. Determine the corresponding transmitted field amplitudes.

(x) A laser beam is incident on the interface between air and some dielectric of index n. For small values of $\theta_i << 1$ show that $\theta_t = \theta_i/n$.

(xi) **Normal incidence**
Assume an electromagnetic plane wave of irradiance $I_1 = 1.2 \times 10^{13}$ W/cm^2 propagating in air (medium '1') incident on water (medium '2') at normal angles. The wavelength of the wave in air is $\lambda_{air} = 800$ nm. The refractive index is $n_{water} = 4/3$. For air $n_{air} = 1$. Air and water are non-magnetic materials ($\mu_{air} = \mu_{water} = \mu_0$).

(a) Using the general expressions for the amplitude coefficients of the reflected (r) and transmitted (t) waves show that:

$$r \equiv \frac{E_{0r}}{E_{oi}} = \frac{k_1 - k_2}{k_1 + k_2}, \qquad t_\perp \equiv \frac{E_{0t}}{E_{oi}} = \frac{2k_1}{k_1 + k_2}.$$

where k_1, k_2 are the wavenumber of the wave in air and water, respectively. Does it matter if the wave hits the water as s–wave or p–wave?
(b) Provide the expressions for the incident $\mathbf{E}_i$, the reflected $\mathbf{E}_r$ and the transmitted wave $\mathbf{E}_t$. Define a proper coordinate system $Oxyz$ for this.
(c) Calculate the reflected (R) and the transmitted (T) power as the wave passes from air to water.

(xii) **Reflection and transmission at Brewster angle**
For the above question, for a not normal incidence case, we define the incidence as $s-$ *incidence* when the electric field is perpendicular to the plane of incidence and as $p-$ *incidence* when it is parallel to the plane of incidence. In this case do the following:

(a) Calculate the Brewster angle, Θ_B, for p–incidence. Calculate the transmitted angle, Θ'_B. Calculate the transmitted amplitude and power coefficients.
(b) At this Brewster angle, Θ_B, calculate the reflected and transmission amplitudes if the wave hits the water as s–wave. What is the transmitted angle in the water?

(c) Calculate the transmission and reflected power coefficients. Compare the found transmission power coefficients with the one found in $p-$incidence.

5 Plane waves in simple matter: non-dispersive conductors

5.1 Maxwell Equations in simple matter

As discussed in the last section (*Dielectrics*) propagation of EM waves through a medium will *polarize* (polarization **P**) and *magnetize* (magnetization **M**) the medium with the result of creating effective electric (**D**) and magnetic fields (**H**) *. In the present chapter the properties of the EM plane waves in *simple* conductors will be discussed. In contrast with dielectric materials, in *conductors* free highly-mobile charged particles † are present in addition to the relatively 'fixed' atoms that constitute the conductor. For example for metals (which normally are considered as *good conductors* these free charges are the electrons, known to be highly mobile. It is also known that applying a voltage difference in a metal (equivalently by applying an external electric field) electric current are appearing due to this high mobility of the electrons. This *forced* coherent motion of the electrons is added to their (always present) thermal, but random, motion. From the above brief discussion should be evident that the exact response of a particular material in presence of external EM fields constitutes a characteristic property which depend on the material itself. A convenient physical quantity to characterize this response is known as *conductivity*. This quantity is used to characterize a medium and provides information about the ability to generate electric currents for given **E** and **B**. In the present treatment, the latter is expressed by the Ohm's law:

$$\mathbf{j}(\mathbf{r}, t) = \sigma \mathbf{E}(\mathbf{r}, t), \tag{5.1}$$

where, in principle, σ is a *dispersive* physical quantity, namely $\sigma = \sigma(\omega)$.

Under this context and from the EM point of view, *simple* mater is fully characterized from the electric permitivity ε, its magnetic permeability μ and its conductivity σ:

$$\boxed{\text{Simple matter EM characterization:,} \qquad \varepsilon, \quad \mu, \quad \sigma}$$

As in the case of the dielectrics similar considerations hold about the description of the EM fields. These fields and their relations (MEs) should be understood as a spatial average of the actual fields in space rather than the actual fields.

For simple conducting media the effective electric and magnetic fields are proportional to the external **E**, **B** fields:

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{H} = \mu^{-1}\mathbf{B}, \qquad \mu = \text{const.}, \quad \varepsilon = \text{const.}, \quad \sigma = \text{const.} \tag{5.2}$$

where the electric permitivity ε and magnetic permeability μ and the conductivity are constants ‡.

If one starts from the known MEs in vacuumn, after taking into account the relations (5.2) one can re-express the so-called macroscopic version of the MEs in a material:

* **D**, **H** named in this text as electric displacement and magnetic intensity, respectively

† With 'free' charge here we mean that these charges are not associated with a particular atom (or molecule). Normally, these charges when external EM fields are applied, are free to move throughout the material, subject of course to collisions with each other or with the atoms/-molecules that constitute the material

‡ Again, this in contrast with general material where these quantities spatially dependent and/or anisotropic. In addition the above quantities may have values that depend on the nature of the interaction of the external EM field and the medium itself (dispersive material).

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho_f, \qquad \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{5.3a}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \qquad \nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{j}_f + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t), \tag{5.3b}$$

where ρ_f, j_f are the *free* charges and currents. The *free* charges are defined as any charges extrinsic to the material under question.

5.2 MEs in simple non-dispersive conductors

Speaking loosely *Conductors* are these conducting materials where free charges are distributed across the medium, apart from the effective bound charges and currents*. These free charges, characterized by the charge density ρ_f are interacting with the external EM waves through the Lorentz force $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$ and are set in motion, thus making up current densities governed by the Ohn's law $\mathbf{j} = \sigma\mathbf{E}$ †. For the simple conductors we examine we assume that the conductivity is not dependent on the frequency of the applied external field. The latter statement refers to the case of a *non-dispersive* conductor.

At this stage it is needed to skip one step of our development without providing the proof. The electric field of an EM wave can always be decomposed in two componets that one of it is perpedincular to the $\mathbf{k}$ propagation vector while the other one is parallel to it:

$$\mathbf{E} = \mathbf{E}_{||} + \mathbf{E}_{\perp}$$

A detailed analysis of the MEs for conductors can show that the MEs (8.8) can also be decomposed as,

$$\nabla \cdot \mathbf{D}_{\perp}(\mathbf{r}, t) = 0, \qquad \nabla \cdot \mathbf{B}_{\perp}(\mathbf{r}, t) = 0, \tag{5.4a}$$

$$\nabla \times \mathbf{E}_{\perp}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}_{\perp}(\mathbf{r}, t), \qquad \nabla \times \mathbf{H}_{\perp}(\mathbf{r}, t) = \mathbf{j}_f^{\perp} + \frac{\partial}{\partial t}\mathbf{D}_{\perp}(\mathbf{r}, t), \tag{5.4b}$$

and

$$\nabla \cdot \mathbf{D}_{||}(\mathbf{r}, t) = \rho_f$$

Solution of the latter equation for $\mathbf{D}_{||} = \varepsilon\mathbf{E}_{||}$ shows that

$$\rho_f(\mathbf{r}, t) = \rho(\mathbf{r}, 0)e^{-\sigma t/\varepsilon}. \tag{5.5}$$

Now, for typical conductors $\sigma \sim 10^7$. This value for conductivity suggests that for times $t >> 10^{-19}s$ the charge density is negligibly small. If $\tau = \varepsilon/\sigma$ all the above means that:

$$\rho_f(\mathbf{r}, t >> \tau) \to 0 \Longrightarrow \mathbf{E}_{II}(\mathbf{r}, t >> \tau) \to 0.$$

These arguments show that the charge $\rho_f(\mathbf{r}, t)$ disappears from the interiors of the conductors to extremely small time with the associated consequence that the component of the field along the propagation vector ($\mathbf{k}$) also becomes zero at the same time scale. So we can conclude that, for all practical purposes, for normal conductors and for times $t >> \tau$ the only EM field present is perpendicular to propagation vector, exactly as it happens in the vacuum and dielectrics cases. Therefore we examine the EM fields under the approximation that:

$$\boxed{\mathbf{E} \simeq \mathbf{E}_{\perp} + 0, \qquad t >> \tau.}$$

* Non-dispersive conductors are those idealized conductors where their conductivity is constant.
† or in terms of charge and current density the Loretz force is expressed as $\mathbf{f} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B}$

With the above clarifications (that the longitudinal part of the fields vanishes extremely fast) we can also drop the notation $\perp$ from all and assume that the EM field is entirely transverse. Therefore for conductors the following MEs are well justified*:

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0, \qquad \nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0, \tag{5.6a}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu \frac{\partial}{\partial t}\mathbf{H}(\mathbf{r}, t), \qquad \nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{j}_f + \varepsilon \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r}, t), \tag{5.6b}$$

We have chosen to express the MEs in terms of $\mathbf{E}$ and $\mathbf{H}$ as the other two fields, $\mathbf{D}$ and $\mathbf{B}$ differ only by a multiplication factor $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$, respectively. It is purely matter of choice to what combination of electric and magnetic fields to use.

Wave equation in simple conductors We start by taking the curl of the **ME3** and using the vector identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ we have,

$$\begin{aligned}
\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H} \quad &\Longrightarrow \quad \nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial_t}(\nabla \times \mathbf{H}), \\
[\nabla \times \mathbf{H} = \mathbf{j}_f + \varepsilon \partial_t \mathbf{E}] \quad &\Longrightarrow \quad \nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial_t}\left[\mathbf{j}_f + \varepsilon \frac{\partial}{\partial t}\mathbf{E}\right], \\
&\Longrightarrow \quad \nabla(\cancelto{0}{\nabla \cdot \mathbf{E}}) - \nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t}\mathbf{j}_f - \mu\varepsilon \frac{\partial^2}{\partial t^2}\mathbf{E}, \\
[\mathbf{j}_f = \sigma \mathbf{E} \;\;\&\;\; \nabla \cdot \mathbf{D} = 0] \quad &\Longrightarrow \quad \left[\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t}\right]\mathbf{E}(\mathbf{r}, t) = \mu \frac{\partial}{\partial t}(\sigma \mathbf{E}) \\
&\Longrightarrow \quad \boxed{\left[\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \epsilon\mu \frac{\partial^2}{\partial t^2}\right]\mathbf{E}(\mathbf{r}, t) = 0.}
\end{aligned}$$

To arrive to the above equation for the electric field we have also used the ME1 (Gauss law) $\nabla \cdot \mathbf{E} = 0$ and the Ohm's conductivity law, $\mathbf{j} = \sigma \mathbf{E}$. Following similar thinking we can also show that the same wave equations holds for the rest of the fields $\mathbf{B}, \mathbf{D}, \mathbf{H}$, to conclude that:

$$\boxed{\left[\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \epsilon\mu \frac{\partial^2}{\partial t^2}\right]\mathbf{F}(\mathbf{r}, t) = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}. \qquad (5.7)}$$

If one compares the above partial differential equation with the form of the 'dielectrics' wave equation, we see that the main difference is the presence of the *damping*' term $-\mu\sigma\partial/\partial t$. We can therefore name this equation as *damped wave equation* (DWE). Ignoring at the moment this term we are led to assume that the EM fields in conducting media are waves propagating at a speed equal to $v = 1/\sqrt{\mu\varepsilon}$ and as such $\omega = vk$. In the below, we'll show that the effect of the damping term is to generate an imaginary part† for the wavenumber k with the physical result that the amplitude of the EM wave attenuates as it propagates along its propagation direction (defined by the direction of $\mathbf{k}$).

* Note that this version results from the general MEs (Eqns 8.8) for the full equations if we had set $\rho_f \equiv 0$ but $\mathbf{j}_f \neq 0$.

† Thus same for the refraction index, $n = (c/\omega)k$ and a number of other quantities, characteristics of the conductor.

5.3 Monochromatic Plane waves

Algebraic form of the MEs (k-space) As known, the monochromatic plane waves (MPWs) of frequency ω are waves of the form:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\,\mathbf{r} \pm \omega t) = \mathrm{Re}(\mathbf{E}_0 e^{\imath(\mathbf{k}\,\mathbf{r} \pm \omega t)}), \qquad \mathbf{E}_0 = const. \tag{5.8}$$

$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}_0 \cos(\mathbf{k}\mathbf{r} \pm \omega t) = \mathrm{Re}(\mathbf{H}_0 e^{\imath(\mathbf{k}\,\mathbf{r} \pm \omega t)}), \qquad \mathbf{H}_0 = const \tag{5.9}$$

with $\mathbf{E}_0, \mathbf{H}_0$ the amplitudes being constant vectors.

Similarly, as in the case of dielectric propagation, For MPWs the MEs can be trasformed to algebraic equations for $\mathbf{E}_0, \mathbf{D}_0$ and $\mathbf{B}_0, \mathbf{H}_0$. For example for the $\mathbf{E}$ and $\mathbf{H}$ the algebraic form of the MEs are as follows:

$$\mathbf{k} \cdot \mathbf{E} = 0, \qquad \mathbf{k} \cdot \mathbf{H} = 0, \tag{5.10a}$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}, \qquad \mathbf{k} \times \mathbf{H} = \mp\omega\varepsilon(1 + \imath\frac{\sigma}{\omega\varepsilon})\mathbf{E}, \tag{5.10b}$$

where the established substitution rule $\nabla \rightarrow \imath\mathbf{k}$ and $\partial/\partial t \rightarrow \pm\imath\omega$ was applied. The proof of the above algebraic equations proceeds identically as in the case of MPWs in dielectrics, provided the replacement,

$$\varepsilon \longrightarrow \tilde{\varepsilon}(\omega) = \varepsilon(1 + \imath\frac{\sigma}{\omega\varepsilon}), \tag{5.11}$$

is made. According to the above propagation in a conducting medium is effectively treated as a dielectric medium with complex, frequency dependent, dielectric constant. To emphasize this difference, we'll be using the tilde symbol,˜, above all the affected quantities and explicitly write the dependence on the frequency of the field, ω. For a more compact set of formulas we also define,

$$\beta \equiv \frac{\sigma}{\omega\varepsilon}. \tag{5.12}$$

Following the substitution rule, $\varepsilon \longrightarrow \tilde{\varepsilon}(\omega) = \varepsilon(1+\imath\beta)$, we then find, for the wavenumbers, $\tilde{k}(\omega)$, the *index of refraction*, $\tilde{n}(\omega)$ and the impedance $\hat{Z}(\omega)$,

$$\boxed{\begin{aligned}
&\tilde{\varepsilon}(\omega) = \varepsilon(1+\imath\beta) = \varepsilon' + \imath\varepsilon'', \\
&\tilde{k}(\omega) = \omega\sqrt{\mu\tilde{\varepsilon}(\omega)} = k\sqrt{1+\imath\beta} = k' + \imath k'', \qquad k = \omega\sqrt{\mu\varepsilon} \\
&\tilde{n}(\omega) = c\sqrt{\mu\tilde{\varepsilon}(\omega)} = n\sqrt{1+\imath\beta} = n' + \imath n'', \qquad n = c\sqrt{\mu\varepsilon} \\
&\tilde{Z}(\omega) = \sqrt{\frac{\mu}{\tilde{\varepsilon}(\omega)}} = \frac{Z}{\sqrt{1+\imath\beta}} = Z' + \imath Z'', \qquad Z = \sqrt{\mu/\varepsilon}
\end{aligned}}$$

In the above we have primed and doubly primed the real and the imaginary parts of the complex numbers.

Question: Using complex arithmetics show that the square root of a complex number, $1 + \imath\beta$, is given by,

$$\sqrt{1+\imath\beta} = \sqrt{\frac{\sqrt{1+\beta^2}+1}{2}} + \imath\sqrt{\frac{\sqrt{1+\beta^2}-1}{2}}$$

First transform the Cartesian form $1 + \imath\beta$ to its polar form of the complex number $1 + \imath\beta = re^{\imath\phi}$, take the square root of it and then return back to the Cartesian plane.

Following further the analogy with the analysis of the MEs in conductors we cam reach to the general conclusion that for a plane EM wave of frequency ω the vectors $(\mathbf{E}, \mathbf{D})$,$(\mathbf{B}, \mathbf{H})$ and $\mathbf{k}$ form an (right-handed) orthogonal triad, while the ratio of the electric and the magnetic field is equal to the velocity:

$$\boxed{(\mathbf{E}, \mathbf{D}) \perp (\mathbf{B}, \mathbf{H}) \perp \mathbf{k}, \qquad \mathbf{E} = \tilde{Z}(\omega)\mathbf{H}}, \tag{5.13}$$

and the energy(power)-related quantities are given as,

$$\boxed{\mathbf{S} = \mathbf{E} \times \mathbf{H}, \qquad U_{EM} = \frac{1}{2}\tilde{\varepsilon}(\omega)\mathbf{E}^2 + \frac{1}{2}\mu\mathbf{H}^2} \tag{5.14}$$

where U_{EM} is the EM energy density of the EM wave and $\mathbf{S}$ is the Poynting vector. Note also that the wavevector $\mathbf{k}$ is now expressed as,

$$\mathbf{k} = \tilde{k}(\omega)\hat{k} = \hat{k}\sqrt{(k')^2 + (k'')^2)},$$

where $\hat{k}$ defines the unit vector along the propagation axis of the field.

Wave Equation algebraic form (k-space) In the above conclusions we could have arrived directly from the DWE derived earlier. Assuming a MPW of the type (7.24) and applying the substitution rules of the nabla and time-derivative operators into DWE (5.7) we again obtain the relation between the frequency and the wavenumber of the field as,

$$k = \omega\mu\varepsilon\sqrt{1+\imath\beta}.$$

Physical meaning of a complex wavenumber We specialize the present analysis to the electric amplitude but the same conclusions hold for all fields $(\mathbf{B}, \mathbf{D}, \mathbf{H})$. The expression for $\mathbf{E}$ maybe rewritten as,

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \mathbf{E}_0 e^{\imath[(k'+\imath k'')\hat{k}\cdot\mathbf{r}-\omega t]} = \mathbf{E}_0 e^{-\imath k''\hat{k}\cdot\mathbf{r}} e^{\imath(k'\hat{k}\cdot\mathbf{r}-\omega t)},$$

and finally we may write:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) e^{\imath(k'\hat{k}\cdot\mathbf{r}-\omega t)}, \qquad \mathbf{E}_0(r) = \mathbf{E}_0 e^{-\imath k''\hat{k}\cdot\mathbf{r}}$$

The above expression for $\mathbf{E}$ in conductors suggest wave propagation that are propagating along the direction of $\mathbf{k}$, with a speed equal to v but with decaying amplitude (along the direction of propagation, $\hat{k}$) with a rate equal to k''. Eventually for propagation distances $r >> 1/k''$ the field will be negligibly small, since,

$$\mathbf{E}(r,t) \longrightarrow 0, \qquad r >> 1/k''.$$

Given the constitutive relations (and the wave equation (5.7) similar conclusions are reached for the remaining EM fields ($\mathbf{B}, \mathbf{D}, \mathbf{H}$). It is then concluded that the EM fields inside a conductor are extremely small with their strength dependent at some distance r inside the conductor dependent on the value of the penetration depth $\delta = 1/k''$ (or skin-depth).

5.4 Special cases: Good and perfect conductors

The study of the propagation of EM waves in arbitrary conductors follows a straigthforward procedure however the final results have complicated expressions. Results can be considerably satisfied if we consider some important special cases, namely those of *good* and *perfect* conductors. These propagation conditions are defined as below:

$$\begin{aligned} \beta &>> 1, && \text{Good conductor} \\ \beta &\longrightarrow \infty && \text{perfect conductor} \end{aligned}$$

Since $\beta = \sigma/(\omega\varepsilon)$ these limits can be reached either for very low frequency fields or for materials with high conductivity. The above limits also suggest that conductors behave differently for low and high frequency EM fields.

Good conductor For good conductor conditions it can be shown* that the real and imaginary parts of the wavenumber k are equal each other:

$$\tilde{k}(\omega) = \frac{1}{\delta}(1 + \imath) = \frac{\sqrt{2}}{\delta} e^{\imath \frac{\pi}{4}}$$

where δ is the skin-depth,

$$\delta = \sqrt{\frac{2}{\mu\omega\sigma}}$$

Perfect conductor ($\sigma \to \infty$ and/or $\omega \to 0$. For *perfect conductor* conditions essentially we have all the fields zero *inside* the conductor:

$$\mathbf{E} = \mathbf{D} = \mathbf{B} = \mathbf{H} = 0.$$

Equivalently, the skin depth is zero $\delta = 0$.

Boundary conditions of MEs at interfacing media

For infinite, linear and isotropic interfacing general media, the extra feature of non-zero value of the conductivity requires a generalization of the matching conditions used in the case of dielectric media. The present case may include media where both are conductors or one of the media being dielectric.

Without giving the proof in these notes we have the following boundary conditions as they derive by direct application of the integral formulation of the MEs:

* Left as tutorial question

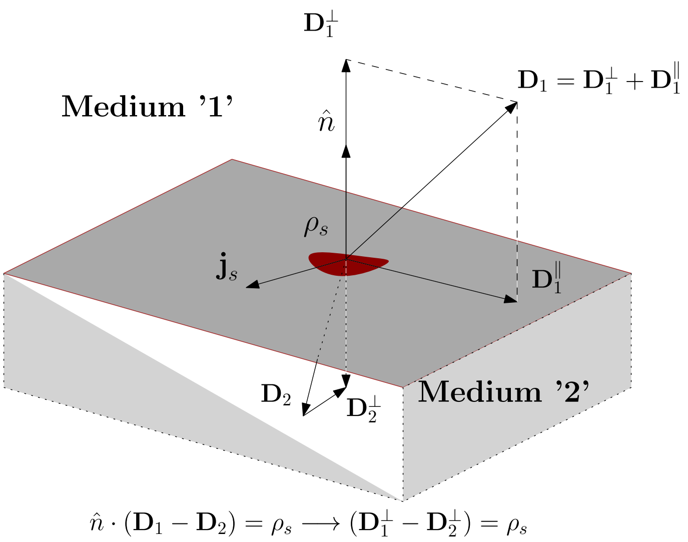

Figure 5.1: Matching conditions for the effective electric field on the interface boundary between two dielectric media. The normal components of $\mathbf{D}_1, \mathbf{D}_2$ to the interface plane should be equal in magnitude.

$$\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s, \quad \xrightarrow{\mathbf{D}_i = \mathbf{D}_i^\perp + \mathbf{D}_i^\parallel, i=1,2} \quad D_1^\perp - D_2^\perp = \rho_s \tag{5.15a}$$

$$\hat{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \xrightarrow{\mathbf{B}_i = \mathbf{B}_i^\perp + \mathbf{B}_i^\parallel, i=1,2} \quad B_1^\perp - B_2^\perp = 0 \tag{5.15b}$$

$$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0, \quad \xrightarrow{\mathbf{E}_i = \mathbf{E}_i^\perp + \mathbf{E}_i^\parallel, i=1,2} \quad E_1^\parallel - E_2^\parallel = 0 \tag{5.15c}$$

$$\hat{n} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{j}_s, \quad \xrightarrow{\mathbf{H}_i = \mathbf{H}_i^\perp + \mathbf{H}_i^\parallel, i=1,2} \quad H_1^\parallel - H_2^\parallel = j_s \tag{5.15d}$$

where ρ_s and j_s are the *surface* charge and current densities of the interface boundary surface,respectively and $\hat{n}$ is the unit vector normal to the interface boundary, towards from medium '1' to medium '2' (see figure (5.1 as an example for the case of the effective electric field $\mathbf{D}$.).

Recalling the constitutive relations $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$ an alternative set of boundary conditions can be written:

$$\begin{aligned}
\varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp &= \rho_s \\
\mu_1 H_1^\perp - \mu_2 H_2^\perp &= 0 \\
\varepsilon_1^{-1} D_1^\parallel - \varepsilon_2^{-1} D_2^\parallel &= 0 \\
\mu_1^{-1} B_1^\parallel - \mu_2^{-1} B_2^\parallel &= j_s,
\end{aligned}$$

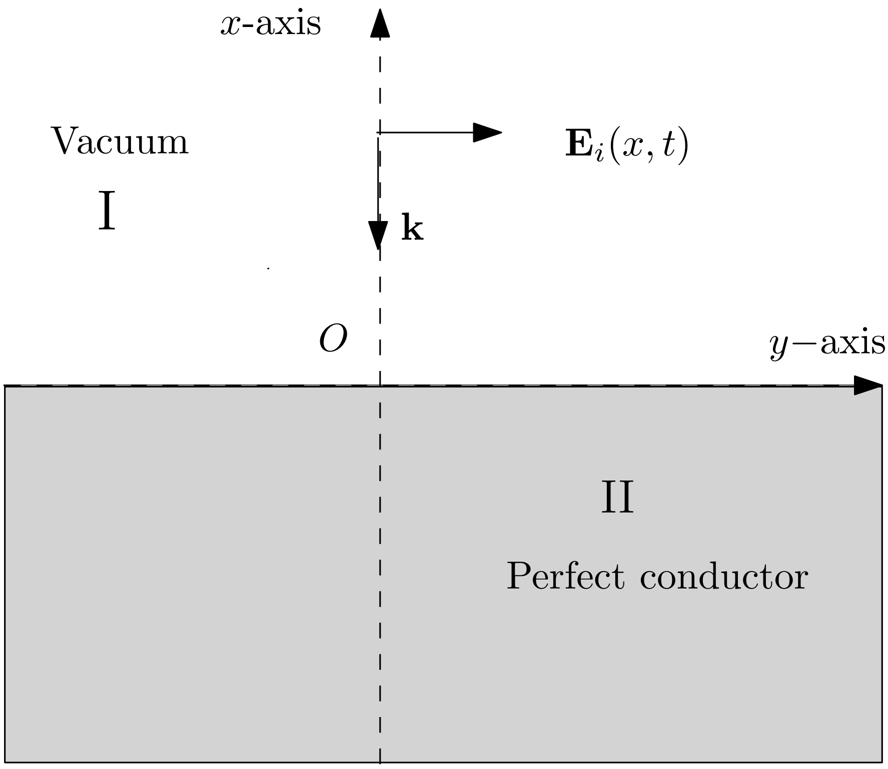

Figure 5.2: Radiation pressure for normal incidence on a perfect conductor

5.5 Tutorial problems

(i) Monochromatic PW in a conductors I
Assume an EM wave in *simple* matter where the electric permiittivity ε, magnetic permeability μ and the ohmic conductivity σ are all constants. Consider the auxilliary fields $\mathbf{D}$ and $\mathbf{H}$ through the relations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ in the case where no free charges are present ($\rho_f = 0$).
(a)Starting from the MEs in matter with the above assumptions and recalling Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ derive the partial differential equation satisfied by the electric $\mathbf{E}(\mathbf{r},t)$ and the auxilliary magnetic field $\mathbf{H}(\mathbf{r},t)$ as:

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\left\{\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right\} = 0$$

(ii) Monochromatic PW in a conductors II
Prove that for MPWs with the electric component expressed as $\mathbf{E}(\mathbf{r},t) = \mathbf{E}e^{\imath(kz\pm\omega t)}$ the wavenumber k is complex and equal to:

$$k^2 = \mu\varepsilon\omega^2 + \imath\mu\sigma\omega \longrightarrow k = \omega\sqrt{\mu\varepsilon}\left(1 + i\frac{\sigma}{\omega\varepsilon}\right)^{1/2}.$$

What is the physical meaning of a complex wavenumber k?
(a) If $\beta = \sigma/\omega\varepsilon$, show that the propagation wavenumber in conductors is further given by:

$$k(\omega) = \omega\sqrt{\mu\varepsilon}\left[\sqrt{\frac{1+r(\omega)}{2}} + \imath\sqrt{\frac{r(\omega)-1}{2}}\right], \qquad r(\omega) = \sqrt{1+\beta^2}.$$

(iii) Sea water skin depth
Sea water has a conductivity of about $\sigma_s = 50\ \mathrm{Ohm}^{-1}\mathrm{m}^{-1}$. For solar

radiation that hits the sea surface give a rough estimate of how deep the TeraHertz (THz), the MHz and the Khz parts of its spectrum transmit into the water before they reach the 1/100 of their power at the sea surface. Which one among the three would support submarine communication better?

(iv) Metal's skin depth
Upon propagation of a monochromatic wave of frequency ω through a medium we say that this medium behave as a *good conductor* if,

$$\beta = \frac{\sigma}{\omega\varepsilon} >> 1.$$

(a) Assuming a *good conductor* conditions show that the real and imaginary parts of the wavenumber k are equal each other:

$$k' = k'' = \sqrt{\frac{\mu\omega\sigma}{2}}.$$

(b) What is the skin depth for an EM wave of frequency $\omega = 900$ MHz when it impinges upon a good conductor of conductivity $\sigma = 5.98 \times 10^7$ Ohm^{-1} m^{-1} and $\mu \sim \mu_0$ (Copper).

(v) Normal incidence of a plane EM wave on a good conductor. Assume the normal incidence of a plane EM wave (of frequency ω, propagation vector $\mathbf{k}_i = -k\hat{x}$ and amplitude $\mathbf{E}_0$) on a conductor as in figure (8.1). Assume that so that the frequency of the wave is such that the approximation $(\sigma/\omega\varepsilon) >> 1$ is valid and the conductor can be considered as a *good conductor*.
(a) Provide the expressions for the incident and the reflected electric components of the EM wave $(x > 0)$.
(b) Find the transmitted $\mathbf{E}_r, \mathbf{H}_r$ fields in the case where the conductor $(x < 0)$ is a good conductor. If $E_0 = 10^2$V/cm, $\omega = 10^{16}$ rad/s, $\varepsilon = 3\varepsilon_0$ and $\mu = \mu_0$ and $\sigma = 10^7\ \Omega/m$ then calculate the ratio $E_r(x = -\delta, 0)/H_r(x = -\delta, 0)$, where δ is the skin-depth of the conductor

(vi) Radiation power loss on a good conductor. Upon normal incidence of a monochromatic PW (of electric amplitude, E_0) on a good conductor show that the time-averaged heat rate per unit area of the conductor's surface is,

$$\frac{d}{dA}\langle P_M \rangle = \frac{\sigma\delta}{4}E_0^2.$$

Most easily this is proved by using the expression for the mechanical power produced in the conductor (Joule heating)

$$P_M = \int dV\, \mathbf{j} \cdot \mathbf{E},$$

where $\mathbf{j}$ is the current density induced on the conductor due to the presence of the EM wave (Note also that Ohm's law is assumed, $\mathbf{j} = \sigma\mathbf{E}$).

6 Plane waves in simple matter: dispersive materials

6.1 EM waves in dispersive conductors

Generally, the propagation properties ofan EM wave (not necessarily monochromatic) through an arbitrary material is not independent on the strength and the frequency content of the pulse while the electric and magnetic properties of the material cannot be fully described by simple scalar constants quantities as we have dealt up to now, namely the ε and μ. In the present case we'll generalize the simpler cases we examined to the case where the response of the material, as expressed from the Ohm's law, is dependent on a frequency-dependent conductivity. However, to keep the discussion as simple as possible, we'll assume isotropic and uniform materials and as such it is expected that all the (material) physical quantities of relevance are spatially and time independent.

In view of this generalization, and from the EM point of view, *simple* mater is (again) fully characterized from the electric permitivity ε and its magnetic permeability μ and its conductivity σ, but the latter is frequency dependent.

Simple matter EM characterization:, $\quad \varepsilon, \quad \mu, \quad \sigma = \sigma(\omega)$

Under the assumption, discussed in the previous chapter, about the longitudinal and transverse part of the EM fields we again set:

$$\mathbf{F} = \mathbf{F}_\perp + \mathbf{F}_\parallel \sim \mathbf{F}_\perp, \qquad \mathbf{F} = \mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}.$$

and solve the general MEs for $\rho_f = 0$ but $\mathbf{j}_f \neq 0$. However, in the present case, the current density is *frequency dependent* as expressed by the Ohm's law:

$$\mathbf{j}_f(\mathbf{r}, t) = \sigma(\omega)\mathbf{E}(\mathbf{r}, t), \tag{6.1}$$

Following the standard procedure our starting point are the macroscopic of the MEs in a material:

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0, \qquad \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{6.2a}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \qquad \nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{j}_f + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t). \tag{6.2b}$$

6.2 Algebraic form of the MEs for monochromatic waves

Following the standard procedure for PWs the MEs can be trasformed from partial-differential equations in time and space to a system of algebraic equations for the amplitudes $\mathbf{E}_0, \mathbf{D}_0$ and $\mathbf{B}_0, \mathbf{H}_0$: If the complex representation of the electric field is as,

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

and use the substitution rules $\nabla \to \imath\mathbf{k}$, $\partial/\partial t \to -\imath\omega$ then the algebraic form of the MEs are expressed as below:

$$\mathbf{k}\cdot\mathbf{E} = 0, \tag{6.3a}$$
$$\mathbf{k}\cdot\mathbf{H} = 0, \tag{6.3b}$$
$$\mathbf{k}\times\mathbf{E} = \omega\mu\mathbf{H}, \tag{6.3c}$$
$$\mathbf{k}\times\mathbf{H} = -\omega\varepsilon(\omega)\mathbf{E}, \tag{6.3d}$$

was applied. Similar expressions that relate the other two EM fields $\mathbf{D}, \mathbf{B}$ can easily be found by considering that,

$$\mathbf{D} = \varepsilon\mathbf{E} \qquad \mathbf{B} = \frac{1}{\mu}\mathbf{H}.$$

As in the case of non-dispersive conductors the assumed expressions for the electric fields are still valid provided that,

$$\varepsilon \longrightarrow \varepsilon(\omega) = \varepsilon' + \imath\varepsilon''(\omega) \tag{6.4a}$$
$$\mu \longrightarrow \mu, \tag{6.4b}$$
$$k \longrightarrow k(\omega) = k'(\omega) + \imath k''(\omega) \tag{6.4c}$$
$$n \longrightarrow n(\omega) = n'(\omega) + \imath n''(\omega) \tag{6.4d}$$
$$Z \longrightarrow Z(\omega) = Z' + \imath Z''(\omega) \tag{6.4e}$$

The main difference now is that the expression for the dielectric constant is,

$$\varepsilon(\omega) = \varepsilon + \imath\frac{\sigma(\omega)}{\omega}, \tag{6.5}$$

where the particular dependence of $\sigma(\omega)$ on ω is material dependent. The determination of $\sigma(\omega)$ requires knowledge of the atomic structure of the materials and in its general form is based on a quantum mechanical description.

As this goes beyond the scope of the present lectures we'll study a much simpler case where the medium is treated classically and as such no quantum mechanics is required for the final expressions. The remarkable observation here is that the final expressions match very well the most accurate quantum mechanical expressions!!

6.3 The Drude model for unbounded conductive matter

Consider a collection of non-interacting electrons of constant number density n_e (number of electrons per unit volume). For reasons explained in the previous sections, the propagation of an EM wave through such medium can be modelled as an unbounded, isotropic material with $\rho(\mathbf{r}, t) = 0$ but with non-vanishing current density $\mathbf{j}(\mathbf{r}, t)$.

The *Drude's model* for a neutral system in the presence of an external EM field, composed by free mobile charges with each particle of having charge q and mass m, is to assume that the motion of an individual particle is governed by the Newton's 2nd law of motion:

$$m\frac{d}{dt}\mathbf{v} = q\mathbf{E}(0, t) - \frac{m}{\tau}\mathbf{v}, \qquad v << c, \tag{6.6}$$

where c is the light's speed in vacuum. The first term in the right hand side (RHS) of the above equation represents the Lorentz electric force that particle is experiencing due to its interaction with the EM field*. The second term in this equation represents a *drag* force being the average result over a large number of collisions suffered by the particle with its neighbor particles. This statistical, in origin, force is characterized uniquely by the mean free path time for the particle, τ. A typical value for τ in metals is 10^{-14} s.

Without loss of generality we assume propagation along the $\hat{z}-$axis, while the electric field lies along the $\hat{x}$-axis of a Cartesian coordinate system:

$$\mathbf{E}(z, t) = \hat{x}E_0 e^{i(kz-\omega t)}$$

Analyzing the Drude's equation (7.32) in its components we keep only the $\hat{x}-$component (as no EM force is applied along the other axes) and obtain:

$$\begin{aligned} &m\ddot{x} = qE_0\cos(\omega t) - \dot{x}(m/\tau) \\ &\Longrightarrow \ddot{x} + \gamma\dot{x} = f_0\cos\omega t, \qquad \gamma \equiv \frac{1}{\tau}, \qquad f_0 \equiv \frac{qE_0}{m}. \end{aligned}$$

Solution of the above equation for the particle's motion is obtained quite straightforward† and for times $t >> 1/\gamma = \tau$ we have the so-called *steady-state* solution,

$$x(t) = -\mathrm{Re}\left[\frac{(f_0/\imath\omega)}{\gamma - \imath\omega}e^{-\imath\omega t}\right],$$

where Re denotes the real part of the above expression. The above solution for $x(t)$ predict for the current density $\mathbf{j}$, by definition, the following expression:

$$\begin{aligned} \mathbf{j} &= nq\mathbf{v}(t) = \hat{x}(nq\dot{x}) = \hat{x}nqRe\left[\frac{f_0}{\gamma - \imath\omega}e^{-\imath\omega t}\right] \\ &= \hat{x}Re\left[\frac{(nq^2/m)}{\gamma - \imath\omega}E_0 e^{-\imath\omega t}\right] \end{aligned}$$

* Here we ignore the magnetic part of the Lorentz force since becomes of importance for relativistic speeds of the particle, $v << c$ (A qualitative proof of it is left as a tutorial problem)
† Left as a tutorial problem

On the other hand from Ohms's law we have the following expression for the current density:

$$\mathbf{j}(t) = \sigma \mathbf{E}(0, t) = \hat{x} Re \left[\sigma E_0 e^{-\imath \omega t} \right].$$

Combining the two latter expressions for the current density, **j**, we end up to the conclusion that the conductivity σ for this medium should be dependent on the EM's wave frequency ω as,

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - \imath \omega \tau}, \qquad \sigma_0 = \frac{n q^2 \tau}{m}, \qquad (6.7)}$$

where the σ_0 is known as the static limit of conductivity. The above expression is the central result of this section. Given this relation one can now write down the expressions for $\varepsilon(\omega), k(\omega), n(\omega)$ and $Z(\omega)$ given in Eqns (6.4).

6.4 Drude's effective material variables in dispersive conductors

As mentioned just above, having calculated the expected conductivity dependence on the EM's frequency, within the Drude's model, we are in position to calculate all the relevant electric and magnetic variables required to describe the propagation of an EM wave through this medium. For example using Eqns (6.5) and (7.30) we obtain:

$$\varepsilon(\omega) = \varepsilon_0 \left[1 - \frac{\omega_p^2 \tau^2}{1+(\omega\tau)^2} + \imath \frac{\omega_p^2 \tau/\omega}{1+(\omega\tau)^2}\right], \qquad \omega_p = \frac{nq^2}{m\varepsilon_0}. \tag{6.8}$$

The quantity ω_p defined in the above equation is known as *plasma frequency* and plays an important role in the propagation of EM waves through dispersive media as it is a cut-off threshold for propagation. In other words it can be shown that EM waves with frequencies below the plasma frequency ($\omega < \omega_p$) cannot be propagated, as will be demonstrated in the next subsection.

Depending on the nature of the medium, as expressed in a combined way from its plasma frequency ω_p*, and the frequency content of an EM wave one can consider some limiting cases for the above effective dielectric constant:

- Low frequency waves in metals:

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \imath \frac{\omega_p^2 \tau}{\omega}\right), \qquad \omega\tau << 1. \tag{6.9}$$

- High frequency, cold plasma

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - (\frac{\omega_p}{\omega})^2\right), \qquad \omega\tau >> 1. \tag{6.10}$$

Cut-off frequency for plasma media ($\omega\tau >> 1$)

In this section we'll examine the propagation properties of an EM wave through a medium where the mean-free collision time and the EM wave's frequency satisfy the following relation:

$$\omega\tau >> 1.$$

A medium that satisfy this condition is the so-called *plasma* medium consisting of completely free (mobile) charges. In this case the conductivity of the plasma as found from the Drude model can be approximated by:

$$\sigma(\omega) = \frac{\sigma_0}{1-\imath\omega\tau} \sim \frac{\sigma_0}{0-\imath\omega\tau} = -\frac{nq^2}{\imath\omega m}, \qquad \omega\tau >> 1 \tag{6.11}$$

while the induced current density is as,

$$\mathbf{j} = \frac{nq^2}{-\imath\omega m}\mathbf{E} \tag{6.12}$$

* In this relation the density of the particles n, their mass m and magnitude of charge q are appearing. Within Drude's model these three quantities are sufficient for the full description of the EM wave propagation through this medium.

Following the standard procedure we combine the Maxwell Equations (6.2) to obtain the wave equation for the electric component of an electromagnetic field propagating through is as below:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\right]\mathbf{E}(\mathbf{r},t) = \mu_0\frac{\partial}{\partial t}\mathbf{j}(\mathbf{r},t). \tag{6.13}$$

From the previous analysis we have for the current density results the following relation for the wave equation,

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\right]\mathbf{E}(\mathbf{r},t) = \mu_0\frac{n_e q_e^2}{m_e}\mathbf{E}(\mathbf{r},t),$$

Now assuming a monochromatic plane wave for the electric field $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)}$ we perform the substitutions, $\nabla \to \imath k$ and $\partial_t \to \pm\imath\omega$, to arrive at,

$$\left[-k^2 + \frac{\omega^2}{c^2}\right]\mathbf{E}(\mathbf{r},t) = \frac{\omega_p^2}{c^2}\mathbf{E}(\mathbf{r},t),$$

where ω_p is the *plasma frequency*, defined as $\omega_p \equiv q_e(n_e/m_e\epsilon_0)^{1/2}$. From the last equation we conclude that the frequency of the EM wave should relate with the plasma frequency as,

$$\omega^2 = \omega_p^2 + c^2k^2,$$

where k is the wavenumber of the field and c is the light's speed in vacuum. For values of ω lower than ω_p the wavenumber of the field becomes purely imaginary as below:

$$\begin{aligned} k &= \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} = \frac{1}{c}\sqrt{-(\omega_p^2 - \omega^2)} = \frac{1}{c}\sqrt{\imath^2(\omega_p^2 - \omega^2)} \\ &= \imath\frac{1}{c}\sqrt{\omega_p^2 - \omega^2} = \imath\frac{k''}{c}, \quad k'' > 0. \end{aligned}$$

The above purely absorptive wavenumber predicts no propagation as the real part of the wavenumber vanishes.

Tutorial Questions

(i) Derive the wave equation satisfied by the electric field starting from the MEs (6.2). If the conductivity is the one given by the Drude's model, how it is approximated if $\omega\tau >> 1$?

(ii) Derive the approximate expressions (6.9) and (6.10) for the effective dielectric constants starting from the Drude's prediction for the $\varepsilon(\omega)$.

(iii) Assume a low-density plasma characterized by its plasma frequency ω_p = 100 MHz. A plane wave of frequency ω = 80 MHz propagates through this medium. Is lossless propagation possible through this plasma or the wave is attenuated? In the second case calculate the *skin depth* of the propagation.

7 Tutorial problems

7.1 Maxwell Equations and Monochromatic waves

Maxwell's Equations Integral form

Assume the differential form of the Maxwell's Equations (MEs) in a dielectric medium, characterized by ε_1, μ_1 electric and magnetic constants.

(1a) [10 Marks]

- Using Gauss and Stokes theorems derive the integral form of the MEs. *(5 Marks)*
- Assume $\rho = j = 0$ and show that MEs predict that electromagnetic fields (EMF) satisfy the wave equation. *(5 Marks)*

(1b) [5 Marks]

Assuming an EMF of frequency ω in an unbounded domain, derive the algebraic form of the MEs.

Solution: (1a) (i) Setting $v = 1/\sqrt{\mu\varepsilon}$ we have from the ME's equations and using the Stokes (for the first 2 equations) and the Gauss (last two equations) identities:

$$
\begin{aligned}
\nabla\cdot\mathbf{E}(\mathbf{r},t) = \frac{\rho(\mathbf{r},t)}{\varepsilon_0} \quad &\xrightarrow{\int_V dV\,\nabla\cdot\mathbf{E} = \oint_A \mathbf{da}\cdot\mathbf{E}} \quad \oint_A \mathbf{da}\cdot\mathbf{E}(\mathbf{r},t) = \frac{q(t)}{\varepsilon_0}\\
\nabla\cdot\mathbf{B}(\mathbf{r},t) = 0 \quad &\xrightarrow{\int_V dV\,\nabla\cdot\mathbf{B} = \oint_A \mathbf{da}\cdot\mathbf{B}} \quad \oint_A \mathbf{da}\cdot\mathbf{B}(\mathbf{r},t) = 0\\
\nabla\times\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \quad &\xrightarrow{\int_A \mathbf{da}\cdot(\nabla\times\mathbf{E}) = \oint_C \mathbf{dr}\cdot\mathbf{E}} \quad \oint_C \mathbf{dr}\cdot\mathbf{E}(\mathbf{r},t) = -\frac{d}{dt}\Phi_B(t)\\
\nabla\times\mathbf{B}(\mathbf{r},t) = \mu\mathbf{j}(\mathbf{r}) + \frac{1}{v^2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t), \quad &\xrightarrow{\int_A \mathbf{da}\cdot(\nabla\times\mathbf{B}) = \oint_C \mathbf{dr}\cdot\mathbf{B}} \quad \oint_C \mathbf{dr}\cdot\mathbf{B}(\mathbf{r},t) = \mu i + \frac{1}{v^2}\frac{d}{dt}\Phi_E(t),
\end{aligned}
$$

where A is defined as the surface that enclosed volume V and C defined to be the line curve that enclosed surface A. In the above integral formulation of the MEs the total charge, $q(t)$, and the current, $i(t)$ are defined as *fluxes* as well as the electric and magnetic fluxes:

$$
\begin{aligned}
q(t) &= \int_V d^3\mathbf{r}\,\rho(\mathbf{r},t), \quad & i(t) &= \int_A \mathbf{da}\cdot\mathbf{j}(\mathbf{r},t),\\
\Phi_B(t) &= \int_A \mathbf{da}\cdot\mathbf{B}(\mathbf{r},t), \quad & \Phi_E(t) &= \int_A \mathbf{da}\cdot\mathbf{E}(\mathbf{r},t)
\end{aligned}
$$

(ii) In the below, the MEs' equations with $\rho = 0$ and $\mathbf{j}_f = 0$ are assumed. We start by taking the curl of the **ME3** and using the vector identity $\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}$ we have,

$$
\begin{aligned}
\nabla\times\mathbf{E} = -\mu\partial_t\mathbf{H} \quad &\Longrightarrow \quad \nabla\times(\nabla\times\mathbf{E}) = -\mu\frac{\partial}{\partial_t}(\nabla\times\mathbf{H}),\\
[\nabla\times\mathbf{H} = \varepsilon\partial_t\mathbf{E}] \quad &\Longrightarrow \quad \nabla\times(\nabla\times\mathbf{E}) = -\mu\frac{\partial}{\partial_t}(\varepsilon\frac{\partial}{\partial t}\mathbf{E}),\\
[\varepsilon\nabla\cdot\mathbf{E} = 0] \quad &\Longrightarrow \quad \nabla(\cancelto{0}{\nabla\cdot\mathbf{E}}) - \nabla^2\mathbf{E} = -\mu\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{E},\\
&\Longrightarrow \quad \boxed{\left[\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right]\mathbf{E}(\mathbf{r},t) = 0.}
\end{aligned}
$$

(1b) In an unbounded domain the solutions of the ME's equations are the plane monochromatic waves (PMW)*. The algebraic form of the MEs are derived when PMW are used:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{\imath(\mathbf{kr}-\omega t)}, \qquad \mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 e^{\imath(\mathbf{kr}-\omega t)}$$

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r},t) &= \frac{\rho(\mathbf{r},t)}{\varepsilon_0} && \xrightarrow{\nabla\to\imath k,\quad \partial_t\to-\imath\omega} && \mathbf{k}\cdot\mathbf{E}_0 = 0\\
\nabla \cdot \mathbf{B}(\mathbf{r},t) &= 0 && \xrightarrow{\nabla\to\imath k,\quad \partial_t\to-\imath\omega} && \mathbf{k}\cdot\mathbf{B}_0 = 0\\
\nabla \times \mathbf{E}(\mathbf{r},t) &= -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) && \xrightarrow{\nabla\to\imath k,\quad \partial_t\to-\imath\omega} && \mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0\\
\nabla \times \mathbf{B}(\mathbf{r},t) &= \mu\mathbf{j}(\mathbf{r}) + \frac{1}{v^2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t), && \xrightarrow{\nabla\to\imath k,\quad \partial_t\to-\imath\omega} && \mathbf{k}\times\mathbf{B}_0 = -\frac{\omega}{c}\mathbf{E}_0
\end{aligned}$$

* and of course any linear combination of their

Properties of a monochromatic plane wave

Assume that the magnetic component of an EM field varies only along the z-axis as $\mathbf{B}(z,t) = B_0 \cos(kz - \omega t)\hat{y}$, with $\omega = ck = 2\pi/\lambda$ (wavelength $\lambda = 800$ nm) and amplitude $B_0 = 10^{-3}$T. Making use of the MEs calculate:

(a) The electric component of the EM field and calculate the values of $\mathbf{E}$ and $\mathbf{B}$ for all times at positions $\mathbf{r}_1 = (0,0,0)$ and $\mathbf{r}_2 = (1,1,0)$. Plot them at these positions for $t_1 = 0$ and $t_2 = \pi/\omega$.

(b) The Poynting vector and its values at $\mathbf{r}_1, \mathbf{r}_2$ for t_1 and t_2.

(c) The density of the electromagnetic energy U_{EM} and the irradiance I. Calculate in Joules the amount of energy incident normal to a surface of area 0.1 m^2 in one day. If the radiation consists of photons, with each one carrying energy $E_\gamma = \hbar\omega$, calculate the number of photons incident in this surface per second. ($\hbar$ is Planck's constant divided by 2π).

(d) If we define the momentum density of the EM wave as $\mathbf{g} = \mathbf{S}/c^2$ what is its magnitude and a angular momentum density the quantity $\mathbf{l} = \mathbf{r} \times \mathbf{g}$ find the their values at the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ at all times.

Solution: (a) Since the EM field is a plane monochromatic field propagating along the z–axis then we have that

$$\mathbf{k} = k\hat{z} = \frac{\omega}{c}\hat{z} \tag{7.1}$$

The frequency and the wavenumber of the EM field can be calculated as:

$$\begin{aligned}
\lambda &= 800 \text{ nm} = 800 \times 10^{-9} \text{ m}, \\
k &= \frac{2\pi}{\lambda} = \frac{2\pi}{800 \times 10^{-9}} = 0.7854 \times 10^{7} \text{ m}^{-1} \\
\omega &= ck = 3 \times 10^{8} \times (0.7854 \times 10^{7}) = 2.3562 \times 10^{15} \text{ rad/sec} \\
f &= \omega/2\pi = 3.75 \times 10^{14} \text{ Hz}.
\end{aligned}$$

The expression for the monochromatic magnetic and the electric components are as below:

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t) = B_0 \cos(kz - \omega t)\hat{y} \tag{7.2}$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t) = \mathbf{E}_0 \cos(kz - \omega t), \tag{7.3}$$

since $\mathbf{k}\cdot\mathbf{r} = kz$. It remains to calculate the amplitude $\mathbf{E}_0$ (direction and magnitude). To this end, since the fields are monochromatic, the algebraic form of the MEs may be recalled, namely:

$$\mathbf{k}\cdot\mathbf{E}_0 = 0, \qquad \mathbf{k}\cdot\mathbf{B}_0 = 0, \tag{7.4a}$$

$$\mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0, \qquad \mathbf{k}\times\mathbf{B}_0 = -\omega\mu_0\varepsilon_0\mathbf{E}_0. \tag{7.4b}$$

where $c^2 = 1/\mu_0\varepsilon_0$ and $\varepsilon_0 = 8.8542\times10^{-12}$ Cb^2/Nm^2, $\mu_0 = 4\pi\times10^{-7}$ N/A^2. From the above equations (bottom-left ME) we have:

$$\mathbf{E}_0 = -\frac{c^2}{\omega}(\mathbf{k}\times\mathbf{B}_0) = -\frac{c^2}{\omega}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & k\\ 0 & B_0 & 0\end{vmatrix} = -\frac{c^2}{\omega}\left[\hat{x}\cdot(-kB_0) - \hat{y}\cdot 0 + \hat{z}\cdot 0\right]$$
$$= \frac{c^2 k}{\omega}B_0\hat{x} = cB_0\hat{x} \Longrightarrow \boxed{\mathbf{E}_0 = cB_0\hat{x}}$$

where the relations $\omega = ck$ and $\mu_0\varepsilon_0 = 1/c^2$ have been used.We then end up to the following expressions for $\mathbf{E}, \mathbf{B}$:

$$\mathbf{E}(\mathbf{r}, t) = cB_0\cos(kz - \omega t)\hat{x}, \qquad \mathbf{B}(\mathbf{r}, t) = B_0\cos(kz - \omega t)\hat{y} \tag{7.5}$$

with $B_0 = 10^{-4}T, k, \omega$ known. The electric field amplitude E_0 is found as:

$$E_0 = cB_0 = (3\times10^8)(10^{-3}) = 3\times10^5 \text{ V/m} = 10^7 \text{ V/cm} \tag{7.6}$$

The fields at the positions $\mathbf{r}_1 = (0,0,0)$ and $\mathbf{r}_2 = (1,1,0)$ for times t_1 and t_2 are given as:

$$\mathbf{E}(\mathbf{r}_1, t_1) = cB_0\cos(k\cdot 0 - \omega\cdot 0)\hat{x} = cB_0\hat{x}$$
$$\mathbf{B}(\mathbf{r}_1, t_1) = B_0\cos(k\cdot 0 - \omega\cdot 0)\hat{x} = B_0\hat{y}$$

While we find for the time $t_2 = \pi/\omega$:

$$\mathbf{E}(\mathbf{r}_1, t_2) = -cB_0\hat{x}, \qquad \mathbf{B}(\mathbf{r}_1, t_2) = -B_0\hat{y}.$$

Similarly for the fields at position $\mathbf{r}_2$ we end up to the following:

$$\mathbf{E}(\mathbf{r}_2, t_1) = cB_0\hat{x}, \qquad \mathbf{B}(\mathbf{r}_2, t_1) = B_0\hat{y}$$
$$\mathbf{E}(\mathbf{r}_2, t_2) = -cB_0\hat{x}, \qquad \mathbf{B}(\mathbf{r}_2, t_2) = -B_0\hat{y}.$$

(b) The Poynting vector may be calculated by the relation $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_0$ as:

$$\mathbf{E}_0 = \frac{1}{\mu_0}(\mathbf{E}_0\times\mathbf{B}_0) = \frac{1}{\mu_0}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ E_0\cos(kz-\omega t) & 0 & k\\ 0 & B_0\cos(kz-\omega t) & 0\end{vmatrix}$$
$$= \frac{1}{\mu_0}\left[\hat{x}\cdot 0 - \hat{y}\cdot 0 + \hat{z}\cdot E_0B_0\cos^2(kz-\omega t)\right] = \frac{E_0B_0}{\mu_0}B_0\hat{z} = \frac{E_0}{\mu_0 c}\hat{z}$$
$$\Longrightarrow \boxed{\mathbf{S}(\mathbf{r}, t) = c\varepsilon_0 E_0^2\cos^2(kz-\omega t)\hat{z}},$$

where the relations $E_0 = cB_0$ and $c\varepsilon_0 = 1/c\mu_0$ have been used.

The Poynting vector at positions $\mathbf{r}_1, \mathbf{r}^2$ at times t_1 and $\mathbf{t}_2$ is easily found to be:

$$\mathbf{S}(\mathbf{r}_1, \mathbf{t}_1) = \mathbf{S}(\mathbf{r}_1, \mathbf{t}_2) = \mathbf{S}(\mathbf{r}_2, \mathbf{t}_1) = \mathbf{S}(\mathbf{r}_2, \mathbf{t}_2) = c\varepsilon_0 E_0^2\hat{z}. \tag{7.7}$$

Substituting the values of the light speed and E_0 we get for the magnitude of the Poynting vector:

$$
\begin{aligned}
S &= c\varepsilon_0 E_0^2 = (3\times 10^8)\times(8.8542\times 10^{-12})\times(3\times 10^5)^2 \\
&= 2.390\times 10^8 \text{ W/m}^2 = 2.390\times 10^{12} \text{ W/cm}^2.
\end{aligned}
$$

(c) The irradiance of the field is given as the time-average of the Poynting vector:

$$
\begin{aligned}
I =& \langle S\rangle = \frac{1}{T}\int_0^T dt S(\mathbf{r},t) = \frac{1}{T}\int_0^T dt c\varepsilon_0 E_0^2\cos^2(kz-\omega t) \\
=& c\varepsilon_0 E_0^2\frac{1}{T}\left(\int_0^T dt\cos^2(kz-\omega t)\right) = c\varepsilon_0 E_0^2\frac{1}{T}(\frac{T}{2}) \\
\implies& \boxed{I = \frac{1}{2}c\varepsilon_0 E_0^2}
\end{aligned}
$$

The magnitude of the irradiance is:

$$I = \frac{1}{2}c\varepsilon_0 E_0^2 = 0.5(8.8542\times 10^{-12})(3\times 10^8)(3\times 10^5)^2 = 1.195\times 10^8 \text{ W/m}^2 = 1.195\times 10^{12} \text{ W/cm}^2.$$

The density of the EM energy is expressed as:

$$
\begin{aligned}
U_{EM}(\mathbf{r},t) &= \frac{\varepsilon_0}{2}E^2(\mathbf{r},t) + \frac{1}{2\mu_0}B^2(\mathbf{r},t) = \frac{\varepsilon_0}{2}E_0^2\cos^2(kz-\omega t) + \frac{1}{2\mu_0}B_0^2\cos^2(kz-\omega t) \\
&\implies \boxed{U_{EM}(z,t) = \varepsilon_0 E_0^2\cos^2(kz-\omega t)},
\end{aligned}
$$

where the relations $E_0 = cB_0$ and $c\varepsilon_0 = 1/c\mu_0$ were used when replaced $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ from Eqns (25). Note that the above two relations confirm that $S = cU_{EM}$. By definition of the irradiance as the radiation power per unit area $I = d\bar{U}_{EM}/dtdA$ immediately we can find the total incident energy (U_{tot}) on a surface of area $A = 0.1$ m^2 in a time interval equal to a day $t_d = 24\times 3600$ sec, as:

$$U_{tot} = \int d\bar{U}_{EM} = \int IdAdt = I\times A\times t_d = c\varepsilon_0 E_0^2\times 0.1\times 24\times 3600 \sim 2\times 10^{12} J,$$

where have substituded $c\varepsilon_0 E_0^2 = 2\times 1.195\times 10^8$ W/m^2. Since the radiation is monochromatic to find the number of photons incident to this area we have to divide the total energy U_{tot} by the amount of energy carried from each individual photon:

$$N_\gamma = \frac{U_{tot}}{E_\gamma} = \frac{U_{tot}}{\hbar\omega} = .. \sim \frac{2\times 10^{12}}{2.5\times 10^{-19}} \sim 8\times 10^{30} \text{ photons},$$

where we have used that $\hbar \sim 1.054\times 10^{-34}$ Jsec

(d) The momentum and angular momentum densities are found by direct substitution as:

$$\mathbf{g}(\mathbf{r},t) = \frac{1}{c^2}\mathbf{S}(\mathbf{r},t) = \frac{\varepsilon_0}{c}E_0^2\cos(kz-\omega t)\hat{z}$$

while the angular momentum density is equal to:

$$\begin{aligned}\mathbf{l}(\mathbf{r},t) =&\mathbf{r}\times\mathbf{S}(\mathbf{r},t) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ 0 & 0 & S(z,t) \end{vmatrix} = \hat{x}yS(z,t) - \hat{y}xS(z,t) \\ =&S(z,t)(\hat{x}y - \hat{y}x) \Longrightarrow \boxed{\mathbf{l}(\mathbf{r},t) = c\varepsilon E_0^2\cos^2(kz-\omega t)(\hat{x}y-\hat{y}x)},\end{aligned}$$

From the above expressions and the equations (25) is easy to find that:

$$\mathbf{g}(\mathbf{r}_1,\mathbf{t}_1) = \mathbf{g}(\mathbf{r}_1,\mathbf{t}_2) = \mathbf{g}(\mathbf{r}_2,\mathbf{t}_1) = \mathbf{g}(\mathbf{r}_2,\mathbf{t}_2) = \frac{\varepsilon_0}{c}E_0^2\hat{z}.$$

and

$$\begin{aligned}\mathbf{l}(\mathbf{r}_1,\mathbf{t}_1) = \mathbf{l}(\mathbf{r}_1,\mathbf{t}_2) = 0,\\ \mathbf{g}(\mathbf{r}_2,\mathbf{t}_1) = \mathbf{g}(\mathbf{r}_2,\mathbf{t}_2) = c\varepsilon_0 E_0^2(\hat{x}-\hat{y}).\end{aligned}$$

Motion of a free charged particle in an EM field I.

Assume in space an electromagnetic field and a dimensionless particle, of charge, q_e, and mass, m_e, initially at rest.

(a) Calculate the motion of the particle due to its interaction with the EM field employing non-relativistic velocities.

(b) Show that not net power transfer from the EM field to the electron exists over a cycle $T = 2\pi/\omega$.

Solution: (a)

In case of non-relativistic speeds, $v << c$, the Newton's second law will be used to find the time-evolution of the mass position:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{r}, \mathbf{p}, t)$$

where $\mathbf{p}$ is the momentum of the particle where for non-relativistic velocities is given by $\mathbf{p} = m_e\mathbf{v} = m_e\dot{\mathbf{r}}$. Recalling the Lorentz force expression, and taking advantage that the magnetic component of this force is much smaller from its electric counterpart, $q_eE >> q_evB$ (see Eq. (1.20), it is sufficient to keep only the electric component. The motion of the electron results from the 2nd Newton's law:

$$\begin{aligned}
&m_e\ddot{\mathbf{r}} = q_e\mathbf{E}(\mathbf{r}, t) + q_e(\mathbf{v} \times \mathbf{B}(\mathbf{r}, t)) \Longrightarrow \\
&m_e\ddot{\mathbf{r}} \approx q_e\mathbf{E}_1(t) + \cancelto{(vB << E)}{q_e(\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))} \qquad \Longrightarrow \\
&m_e\ddot{x} = q_eE_0\cos(k \cdot 0 - \omega t) = q_eE_0\cos(-\omega t) \Longrightarrow \\
&m_e\ddot{x} = q_eE_0\cos\omega t,
\end{aligned}$$

In the above we assumed $z = 0$ by assuming that the particle is located at the origin of the coordinate system and that the particle is propagating along the $z-$axis and is polarized along the $x-$axis, namely $\mathbf{E} = \hat{x}E(z, t)$. Then force on the particle is along the x-axis component (no motion to the other axes $y(t) = z(t) = 0$). Integration of the above differential equation for $\ddot{x}(t)$ provides velocity (with the initial condition that $\mathbf{v}(0) = 0$):

$$\begin{aligned}
&\ddot{x}(t) = \frac{dv_x}{dt} \longrightarrow \int_0^{v_x} dv_x = \int_0^t dt'\,\ddot{x}(t') \Longrightarrow v_x(t) - 0 = \int_0^t dt'\frac{q_eE_0}{m}\cos\omega t' \\
&\Longrightarrow v_x(t) = \frac{q_eE_0}{m_e\omega}\sin\omega t. \qquad (7.8)
\end{aligned}$$

Eventually we find for the $x-$position of the particle,$v_x = dx/dt$,

$$x(t) = \cancelto{0}{x(0)} + \int_0^t dt' v(t') = -\frac{q_eE_0}{m_e}\cos\omega t, \qquad (7.9)$$

(b)

The EM power transformed to mechanical power is given by:

$$P_M = \frac{dW_M}{dt} = \int d^3\mathbf{r}\ j(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = \int d^3\mathbf{r}\ \rho(\mathbf{r},t)\mathbf{v}(t)\cdot\mathbf{E}(\mathbf{r},t)$$
$$\simeq \mathbf{v}(t)\cdot E(0,t)\int d^3\mathbf{r}\rho(\mathbf{r},t) = q\mathbf{v}(t)\cdot\mathbf{E}(0,t) = q_e\mathbf{v}(t)\cdot\mathbf{E}(t) \quad (7.10)$$

where $q_e \equiv \int d^3\mathbf{r}\rho(\mathbf{r},t)$ and $\mathbf{E}(0,t) = \mathbf{E}(t)$. Since the electric field is known then we only need to calculate the velocity of the electron $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$, which by using Eq. (7.8) we have $\mathbf{v}(t) = (v_x(t), 0, 0)$.

The averaged power transferred to the electron will be calculated using Eq. (7.10) as

$$\bar{P}_M \equiv \frac{1}{T}\int_0^T dtP_M(t) = \frac{1}{T}\int_0^T dtq_e\mathbf{v}(t)\cdot\mathbf{E}(t) = \langle q_eE_x(t)v_x(t)\rangle$$
$$= \langle q_eE_0\cos\omega t\frac{q_eE_0}{m_e\omega\rangle\sin\omega t)}\rangle = \frac{q_e^2E_0^2}{m_e\omega}\langle\cos\omega t\ \ \sin\omega t\rangle$$
$$= \frac{q_e^2E_0^2}{2m_e\omega}\langle\sin 2\omega t\rangle = \frac{q_e^2E_0^2}{2m_e\omega}\left(\frac{1}{T}\int_0^T dt\sin(2\omega t)\right)$$
$$= \frac{q_e^2E_0^2}{2m_e\omega}(0) = 0.$$

Thus the averaged power transferred to the particle equals to zero. We conclude then that as the EM propagates through a gas of non-interacting electrons is not attenuated since the amount of EM energy that is transformed to mechanical energy is returned back from the particle to the EM field*. For now, it sufficient to comment that the obtained result is in accordance with the requirement of a finite collision time τ to have a net transfer of power to the electron(s) in a gas. The event of a collision interrupts the coherent exchange of energy between the EM field and the charged particle by introducing an another channel of energy transport. When the number of collisions is large (within the period of the field), which is equivalent to say for short collision mean time, then this channel contributes significantly to the a net energy flow from the EM to the 'medium', observed as EM wave attenuation. The assumption of a free-electron is equivalent with the assumption of taking the τ infinitely large ($\tau \longrightarrow \infty$).

* A quantitative proof of this statement requires consideration of the radiation generated by a moving charged particle (dipole radiation) developed to later sections

Monochromatic plane waves in vacuum II.

Assume in vacuum an electromagnetic, monochromatic, plane wave, of frequency $f = 3.75 \times 10^{14}$ Hz, with the amplitude of its magnetic component equal to $B_0 = 10^{-4}$ T. Using the Maxwell equations (ME):

(a)
 i. Provide the analytical expression for the magnetic component of the electromagnetic field.
 ii. Calculate the electric component and plot the electric and magnetic fields at the point $\mathbf{r}_1 = (0, 0, 0)$ at times $t_1 = 0$ and $t_2 = \pi/\omega$ sec.

(b)
 i. Provide the expressions for the electromagnetic energy density U and the Poynting vector $\mathbf{S}$ and make a plot of the Poynting vector at $\mathbf{r} = 0$ for t_1 and t_2.
 ii. Prove that the irradiance of the field is given by, $I = c\frac{\varepsilon_0 E_0^2}{2} = c\frac{B_0^2}{2\mu_0}$, where ε_0 and μ_0 are vacuum's electric permitivity and magnetic permeability, respectively.

(c) Calculate in Joules the amount of energy incident normal to a surface of area 1 m^2 in one day.

Solution: (a) Since the EM field is a plane monochromatic field propagating along the z−axis then when the propagation vector $\mathbf{k}$ is along the z−axis,

$$\mathbf{k} = k\hat{z} = \frac{\omega}{c}\hat{z}$$

The angular frequency $\omega = 2\pi f$, wavelength (λ) and the wavenumber $k = 2\pi/\lambda$, of the EM field are calculated as:

$$\begin{aligned}
\lambda &= \frac{c}{f} = 800\text{ nm} = 800 \times 10^{-9}\text{ m},\\
k &= \frac{2\pi}{\lambda} = \frac{2\pi}{800 \times 10^{-9}} = 0.7854 \times 10^{7}\text{ m}^{-1}\\
\omega &= ck = 3 \times 10^{8} \times (0.7854 \times 10^{7}) = 2.3562 \times 10^{15}\text{ rad/sec}
\end{aligned}$$

By assuming a Cartesian coordinate system where the magnetic field lies along the $\hat{y}$ axis, we obtain the following expression:

$$\mathbf{B}(\mathbf{r}, t) = \hat{y} B_0 \cos(kz - \omega t), \quad \omega = ck.$$

For plane monochromatic waves the electric field and magnetic fields differ only in their amplitude vectors $\mathbf{E}_0$ and $\mathbf{B}_0$. Therefore,

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(kz - \omega t).$$

It remains to calculate the amplitude $\mathbf{E}_0$ (direction and magnitude). To this end, since the fields are monochromatic, the algebraic form of the MEs (inverse space MEs) may be used:

$$\mathbf{k} \cdot \mathbf{E}_0 = 0, \qquad \mathbf{k} \cdot \mathbf{B}_0 = 0, \tag{7.11a}$$

$$\mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0, \qquad \mathbf{k} \times \mathbf{B}_0 = -\frac{\omega}{c^2}\mathbf{E}_0. \tag{7.11b}$$

where $c^2 = 1/\mu_0\varepsilon_0$ and $\varepsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{Cb^2/Nm^2}$, $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$.
From the above equations (bottom-left ME) we have:

$$\mathbf{E}_0 = -\frac{c^2}{\omega}(\mathbf{k}\times\mathbf{B}_0) = -\frac{c^2}{\omega}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & k \\ 0 & B_0 & 0 \end{vmatrix} = -\frac{c^2}{\omega}\left[\hat{x}\cdot(-kB_0) - \hat{y}\cdot 0 + \hat{z}\cdot 0\right]$$
$$= \frac{c^2 k}{\omega}B_0\hat{x} = \frac{c^2 k}{ck}B_0\hat{x} = cB_0\hat{x} \Longrightarrow \boxed{\mathbf{E}_0 = cB_0\hat{x}}$$

where the relation $\omega = ck$ was used.We then end up to the following expressions for $\mathbf{E}, \mathbf{B}$:

$$\mathbf{E}(\mathbf{r}, t) = E_0\cos(kz - \omega t)\hat{x}, \qquad \mathbf{B}(\mathbf{r}, t) = B_0\cos(kz - \omega t)\hat{y}$$

with $B_0 = 10^{-4}T, k, \omega$ known and the electric field amplitude E_0 calculated as:

$$E_0 = cB_0 = (3\times 10^8)(10^{-4}) = 3\times 10^4\ \mathrm{V/m}.$$

The fields at the point $\mathbf{r}_1 = (0, 0, 0)$ for times t_1 and t_2 are given as:

$$\mathbf{E}(\mathbf{r}_1, t_1) = E_0\cos(k\cdot 0 - \omega\cdot 0)\hat{x} = cB_0\hat{x}, \qquad \mathbf{B}(\mathbf{r}_1, t_1) \ = B_0\cos(k\cdot 0 - \omega\cdot 0)\hat{x} = B_0\hat{y}$$

While we find for the time $t_2 = \pi/\omega$:

$$\mathbf{E}(\mathbf{r}_1, t_2) = -E_0\hat{x}, \qquad \mathbf{B}(\mathbf{r}_1, t_2) = -B_0\hat{y}.$$

(b) The EM energy density of this field is calculated as:

$$U(\mathbf{r}, t) = \frac{\varepsilon_0}{2}E^2(\mathbf{r}, t) + \frac{1}{2\mu_0}B^2(\mathbf{r}, t) = \frac{\varepsilon_0}{2}E_0^2\cos^2(kz - \omega t) + \frac{1}{2\mu_0}B_0^2\cos^2(kz - \omega t)$$
$$\Longrightarrow \boxed{U(z, t) = \varepsilon_0 E_0^2\cos^2(kz - \omega t)},$$

where the relations $E_0 = cB_0$ and $c\varepsilon_0 = 1/c\mu_0$ were used.

Similarly, the Poynting vector may be calculated using it's definition relation $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_0$ as:

$$\mathbf{S}(z, t) \equiv \frac{1}{\mu_0}(\mathbf{E}_0\times\mathbf{B}_0) = \frac{1}{\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_0\cos(kz - \omega t) & 0 & k \\ 0 & B_0\cos(kz - \omega t) & 0 \end{vmatrix}$$
$$= \frac{1}{\mu_0}\left[\hat{x}\cdot 0 - \hat{y}\cdot 0 + \hat{z}\cdot E_0 B_0\cos^2(kz - \omega t)\right] = \frac{E_0 B_0}{\mu_0}B_0\hat{z} = \frac{E_0}{\mu_0 c}\hat{z}$$
$$\Longrightarrow \boxed{\mathbf{S}(\mathbf{r}, t) = c\varepsilon_0 E_0^2\cos^2(kz - \omega t)\hat{z}},$$

where the relations $E_0 = cB_0$ and $c\varepsilon_0 = 1/c\mu_0$ were used.

The Poynting vector at the point $\mathbf{r}_1$ and at times t_1 and t_2 is then easily found to be:

$$\mathbf{S}(\mathbf{r}_1, t_1) = \mathbf{S}(\mathbf{r}_1, t_2) = c\varepsilon_0 E_0^2\hat{z}.$$

(c) The irradiance of the field is given as the time-averaged value of the Poynting vector amplitude:

$$I = \frac{1}{2}c\varepsilon_0 E_0^2 = 0.5(8.8542\times10^{-12})(3\times10^8)(3\times10^4)^2 = 1.1953\times10^6\ \mathrm{W/m^2} = 1.1953\times10^6\ \mathrm{W/cm^2}.$$

By definition of the irradiance as the radiation power per unit area, the total incident energy on a surface of area $A = 0.1\ \mathrm{m}^2$ in a time interval equal to a day $t_d = 24\times3600$ sec, is calculated as:

$$I = \frac{d\bar{U}}{dtdA} \Longrightarrow d\bar{U}_{EM} = \int IdAdt = I\times A\times t_d = 1.1953\times10^6\times1\times24\times3600 \simeq 1.03\times10^{11}\,J,$$

Monochromatic plane waves III.

A linearly-polarized monochromatic plane wave of wavelength $\lambda = 800$ nm and amplitude $E_0 = 3.635 \times 10^9$ Volt/cm. A measurement of the electric field at time $t = 0$ and at position $\mathbf{r} = (0, 0, 0)$ provides a value equal to E_0.

(a) In an orthogonal coordinate system $Oxyz$ provide the expressions for the electric $\mathbf{E}$ and the magnetic field $\mathbf{B}$ of this plane wave.

(b) Calculate the Poynting vector, the energy and momentum density.

(c) Calculate the average number of photons contained in a cube of dimensions $V = 1$ cm^3 in a region where the EM field is non-zero.

(a) Since the EM field is a monochromatic plane wave propagating freely in an unbounded domain then the $\mathbf{E}, \mathbf{B}$ vectors are necessarily normal to each other and to their propagation direction. Without loss of generality we can choose the orthogonal coordinate system such that the z−axis to be the propagation direction and the electric field pointing along the positive direction of the x−axis.

In this coordinate system the electric field is expressed as:

$$\mathbf{E}(z, t) = \hat{x} E_0 \cos(kz - \omega t)$$

The angular frequency and the wavenumber of the EM field can be calculated as:

$$\begin{aligned}
\lambda &= 800 \text{ nm} = 800 \times 10^{-9} \text{ m}, \\
k &= \frac{2\pi}{\lambda} = \frac{2\pi}{800 \times 10^{-9}} = 7.854 \times 10^6 \text{ m}^{-1} \\
\omega &= ck = 3 \times 10^8 \times (0.7854 \times 10^7) = 2.3562 \times 10^{15} \text{ rad/sec}
\end{aligned}$$

Note in addition that the following relations hold in the vacuumn:

$$c^2 = \frac{1}{\mu_0 \varepsilon_0} = 299792458 \text{ m/s} \tag{7.12}$$

$$\varepsilon_0 = 8.8542 \times 10^{-12} \text{ Cb}^2/\text{Nm}^2, \qquad \mu_0 = 4\pi \times 10^{-7} \text{N/A}^2 \tag{7.13}$$

Since it is plane wave the algebraic MEs can be employed to calculate the magnetic component as:

$$\begin{aligned}
\mathbf{B}_0 &= \frac{1}{\omega}(\mathbf{k} \times \mathbf{E}_0) = \frac{1}{\omega} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & k \\ E_0 \cos(kz - \omega t) & 0 & 0 \end{vmatrix} \\
&= \frac{1}{\omega} \left[\hat{x} \cdot (0) - \hat{y}(0 - E_0 \cos(kz - \omega t)) + \hat{z} \cdot (0) \right] = \hat{y} B_0 \cos(kz - \omega t),
\end{aligned} \tag{7.14}$$

with the magnetic field amplitude B_0 given by:

$$B_0 = \frac{E_0}{\sqrt{\mu_0 \varepsilon_0}} = \frac{3.635 \times 10^7}{\sqrt{4\pi \times 10^{-7} \times 8.8542 \times 10^{-12}}} = 1.2125 \text{ T}. \tag{7.15}$$

The expression for the monochromatic magnetic and the electric components $(\mathbf{E}, \mathbf{B})$:

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(kz - \omega t)\hat{x}, \qquad \mathbf{B}(\mathbf{r}, t) = B_0 \cos(kz - \omega t)\hat{y}$$

(b) The EM energy density is given by:

$$\boxed{U(\mathbf{r}, t) = \frac{\epsilon_0}{2} E^2(z, t) + \frac{1}{2\mu_0} B^2(z, t) = \epsilon_0 E^2(z, t) = \varepsilon_0 E_0^2 \cos^2(kz - \omega t)}$$

where the relations $E_0 = cB_0$ and $c\varepsilon_0 = 1/c\mu_0$ have been used. The Poynting vector is calculated as:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_0 \cos(kz - \omega t) & 0 & 0 \\ 0 & B_0 \cos(kz - \omega t) & 0 \end{vmatrix}$$

$$= \frac{1}{\mu_0} \left[\hat{x} \cdot 0 - \hat{y} \cdot 0 + \hat{z} \cdot E_0 B_0 \cos^2(kz - \omega t) \right] = \frac{E_0 B_0}{\mu_0} B_0 \hat{z} = \frac{E_0^2}{\mu_0 c} \hat{z}$$

$$\Longrightarrow \boxed{\mathbf{S}(\mathbf{r}, t) = c\varepsilon_0 E_0^2 \cos^2(kz - \omega t)\hat{z}}, \tag{7.16}$$

Substituting the values of the light speed and E_0 we get for the magnitude of the Poynting vector:

$$S = c\varepsilon_0 E_0^2 = (3 \times 10^8) \times (8.8542 \times 10^{-12}) \times (3.635 \times 10^9)^2$$
$$= 3.509 \times 10^{16}\ \mathrm{W/cm^2}.$$

(c) The link between the photon's average number, which is a quantum mechanical concept and the EM energy density is the Einstein's relation $E_\gamma = \hbar\omega$. First we have to calculate the amount of EM energy per unit area (m) per unit time (sec) which is given by the intensity of the field:

The *irradiance* of the EM field is given as the time-average of the amplitude of the Poynting vector. This has been shown in the previous chapter to be:

$$I = \frac{1}{2} c\varepsilon_0 E_0^2 \equiv \frac{d\bar{U}}{dt dA} \qquad \Longrightarrow I = 1.7545 \times 10^{16}\ \mathrm{W/cm^2}.$$

Therefore we can find the total EM energy (U_{tot}) carried by the plane wave in the time interval required to cover a distance, parallel to z-axis, equal to $d = 1$ cm, as

$$dt = \frac{d}{c} = \frac{1}{3 \times 10^{10}} = 3.3333\ \mathrm{ns}$$

Then the EM energy contained in the cube of dimensions $A \times d = d^2 \times d$ is calculated as:

$$U_{tot} = \int d\bar{U}_{EM} = \int I dA dt = I \times \int_A dA \times dt$$
$$= I \times A \times dt = 1.7545 \times 10^{16} \times 1 \times 3.333 \times 10^{-9} = 5.8483 \times 10^7 J$$

Since the radiation is monochromatic to calculate the number of photons incident to this area we need to divide the total energy U_{tot} by the amount of energy carried from each individual photon:

$$N_\gamma = \frac{U_{tot}}{E_\gamma} = \frac{U_{tot}}{\hbar\omega} \simeq \frac{5.843 \times 10^5}{2.5 \times 10^{-19}} \simeq 2.34 \times 10^{26} \text{ photons,}$$

where we have used that $\hbar \sim 1.054 \times 10^{-34}$ Jsec

Monochromatic plane waves in vacuum IV

Assume in vacuum an electromagnetic, monochromatic plane wave, travelling along the z–axis of an $Oxyz$ Cartesian coordinate system, with its magnetic field component expressed as,

$$\mathbf{B}(z,t) = \hat{e}\mathbf{B}_0 \sin(kz - \omega t), \qquad \hat{e} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y}\sqrt{3}),$$

where $\omega = 10^{14}$ Hz is its angular frequency, k is its wavenumber and $\hat{x}, \hat{y}$ the unit vectors along the $x, y-$ axes of the given coordinate system, respectively. The amplitude of the electric component of the field is equal to $E_0 = 5.142 \times 10^7$ V/cm.

(a) Calculate the electric component, $\mathbf{E}(z,t)$, of the field. Verify that $\mathbf{E} \cdot \mathbf{B} = 0$ and that $\mathbf{E} \cdot \mathbf{k} = 0$. Plot them at $z = 0$ and $t = -\pi/(4\omega)$.
(b) Starting from the definition of the Poynting vector show that the irradiance of the field $I \equiv \langle S \rangle$ (the brackets denote time-averaging) is given by, $I = cB_0^2/(2\mu_0)$ where μ_0 are vacuum's magnetic permeability.
(c) Calculate the averaged power, passing through a flat surface, of area $A = 10$ cm^2 with its normal along the direction of the unit vector $\hat{n} = (\hat{y} + \hat{z})/\sqrt{2}$.

Solution: (a) Since the EM field is a monochromatic plane wave (PW) propagating along the z–axis then the propagation vector $\mathbf{k}$ also lies along the z–axis,

$$\mathbf{k} = k\hat{z} = \frac{\omega}{c}\hat{z}.$$

Since the field is monochromatic, to find the electric component we can use the algebraic form of the MEs (inverse space MEs):

$$\mathbf{k} \times \mathbf{B}(z,t) = -\frac{\omega}{c^2}\mathbf{E}(z,t) \quad \Longrightarrow \quad \mathbf{E}(z,t) = -\frac{c^2}{\omega}\mathbf{k} \times \mathbf{B}(z,t)$$

where $B(z,t) = B_0 \sin(kz - \omega t)$ and $c^2 = 1/\mu_0\varepsilon_0$ and $\varepsilon_0 = 8.8542 \times 10^{-12}$ Cb^2/Nm^2, $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 From the above equations (bottom-left ME) we have:

$$\begin{aligned}
\mathbf{E}(z,t) &= -\frac{c^2}{\omega}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & k \\ B(z,t)/\sqrt{2} & B(z,t)\sqrt{3/2} & 0 \end{vmatrix} \\
&= -\frac{c^2}{\omega}\left[\hat{x}(-B(z,t)k/\sqrt{2}) - \hat{y}(-B(z,t)k\sqrt{3/2}) + \hat{z}\cdot 0\right] \\
&= \frac{c^2 k}{\omega}B(z,t)\frac{1}{\sqrt{2}}(\hat{x}\sqrt{3} - \hat{y}) = cB_0\frac{1}{\sqrt{2}}(\hat{x}\sqrt{3} - \hat{y})\sin(kz - \omega t),
\end{aligned}$$

where the relation $\omega = ck$ was used. To summarize we have for $\mathbf{E}, \mathbf{B}$:

$$\mathbf{B}(\mathbf{r},t) = \hat{e}\frac{E_0}{c}\sin(kz - \omega t) \qquad \mathbf{E}(\mathbf{r},t) = \hat{e}' E_0 \sin(kz - \omega t),$$

with $E_0 = 5.142 \times 10^7$ V/cm, $\omega = 10^{14}$ Hz and the EM speed in vacuum, $c = 3 \times 10^8$ m/s known. In vacuum, the wavenumber k is related with the angular frequency by,

$$k = \frac{\omega}{c} = \frac{10^{14}}{3 \times 10^8} = 0.333 \times 10^6 \text{ m}^{-1}$$

. The unit vectors define the directions of the electric and magnetic components,

$$\hat{e} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y}\sqrt{3}) \qquad \hat{e}' = \frac{1}{\sqrt{2}}(\hat{x}\sqrt{3} - \hat{y}).$$

Given that

$$\hat{e} \cdot \hat{e}' = \sqrt{1}2(\sqrt{3} - \sqrt{3}) = 0$$

and $\hat{e} \cdot \mathbf{k} = \hat{e}' \cdot \mathbf{k} = 0$ it is ensured that $\underline{\mathbf{E} \cdot \mathbf{B} = \mathbf{E} \cdot k = \mathbf{B} \cdot \mathbf{k} = 0}$.

For $z = 0$ and $t = -\pi/(4\omega)$ we have,

$$\mathbf{E}(0, -\frac{\pi}{4\omega}) = \hat{e}' E_0 \sin(0 + \frac{\pi}{4\omega}\omega) = \hat{e}' E_0 \sin(\frac{\pi}{4}) = \underline{\frac{E_0}{2}(\hat{x}\sqrt{3} - \hat{y})}.$$

where $\sin(\pi/4) = 1/\sqrt{2}$ and $E_0 = 5.142 \times 10^7$ V/cm $= 5.142 \times 10^9$ V/m in SI units. Similarly we find for the magnetic component:

$$\underline{\mathbf{B}(0, -\frac{\pi}{4\omega}) = \frac{E_0}{2c}(\hat{x} + \hat{y}\sqrt{3})},$$

with $E_0/c = 5.142 \times 10^9/(3 \times 10^8) = 17.14$ T.

(b) The Poynting vector may be calculated using it's definition relation $\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu_0$ as:

$$\begin{aligned} \mathbf{S}(z,t) \equiv & \frac{1}{\mu_0}(\mathbf{E}(z,t) \times \mathbf{B}(z,t)) = \frac{1}{\mu_0}(\hat{e}' E_0 \sin(kz - \omega t)) \times (\hat{e}\frac{E_0}{c} \sin(kz - \omega t) = \\ = & \frac{1}{\mu_0}\frac{E_0^2}{c} \sin^2(kz - \omega t) \underbrace{(\hat{e}' \times \hat{e})}_{\hat{z}} = \frac{E_0^2}{\mu_0 c} \sin^2(kz - \omega t)\hat{z} = \hat{z}S(z,t). \end{aligned} \tag{7.17}$$

Integrating over a period the magnitude of the Poynting vector $S(z,t)$ we get,

$$I = \langle S(z,t) \rangle \equiv \frac{1}{T}\int_0^T dt S(z,t) = \underbrace{\frac{E_0^2}{\mu_0 c}}_{B_0/\mu_0} \left[\underbrace{\frac{1}{T}\int_0^T dt \sin^2(kz - \omega t)}_{=\frac{1}{2}} \right] = \underline{\frac{cB_0^2}{2\mu_0}}$$

where $T = 2\pi/\omega$ and $E_0 = cB_0$ were used.

(c) Using Eq. (7.17) the averaged electromagnetic power crossing a surface $da = \hat{n}da$ is calculated by first calculating the power density*,

$$P_{\hat{n}} = \langle \hat{n} \cdot \mathbf{S}(z,t) \rangle = \langle \hat{n} \cdot \hat{z}S(z,t) \rangle = \hat{n} \cdot \hat{z}\langle S(z,t) \rangle = [\frac{1}{\sqrt{2}}(\hat{y} + \hat{z}) \cdot \hat{z}] \times \frac{cB_0^2}{2\mu_0} = \frac{cB_0^2}{2\sqrt{2}\mu_0},$$

* Both $\hat{n}$ and $\hat{z}$ are constant in time.

since $\hat{z} \cdot \hat{z} = 1$ and $\hat{z} \cdot \hat{y} = 0$. Note that the average power density is independent on time and its actual position in the field. All what matters (consistent with what is physically observed) is its relative direction with the Poynting vector $\hat{k}$. Then the power crossing the surface, A, is,

$$P_A = \int_A da P_{\hat{n}} = \frac{cB_0^2}{2\mu_0} \underbrace{\int_A da}_{=A} = A\frac{cB_0^2}{2\sqrt{2}\mu_0} = A\frac{I}{\sqrt{2}}.$$

7.2 Dielectrics and Fresnel relations

ME boundary conditons

Assume the differential form of the Maxwell's Equations (MEs) in a dielectric medium, characterized by ε_1, μ_1 electric and magnetic constants. Now assume a second dielectric medium, with constants, ε_2, μ_2. The two media are separated with an uncharged, flat, interface boundary plane. Prove the boundary conditions:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, \qquad B_1^{\perp} = B_2^{\perp}$$

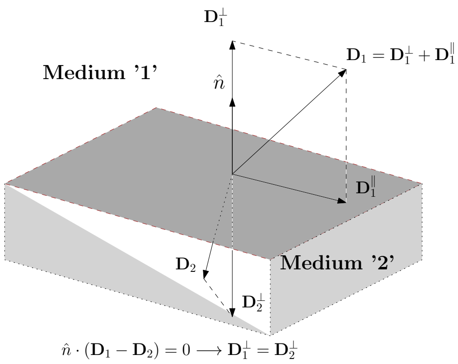

Figure 7.1: The normal components of $\mathbf{D}_1, \mathbf{D}_2$ to the interface plane should be equal in magnitude, namely, $\epsilon_1 E_1 = \epsilon_2 E_2$

Solution: Considering the integral form of the ME's equations, (2.1a) (for $\rho = 0$) one can perform the surface integral for the elementary cuboid of Fig. (7.1) breaking it to 6 surface integrals, 4 for the side rectangles as well as the upper and the bottom rectangles, with the electric field taken constant on the respective surfaces. Assuming that the z−axis of a Cartesian coordinate system aligns with the $\hat{n}$ direction, $\hat{z} = \hat{n}$, the elementary volume of the cuboid is given by $\Delta V = \Delta x \Delta y \Delta z$. Then, by assuming that $\Delta z \to 0$, we have:

$$\int \mathbf{da} \cdot \mathbf{D}(x, y, z \to 0, t) = \int \mathbf{da}_1 \cdot \mathbf{D}_1 + \int \mathbf{da}_2 \cdot \mathbf{D}_2 + \lim_{\Delta z \to 0} \sum_{i=1-4} \int \mathbf{da}_i \cdot \mathbf{D}_i$$
$$= \hat{z} \cdot \mathbf{D}_1 - \hat{z} \cdot \mathbf{D}_2 = D_1^{\perp} - D_2^{\perp} = \boxed{\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0}.$$

We end up to $\boxed{B_1^{\perp} = B_2^{\perp}}$, by replacing in the above $\mathbf{D}$ by $\mathbf{B}$.

Normal incidence Fresnel Equations

Assume an electromagnetic plane wave of irradiance $I_1 = 1.2 \times 10^{13}$ W/cm^2 propagating in water (medium '1') surrounded by a glass-like material (medium '2'). Suppose the electromagnetic wave hits the interface boundary of these two media with its propagation wavevector $\mathbf{k}_1$ normal to the interface plane of the two media (water-glass). For a wavelength of the wave equal to $\lambda_1 = 800$ nm and at around room temperature 20^o the propagation speed of the light in the water is circa 3/4 of the light speed in the vacuum, c. The corresponding fraction in glass is about 2/3, so that $v_2 = 2c/3$. Both the glass and water are non-magnetic materials (meaning that we can take $\mu_w = \mu_g = \mu_0$), where μ_w and μ_g the magnetic permeabilities of the water and glass respectively.

(a) Assume a Cartesian coordinate system $Oxyz$, with the glass occupying the $x > 0$ region, yz the interface plane between the water and the glass (see figure 1) and the electric component of the incident plane wave in the water expressed as, $\mathbf{E}_i(\mathbf{r}, t) = \hat{y}E_{0i}\cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t)$,

i. Calculate the reflection and transmission angles θ_r and θ_t by applying the reflection and Snell's laws.

ii. Provide the corresponding expressions for the reflected ($\mathbf{E}_r$) and transmitted waves ($\mathbf{E}_t$).

(b)

i. Show that the reflected power can be found as $R_\perp = r_\perp^2$.

ii. Calculate the reflected ($R_\perp$) and the transmitted ($T_\perp$) power as the wave passes from water to the glass.

(c) Now consider the opposite case, where the medium '1' is the glass and the medium '2' is the water. Calculate again the percentage of the light power that will be reflected and transmitted.

Solution: (a) We assume a coordinate system $Oxyz$ where the axes $\hat{y}$ and $\hat{z}$ define the interface boundary of the two media and the $\hat{x}$ axis is normal to it. According to this for normal incidence we can express the wavevector $\mathbf{k}_1$ as $\mathbf{k}_1 = k_1\hat{x}$, to re-express the incident field as,

$$\mathbf{E}(x, t) = \hat{y}E_{0i}\cos(k_1 x - \omega t).$$

In the above $k_1 = 2\pi/\lambda_1 = 7.854 \times 10^6/\mathrm{m}^{-1}$, $\omega = v_1 k_1 = (3c/4) \times k_1 = 1.767 \times 10^{15}$ Hz. The amplitude of the electric field can be found from the relation for the intensity (irradiance):

$$\begin{aligned} I_i &= v_1\epsilon_1\frac{E_{0i}^2}{2} = \frac{1}{v_1\mu_0}\frac{E_{0i}^2}{2} \Longrightarrow E_{0i} = \sqrt{2\mu_0 v_1 I_1} = \sqrt{2\mu_0 v_1 I_1} \\ &= \sqrt{2 \times 4\pi \times 10^{-7} \times 0.75 \times 3 \times 10^8 \times 1.2 \times 10^17} = ... = 8.23 \times 10^9 \text{ V/m}. \end{aligned}$$

In the above we have used $v_1 = 1/\sqrt{\mu_1\epsilon_1} = 1/\sqrt{\mu_0\epsilon_1} \longrightarrow \epsilon_1 v_1 = 1/(\mu_0 v_1)$.

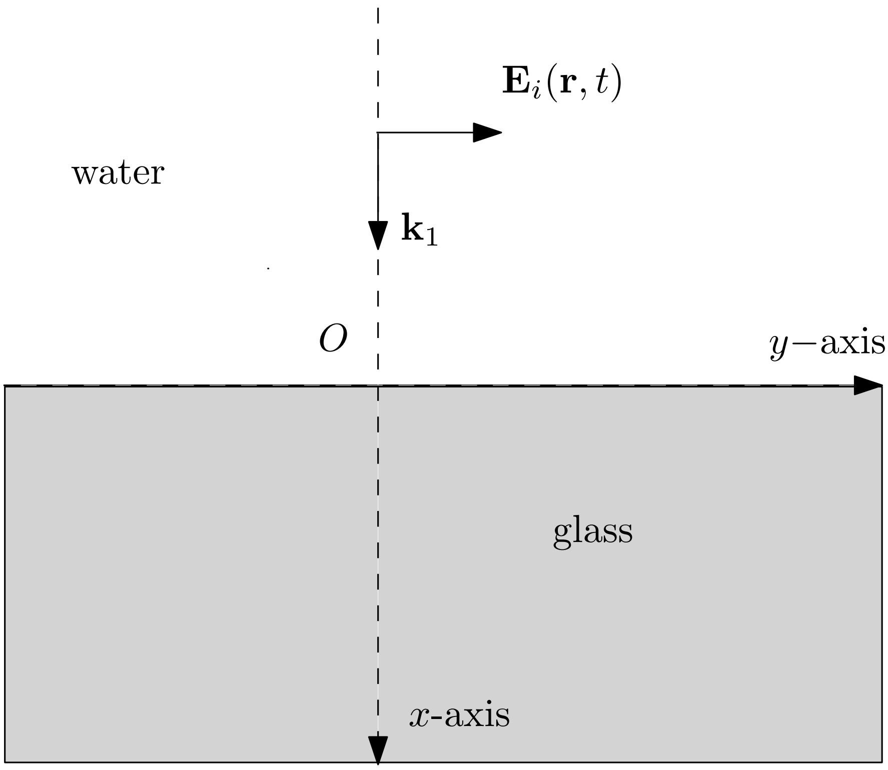

Figure 7.2: Normal incidence

Next, using the reflection and Snell's laws, given the incidence angle $\theta_1 \equiv \theta_i$, we calculate the reflection and transmission angles, θ_r and $\theta_2 \equiv \theta_t$, respectively, as,

$$\begin{aligned}\theta_r &= \qquad \theta_i \longrightarrow \theta_r = 0. \\ \sin\theta_t &= \frac{n_i}{n_t}\sin\theta_i = \frac{n_1}{n_2}\sin\theta_1 = \frac{v_2}{v_1}\sin\theta_i = \frac{8}{9}\sin 0 = 0 \longrightarrow \theta_t = 0,\end{aligned}$$

In the above since $n_i \equiv c\sqrt{\mu_0\epsilon_i} = c/v_i$, $i = 1, 2$ we have used the following relation,

$$\frac{v_1}{v_2} = \frac{n_1}{n_2}.$$

In this case the reflected and the transmitted fields are given as,

$$\mathbf{E}_r(x,t) = \hat{y}E_{0r}\cos(k_1x + \omega t) = \hat{y}rE_{0i}\cos(k_1x + \omega t) \tag{7.18}$$

$$\mathbf{E}_t(x,t) = \hat{y}E_{0t}\cos(k_2x - \omega t) = \hat{y}tE_{0i}\cos(k_2x - \omega t) \tag{7.19}$$

where $k_i = \omega/v_i, i = 1, 2$, namely, $k_1 = 4\omega/3c$ and $k_2 = 3\omega/2c$. The amplitude coefficients r and r can be found by substituting the values $\theta_1 = \theta_2 = 0$ in the Fresnel equations (given in the appendix). The above choice of the incident and reflected electric fields dictate that the Fresnel equations that should be used are those that corresponds to the perpedicular plane-of-incidence case $\mathbf{r}_\perp$ and $t_\perp$:

$$r_\perp \equiv \left[\frac{E_{0r}}{E_{0i}}\right]_\perp = \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_1\cos\theta_2}, \qquad t_\perp = \left[\frac{E_{0r}}{E_{0i}}\right]_\perp = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2}.$$

We now note that, since $Z_i \equiv \sqrt{\mu_0/\epsilon_i}$, $i = 1, 2$ the following relation holds,

$$\frac{Z_1}{Z_2} = \frac{v_1}{v_2}.$$

Using the above relations we find for the reflection and transmitted amplitudes in the normal incidence case $r \equiv r_\perp(\theta_i = 0)$ $t \equiv \mathbf{t}_\perp(\theta_i = 0)$:

$$r = \frac{E_{0r}}{E_{oi}} = \frac{Z_2 - Z_1}{Z_1 + Z_2} = \frac{1 - Z_1/Z_2}{1 + Z_2/Z_1} = \frac{1 - v_1/v_2}{1 + v_1/v_2} = \frac{1 - 9/8}{1 + 9/8} = -\frac{1}{17} \quad (7.20)$$

$$t = \frac{E_{0t}}{E_{oi}} = \frac{2Z_2}{Z_1 + Z_2} = \frac{2}{1 + Z_1/Z_2} = \frac{2}{1 + v_1/v_2} = \frac{2}{1 + 9/8} = \frac{16}{17}. \quad (7.21)$$

(b) The *power reflection coefficient* (R)) and the *power transmission coefficient* (T) are defined as:

$$R = \left| \frac{\hat{n} \cdot \langle \mathbf{S}_r \rangle}{\hat{n} \cdot \langle \mathbf{S}_i \rangle} \right|, \qquad T = \left| \frac{\hat{n} \cdot \langle \mathbf{S}_t \rangle}{\hat{n} \cdot \langle \mathbf{S}_i \rangle} \right|,$$

where $\langle S_i \rangle, \langle S_r \rangle$ and $\langle S_t \rangle$ are the time-average of the corresponding Poynting vector and $\hat{n}$ is the unit vector normal to the interface plane (in the particular case it is chosen $\hat{n} = \hat{x}$. Since the Poynting vector is equal to $\mathbf{S}_j = \hat{k}_j \epsilon E_0^2/2$, $j = i, r$ the power reflection coefficient is calculated as,

$$R = \left| \frac{\hat{x} \cdot (-\hat{x}) \epsilon_1 E_{0r}^2/2}{\hat{x} \cdot \hat{x} \epsilon_1 E_{0i}^2/2} \right| = \left| -\frac{E_{0r}^2}{E_{0i}^2} \right| = r^2 = \frac{1}{17^2}.$$

since $\hat{k}_i = k_1 \hat{x}$ and $\hat{k}_r = -\hat{x}$.

The transmission power coefficient can be found either by employing the definition or by using the energy-conversing relation $T = 1 - R = 1 - 1/17^2 =$ ....

(c) By inspection of the expressions of the power coefficients we can see that they are unaffected by this change and as such the calculated values above remain the same.

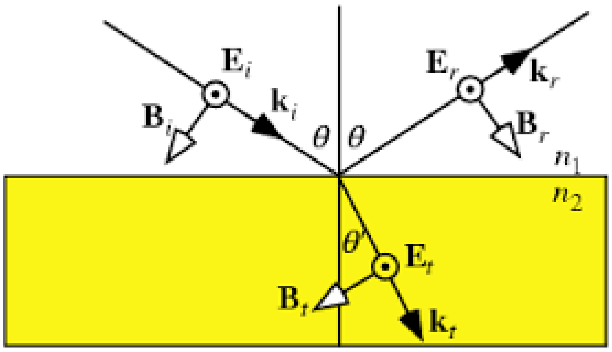

Figure 7.3: Sketch for the s-polarized ($\mathbf{E}_i \perp$ PoI) incidence of an EM wave on a infinitely flat interface boundary between two dielectrics. In the text the refraction and Snell laws that relate the incidence and reflections angles as $\theta_i = \theta_r = \theta$ and $n_1 \sin\theta = n_2 \sin\theta'$, where $\theta_t = \theta'$.

Fresnel Equations for dielectric media

Assume a monochromatic plane wave (frequency ω_i) propagating in the dielectric medium $'i'$ ($\varepsilon_i, \mu_i, n_i = c\sqrt{\mu_i \varepsilon_i}$) incident on dielectric medium $'t'$ ($\varepsilon_t, \mu_t, n_t = c\sqrt{\mu_t \varepsilon_t}$). The interface boundary surface, S, between the media is flat. The angle of incidence with the normal to the interface boundary is θ_i. If the electric field amplitude of the incident field is $\mathbf{E}_{0i}$ and *parallel* to the interface boundary, S, and by making use of the Maxwell boundary conditions answer the below questions:

(a) Prove that the frequencies of the reflected and transmitted waves are equal to ω_i. Prove the reflection ($\theta_r = \theta_t$) and Snell's kinimatic laws ($n_i \sin\theta_i = n_t \sin\theta_t$), where θ_t is the transmission angle.
(b) Prove the Fresnel equations for the reflection and transmitted coefficients amplitudes.

Solution: (a) The *incident*, *reflected* and *transmitted* electric fields defined as :

$$\mathbf{E}_i(\mathbf{r}, t) = \mathbf{E}_{0i} \cos(\mathbf{k}_i \cdot \mathbf{r} - \omega_i t), \qquad \text{incident wave} \tag{7.22}$$
$$\mathbf{E}_r(\mathbf{r}, t) = \mathbf{E}_{0r} \cos(\mathbf{k}_r \cdot \mathbf{r} - \omega_r t) \qquad \text{reflected wave} \tag{7.23}$$
$$\mathbf{E}_t(\mathbf{r}, t) = \mathbf{E}_{0t} \cos(\mathbf{k}_t \cdot \mathbf{r} - \omega_t t) \qquad \text{transmitted wave} \tag{7.24}$$

with $\mathbf{E}_{0i}, \mathbf{E}_{0r}$ and $\mathbf{E}_{0t}$ being constant vectors, while $\omega_i = v_i k_i$, $\omega_r = v_r k_r$ and $\omega_t = v_t k_t$ (see caption in Fig (4.2)).

We also have for the fields in the two media

$$\mathbf{E}_1(\mathbf{r}, t) = \mathbf{E}_i(\mathbf{r}, t) + \mathbf{E}_r(\mathbf{r}, t), \qquad \text{medium '1'},$$
$$\mathbf{E}_2(\mathbf{r}, t) = \mathbf{E}_t(\mathbf{r}, t), \qquad \text{medium '2'}$$

A Cartesian coordinate system is placed with the interface boundary between the two media to lie in the xz–plane. Then the y–axis is perpendicular to the interface plane. We define as $0 \le \theta_i, \theta_r, \theta_t \le \pi/2$ the angles of the wavevectors $\mathbf{k}_i, \mathbf{k}_r, \mathbf{k}_t$, respectively, with the y–axis.

The MEs boundary conditions for the parallel components of the electric fields [Eq. (5.15c)] at the *interface plane* ($y = 0$) of two media '1' and '2' at all instants, are as below:

$$E_2^{\|}|_{y=0} = E_1^{\|}|_{y=0} \Longrightarrow$$
$$\Longrightarrow E_t^{\|}(\mathbf{r},t)|_{y=0} = E_i^{\|}(\mathbf{r},t)|_{y=0} + E_r^{\|}(\mathbf{r},t)|_{y=0}$$
$$\Longrightarrow E_{t0}^{\|}\cos(\mathbf{k}_t\cdot\mathbf{r}-\omega_t t)|_{y=0} = E_{i0}^{\|}\cos(\mathbf{k}_i\cdot\mathbf{r}-\omega_i t)|_{y=0} + E_{r0}^{\|}\cos(\mathbf{k}_r\cdot\mathbf{r}-\omega_r t)|_{y=0}$$
$$\Longrightarrow (\mathbf{k}_t\cdot\mathbf{r}-\omega_t t)|_{y=0} = (\mathbf{k}_i\cdot\mathbf{r}-\omega_i t)|_{y=0} = (\mathbf{k}_r\cdot\mathbf{r}-\omega_r t)|_{y=0}$$

From the last equalities we conclude the separate conditions by applying it twice for $\mathbf{r} = 0$ and for $t = 0$:

- $\mathbf{r} = (x, y, z) = 0$:

$$\omega_i t = \omega_r t = \omega_t t \qquad \Longrightarrow \boxed{\omega_r = \omega_i = \omega_t} \qquad \textit{frequency law}$$

- $t = 0$:

$$(\mathbf{k}_t\cdot\mathbf{r})|_{y=0} = (\mathbf{k}_i\cdot\mathbf{r})|_{y=0} = (\mathbf{k}_r\cdot\mathbf{r})|_{y=0} \Longrightarrow xk_{tx} = xk_{ix} = xk_{rx}$$
$$\Longrightarrow k_t\sin\theta_t = k_i\sin\theta_i = k_r\sin\theta_r$$

Finally we have from the above equations, since $k_i = k_r$:

$$k_i\sin\theta_i = k_r\sin\theta_r \qquad \Longrightarrow \boxed{\theta_i = \theta_r} \qquad \textit{Refraction law}$$
$$k_i\sin\theta_i = k_t\sin\theta_t \qquad \Longrightarrow \boxed{n_i\sin\theta_i = n_t\sin\theta_t} \qquad \textit{Snell's law,}$$

where n_i, n_t are the refraction indices of the two media,

$$n_i = \frac{ck_i}{\omega} = \frac{c}{v_i}, \qquad n_t = \frac{ck_t}{\omega} = \frac{c}{v_t}.$$

and ω defined as $\omega = \omega_t = \omega_i$.

(b) In the present case, only the $z-$component of the electric field are non-zero. We'll need the $x-$components of the magnetic field B_{ix}, B_{rx}, B_{tx}. From standard trigronometry we have:

$$B_i^{||} = -B_{0i}\cos\theta_i \quad B_i^{||} = B_{0r}\cos\theta_r \quad B_t^{||} = -B_{0t}\cos\theta_t$$

In the above we used the fact that the parallel component of $\mathbf{B}$ to the interface plane is the component along the $x-$axis. In other words $\mathbf{B}_i^{\|} \equiv B_{ix}$,$\mathbf{B}_r^{\|} \equiv B_{rx}$,$\mathbf{B}_t^{\|} \equiv B_{tx}$.

Next step is to calculate the amplitudes of the reflected (E_{r0}) and transmitted (E_{t0}) electric field in terms of the amplitude of the incident field E_{i0}. Given that the uknown amplitudes are two (E_{r0}, E_{t0}) we need to formulate two equations that include these amplitudes. To this end, again the boundary conditions for the electric component will be recalled (at time $t = 0$ since these relations hold at all times):

$$E_1^{\|}(0,t) = E_2^{\|}(0,t) \Longrightarrow E_{i0}\cos(\omega t) + E_{r0}\cos(\omega t) = E_{t0}\cos(\omega t) \Longrightarrow \boxed{E_{i0} + E_{r0} = E_{t0}}$$

where the *frequency law* ($\omega_i = \omega_r = \omega_t$) was used. For the magnetic components we have:

$$H_1^{\parallel}(0,t) = H_2^{\parallel}(0,t) \Longrightarrow \frac{B_1^{\parallel}(0,t)}{\mu_1} = \frac{B_2^{\parallel}(0,t)}{\mu_2} \Longrightarrow \boxed{-\frac{B_{i0}\cos\theta_1}{\mu_1} + \frac{B_{r0}\cos\theta_1}{\mu_1} = -\frac{B_{t0}\cos\theta_2}{\mu_2}}.$$

The latter expression, by substituting the expressions for $\mathbf{B}_{ix}, \mathbf{B}_{rx}, \mathbf{B}_{tx}$ with those we derived earlier (in terms of the incident, transmission and reflection angles) is rewritten as:

From the above relations and considering that $B_{j0} = E_{j0}/v_j$ for $j = i, r, t$ and $Z_i = \mu_i v_i, i = 1, 2$t we get:

$$E_{i0} + E_{r0} = E_{t0}, \qquad -\frac{E_{i0}\cos\theta_1}{v_1\mu_1} + \frac{E_{r0}\cos\theta_1}{v_1\mu_1} = -\frac{E_{t0}\cos\theta_2}{v_2\mu_2}$$

The above system of equations is a 2×2 algebraic system for the unknowns E_{r0} and E_{t0} and is solved to provide:

$$\underline{E_{0r} = \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_1\cos\theta_2}E_{0i}, \qquad E_{0t} = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2}E_{0i}.}$$

where $Z_i = Z_r = \sqrt{\mu_1/\varepsilon_1}$ and $Z_r = \sqrt{\mu_2/\varepsilon_2}$. Also note that $v_i = v_r = v_1$ and $v_t = v_2$.

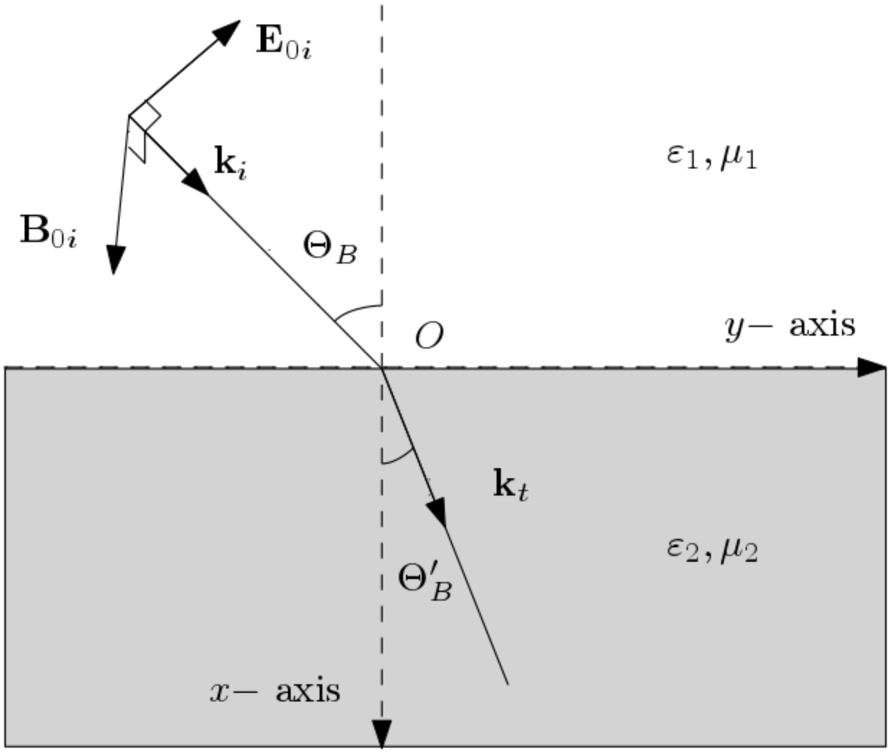

Figure 7.4: Question 2

Brewster angle incidence

Assume an electromagnetic plane wave with electric field amplitude $E_{0i} = 10^7$ V/cm propagating in air (medium '1') incident on glass (medium '2'). The wavelength of the wave in air is $\lambda_i = 800$ nm. We define the incidence as $p-$ *wave incidence* when the electric field component of the field is parallel to the plane of incidence ($xy-$ plane in figure 1).

[(a)]

$n_{glass} = 1.56$ and $n_{air} = 1$ are the refraction indices of glass and air, respectively. Air and glass are non-magnetic materials ($\mu_{air} = \mu_{glass} = \mu_0$). First, using the general Fresnel equations show that the Brewster angle, Θ_B and the transmission angle Θ'_B are given (in rad units) as, $\tan\Theta_B = n_{glass}/n_{air}$ and $\Theta'_B = \pi/2 - \Theta_B$.

[(b)]

Calculate the electric components of the incident and the transmitted waves, if the incident electric field is given as $\mathbf{E}_i = \hat{e}E_{0i}\cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t)$. Calculate the transmission power coefficient. [(c)]

How do the transmission properties change if the incident wave hits the interface boundary from glass towards air? (Recalculate Brewster angle, Θ_B, transmission angle, Θ'_B and transmission power coefficient)

Solution: (a) Incident and reflected electric fields are related through the Fresnel equations. Zero reflection is allowed only for the parallel to the plane-of-incidence case $\mathbf{r}_{||}$:

$$r_{\|} \equiv \left[\frac{E_{0r}}{E_{0i}}\right]_{\perp} = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2},$$

Requiring, $r_{\|} = 0$ at the Brewster angle, $\theta_1 = \Theta_B$ is equivalent to set $n_2\cos\Theta_B - n_1\cos\Theta'_B = 0$. At the same time refractive indices and incident and transmission angles are related through the Snell's law, $n_1\sin\Theta_B = n_2\sin\Theta'_B$. We thus have two equations and two unknowns Θ_B, Θ'_B. This system can be solved relatively straightforward. Squaring the two equations, followed by some trigonometric manipulations we can eliminate the angle θ'_B,

$$n_2^2\cos^2\Theta_B = n_1^2\cos^2\Theta'_B \xrightarrow[1+\tan^2\Theta_B = 1/\cos^2\Theta_B]{n_1^2\sin^2\Theta_B = n_2^2\sin^2\Theta'_B} \tan\Theta_B = \frac{n_2}{n_1}$$

$$\Theta_B = \tan^{-1}\left(\frac{n_{glass}}{n_{air}}\right) \underline{\simeq 57.34^o}$$

From $\tan\theta_B = n_1/n_2$ and the Snell's law we have,

$$\cos(\Theta_B) = \sin\Theta'_B \Longrightarrow \underline{\Theta'_B = \frac{\pi}{2} - \Theta_B \simeq 32.66^o}.$$

(b) We assume a coordinate system $Oxyz$ where the axes $\hat{y}$ and $\hat{z}$ define the interface boundary of the two media and the $\hat{x}$ axis is normal to it. According to this for the Brewster angle incidence we can express the wavevector $\mathbf{k}_1$ as $\mathbf{k}_1 = k_1\cos\Theta_B\hat{x} + k_1\sin\Theta_b\hat{y}$. We also express the amplitude $\mathbf{E}_0$ as,

$$\mathbf{E}_{0i} = -E_{0i}\sin\Theta_B\hat{x} + E_{0i}\cos\Theta_B\hat{y}$$

We can then write for the incident field, $\mathbf{E}_i(x,t) = \mathbf{E}_{0i}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$ (with $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$ as

$$\mathbf{E}_i(x,t) = \hat{e}_i E_{0i}\cos(xk_1\cos\Theta_B + yk_1\sin\Theta_B - \omega t),$$
$$\hat{e}_i = -\sin\Theta_B\hat{x} + \cos\Theta_B\hat{y}$$

Along similar lines, the transmitted field,$\mathbf{E}_t(x,t)$, is written in terms of the transmission angle, Θ'_B as,

$$\mathbf{E}_t(x,t) = \hat{e}_t E_{0t}\cos(xk_2\cos\Theta'_B + yk_2\sin\Theta'_B - \omega t),$$
$$\hat{e}_t = -\sin\Theta'_B\hat{x} + \cos\Theta'_B\hat{y}$$

with $E_{0t} = 1.56E_{0i}$. The coefficient transmission amplitude, t, is calculated as follows; Using $n_2\cos\Theta_B = n_1\cos\Theta'_B$ and the definition for t we have,

$$t = \frac{2n_1\cos\Theta_B}{n_2\cos\Theta_B + n_1\cos\Theta'_B} = \frac{2n_1\cos\Theta_B}{n_1\cos\Theta'_B + n_1\cos\Theta'_B} = \frac{\cos\Theta_B}{\cos\Theta'_B}$$
$$= \frac{\cos\Theta_B}{\sin(\frac{\pi}{2} - \Theta_B)} = \frac{\cos\Theta_B}{\sin\Theta_B} = \frac{1}{\tan\Theta_B} = \frac{n_1}{n_2} = 1.56$$

Since there is no reflected wave all the incident power is transmitted to glass and therefore by definition,

$$T = \frac{I_t}{I_i} = 1.$$

Alternatively, for a more formal answer, the transmission power coefficient is given by,

$$T = \frac{\hat{n} \cdot \langle \mathbf{S}_t \rangle}{\hat{n}\langle \mathbf{S}_i \rangle} = \frac{v_2 \varepsilon_2 \cos\theta_2 E_{0t}^2/2}{v_1 \varepsilon_1 \cos\theta_1 E_{0i}^2/2} = \frac{(1/\mu_0 v_2)\cos\theta_2 (tE_{0i})^2}{(1/\mu_0 v_1)\cos\theta_1 E_{0i}^2} = \frac{n_2}{n_1}\frac{\cos\theta_2}{\cos\theta_1} t^2,$$

where we used $E_{0t} = tE_{0t}$, $n \equiv c/v$. Also, for non-magnetic materials, $\mu_i = \mu_0$ we have $v_i = 1/\sqrt{\mu_0 \epsilon_i} \longrightarrow v_i \varepsilon_i = 1/\mu_0 v_i$. In the present particular case of Brewster-angle incidence we have,

$$T = \frac{n_2}{n_1}\frac{\cos\Theta_B'}{\cos\Theta_B} t^2 = \frac{n_2}{n_1}\frac{\sin\Theta_B}{\cos\Theta_B}(\frac{n_1}{n_2})^2 = \frac{n_2}{n_1}\frac{1}{\tan\Theta_b}(\frac{n_1}{n_2})^2 = \frac{n_2}{n_1}\frac{n_2}{n_1}(\frac{n_1}{n_2})^2 = 1!$$

(c) In all the above relations we only need to interchange n_2 and n_1 and set $n_1 = n_{glass}$ and $n_2 = n_{air}$. Accordingly we swap Θ_B and Θ_B'. Then,

$$\underline{\Theta_B = \tan^{-1}(\frac{n_{air}}{n_{glass}}) = 32.66^o \qquad \Theta_B' = 57.34^o.}$$

Accordingly the power coefficient, following the same reasoning as in (c) is equal to one, $T = 1$.

7.3 Wave propagation in conducting media

Wave propagation in a flat conductor (e.g. copper)

Assume a monochromatic plane wave in a non-magnetic conductor where the electric permittivity ε, the magnetic permeability $\mu = \mu_0$ and the ohmic conductivity σ are all constants and no free charges are present, $\rho_f = 0$. The mathematical expression of a wave, propagating along the z-axis of a Cartesian coordinate system inside a semi-infinite flat shape conductor ($z > 0$), is $\mathbf{E}(z,t) = \hat{x}E_0 e^{\imath(kz-\omega t)}$.

(a) Starting from the Maxwell Equations with the above assumptions, supplemented by the Ohm's law, $\mathbf{j} = \sigma\mathbf{E}$, show that the electric component $\mathbf{E}(\mathbf{r},t)$ of the EM field satisfies:

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(z,t) = 0.$$

(b) The wavenumber k of an EM field propagating in a *good* conductor is a complex quantity equal to, $k \simeq (1+\imath)/\delta$, where $\delta = \sqrt{2/(\mu\sigma\omega)}$. Provide the expressions for the electric, $\mathbf{E}$, and magnetic, $\mathbf{H}$, components of the field, if $\omega = 3141.59$ rad/s and $\sigma = 5.952\times10^7/$(Ohm m).

(c) Considering only the spatial dependent part of the electric field, namely $\mathbf{E}(z) \equiv \mathbf{E}(z,0)$, provide a rough plot of $E(z)$ into the conducting material, ($z > 0$). In this plot the skin depth should also appear clearly.

(d) Show that the time-averaged heat rate per unit area of the conductor's surface is,

$$\langle\frac{dP}{dA}\rangle = \frac{\sigma\delta}{4}E_0^2.$$

Solution: (a) We start by taking the curl of the ME $\nabla\times\mathbf{E} = -\partial_t B$,

$$\begin{aligned}
\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \quad &\Longrightarrow \nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}\overbrace{(\nabla\times\mathbf{B})}^{\mu\mathbf{j}+\partial_t\mathbf{E}/c^2} \\
&\Longrightarrow \nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}\left[\mu\mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}\right] \\
&\Longrightarrow \nabla(\cancelto{0}{\nabla\cdot\mathbf{E}}) - \nabla^2\mathbf{E} = -\mu\frac{\partial}{\partial t}\overbrace{\mathbf{j}}^{=\sigma\mathbf{E}} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}.
\end{aligned}$$

The divergence ($\nabla\cdot\mathbf{E} = 0$) vanishes according to Gauss law when $\rho = 0$. Rearranging the terms and noting that $c^2 = 1/(\mu\varepsilon)$ we obtain the required inhomogeneous wave equation for the electric field $\mathbf{E}$:

$$\left[\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right]\mathbf{E}(z,t) = 0$$

(b) We have for the wave the following

The wavevector of the monochromatic plane wave (propagating along the z-axis $z > 0$) of a good conductors is expressed as,

$$\mathbf{k} = (k_r + \imath k_i)\hat{z} = \tilde{k}\hat{z} = \frac{1}{\delta}(1+\imath)\hat{z}, \tag{7.25}$$

where $\tilde{k}$ is the (complex number) wavenumber and k_r, k_i the corresponding real and imaginary parts. The wavenumber for a good conductor is expressed via the so-called *skin depth*, δ. In the present case where $\mu = \mu_0$ the skin-depth equals to,

$$\delta = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 5.952 \times 10^{7} \times 10^{3}\pi}} = \frac{1}{109.105\pi} = 2.917\ mm.$$

Then we have $\mathbf{k}\cdot\mathbf{r} = \tilde{k}z$. The electric component of the field is expressed as,

$$\begin{aligned}\mathbf{E}(z,t) &= \hat{x}E_0 e^{\imath(kz-\omega t)} = \hat{x}E_0 e^{\imath((1+\imath)z/\delta - \omega t)} \\ &= \hat{x}E_0 e^{-z/\delta} e^{\imath(z/\delta-\omega t)} = \hat{x}E_0 e^{-z/\delta}\cos(z/\delta - \omega t)\end{aligned} \tag{7.26}$$

The magnetic field component, $\mathbf{H}$ is calculated as below:

$$\begin{aligned}\mathbf{H} &= \frac{1}{\mu_0\omega}\mathbf{k}\times\mathbf{E} = \overbrace{\frac{1}{\delta\mu_0\omega}}^{\sqrt{\sigma/2\mu_0\omega}}(1+\imath)\overbrace{(\hat{z}\times\mathbf{E}(z,t))}^{\hat{y}E(z,t)} = \hat{y}\sqrt{\frac{\sigma}{\mu_0\omega}}\overbrace{\frac{1+\imath}{\sqrt{2}}}^{e^{\imath\pi/4}}E(z,t) \\ &= \underline{\hat{y}\sqrt{\frac{\sigma}{\mu_0\omega}}E_0 e^{-z/\delta} e^{\imath(\frac{z}{\delta}-\omega t+\pi/4)}}\end{aligned}$$

(c) Setting $t = 0$ we obtain for the (real part) spatial dependence of the electric component [1],

$$E(z) = Re\left[E_0 e^{-z/\delta} e^{\imath z/\delta}\right] = \underline{E_0 e^{-\frac{z}{\delta}}\cos(\frac{z}{\delta})}, \qquad z > 0.$$

1: Notice that the time-averaged squared-field is,

$$\langle \mathbf{E}^2(z,t)\rangle_t = \frac{1}{2}E_0^2 E^{-2z/\delta} \neq E_z^2(t,0).$$

The above expression can be plotted and gives the information that the field decays exponentially inside the conductor (in a sinusoidal way) with the skin-depth δ a measure of the rate of the attenuation ($\simeq e^{-z/\delta}$).

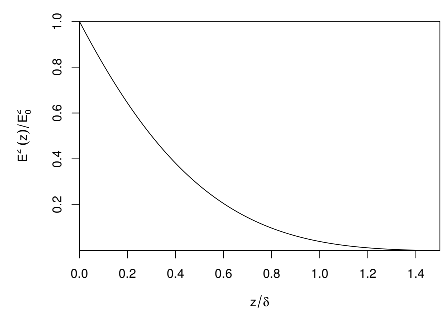

Figure 7.5: Plot of $E^2(z)/E_0^2$ as a function of z/δ, $\delta = 2.917$ mm.

(4c) Setting $t = 0$ we obtain for the (real part) spatial dependence of the electric component,

$$E(z) = Re\left[E_0 e^{-z/\delta} e^{\imath z/\delta}\right] = \underline{E_0 e^{-\frac{z}{\delta}}\cos(\frac{z}{\delta})}, \qquad z > 0.$$

The above expression can be plotted and gives the information that the field decays exponentially inside the conductor (in a sinusodial way) with the skin-depth δ a measure of the rate of the attenuation ($\simeq e^{-z/\delta}$) (see Fig (7.5).

(4d) The radiation power loss on a good conductor may be calculated by using the expression for the mechanical power produced in the conductor (Joule heating)

$$P(t) = \int dV\, \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) \qquad \rightarrow \qquad dP(t) = dA \int dz\, \mathbf{j}(z, t) \cdot \mathbf{E}(z, t)$$

where $\mathbf{j}$ is the current density induced on the conductor due to the presence of the EM wave (Here $\mathbf{j} = \sigma \mathbf{E}$). Propagation is along the $z-$axis (normal incidence). Since we don't integrate over the whole surface of the conductor dP/dA is the power loss per unit area. For conductors we have, $\mathbf{j}(z, t) = \sigma \mathbf{E}(z, t)$ and the time-averaged power loss is given by [2],

$$\begin{aligned}\langle \frac{dP}{dA} \rangle &= \int_0^\infty dz\, \sigma \langle \mathbf{E}^2(z, t) \rangle_t = \sigma \int_0^\infty dz \frac{E_0^2(z)}{2} \\ &= \sigma \frac{E_0^2}{2} \int_0^\infty dz\, e^{-2\frac{z}{\delta}} = \frac{\sigma\delta}{4} E_0^2\end{aligned}$$

where Eq. (7.26) was used (after time-integration).

2: In this expression we must use the real expression of the electric field When EM expressions involve multiplications then the trick where the real part is substituted by a complex expression does not work, unless special care is taken into account. More specifically, if $\mathbf{A}, \mathbf{B}$ are real then their complex extensions, $\mathscr{A}$ and $\mathscr{B}$ must be multiplied as below,

$$\mathbf{A} \cdot \mathbf{B} = \frac{1}{2} \left(\mathscr{A} \cdot \mathscr{B}^* \right)$$

Same rule applies for the cross product,

$$\mathbf{A} \times \mathbf{B} = \frac{1}{2} Re \left(\mathscr{A} \times \mathscr{B}^* \right)$$

Radiation pressure for water/glass and water/copper

Assume an electromagnetic plane wave with electric field amplitude $E_{0i} = 10^7$ V/cm propagating in water (medium '1') incident on glass (medium '2'). The wavelength of the field in water is $\lambda_i = 800$ nm and $n_{water} = 4/3$. Consider as plane of incidence the $xy-$ plane [see Fig. (7.2)].

Take as medium '2' to be glass with $n_{glass} = 3/2$ and for the refractive indices of the glass and water, respectively.

Calculate the time-averaged radiation pressure on glass.

Water and glass are non-magnetic materials.

Solution: First, using the general Fresnel laws, we need to express analytically all the electromagnetic fields in water and glass, $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$. To this end we follow the steps in (1a) of Question '*Normal incidence Fresnel Equations*'.

3

3: Note that the fields $\mathbf{D}, \mathbf{H}$ and $\mathbf{B}$ (if needed) are calculated by the relations $\mathbf{H}_j = \mathbf{B}_j/\mu_q$, $\mathbf{D}_j = \varepsilon_q \mathbf{E}_j$ and $\mathbf{H}_j = \sqrt{\varepsilon_q/\mu_q}\mathbf{E}_j$ for $j = r, t$ (when $j = i, r$ then $q = 1$ and when $j = t$ then $q = 2$).

(3b) As the incident EM wave ($\mathbf{E}_i = \hat{y}E_{0i}\cos(kx - \omega t)$) hits the interface boundary we have a reflection ($\mathbf{E}_r = -\hat{y}E_{0r}\cos(kx + \omega t)$ and a trasmitted wave as above. Since we relate the corresponding Poynting vectors with the momentum carried by the EM waves,

$$\mathbf{g}_q(\mathbf{r}, t) = \frac{\mathbf{S}_q(\mathbf{r}, t)}{v_q^2} = \frac{\epsilon_q}{v_q}|\mathbf{E}_q(\mathbf{r}, t)|^2\hat{k}_q, \qquad q = i, r, t \tag{7.27}$$

we expect that a net momentum change, carried by the EM wave. This momentum change can be interpreted 'loosely' that the dielectric '2' 'exerts' a force on the EM wave, $\mathbf{F}_{em}$ (since a force is the rate of change of a momentum). Then, from Newton's reaction law we expect that the EM exerts a (reaction) force *on the dielectric '2'* which is the negative of the force exerted by the dielectric on the field ($\mathbf{F}_R = -\mathbf{F}_{em}$). Now assuming an exposed area ΔA on we can define the so-called *radiation pressure*, $\mathscr{P}_R$ as the time average of this force devided by the area ΔA, namely:

$$\mathscr{P}_R \equiv \frac{|\langle F_R \rangle|}{\Delta A}. \tag{7.28}$$

In the present case of plane monochromatic wave, radiation pressure has a particular simple expression in terms of its Poynting vector. To this end, we first recall that the time-averaged force from the change of the momentum in EM momentum, is given by

$$\begin{aligned}\langle F_{em} \rangle &\equiv \lim_{\Delta t \to 0}\langle \frac{\Delta P_{em}}{\Delta t}\rangle = \lim_{\Delta t \to 0}\frac{\langle P_{final} - P_{initial}\rangle}{\Delta t} = \lim_{\Delta t \to 0}(\langle -g_r\rangle\frac{\Delta V_1}{\Delta t} + \langle g_t\rangle\frac{\Delta V_2}{\Delta t} - \langle g_i\rangle\frac{\Delta V_1}{\Delta t}) \\ &= (-\frac{S_r}{v_1^2}\frac{\Delta x_1}{\Delta t} + \frac{S_t}{v_2^2}\frac{\Delta x_2}{\Delta t} - \frac{S_i}{v_1^2}\frac{\Delta x_1}{\Delta t})\Delta A = \frac{1}{2}E_{0i}^2\left[\varepsilon_1(1 + r^2) - \varepsilon_2 t^2\right]\Delta A\end{aligned}$$

In the above derivation we have used the definition of the momentum density $\mathbf{g}$ of an EM wave (Eq. 7.27) which provides for the time-averaged momentum enclosed on a volume ΔV. Finally, the volume ΔV_i for each of the media was re-written as $\Delta V_i = \Delta x_i \Delta A = (v_i \Delta t)\Delta A, \quad i = 1, 2$. The idea is that the EM wave propagates along the axis $\hat{x}$ with a speed v_i then during time Δt, given an exposing area A on the plane yz, the dielectric interacts with the part of the wave contained in a volume equal to $\Delta V_i = \Delta x_i \Delta A = v_i \Delta t \Delta A$. r and t are the reflection and transmission amplitude coefficients. Now gathering all the above results we have,

$$\boxed{\mathscr{P}_R \equiv \frac{|\langle F_R \rangle|}{\Delta A} = \frac{|-\langle F_{em} \rangle|}{\Delta A} = \frac{I_1}{v_1}(1 + |r|^2 - \frac{\varepsilon_2}{\varepsilon_1}|t|^2), \qquad I_1 = \frac{1}{2} v_1 \varepsilon_1 E_{0i}^2}$$

For normal incidence ($\theta_1 = \theta_2 = 0^o$) and non-magnetic materials we have, $R = |r|^2$ and (4.24)

$$T = \frac{Z_1}{Z_2}|t|^2 = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}|t|^2, \quad \text{since} \quad \frac{Z_1}{Z_2} = \frac{\sqrt{\mu_0/\varepsilon_1}}{\sqrt{\mu_0/\varepsilon_2}} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}.$$

We also have,

$$\frac{n_1}{n_2} = \frac{c/v_1}{c/v_2} = \frac{v_2}{v_1} = \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}$$

Then we can replace the radiation pressure expression by,

$$\boxed{\mathscr{P}_R \equiv \frac{|\langle F_R \rangle|}{\Delta A} = \frac{|-\langle F_{em} \rangle|}{\Delta A} = \frac{I_1}{v_1}(1 + R - \frac{n_2}{n_1}T), \qquad I_1 = \frac{1}{2} v_1 \varepsilon_1 E_{0i}^2 \quad (7.29)}$$

Drude model

Consider propagation of a plane wave in Sodium (Na) with static conductivity $\sigma_0 = (1/50) \times 10^9 /$ (Ohm m) and $\tau \sim 10^{-14}$s the mean free time.

(5a) Calculate the plasma frequency and the corresponding cut-off wavelength.

(5b) Discuss the Drude model and find, within this model, the Sodium's conductivity for $\omega_1 = 10^{17}$ Hz and $\omega_2 = 10^{14}$ Hz. It is required to *derive* the relation from Drude's model assumptions,

$$\sigma(\omega) = \frac{\sigma_0}{1 - \imath\omega\tau}$$

and then to calculate the values for the two frequencies.

(5c) What is the physical meaning of a complex conductivity and what are the observable effects on the electric and magnetic field of such wave? Under what physical conditions we have purely imaginary or real conductivity? Explain your answers in detail.

(5a) The frequency of the EM wave is related with the plasma frequency as,

$$\omega^2 = \omega_p^2 + c^2k^2,$$

where k is the wavenumber of the field and c is the light's speed in vacuum. The plasma frequency is given by,

$$\omega_p^2 = \frac{n_e q_e^2}{m_e \epsilon_0}, \qquad \sigma_0 = \frac{n_e q_e^2 \tau}{m_e}.$$

Since σ_0, τ are known then $q_e n_e / m_e$ can be calculated from the second equation and be replaced to the first:

$$\boxed{\omega_p = \sqrt{\frac{\sigma_0}{\varepsilon_0 \tau}} \sim 1.49 \times 10^{16}\ \text{Hz} \qquad \Longrightarrow \qquad \lambda_p = \frac{2\pi c}{\omega_p} \sim 126.5\ \text{nm.}}$$

(5b) Consider a collection of non-interacting electrons of constant number density n_e (number of electrons per unit volume).

Without loss of generality we assume propagation of a wave along the $\hat{z}$−axis, while the electric field lies along the $\hat{x}$-axis of a Cartesian coordinate system:

$$\mathbf{E}(z,t) = \hat{x}E_0 e^{i(kz-\omega t)}$$

The Drude's equation along the x-axis (polarization axis of the field) we keep only the $\hat{x}$−component (as no EM force is applied along the other axes) and obtain:

$$m\ddot{x} = qE_0\cos(\omega t) - \dot{x}(m/\tau) \Longrightarrow \ddot{x} + \gamma\dot{x} = f_0 \cos\omega t, \qquad \gamma \equiv \frac{1}{\tau}, \qquad f_0 \equiv \frac{qE_0}{m}.$$

Solution of the above equation for the particle's motion is obtained quite straightforward and for times $t >> 1/\gamma = \tau$ we have the so-called *steady-state* solution,

$$x(t) = -\mathrm{Re}\left[\frac{(f_0/\imath\omega)}{\gamma - \imath\omega} e^{-\imath\omega t}\right],$$

where Re denotes the real part of the above expression. The above solution for $x(t)$ predicts for the current density $\mathbf{j}$, by definition, the following expression:

$$\mathbf{j} = nq\mathbf{v}(t) = \hat{x}(nq\dot{x}) = \hat{x}nqRe\left[\frac{f_0}{\gamma - \imath\omega} e^{-\imath\omega t}\right] = \hat{x}Re\left[\frac{(nq^2/m)}{\gamma - \imath\omega} E_0 e^{-\imath\omega t}\right]$$

On the other hand from Ohms's law we have the following expression for the current density:

$$\mathbf{j}(t) = \sigma\mathbf{E}(0,t) = \hat{x}Re\left[\sigma E_0 e^{-\imath\omega t}\right].$$

Combining the two latter expressions for the current density, $\mathbf{j}$, we end up to the conclusion that the conductivity σ for this medium should be dependent on the EM's wave frequency ω as,

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - \imath\omega\tau}, \qquad \sigma_0 = \frac{nq^2\tau}{m},} \tag{7.30}$$

where the σ_0 is known as the static limit of conductivity. The above expression is the central result of this section. Given this relation one can now write down the expressions for $\varepsilon(\omega), k(\omega), n(\omega)$ and $Z(\omega)$ given in Eqns (6.4).

Now, considering the two frequencies we have for $\omega_1\tau = 1$ while for the other frequency, $\omega_2\tau = 10^3 >> 1$ then get:

$$\boxed{\sigma(\omega_1) = \frac{1}{2}\sigma_0(1+\imath) = 10^7(1+\imath), \qquad \sigma(\omega_2) \sim \imath 10^{-3}\sigma_0 = \imath 2\times 10^5, \quad \text{SI units,}}$$

where for the ω_2 the unity was ignored (relative to $\omega_2\tau$).

(5c) Conductivity, complex or not, generally indicates field's absorption during propagation through the medium. There are though situations where the conductivity is either purely real or imaginary. By inspecting the conductivity relation, the real part dominates when $\omega\tau << 1$ and this is a typical case of low-frequency waves in metals. The dielectric constant in this case is complex and as a result absorption is an inevitable effect.

On the other hand, dominance of the imaginary part of the conductivity is possible when $\omega\tau >> 1$, with a typical case of high-frequency waves in fully ionized (non-very energetic) medium (e.g. cold plasma). Then the dielectric constant is real, however depends on the wave frequency. This dependence results to the existence of a cut-off frequency (plasma frequency), where below this frequency no waves can be propagated, while waves with frequency above this, can be propagated freely. According to the above numbers it is only the wave with frequency ω_2 that will be propagated since $\omega_2 > \omega_p$ while the other one will decay as soon as enters the medium's region.

Wave propagation in plasma

Consider a collection of non-interacting electrons of constant number density n_e (number of electrons per unit volume). This state of the matter can be modelled as an unbounded, isotropic material with $\rho(\mathbf{r}, t) = 0$ but with non-vanishing current density $\mathbf{j}(\mathbf{r}, t)$.

(a) Using the Maxwell Equations (ME) show that the wave equation for the electric component of an electromagnetic field propagating through is as below:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\right]\mathbf{E}(\mathbf{r}, t) = \mu_0 \frac{\partial}{\partial t}\mathbf{j}(\mathbf{r}, t).$$

(b) Give a physical justification to employing the above model for the non-interacting electrons. Assuming propagation along the $\hat{z}-$axis, employ the Newton's second law for the electron's reaction to the electric field to show that the above equation has a plane wave solution of the type $\mathbf{E}(z, t) = \mathbf{E}_0 e^{i(kz-\omega t)}$ where the frequency ω is given by,

$$\omega^2 = \omega_p^2 + c^2 k^2,$$

where k is the wavenumber of the field and c is the light's speed in vacuum. In doing this, derive ω_p, known as the *plasma frequency*.

(c) For a low-density plasma (e.g. ionosphere) the plasma frequency is in the regime of the radio frequency ($\sim MHz$). Given the above conclusions assume a plane wave of frequency $\omega = 4\omega_p/5$ propagating through this medium. Calculate the *skin depth* of this wave for $\omega_p = 100$ MHz.

Solution: (a) We start by combining the curl of the ME $\nabla \times \mathbf{E} = -\partial_t B$ and using $\nabla \times B = \mu_0 \mathbf{j} + \partial_t \mathbf{E}/c^2$

$$\nabla \times (\nabla \times \mathbf{E}) = -\partial_t \left[\mu_0 \mathbf{j} + \frac{1}{c^2}\partial_t \mathbf{E}\right] \Longrightarrow \nabla(\cancelto{0}{\nabla \cdot \mathbf{E}}) - \nabla^2 \mathbf{E} = -\mu_0 \partial_t \mathbf{j} - \frac{1}{c^2}\partial_{tt}\mathbf{E}.$$

The divergence $\nabla \cdot \mathbf{E} = 0$ according to the ME (Gauss law) when $\rho = 0$. Rearranging the terms we obtain the required inhomogeneous wave equation for the electric field $\mathbf{E}$:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\right]\mathbf{E}(\mathbf{r}, t) = \mu_0 \frac{\partial}{\partial t}\mathbf{j}(\mathbf{r}, t) \tag{7.31}$$

(b) The Drude's model for a neutral system composed by free mobile charges with n number density and each particle of charge q and mass m is to assume that each of the particle is governed by the following equation of motion,

$$m\frac{d}{dt}\mathbf{v} = q\mathbf{E} - m\mathbf{v}/\tau, \tag{7.32}$$

where τ is the mean free path time for the particle. The second term in this equation acts as *drag* force and it is the average result over a large number of collisions suffered by the particle with its neighbor particles. Since in the present case we assume that the electrons are non-interaction (e.g. this situation can represent a low density plasma) we can remove this term from the following analysis. Therefore, since by definition we have $\mathbf{j} = nq_e\mathbf{v}$ and with the help of Eq. (7.32) we can eliminate $\mathbf{j}$ from Eqns (7.31) to arrive at,

$$\dot{\mathbf{j}} = nq_e\dot{\mathbf{v}} = \frac{nq_e}{m_e}\mathbf{E} \Longrightarrow \left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t}\right]\mathbf{E}(\mathbf{r},t) = \mu_0\frac{n_e q_e^2}{m_e}\mathbf{E}(\mathbf{r},t),$$

where in the above we assumed the n_e is a constant.

From the last relation and recalling that for any wave field $\mathbf{F}(\mathbf{r},t) = \mathbf{F}_0 e^{i(\mathbf{k}\cdot\mathbf{r}\pm\omega t)}$ (given in the appendix) we can make the substitutions, $\nabla \to \imath k$ and $\partial_t \to \pm\imath\omega$, to obtain,

$$\left[-k^2 + \frac{\omega^2}{c^2}\right]\mathbf{E}(\mathbf{r},t) = \frac{\omega_p^2}{c^2}\mathbf{E}(\mathbf{r},t),$$

where ω_p is the *plasma frequency*, defined as $\omega_p \equiv q_e(n_e/m_e\epsilon_0)^{1/2}$. From the last equation we conclude that the frequency of the EM wave should satisfy,

$$\omega^2 = \omega_p^2 + c^2k^2,$$

(c) If the frequency of the EM wave is below the plasma frequency ω_p (here $\omega = 4\omega_p/5$) then the wavenumber becomes purely imaginary since,

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} = \frac{1}{c}\sqrt{(4\omega_p/5)^2 - \omega_p^2} = \frac{\omega_p}{c}\frac{3}{5}\sqrt{-1} = \imath\frac{3k_p}{5},$$

where k_p,

$$k_p \equiv \frac{\omega_p}{c} = \frac{10^8}{3\times 10^8} \sim 0.3333\,\mathrm{m}^{-1}$$

As it is known, the imaginary part of the wavenumber contributes to the attenuation of the fields during their propagation and the skin-depth is defined as,

$$\delta = \frac{1}{k_I} = \frac{5}{3k_p} = 5\,\mathrm{m}.$$

Radiation pressure 8

8.1 Radiation pressure (normal incidence) on a perfect conductor

As the incident EM wave ($\mathbf{E}_i = \hat{y}E_0\cos(kx - \omega t)$) (see Fig. (8.1) hits the *perfect* conductor we have only a reflection ($\mathbf{E}_r = -\hat{y}E_0\cos(kx + \omega t)$ since inside the perfect conductor (region-II) no EM fields are allowed ($\mathbf{E}_{II} = \mathbf{B}_{II} = 0$). Since the wavevector of the EM $\mathbf{k}_i \to \mathbf{k}_r = -\mathbf{k}_i$ changes direction (although its magnitude stays the same) we expect that a net momentum change, carried by the EM wave, occurs*. This can be interpreted 'loosely' that the conductor 'exerts' a force on the EM wave, $\mathbf{F}_{em}$ (since a force is the rate of change of a momentum). Then, from Newton's reaction law we expect that the EM exerts a (reaction) force *on the conductor* which is the negative of the force exerted by the conductor on the field ($\mathbf{F}_R = -\mathbf{F}_{em}$). Now assuming an exposing area ΔA on the conductor's surface we can define the so-called *radiation pressure*, $\mathcal{P}_R$ as the time average of this force devided by the area ΔA, namely:

$$\mathcal{P}_R \equiv \frac{|\langle F_R \rangle|}{\Delta A}. \tag{8.1}$$

In the present case of plane monochromatic wave, radiation pressure has a particular simple expression in terms of its Poynting vector.

To this end, we first recall that the time-averaged force from the change of the momentum in EM momentum, is given by

$$\langle F_{em} \rangle \equiv \lim_{\Delta t \to 0} \langle \frac{\Delta P_{em}}{\Delta t} \rangle = \lim_{\Delta t \to 0} \frac{\langle P_r - P_i \rangle}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\langle g_r \rangle - \langle g_i \rangle)\Delta V}{\Delta t}$$

In the above derivation we have used the definition of the momentum density $\mathbf{g}$ of an EM wave (Eq. 3.8) which provides for the time-averaged momentum enclosed on a momentum ΔV

$$P_{em} = \int_{\Delta V} dV\, g(\mathbf{r}, t) \Longrightarrow \langle g \rangle = \frac{\langle P_{em} \rangle}{\Delta V}$$

In addition from Eq. (3.6) we know that,

$$\mathbf{g}(\mathbf{r}, t) = \frac{\mathbf{S}(\mathbf{r}, t)}{c^2} = \frac{\epsilon_0}{c}|\mathbf{E}(\mathbf{r}, t)|^2 \hat{k}$$

The latter two equation give the following relations:

$$\langle g_r \rangle - \langle g_i \rangle = \frac{\varepsilon_0 E_0^2}{2c}(\hat{k}_r - \hat{k}_i)\Delta A = \frac{\varepsilon_0 E_0^2}{2c} \times (2\hat{x}) = \frac{\varepsilon_0}{c} E_0^2 \hat{x} = \frac{2I}{c^2}\hat{x},$$

since it is known that for plane monochromatic waves $I = \langle S \rangle = c\varepsilon_0 E_0^2/2$. Finally, the volume ΔA was re-written as $\Delta V = c\Delta t \Delta A$. Since the EM wave propagates along the axis $\hat{x}$ with a speed (light's vacuum speed) is c then during time Δt, given an exposing area A on the plane yz, the conductors interacts with the part of the wave contained in a volume equal to $\Delta V = dx\Delta A = c\Delta t\Delta A$.

* $d\mathbf{k} = \mathbf{k}_r - \mathbf{k}_i = -\mathbf{k}_i - \mathbf{k}_i = -2\mathbf{k}_i$, with magnitude $|\delta k| = 2k_i = 2k$

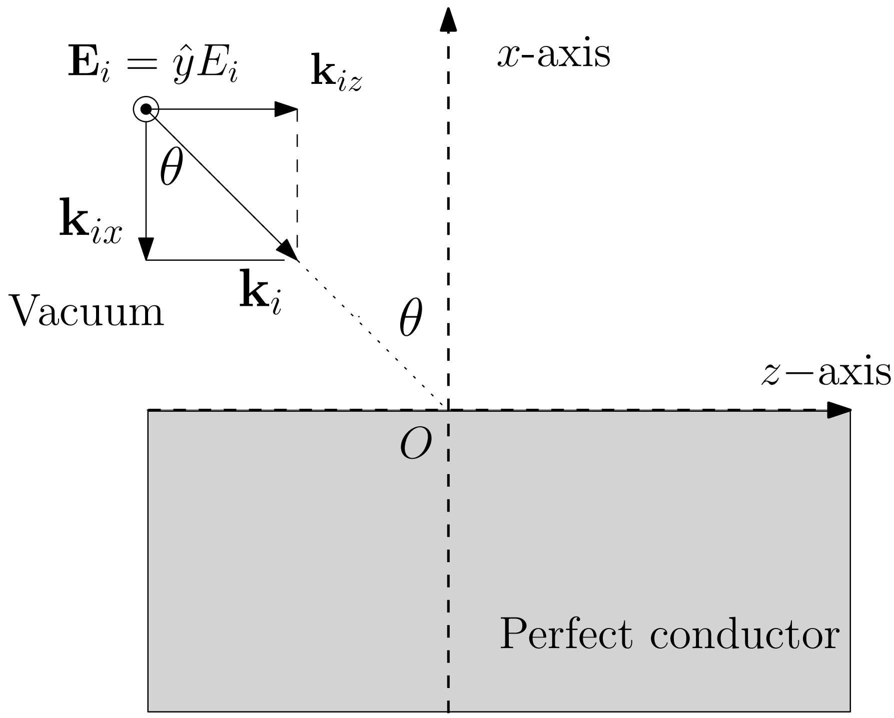

Figure 8.1: Figure for *Radiation Pressure on a perfect conductor* section. EM plane wave is incident on a perfect conductor with an incident angle θ with the normal. Electric field is perpedincular to the plane of incidence (PoI=zx-plane).

Now gathering all the above results we have,

$$\mathcal{P}_R \equiv \frac{|\langle F_R \rangle|}{\Delta A} = \frac{|-\langle F_{em} \rangle|}{\Delta A} = \frac{|(-2I/c^2)\hat{x} \times (c\Delta t \Delta A)}{\Delta t \Delta A} = \frac{2I}{c}.$$

Given the above we have,

$$\begin{aligned}
\langle g_i \rangle &= \frac{1}{2c}\varepsilon_0 |\mathbf{E}_0|^2 \hat{k}_i, \\
\langle g_r \rangle &= \frac{1}{2c}\varepsilon_0 |\mathbf{E}_0|^2 \hat{k}_r \\
dV &= c\Delta t A \cos\theta
\end{aligned}$$

Combining the above we have,

$$\langle F \rangle_{em} = |\langle g \rangle_r - \langle g \rangle_i| cA\cos\theta = \frac{1}{2c}\varepsilon_0 |\mathbf{E}_0|^2 |\hat{k}_i - \hat{k}_r| cA\cos\theta$$

From the expressions for the $\mathbf{k}_i$ and $\mathbf{k}_r$ (Eqns 8.2-8.4) wavevectors we obtain

$$|\hat{k}_i - \hat{k}_r| = 2\cos\theta\hat{x}$$

Gathering all the above expressions we end up for the radiation pressure:

$$\mathcal{P}_R = \frac{|\langle F \rangle_R|}{A} = \frac{(2\cos\theta)(\varepsilon_0|\mathbf{E}_0|^2/2c)(cA\cos\theta)}{A} = \varepsilon_0 E_0^2 \cos^2\theta$$

which is in agreement with the findings of the first method.

8.2 Application: Radiation Pressure on a perfect conductor with arbitrary angle incidence.

We assume a Cartesian coordinate system $Oxyz$ and a *perfect* flat conductor which occupies the half-space (region II) $x < 0$. A monochromatic plane wave, propagating in region I ($x > 0$), of frequency ω with electric field $\mathbf{E}_i(\mathbf{r}, t)$ along the $\hat{y}$-axis and amplitude E_0, impinges the conductor with an angle θ with the normal.

We are going to calculate the radiation pressure 'felt' by the conductor with two different methods. First, we'll evaluate the Lorentz force on the conductor's surface (applied to the surface electrons) and the second method will consist to calculate the change of the EM momentum carried by the incident and the reflected waves.

since these equations are so general to include the case of dielectric-conductor interface boundary.

(i) Method 1: Lorentz Force evaluation
Our strategy will be as follow:

- First is to find the expressions for the incident ($\mathbf{E}_i$) and the reflected ($\mathbf{E}_r$) waves, which are the only non-vanishing EM fields (since the conductor is *perfect* the fields *inside* the conductor (by definition) are all zero). These expressions will provide the total electric field in vacuum (region-I)as,

$$\mathbf{E}_I(\mathbf{r}, t = \mathbf{E}_i(\mathbf{r}, t) + \mathbf{E}_r(\mathbf{r}, t)$$

and for the region II we'll have,

$$\mathbf{E}_{II} = 0, \qquad \mathbf{B}_{II} = 0, \qquad \text{for perfect conductor}$$

- Then by using the semi-algebraic form of the ME we'll find the magnetic field on the outer part of the conductor's surface as,

$$\mathbf{B}_I(\mathbf{r}, t) = \frac{1}{\imath\omega}\nabla \times \mathbf{E}_I(\mathbf{r}, t)$$

- Finally, by considering that the Lorentz force per unit area *on the conductor* ($x = 0$) as,

$$\mathbf{f}(\mathbf{r}, t) = \mathbf{j}_s(\mathbf{r}, t) \times (\mathbf{B}_I(\mathbf{r}, t) + \mathbf{B}_{II}(\mathbf{r}, t))|_{x=0}, \qquad \mathbf{j}_s \equiv \frac{1}{\mu_0}\hat{x} \times \mathbf{B}_I(\mathbf{r}, t)|_{x=0},$$

where the expression for the *surface current density* $\mathbf{j}_s(0,$ is obtained from the boundary conditions. Then, it only remains to calculate the radiation pressure as the half time-averaged Lorentz force*:

$$\mathscr{P} \equiv \frac{1}{2}\langle|\mathbf{f}|\rangle$$

* The number 2 in this expression results from the fact that the conductor is perfect and as such its skin depth is exactly zero ($\delta = 0$). This expression is taken as granted and is left without proof.

We start by invoking the complex-representation formulation of a plane wave where the incident field is expressed as:

$$\mathbf{E}_i(\mathbf{r}, t) = \hat{y} E_0 e^{i(\mathbf{kr}-\omega t)},$$

Using standard trigonometry we expand the wavevector $\mathbf{k}_i$ to its Cartesian components:

$$\mathbf{k}_i = k_{ix}\hat{x} + k_{iy}\hat{y} + k_{iz}\hat{z} = \hat{x}(-k\cos\theta) + \hat{z}(k\sin\theta) \tag{8.2}$$

Then the EM wave is rexpressed as,

$$\mathbf{E}_i(x, z, t) = \hat{y} E_0 e^{i(kz\sin\theta - kx\cos\theta - \omega t)}, \tag{8.3}$$

For the reflected wave $\mathbf{E}_r$ the propagation wave vector $\mathbf{k}_r$ in Cartesian coordinates should be expressed as,

$$\mathbf{k}_r = k_{rx}\hat{x} + k_{ry}\hat{y} + k_{rz}\hat{z} = \hat{x}(k\cos\theta) + \hat{z}(k\sin\theta) \tag{8.4}$$

Then, given the above equation and the incident wave expression (8.3) above, the reflected electric field will be as below

$$\mathbf{E}_i(x, z, t) = -\hat{y} E_0 e^{i(kz\sin\theta + kx\cos\theta - \omega t)}.$$

The above expression can be obtained either by applying the Fresnel equations for a dielectric-conductor interface boundary or by recalling the ME boundary conditions that require the total tangential field to be zero at the interface boundary ($x = 0$).
The total field is the sum of the incident and the reflected waves,

$$\begin{aligned}\mathbf{E}_I = \mathbf{E}_i + \mathbf{E}_r &= \hat{y} E_0 e^{ik(z\sin\theta - \omega t)} \left[e^{-\imath x\cos\theta} + e^{ikx\cos\theta}\right] \\ &= -2\imath E_0 e^{ik(z\sin\theta - \omega t)} \sin(kx\cos\theta)\hat{y} \end{aligned}\tag{8.5}$$

This field is indeed zero at the interface plane ($x = 0$) since,

$$\mathbf{E}_I(x = 0, z, t) = -2\imath E_0 e^{ik(z\sin\theta - \omega t)} \sin(0)\hat{y} = 0.$$

Note that there is no dependance on the y-variable for all the above expressions. So all the conclusions above hold for all values of y (this justifies the present plane-wave treatment). For simplicity we can assume y fixed at some value and omit from the subsequent relations.
The direct evaluation of the semi-algebraic ME provides the associated magnetic field in region I as,

$$\begin{aligned}\mathbf{B}_I &= -\frac{\imath}{\omega}\nabla\times\mathbf{E}_I = -\frac{\imath}{\omega}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_I & 0\end{vmatrix} = -\frac{\imath}{\omega}(\hat{x}\partial_z E_I - \hat{z}\partial_z E_I) \\ &= -2\frac{E_0}{c} e^{i(kz\sin\theta - \omega t)} \left[\cos\theta\cos(kx\cos\theta)\hat{z} - \imath\sin\theta\sin(kx\cos\theta)\hat{x}\right]\end{aligned}$$

It results that the magnetic field in the conductor's outer surface (region I) is,

$$\mathbf{B}_I(x = 0, z, t) = -2\frac{E_0}{c}\cos(kz\sin\theta - \omega t)\cos\theta\hat{z}$$

Now we are at the stage where we can calculate the radiation pressure is defined as the half of the averaged force:

$$\mathcal{P}_R = \frac{1}{2}\langle f\rangle \equiv \frac{1}{2T}\int_0^T dt|\mathbf{f}|, \qquad T = 2\pi/\omega.$$

Taking now the average of the density force, **f** we have,

$$\begin{aligned}
\mathcal{P}_R = \frac{1}{2T}\int_0^T dt|\mathbf{f}| &= \frac{1}{2}\langle|\mathbf{j}_s \times \mathbf{B}_I(0,z,t)|\rangle = \frac{1}{2\mu_0}\langle|\hat{x}\times\mathbf{B}_I(0,z,t)\times\mathbf{B}_I(0,z,t)|\rangle \\
&= \hat{x}\langle|-\mathbf{B}^2(0,z,t)|\rangle = \frac{1}{2\mu_0}\frac{4E_0^2}{c^2}\cos^2\theta \\
&= \epsilon_0 E_0^2\cos^2\theta,
\end{aligned}$$

where in the last expression we have used $c^2 = 1/\mu_0\epsilon_0$ and that $\langle\cos(\omega t)\rangle = 1/2$. Then we have for the radiation pressure the known expression,

$$\mathcal{P}_R = \epsilon_0 E_0^2\cos^2\theta$$

(ii) Method 2: Momentum change of the EM wave
In the present approach, the strategy will be as follows:

- First we'll calculate the time-averaged force from the change of the momentum in EM momentum, as

$$\langle F_{em}\rangle \equiv \lim_{\Delta t\to 0}\frac{\langle P_r\rangle - \langle P_i\rangle}{\Delta t}, \qquad \langle P\rangle = \langle g_{em}\rangle\Delta V$$

- Then the force exerted on the conductor by the field is the negative of the force exerted by the conductor on the field

$$\langle F_R\rangle = -\langle F_{em}\rangle$$

- Finally if the normal incidence exposes an area A of the conductor then the pressure is evaluated as,

$$\mathcal{P}_R = \frac{|\langle F_R\rangle|}{A}$$

Given the above we have,

$$\begin{aligned}
\langle g_i\rangle &= \frac{1}{2c}\varepsilon_0|\mathbf{E}_0|^2\hat{k}_i, \\
\langle g_r\rangle &= \frac{1}{2c}\varepsilon_0|\mathbf{E}_0|^2\hat{k}_r \\
dV &= c\Delta t A\cos\theta
\end{aligned}$$

Combining the above we have,

$$\langle F_{em}\rangle = |\langle g_r\rangle - \langle g_i\rangle|cA\cos\theta = \frac{1}{2c}\varepsilon_0|\mathbf{E}_0|^2|\hat{k}_i - \hat{k}_r|cA\cos\theta$$

From the expressions for the $\mathbf{k}_i$ and $\mathbf{k}_r$ (Eqns 8.2-8.4) wavevectors we obtain

$$|\hat{k}_i - \hat{k}_r| = 2\cos\theta\hat{x}$$

Gathering all the above expressions we end up for the radiation pressure:

$$\mathcal{P}_R = \frac{|\langle F_R \rangle|}{A} = \frac{(2\cos\theta)(\varepsilon_0 |\mathbf{E}_0|^2/2c)(cA\cos\theta)}{A} = \varepsilon_0 E_0^2 \cos^2\theta$$

which is in agreement with the findings of the first method.

8.3 Radiation pressure

Radiation pressure on a perfect flat conductor.

Assume a Cartesian coordinate system $Oxyz$ and a *perfect* flat conductor which occupies the half-space (region II) $x < 0$. A monochromatic plane wave, propagating in region I ($x > 0$) (vacuum), of frequency ω with electric field $\mathbf{E}_i(\mathbf{r}, t)$ along the $\hat{y}$-axis and amplitude E_0, hits the interface boundary at normal incidence conditions.

(a) If, within the complex-representation formulation of a plane wave, the incident field is expressed as:

$$\mathbf{E}_i(x, t) = \hat{y} E_0 e^{-i(kx+\omega t)},$$

provide the corresponding expression for the reflected ($\mathbf{E}_r$) wave. If the electric fields in the regions I ($x > 0$) and II ($x < 0$) are denoted by $\mathbf{E}_I$ and $\mathbf{E}_{II}$, respectively, provide their mathematical expressions $\mathbf{E}_I(x, t)$ and $\mathbf{E}_{II}(x, t)$. If $\mathbf{E}(0, t) = \mathbf{E}_I(0, t) + \mathbf{E}_{II}(0, t)$ is the total electric field on the conductor's surface, provide accordingly the fields $\mathbf{E}_I(0, t)$, $\mathbf{E}_{II}(0, t)$ and $\mathbf{E}(0, t)$.

(b) Using the semi-algebraic form of the Maxwell Equations ($\nabla \times \mathbf{E}_I = \imath\omega \mathbf{B}_I$), show that the magnetic field on the outer part of the conductor's surface, is given by,

$$\mathbf{B}_I(0, t) = (-\frac{2E_0}{c} \cos \omega t)\ \hat{z}.$$

Similarly find the magnetic field in the conductor's region $\mathbf{B}_{II}(x, t)$, $x < 0$. If $\mathbf{B}(0, t) = \mathbf{B}_I(0, t) + \mathbf{B}_{II}(0, t)$ is the total magnetic field on the interface boundary provide $\mathbf{B}_I(0, t)$,$\mathbf{B}_{II}(0, t)$ and $\mathbf{B}(0, t)$.

(c) The Lorentz force per unit area (surface force density) on the conductor is given by, $\mathbf{f}(t) = \rho_s \mathbf{E}(0, t) + \mathbf{j}_s \times \mathbf{B}(0, t)$, where ρ_s is the isurface charge density and $\mathbf{j}_s$ is the induced surface current density, calculated as, $\mathbf{j}_s(0, t) = \frac{1}{\mu_0} \hat{x} \times \mathbf{B}(0, t)$. Using the above expressions, calculate the radiation pressure as the following averaged quantity (within the perfect conductor assumption):

$$\mathscr{P}_R = \frac{1}{2}\langle f \rangle \equiv \frac{1}{2T} \int_0^T dt\, |\mathbf{f}|, \qquad T = 2\pi/\omega.$$

Solution: (a) Given the incident wave expression the reflected electric field is as,

$$\mathbf{E}_r(x, t) = -\hat{y} E_0 e^{-i(-kx+\omega t)} = -\hat{y} E_0 e^{i(kx-\omega t)}.$$

The above expression can be obtained either by applying the Fresnel equations for a dielectric-conductor interface boundary or by recalling that the boundary conditions for the EM fields require the total tangential electric field to be zero at the interface boundary ($x = 0$).

The total field in region I is the sum of the incident and the reflected waves,

$$\mathbf{E}_I = \mathbf{E}_i + \mathbf{E}_r = \hat{y} E_0 e^{-i\omega t} \left[e^{-\imath kx} - e^{ikx} \right] = -2\imath E_0 e^{-i\omega t} \sin(kx) \hat{y} = E_I \hat{y}.$$

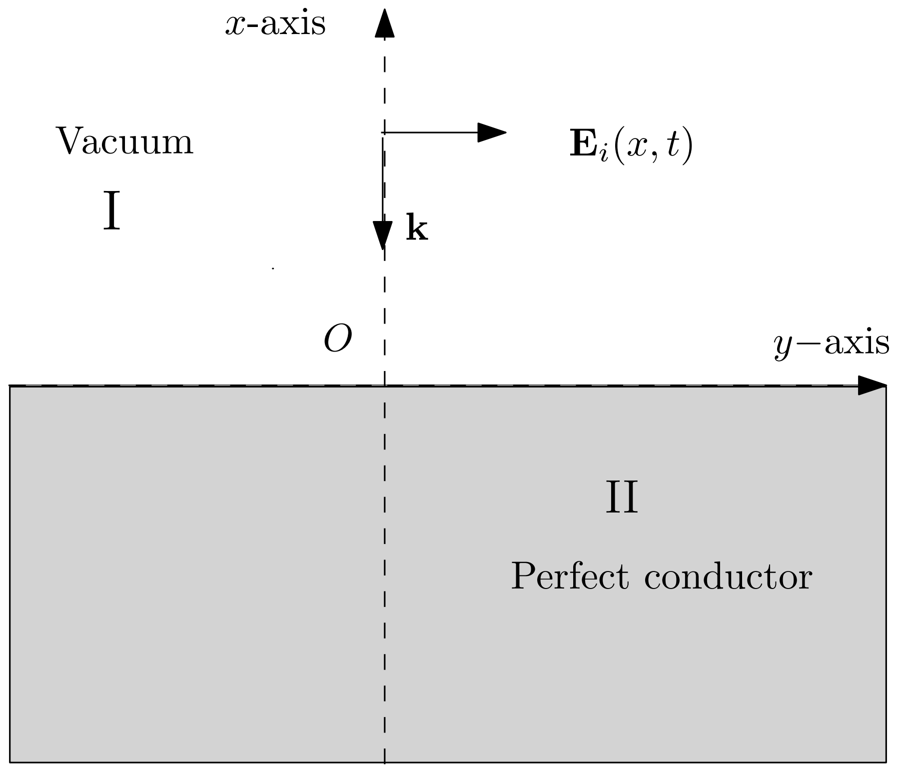

Figure 8.2

So,

$$\mathbf{E}_I(x,t) = -2\imath E_0 e^{-\imath\omega t}\sin(kx)\hat{y} = E_I\hat{y}, \qquad x \leq 0. \tag{8.6}$$

As expected, for perfect reflection a standing EM wave is formed. This field is indeed zero at the interface boundary since,

$$\mathbf{E}_I(0,t) = 0.$$

For the perfect conductor is known that $\mathbf{E}_{II}(x,t) = 0,\ x \leq 0$. This will result for the total electric field,

$$\mathbf{E}(0,t) = \mathbf{E}_I(0,t).$$

The latter field has calculated in the Eq. (8.6) above.

(b) Direct evaluation of the semi-algebraic ME provides the associated magnetic field in region I as,

$$\begin{aligned}\mathbf{B}_I(x,t) &= -\frac{\imath}{\omega}\nabla\times\mathbf{E}_I = -\frac{\imath}{\omega}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ 0 & E_I & 0\end{vmatrix} = -\frac{\imath}{\omega}(\hat{x}\partial_z E_I - \hat{z}\partial_z E_I)\\ &= -2\frac{E_0}{c}e^{-\imath\omega t}\cos(kx)\hat{z}\end{aligned}$$

It results that the magnetic field in the conductor's outer surface (region I) is the real part of,

$$\mathbf{B}_I(x=0,t) = -2\frac{E_0}{c}e^{-\imath\omega t}\cos\omega t\,\hat{z}$$

(c) Taking now the average of the density force, $\mathbf{f}(t) = \rho_s \mathbf{E}(0,t) + \mathbf{j}_s \times \mathbf{B}(0,t)$ we have,

$$
\begin{aligned}
\mathcal{P}_R = \frac{1}{2}\langle f \rangle = \quad & \frac{1}{2T}\int_0^T dt\, f(0,t) = \frac{1}{2}\left[\langle |\rho_s \cancelto{0}{\mathbf{E}}(0,t)|\rangle + \langle \mathbf{j}_s \times \mathbf{B}(0,t)\rangle\right] \\
= \quad & 0 + \frac{1}{2}\langle |\mathbf{j}_s \times (\mathbf{B}_I(0,z,t) + \cancelto{0}{\mathbf{B}_{II}}(0,t)|\rangle \\
= \quad & \frac{1}{2\mu_0}\langle |\hat{x} \times \mathbf{B}_I(0,t) \times \mathbf{B}_I(0,t)|\rangle = | - \hat{x}\langle \mathbf{B}^2(0,t)\rangle| = | - \frac{E_0^2}{\mu_0 c^2}| = \epsilon_0 E_0^2 \hat{x}.
\end{aligned}
$$

where in the last expression we have used $c^2 = 1/\mu_0\epsilon_0$ and that $\langle \cos^2 \omega t\rangle = 1/2$. From the above expression it results that the radiation pressure is equal to

$$\mathcal{P}_R = \epsilon_0 E_0^2.$$

The above expression is consistent with the relation $\mathcal{P}_R = 2I/c$ where $I = c\epsilon_0 E^2/2$ is the intensity of the pulse for the radiation pressure on totally reflective media.