# Accurate predictions of density field statistics with fast simulations

**Lena Einramhof**

**Oliver Hahn**

# Early Universe & CMB

Inflation

↓

Primordial fluctuations

↓

Structure

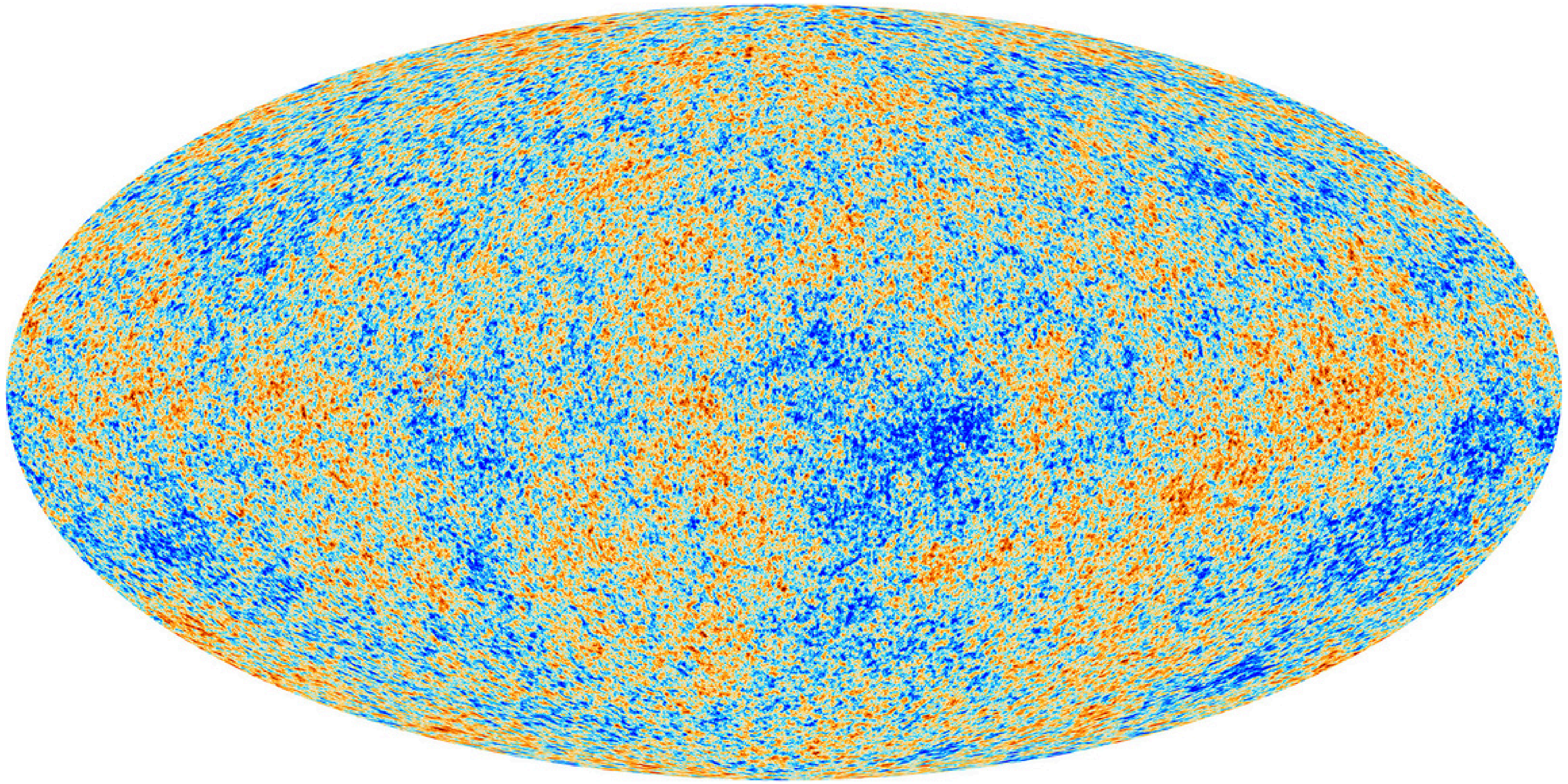


Image: @ESA and the Planck Collaboration

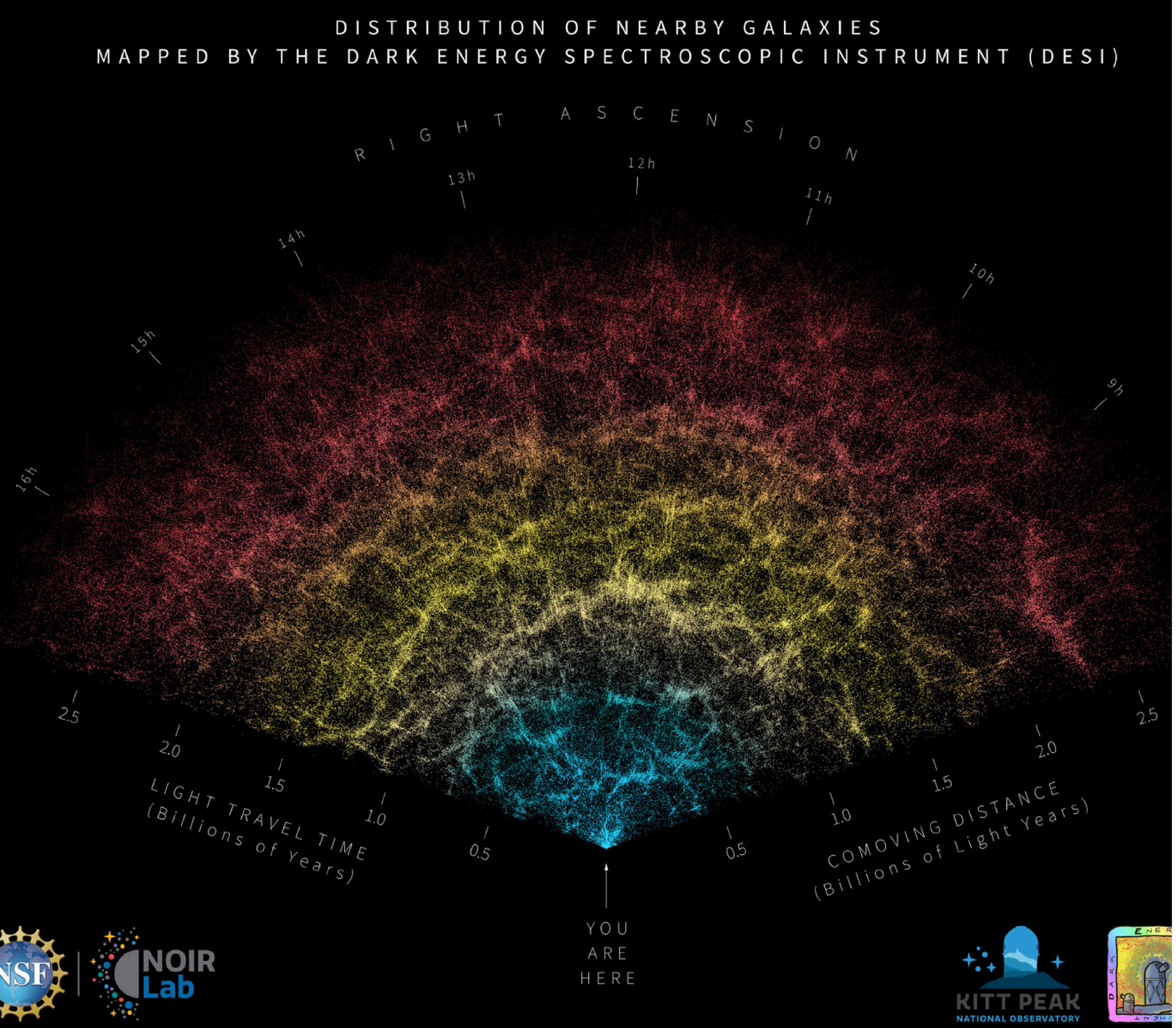



Image: DESI Collaboration / NOIRLab / NSF / AURA / R. Proctor.

# Large Scale Structure

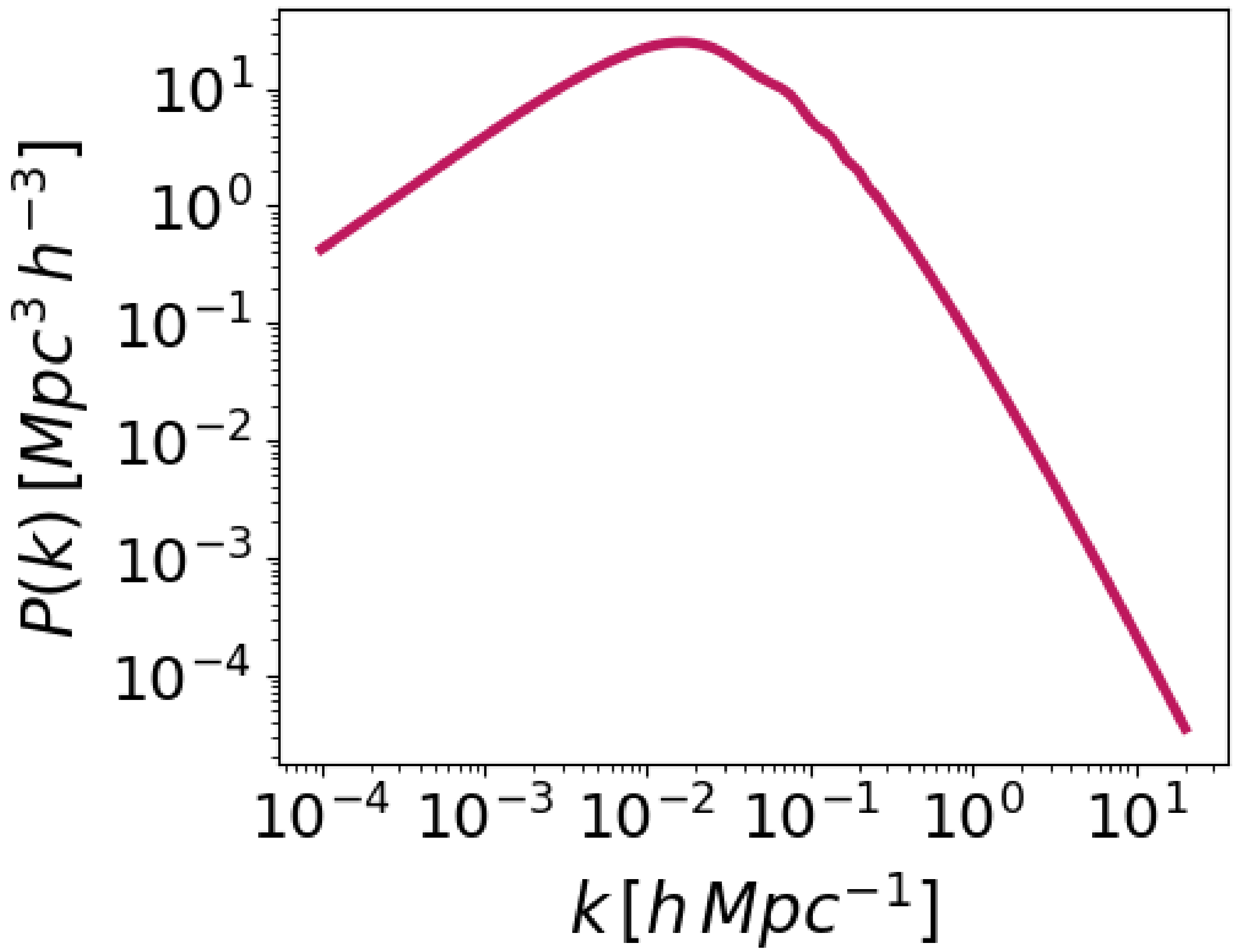
**Density contrast** $\delta(\vec{x},t)$

$\delta(\vec{x},t) \ll 1$ → linear growth (large scales)

$\delta(\vec{x},t) \gtrsim 1$ → non-linear growth (small scales)

# Matter Power Spectrum

How much power/variance is in a certain scale?

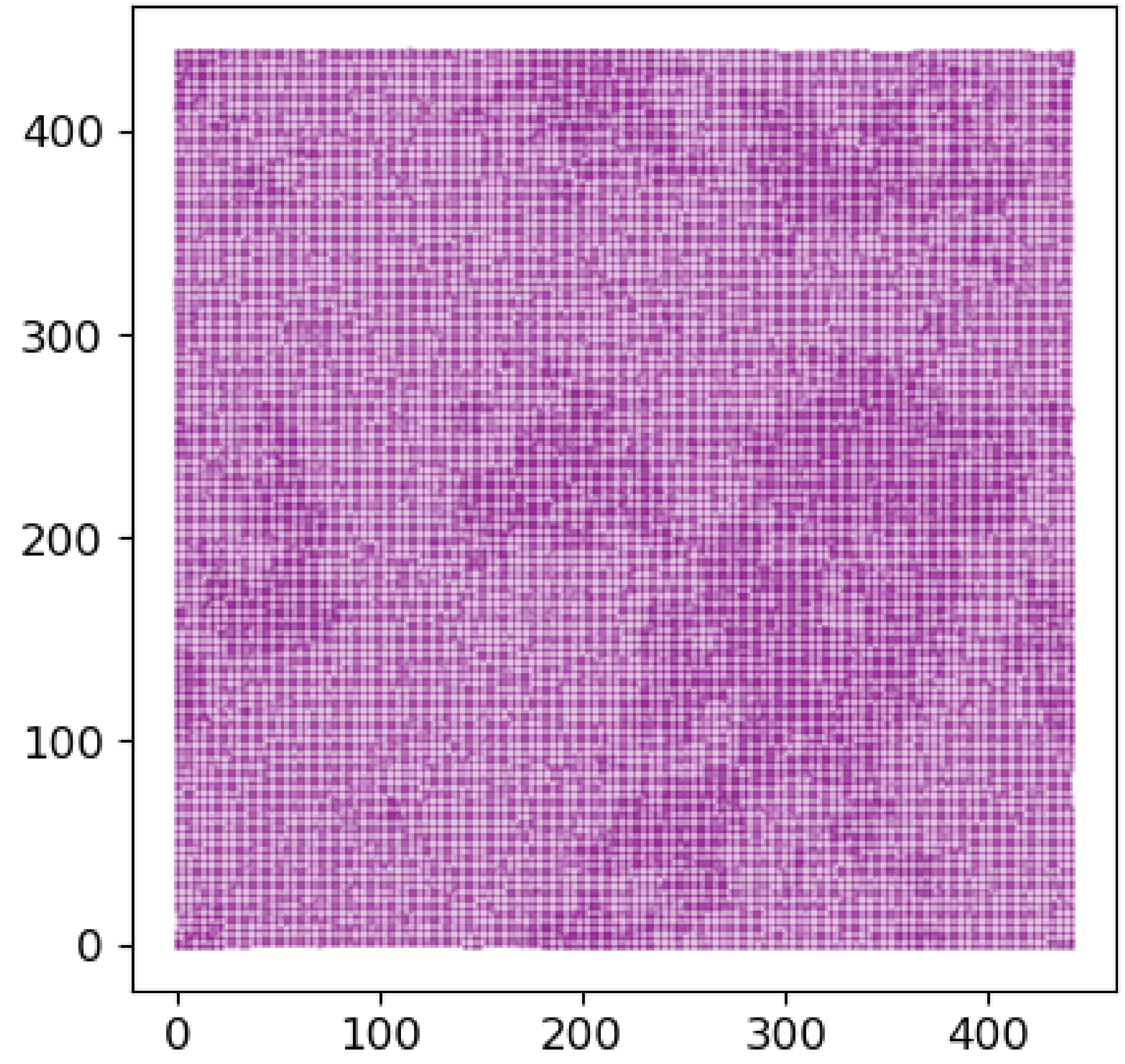



large scales small scales

→ fully characterizes the density fluctuations in early universe

# Cosmological Simulations

$z = 24$




Initial conditions:
Gaussian fluctuations

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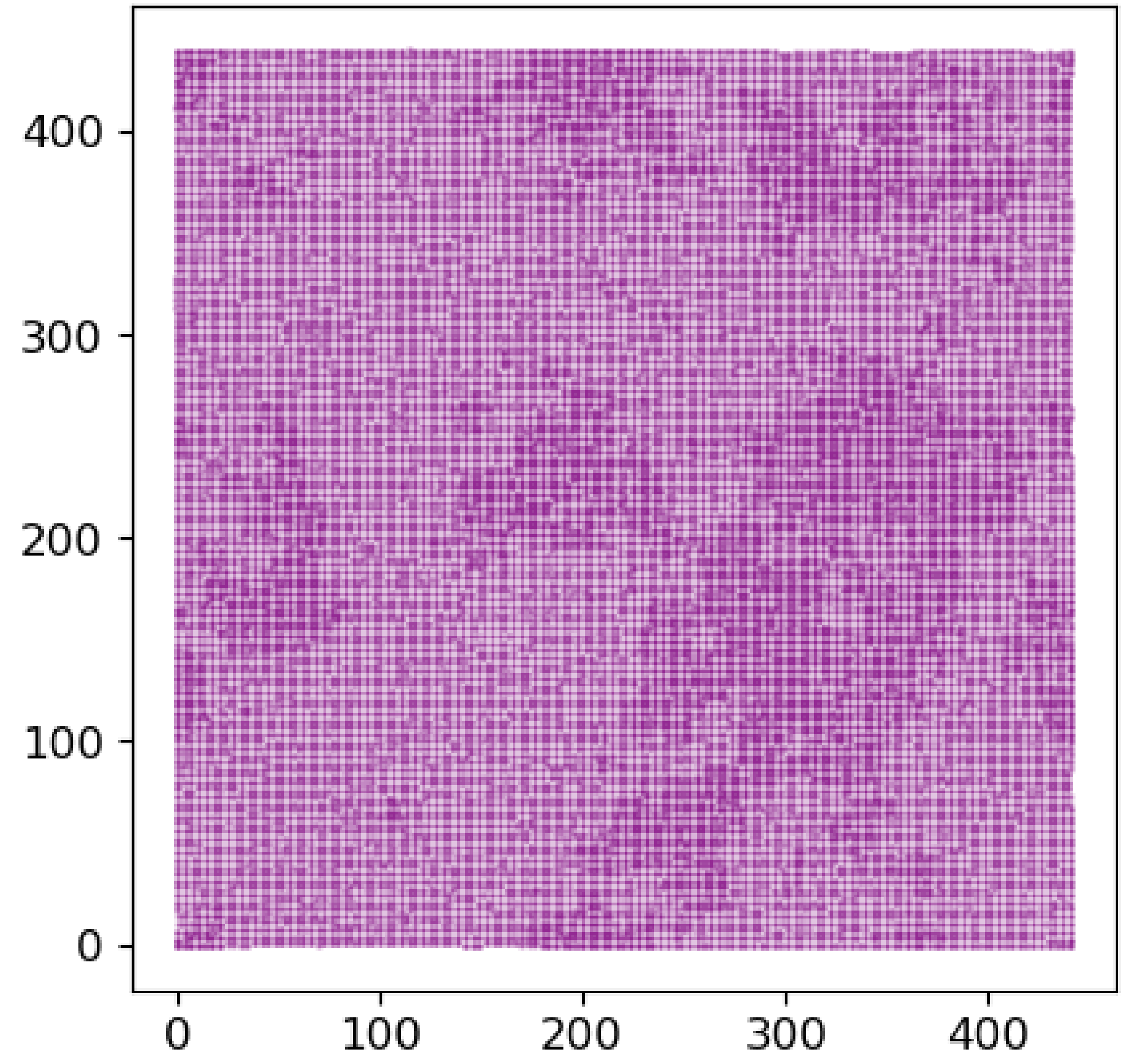


simulation

$\longrightarrow$

Initial conditions:
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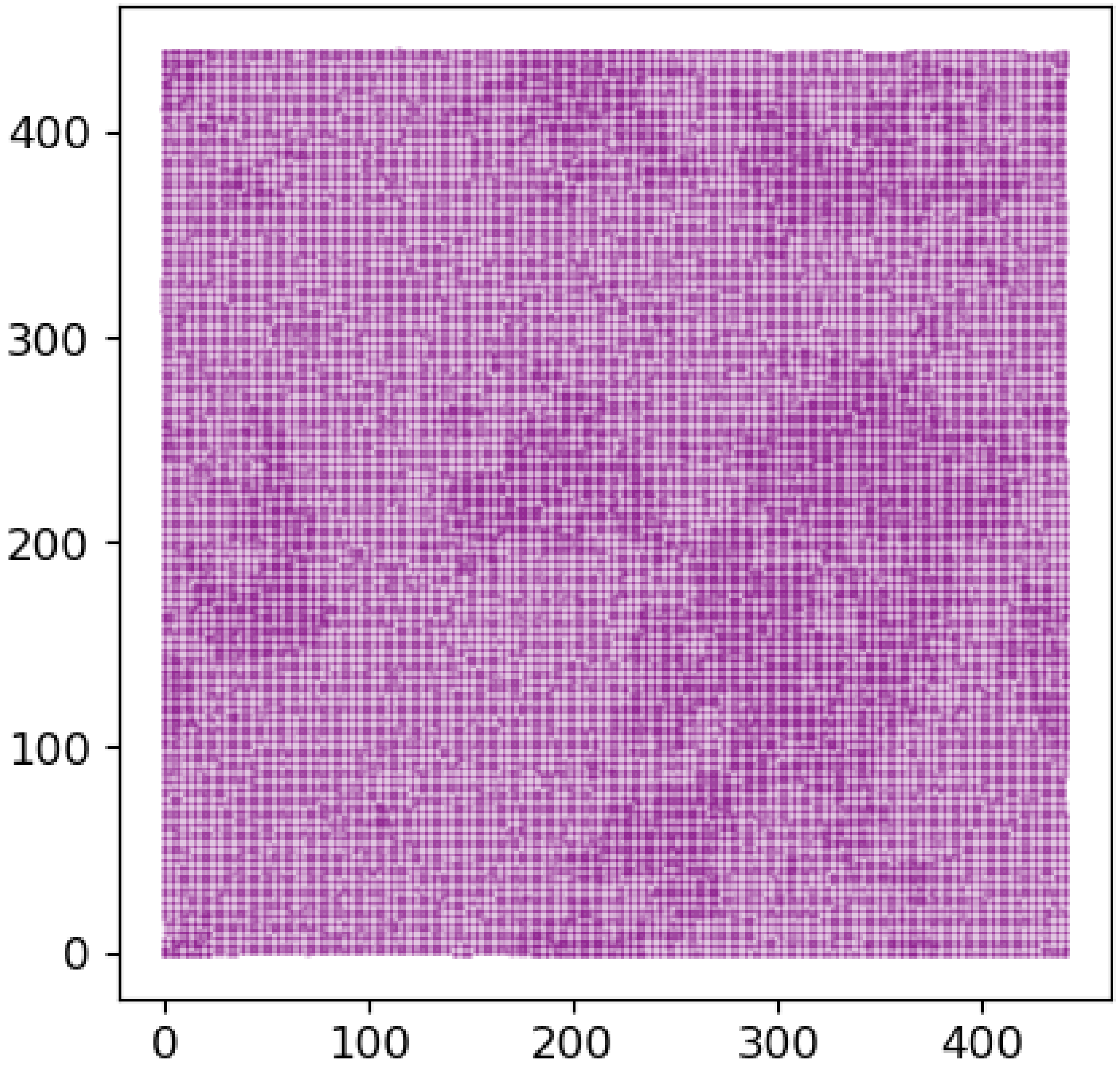
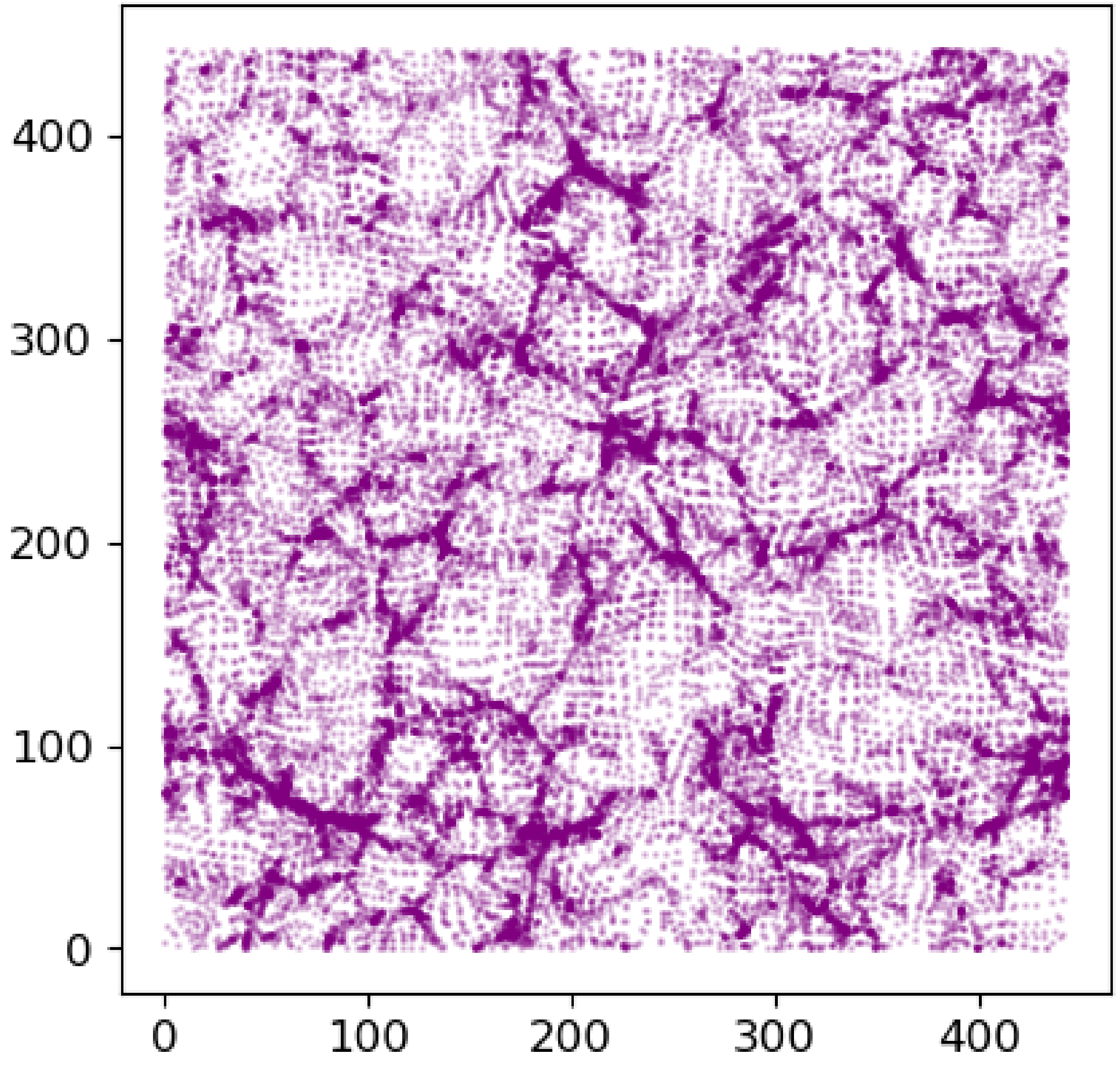



Initial conditions:
Gaussian fluctuations

$\xrightarrow{\text{simulation}}$

$z \approx 0$




Cosmic web:
filaments, voids, clusters

# Cosmological Simulations

Observational data $\longleftrightarrow$ Simulation

# Cosmological Simulations

Observational data ⟷ Simulation

→ **fast**

# Cosmological Simulations

Observational data ⟷ Simulation

→ **fast**

→ **accurate**

**How can the accuracy of cosmological simulations be maximized with minimal computational costs?**

# DISCO

## DIfferentiable Simulations for COsmology

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Cosmic Linear Anisotropy Solving System (CLASS)

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cosmology

$$\downarrow$$

$$P(k, z_{\mathrm{target}})$$

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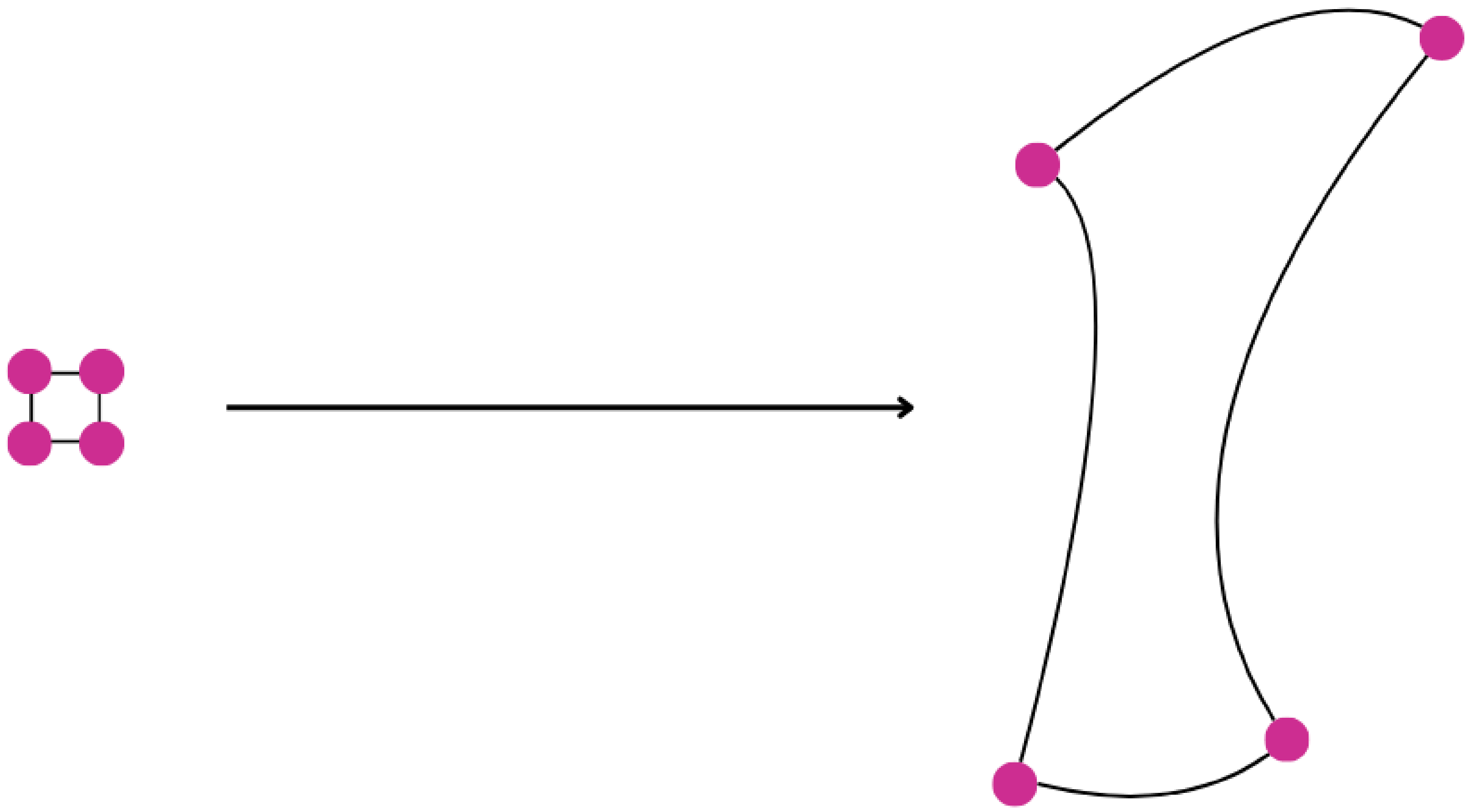
**n-th order Lagrangian PT**

# Lagrangian Perturbation Theory

Fluid approximation of particles

$$\mathbf{x}(\mathbf{q},t) = \mathbf{q} + \mathrm{D}_+^1 \Psi^{(1)}(\mathbf{q},t) + \mathrm{D}_+^2 \Psi^{(2)}(\mathbf{q},t) + \mathrm{D}_+^3 \Psi^{(3)}(\mathbf{q},t)$$



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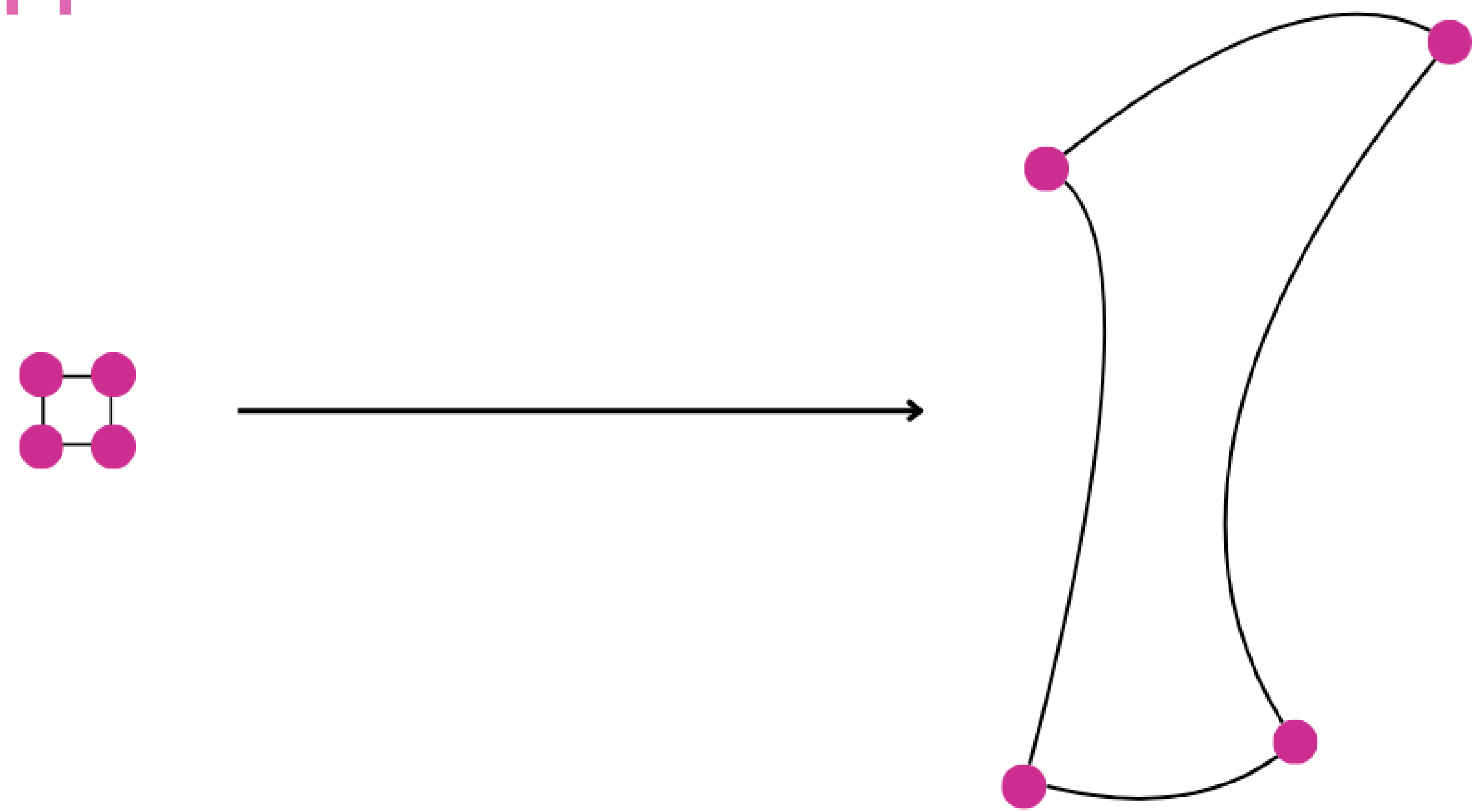
**Zel'dovich approximation**



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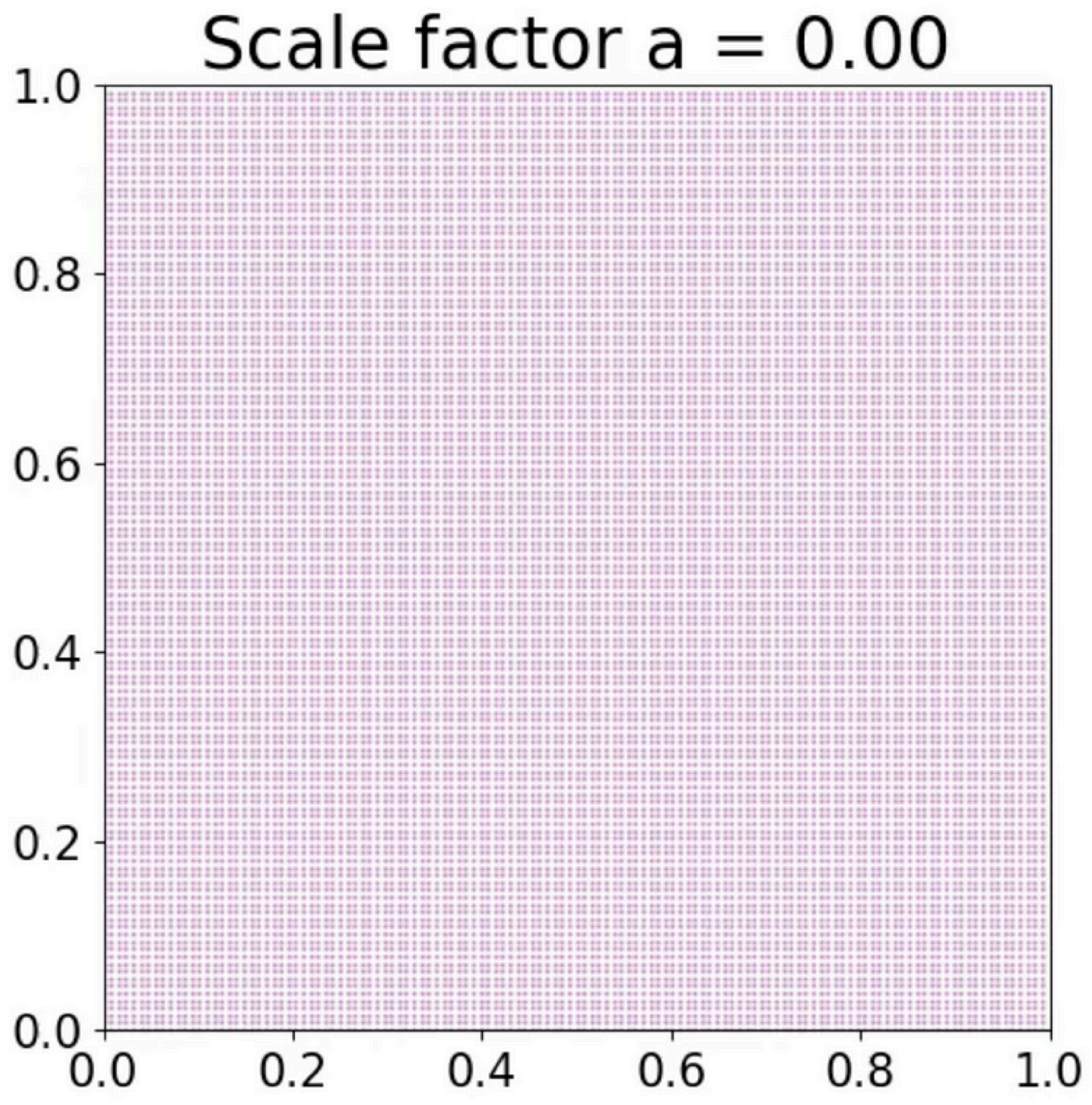

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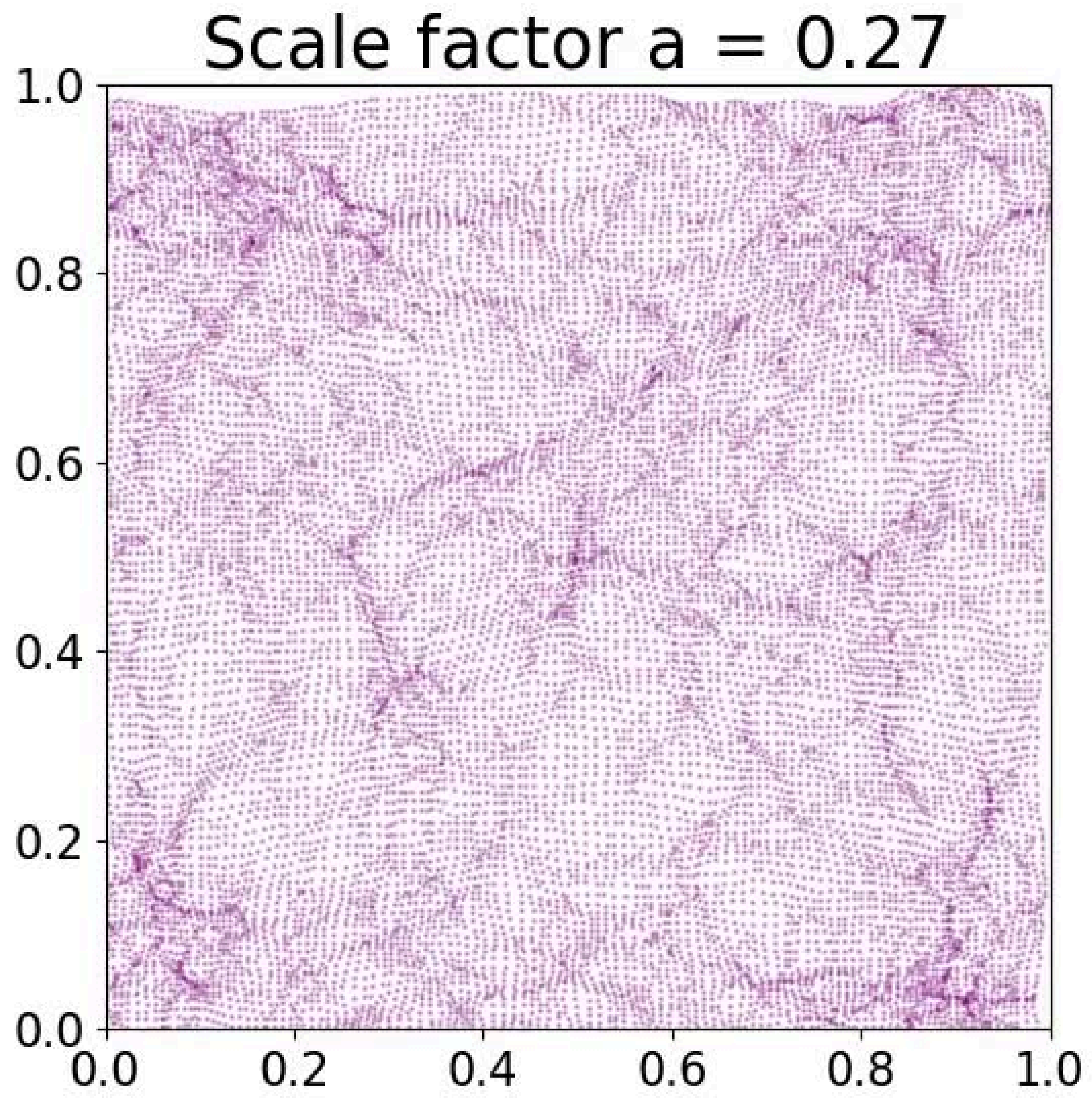

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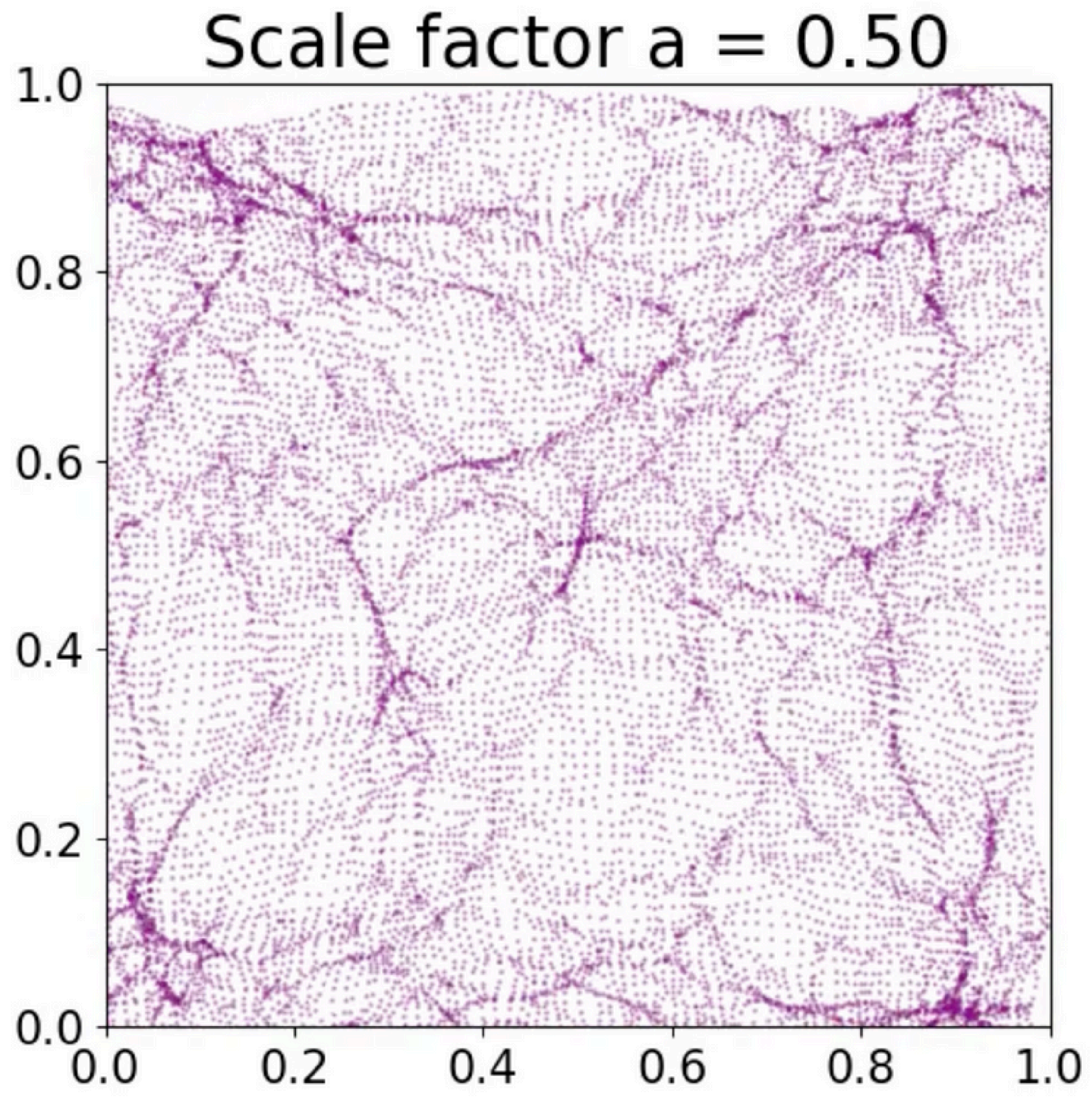
Zel’dovich approximation

→ **NOT** valid after shell crossing!



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cosmology

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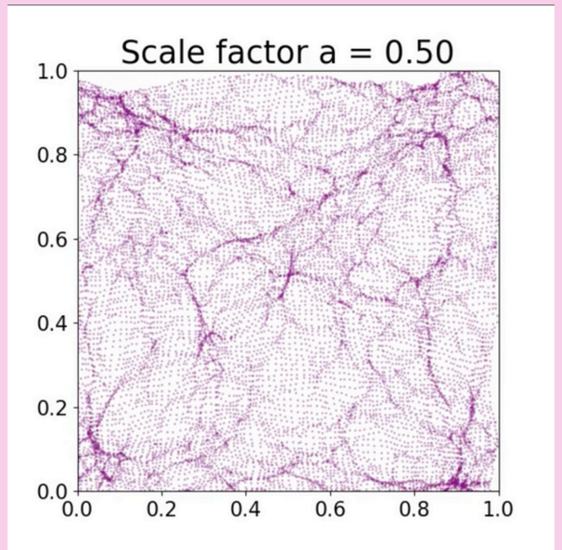
**n-th order Lagrangian PT**



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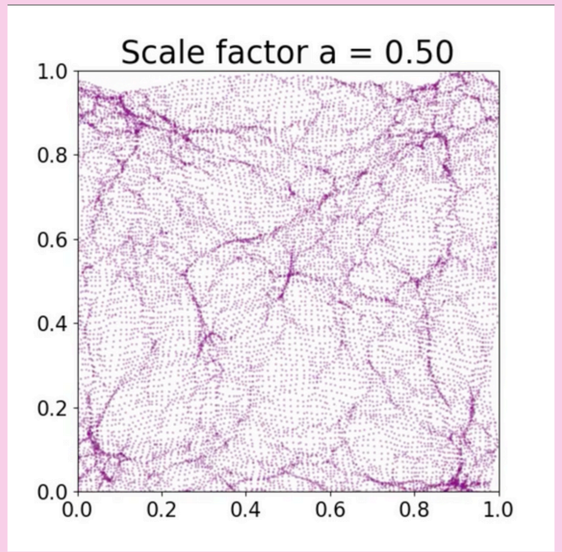


### N-Body - NUFFT

# N-Body Simulations

Discrete particles

1. Implementation of initial conditions
2. Calculation of force
3. Update position and velocities
4. Diagnostics
5. Repeat from 2. until simulation ends

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**Poisson equation**

$$\boxed{\Delta\Phi(\vec{x}) = \kappa\rho}$$

**Relation acceleration and potential**

$$\boxed{\vec{a} = -\nabla\Phi}$$

# N-Body Simulations

Discrete particles

$$\boxed{\Delta\Phi(\vec{x}) = \kappa\rho}$$

$$\boxed{\vec{a} = -\nabla\Phi}$$

$$\xrightarrow{\mathcal{F}} \quad \hat{\vec{a}} = \kappa \frac{i\vec{k}}{||\vec{k}||^2}\hat{\rho} \quad \xrightarrow{\mathcal{F}^{-1}} \quad \vec{a}$$

→ Non-Uniform Fast Fourier Transform (NUFFT)

(Barnett et al., 2019)

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**Drift-Kick-Drift**

$$\vec{x}_{n+\frac{1}{2}} = \vec{x}_n + \frac{1}{2}\tau\vec{v}_n$$

$$\vec{v}_{n+1} = \vec{v}_n + \tau\vec{a}(\vec{x}_{n+\frac{1}{2}})$$

$$\vec{x}_{n+1} = \vec{x}_{n+\frac{1}{2}} + \frac{1}{2}\tau\vec{v}_{n+1}$$

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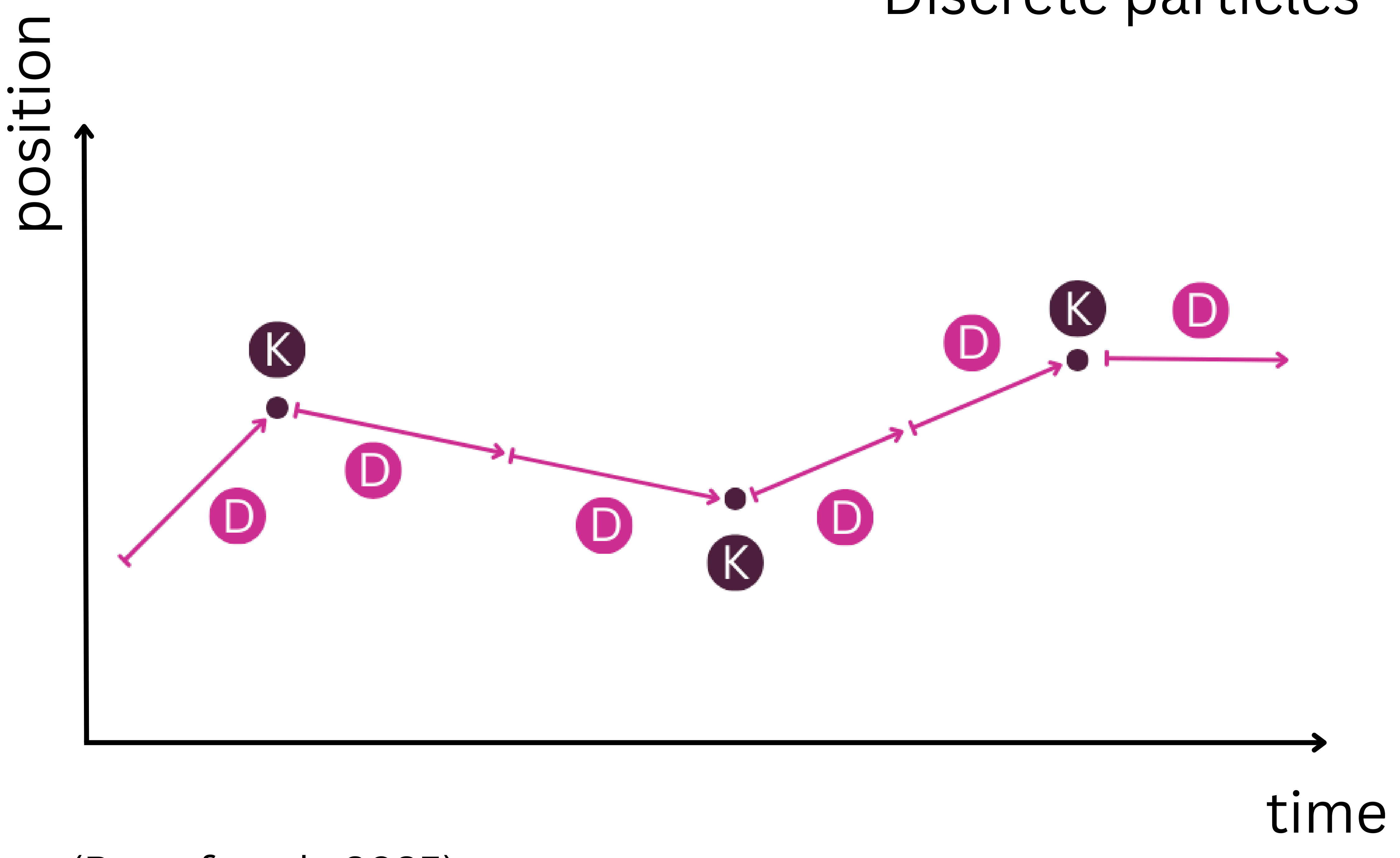


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(Rampf et al., 2025)

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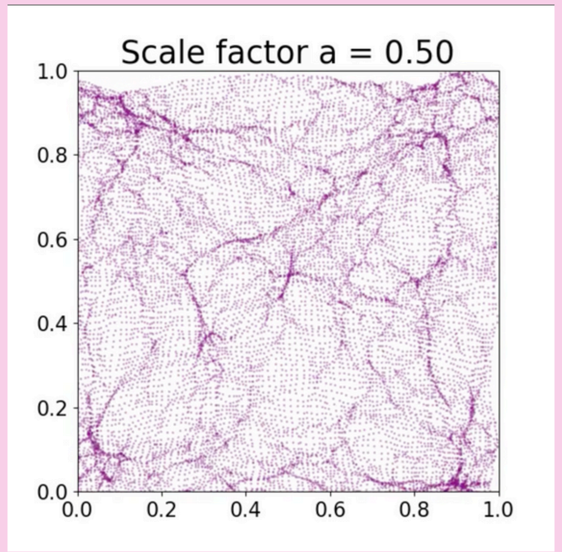


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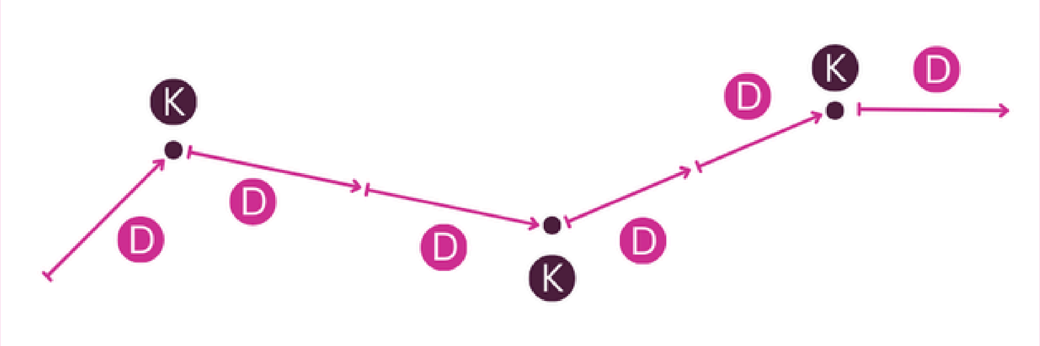

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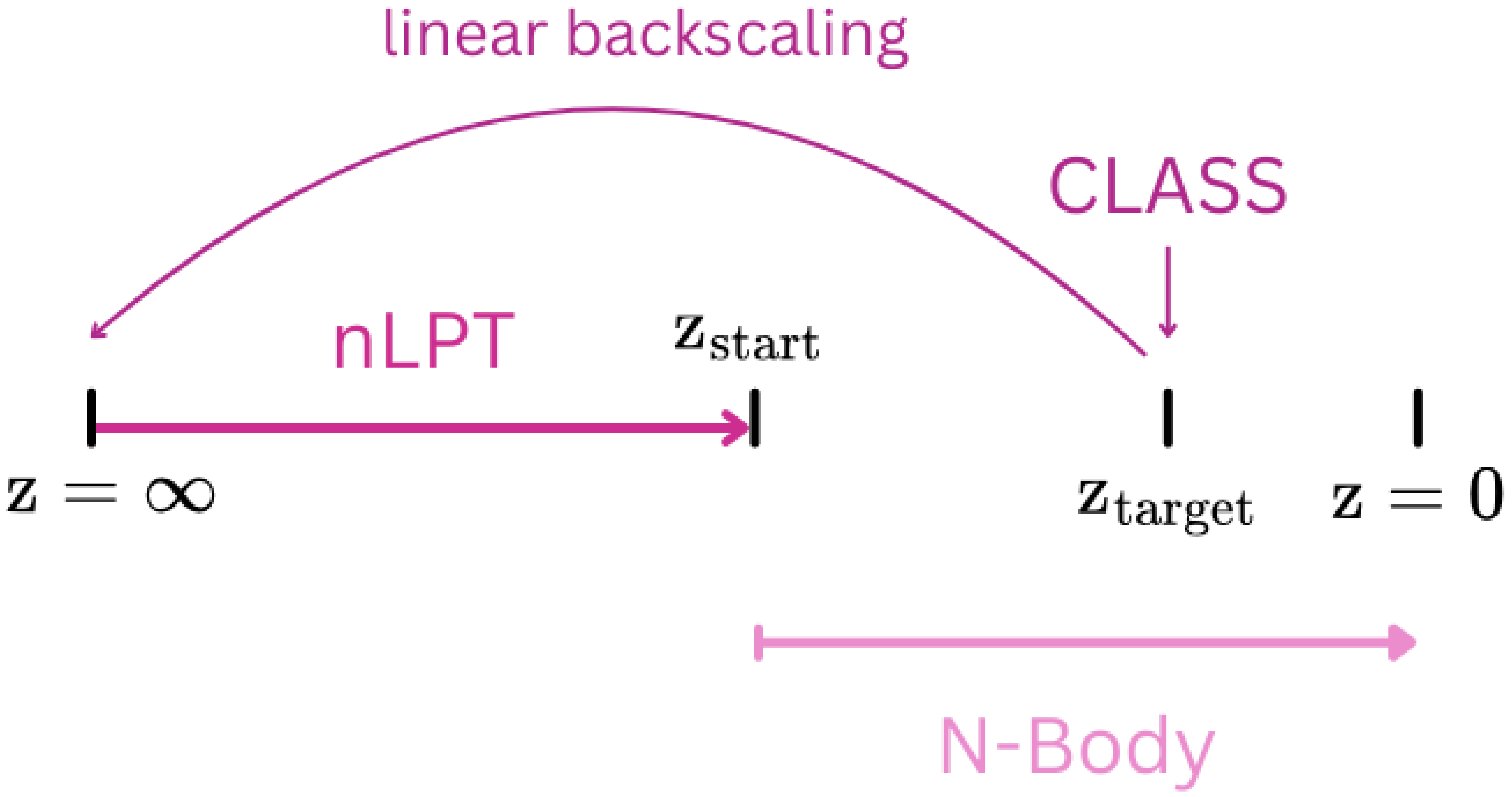
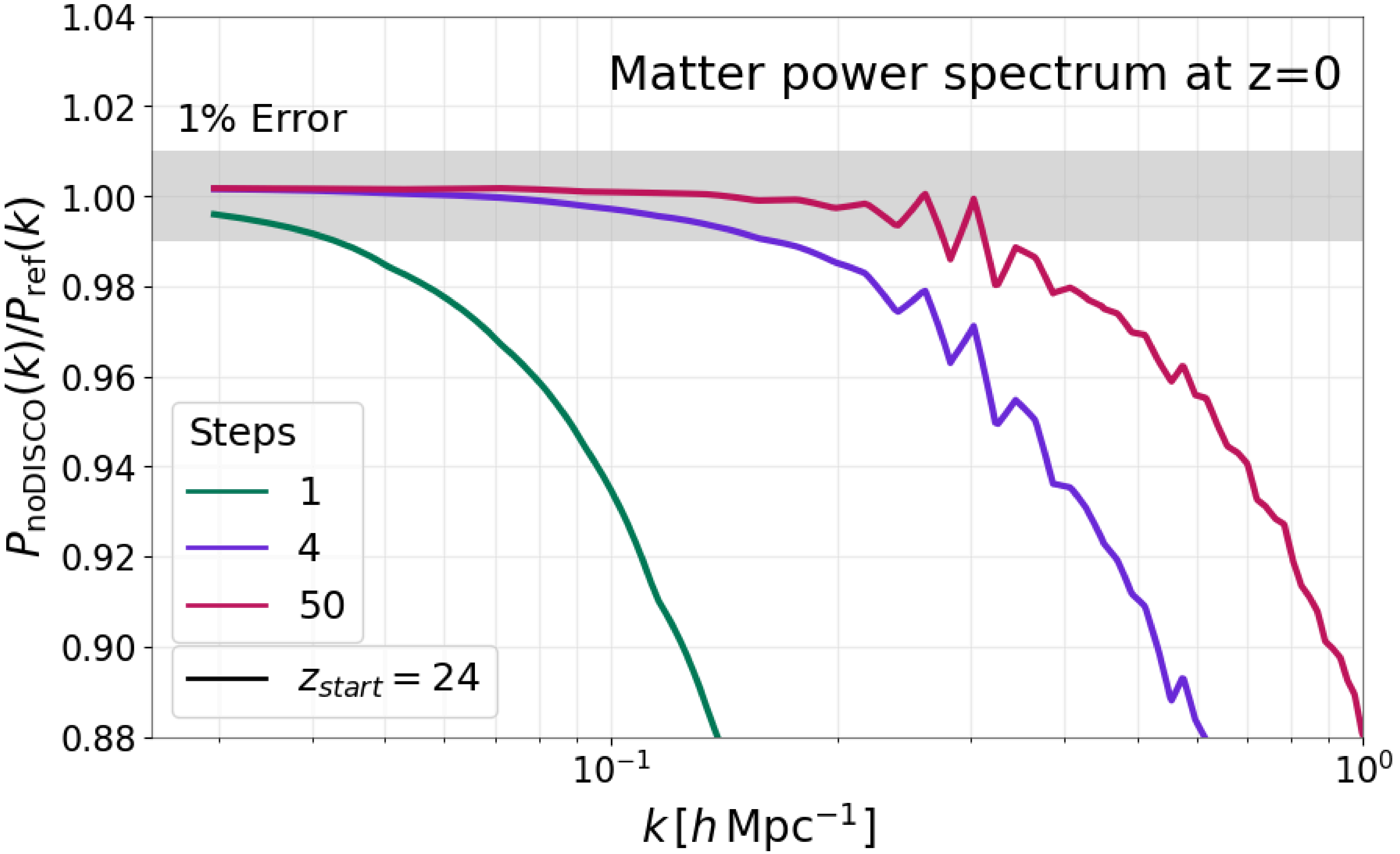
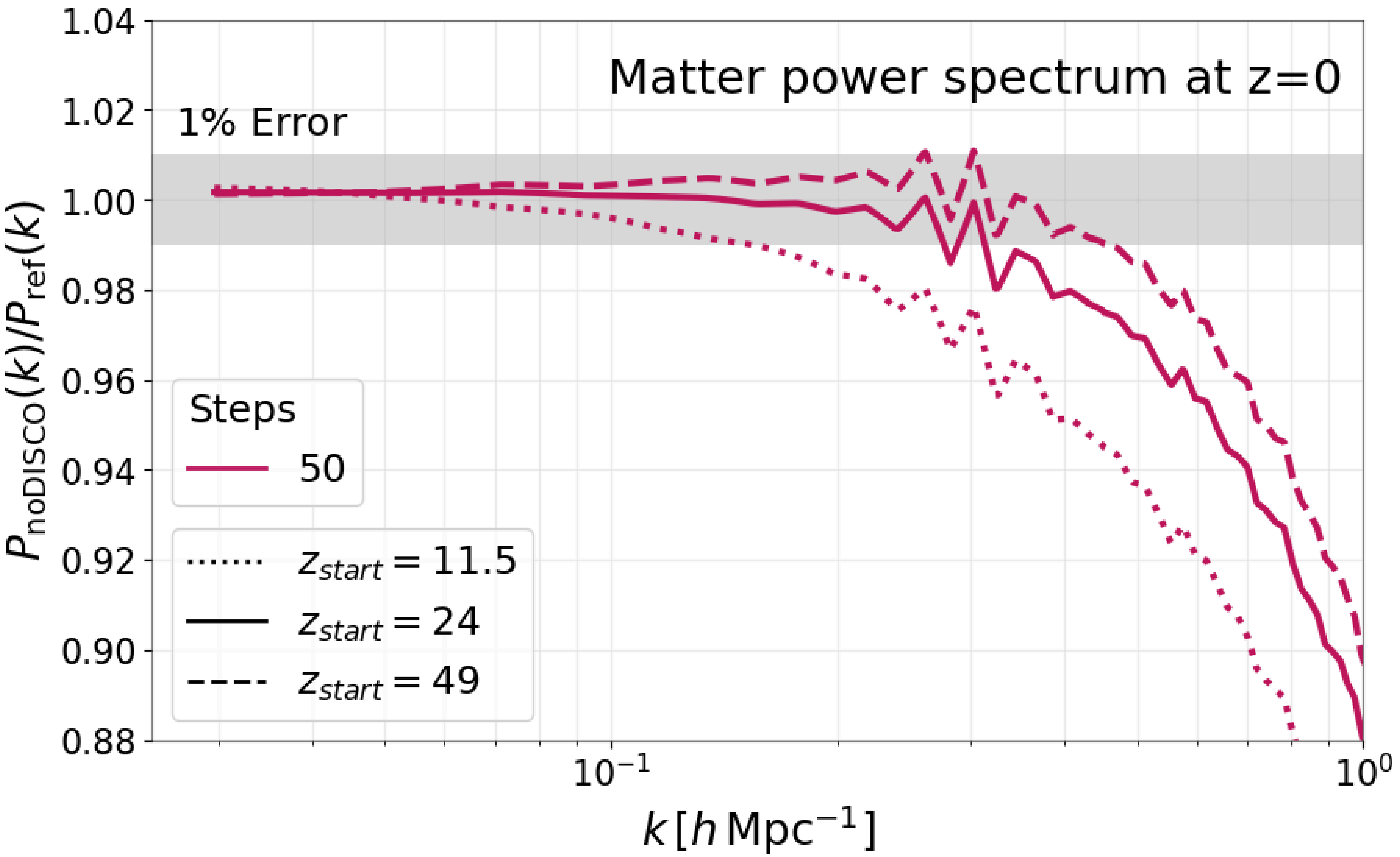


# 1% Accuracy? - Time Steps



# 1% Accuracy? - Starting Redshift



# Challenge

**Computational cost vs. accuracy**

# What else could be done?

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**Initial condition accuracy**

- Compare power spectra for different LPT orders and starting redshifts
- Determine shell crossing time

(Michaux et al., 2020)

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**Initial condition accuracy**

- Compare power spectra for different LPT orders and starting redshifts
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**Beyond the power spectrum**

- Higher order statistics: bispectrum/trispectrum
- One-point statistics: cumulants

(Michaux et al., 2020)

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1. Fast simulations are crucial for **modeling them to observations** and thus understanding **complex non-linear structures**

2. More steps & higher starting redshift → greater range of scales/ smaller scales covered more accurately

3. Challenge: **computational cost vs. accuracy**

# Key literature

noDISCO source code (not publically available yet)

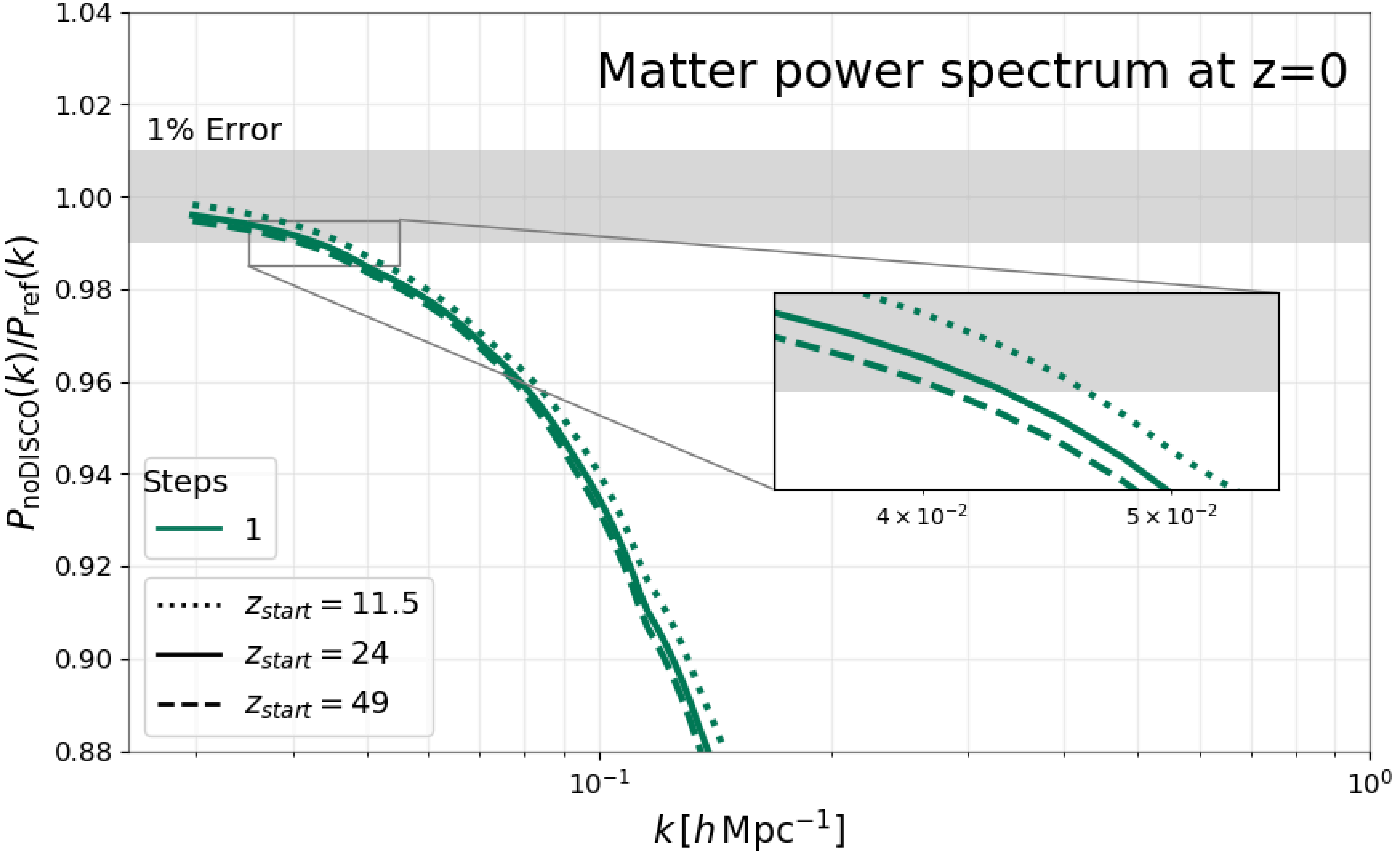

Rampf, C., List, F., and Hahn, O. (2025). BullFrog: Multi-step perturbation theory as a time integrator for cosmological simulations. Journal of Cosmology and Astroparticle Physics, 2025(02):020. arXiv:2409.19049 [astro-ph]

Michaux, M., Hahn, O., Rampf, C., and Angulo, R. E. (2020). Accurate initial conditions for cosmological N-body simulations: Minimizing truncation and discreteness errors. Monthly Notices of the Royal Astronomical Society, 500(1):663–683. arXiv:2008.09588 [astro-ph].

Barnett, A. H., Magland, J. F., and Klinteberg, L. a. (2019). A parallel non-uniform fast Fourier transform library based on an "exponential of semicircle" kernel. arXiv:1808.06736 [math].

# Additional slides

# 1% Accuracy? - Starting Redshift



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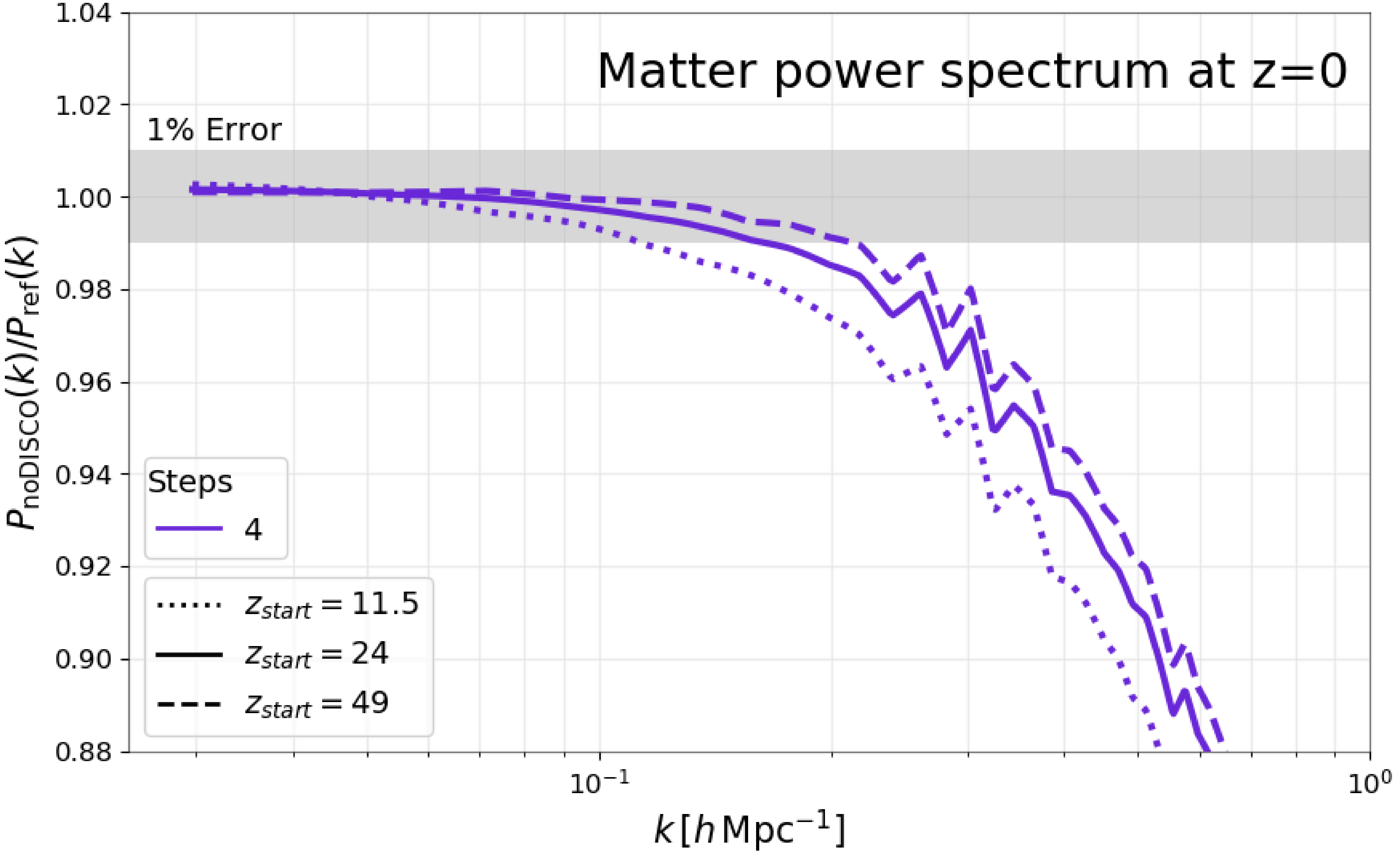

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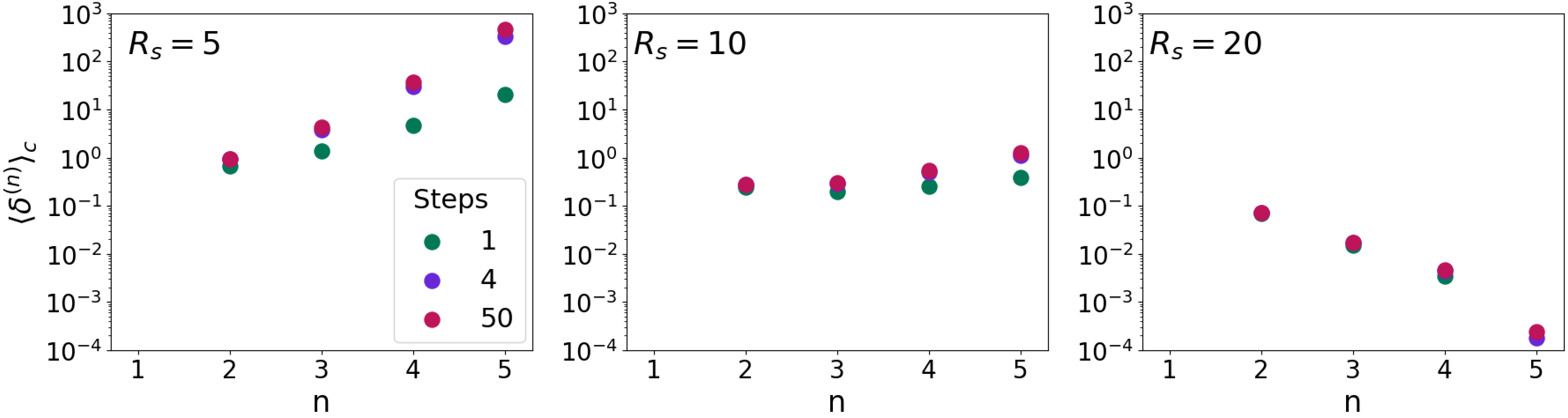
## One-point statistics: cumulants



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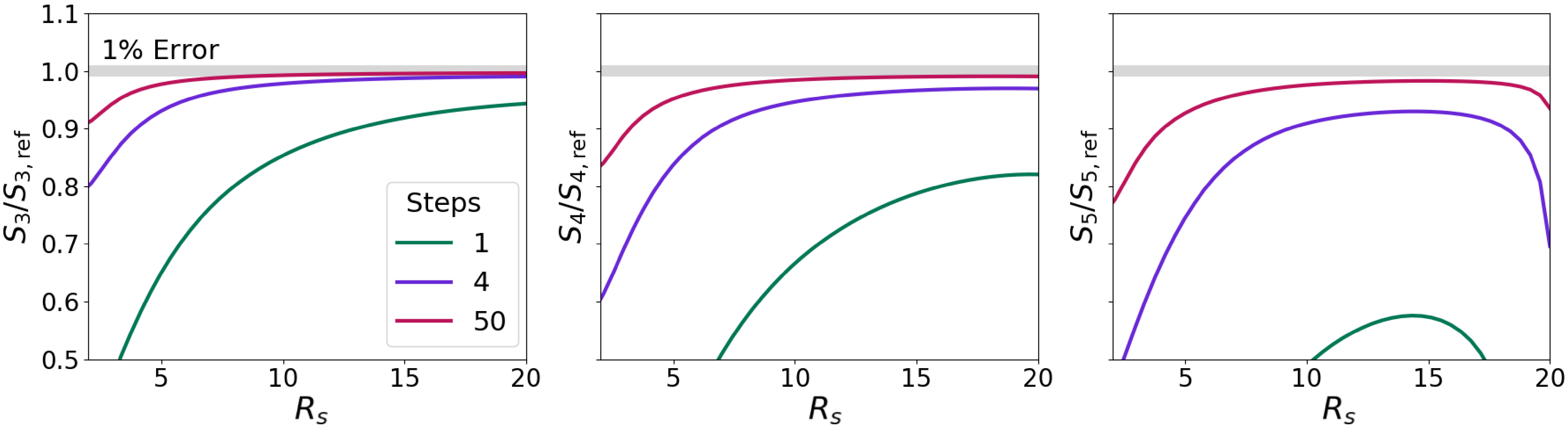

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