Quadratic Programming (QP)

Big Picture

1. QP is a special case of nonlinear programming
2. The objective function is quadratic, a second order polynomial of decision variables
3. Global minimum exists <u>if</u> the quadratic form is positive definite (or the function is strictly convex)
4. There may be constraints, which may or may not be binding

Example 1: Unconstrained QP

1. For example, consider minimizing a quadratic function without constraints
$$\min\ -8x_1 - 16x_2 + x_1^2 + 4x_2^2 \tag{1}$$
2. To see why this function has a minimum, we complete the square, and rewrite it as
$$-8x_1 - 16x_2 + x_1^2 + 4x_2^2 = (x_1 - 4)^2 + 4(x_2 - 2)^2 - 32 \geq -32 \tag{2}$$
where the last equality holds when $x_1 = 4, x_2 = 2$. The global minimum value is -32.
3. Using jargon, there is a global minimum because the objective function is strictly convex.
4. Alternatively, we can solve this problem by taking (partial) derivatives, and setting them to zero
$$\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0 \Rightarrow x_1 = 4, x_2 = 2 \tag{3}$$
5. Next we show how R uses matrix form to solve QP problem

Inner Product and Quadratic Form I

From matrix algebra, we define inner product as a row vector multiplied by a column vector. Inner product is a scalar (a 1×1 matrix)

$$(x_1, x_2) \begin{pmatrix} a \\ b \end{pmatrix} = ax_1 + bx_2 \tag{4}$$

A quadratic form is also a scalar, and it is a row vector multiplied by a squared matrix multiplied by the transpose of the row vector

$$(x_1, x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2 \tag{5}$$

Exercise: use definition of inner product (4) to prove (5)

Inner Product and Quadratic Form II

1. We can use inner product to rewrite

$$-8x_1 - 16x_2 = -(8, 16)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} 8 \\ 16 \end{pmatrix}'\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{6}$$

   where A' is the transpose of A. The transpose of a column vector is a row vector.

2. Moreover, we can use quadratic form to rewrite

$$x_1^2 + 4x_2^2 = (x_1, x_2)\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7}$$

3. We organize our matrix in such a special way in order to use R package quadprog

R Package "quadprog"

The R package "quadprog" can solve the QP with the form of

$$\min\ -d'b + \frac{1}{2}b'Db \quad (\texttt{with constraint } A'b \geq b_0) \tag{8}$$

where

1. b is the column vector of decision variables
2. D is the square matrix in the middle of quadratic form <u>multiplied by 2</u>
3. d is the <u>column</u> vector specifying the linear part in the objective function
4. The constrain is given by $A'b \geq b_0$

For Example 1, $b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}, d = \begin{pmatrix} 8 \\ 16 \end{pmatrix}, A = 0, b_0 = 0.$

R codes

```
library(quadprog)
Dmat = matrix(c(2,0,0,8),2,2,byrow=T)
dvec = c(8,16)
Amat = matrix(c(0,0,0,0),2,2,byrow=T)
bvec = c(0,0)
solve.QP(Dmat,dvec,Amat,bvec)

$`solution`
[1] 4 2

$value
[1] -32
```

Note that Amat is a zero matrix, and bvec is a zero column vector. We get the same answer as using the calculus.

Example 2: a QP problem that has no minimum

Let's change D and see what happens

```
Dmat = matrix(c(2,4,4,8),2,2,byrow=T)
solve.QP(Dmat,dvec,Amat,bvec)


Error in solve.QP(Dmat, dvec, Amat, bvec) :
matrix D in quadratic function is not positive definite!
```

Exercise:

1. write down the objective function
2. is there constraint?
3. take partial derivatives, and set them to zero. Can you solve them?
4. we have trouble here because D is not positive definite!

Positive Definite Matrix–I

To investigate this issue, let's complete the square for a general second-order polynomial

$$ax_1^2 + bx_1x_2 + cx_2^2 = a\left(x_1 + \frac{b}{2a}x_2\right)^2 + \left(c - \frac{b^2}{4a}\right)x_2^2 \tag{9}$$

It is positive for all x_1, x_2 (so a global minimum exists) only if

$$a > 0,\ \texttt{and}\ c - \frac{b^2}{4a} > 0 \tag{10}$$

Using quadratic form we have

$$ax_1^2 + bx_1x_2 + cx_2^2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{11}$$

Positive Definite Matrix–II

1. The matrix $\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ is <u>positive definite</u> if and only if all its <u>leading principal minors</u> are strictly positive, i.e.,

$$a > 0, \texttt{ and } ac - \frac{b}{2}\frac{b}{2} > 0,$$

   which is the same as (10). In short, there is global minimum only if the square matrix in the quadratic form is positive definite.

2. Exercise: show the square matrix is NOT positive definite for Example 2

Example 3: Constrained QP with non-binding constraints

1. Still consider

$$\min \; -8x_1 - 16x_2 + x_1^2 + 4x_2^2$$

2. but now we add constraints

$$x_1 + x_2 \geq 5, x_1 \geq 3, x_1 \geq 0, x_2 \geq 0 \tag{12}$$

3. Note that the <u>unconstrained</u> solution $x_1 = 4, x_2 = 2$ shown on page 7 satisfies (12). So constraint (12) has no effect, so is <u>non-binding</u>

Example 3: R

```
Dmat = matrix(c(2,0,0,8),2,2,byrow=T)
dvec = c(8,16)
Amat = matrix(c(1,1,1,0),2,2,byrow=T)
bvec = c(5,3)
solve.QP(Dmat,dvec,Amat,bvec)
```

```
$`solution`
[1] 4 2
$value
[1] -32
```

Pay attention how we define Amat and bvec.

Amat and bvec

The way we define Amat and bvec is based on the matrix form of the constraint: The first two constraints in (12) can be rewritten as

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 5 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Exercise: use definition of transpose matrix and inner product (4) to simplify the above inequality

Example 4: Some constraints are binding

```
Dmat = matrix(c(2,0,0,8),2,2,byrow=T)
dvec = c(8,16)
Amat = matrix(c(1,1,1,0),2,2,byrow=T)
bvec = c(5,4.5)
solve.QP(Dmat,dvec,Amat,bvec)

$`solution`
[1] 4.5 2.0
$value
[1] -31.75
```

Can you write down the constraints (Hint: look at Amat and bvec)? Here only x_1 changes to satisfy the new constraint. Note that the unconstrained minimum value -32 is <u>less</u> than the constrained minimum value -31.75. Does that make sense?

Example 5: Some constraints are binding

```
Dmat = matrix(c(2,0,0,8),2,2,byrow=T)
dvec = c(8,16)
Amat = matrix(c(1,1,1,0),2,2,byrow=T)
bvec = c(7,3)
solve.QP(Dmat,dvec,Amat,bvec)
```

```
$`solution`
[1] 4.8 2.2
$value
[1] -31.2
```

Can you write down the constraints? Here both x_1 and x_2 change to satisfy the new constraint.

Homework:

Still consider

$$\min \ -8x_1 - 16x_2 + x_1^2 + 4x_2^2$$

but the new constraint is

$$x_1 + x_2 \leq 5, x_1 \leq 3, x_1 \geq 0, x_2 \geq 0 \tag{13}$$

Please use R to solve this QP problem. (Hint: $a < b$ is the same as $-a > -b$)