Best synthetic control for VAR models

Consider the VAR(1) model

$$y_t = \mu + A_1 y_{t-1} + u_t$$

where $u_t \sim (0, \Sigma)$ and $\Sigma = RDR'$ such that R is lower diagonal with ones on the main diagonal and D is a diagonal matrix. Accordingly,

$$\begin{aligned} R^{-1} y_t &= \mu^* + C y_{t-1} + \varepsilon_t \\ y_t &= \mu^* + \underbrace{(I - R^{-1})}_{B} y_t + C y_{t-1} + \varepsilon_t \ . \end{aligned}$$

Assume an ordering such that the treated unit occors at the end. It follows that the optimal forecast of the counterfactual at $t = T_0 + 1$ given the information before the intervention time T_0 is

$$\widehat{y}_{K,T_0+1} = \mu_K^* + \sum_{i=1}^{K-1} b_i y_{i,T_0+1} + \sum_{i=1}^{K} c_i y_{i,T_0}$$

For $t = T_0 + 2$ the optimal prediction is

$$\begin{aligned} \widehat{y}_{K,T_0+2} &= \mu_K^* + \sum_{i=1}^{K-1} b_i y_{i,T_0+2} + \sum_{i=1}^{K-1} c_i y_{i,T+1} + c_K \widehat{y}_{K,T_0+1} \\ &= \mu_K^* + \sum_{i=1}^{K-1} b_i y_{i,T_0+2} + \sum_{i=1}^{K-1} c_i y_{i,T+1} + \sum_{i=1}^{K} d_i y_{i,T_0} \end{aligned}$$

It is important to notice, that the lag of the dependent variable needs to increase as for this variable we can only include the information up to period T_0. Since the forecast of this variable depends on the lagged value of all other variables, we need to include all variables at period T_0. Accordingly the best forecast needs to include more and more variables the higher the forecast horizon is.

It follows that the optimal SC for $T_0 + 1$ can be obtained by minimizing

$$\sum_{t=2}^{T_0}(y_{K,t} - \mu_K^* - \sum_{i=1}^{K-1} b_i y_{i,t} - \sum_{i=1}^{K} c_i y_{i,t-1})^2$$

It follows that the standard SC method ignores the lagged values $y_{i,t-1}$ (and the constant μ_K^*). Similar conclusions can be obtained for $t > T_0 + 1$.