Finite N Sachdev-Ye-Kitaev model

Subject: Matrix diagonalization, sparse arrays, function compilation.

Introduction

The Sachdev-Ye-Kitaev (SYK) model is a $0+1$ dimensional model of N Majorana fermions ψ_i for $i = 1, 2, \ldots, N$, satisfying the anti-commutation relations

$$\{\psi^i, \psi^j\} = \delta^{ij}\,, \tag{1}$$

with random, all-to-all quartic interactions. The Hamiltonian is

$$H_N = \sum_{i<j<k<l} J_{ijkl}\psi^i\psi^j\psi^k\psi^l\,, \tag{2}$$

where J_{ijkl} are $\binom{N}{4}$ couplings, taken to be independent random variables drawn from a Gaussian distribution with zero mean and fixed variance[1] $\sigma^2 = 6J^2/N^3$. The quantity J appearing here is usually called "the coupling" of the theory, and we will hereafter set it to $J = 1$ thus fixing the energy units.

The model is defined with quenched disorder, so that expectation values are computed by averaging over the results obtained for specific realizations of the coupling,

$$\langle\cdots\rangle = \int \mathcal{D}J\,\langle\cdots\rangle_{\{J_{ijkl}\}} \qquad \text{with} \qquad \mathcal{D}J = \prod_{i<j<k<l} \frac{e^{-\frac{J_{ijkl}^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dJ_{ijkl}\,. \tag{3}$$

It is in this sense that $\langle J_{ijkl}\rangle = 0$ and $\langle J_{ijkl}^2\rangle = \sigma^2$.

[1] The N^{-3} scaling is used here to ensure a well-defined large-N limit, with a factor of 6 included for convenience.

The Hilbert space of N Majorana fermions in $0+1$ dimensions is $L = 2^{\lfloor N/2 \rfloor}$ dimensional, so that for small values of N we can explicitly write down Hamiltonians such as (2) and diagonalize them exactly to analyze their spectral properties. In this problem, we will reproduce some of the results obtained in this way and first presented in [1, 2].

Problem statement

In order to write down the SYK Hamiltonian for finite values of N, we will first need to construct a matrix representation of the Majorana fermions satisfying the anti-commutation relations (1). Up to a normalization, we can use any representation of the higher-dimensional generalization of Dirac's gamma matrices, which satisfy $\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$. Once the matrix representation of Majorana fermions is available, building specific realizations of the Hamiltonian (2) is straightforward.

Before diagonalizing the Hamiltonians we construct, one should notice that for even values of N there is a residual particle-hole symmetry.[2] It is therefore convenient to *deflate* the $L \times L$ matrix by rearranging its rows and columns to put it into block-diagonal form. Then one may diagonalize each block separately for increased efficiency (why is this effective?).

What is the largest value of N your implementation can attain? Try to identify the various bottlenecks you may run into during the computation, and then optimize your code accordingly. There are various tricks you can use to speed-up intermediate calculations, but there is little one can do to improve the actual diagonalization step performed by `Eigenvalues`.

To test your implementation, we can now try to reproduce results such as Fig. (13) of [1]. Computing the spectrum of the SYK Hamiltonian for various values of N and many realizations, plot the density of states $\rho(E)$ to obtain something similar to Fig. 1. Note that the "accuracy" of your results for any given value of N depends not on the total number of realizations you average over, but on the number of energy levels you actually sample (so that for smaller values of N a larger number of realizations will be required for the density of states to converge).

[2] This symmetry can be made explicit by rewriting the Hamiltonian in terms of $N/2$ Dirac fermions defined as complex combinations of the original Majorana fermions.

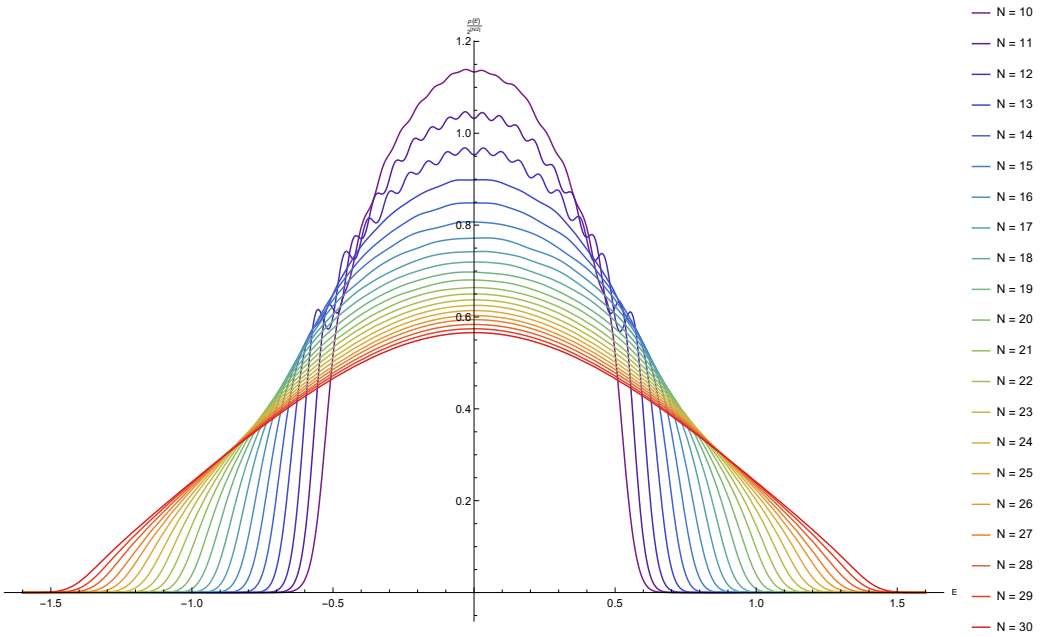

Figure 1: Normalized density of states for the SYK model with $N = 10, \ldots 30$ (using $2^{18} \sim 2 \times 10^5$ energy levels per value of N).

The central region of the density of states essentially captures the infinite-temperature entropy $S_\infty = \frac{N}{2} \log 2$, so that $\rho(E \sim 0) \simeq 2^{N/2} \approx e^{0.35N}$. On the other hand, the low-energy tail of the density of states captures the zero-temperature entropy S_0, which can be calculated in the large-N limit to be[3]

$$S_0 = \frac{\text{Cat}}{2\pi} + \frac{\log 2}{8} \approx 0.23\,, \tag{4}$$

meaning we expect $\rho(E \sim E_0) \approx e^{0.23N}$. To verify this, write a function to count the number of states lying within an interval of fixed width ΔE above the groundstate or around $E = 0$, averaging over all the realizations you have diagonalized. Repeating this for various values of N, you can fit the coefficient S in the log-plot of $\rho(E)\Delta E \propto e^{SN}$ for the largest system sizes you can achieve, as seen in Fig. 2.

Chaos in the SYK model was studied in [2] through the spectral form factor, defined in terms of the analytic continuation of the partition function

[3]In this expression Cat is the Catalan number, computed in `Mathematica` by `Catalan`.

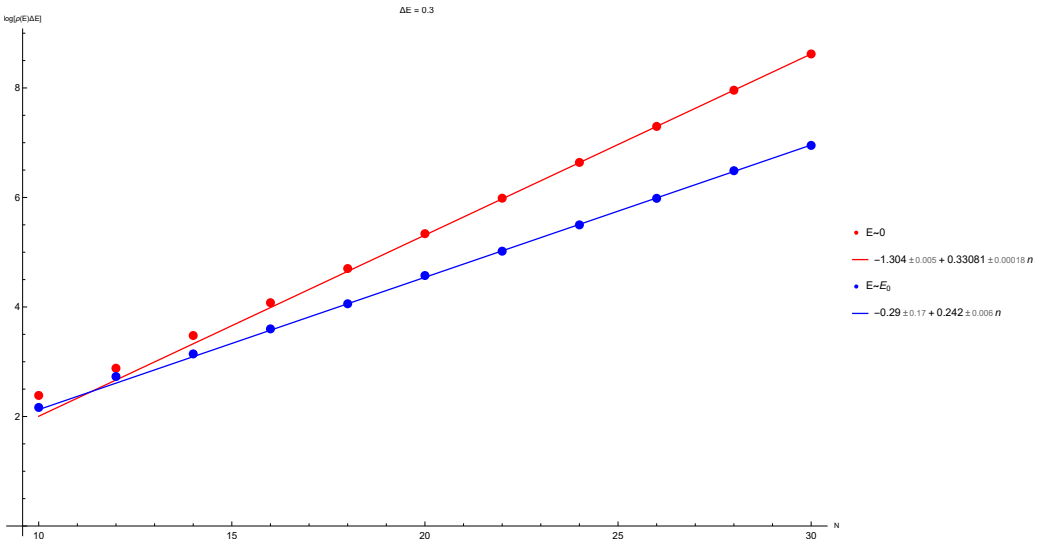

Figure 2: Groundstate (blue) and infinite-temperature (red) entropy of the SYK model, for various values of N, computed using an energy interval of width $\Delta E = 0.3$.

as

$$f_\beta(t) = \left\langle \frac{Z(\beta+it)Z(\beta-it)}{Z(\beta)^2} \right\rangle \qquad \text{with} \qquad Z(\beta) = \mathrm{Tr}\, e^{-\beta H}\,. \tag{5}$$

The spectral form factor is essentially the same as (but not quite exactly)

$$f_\beta(t) \simeq \frac{1}{\langle Z(\beta)^2\rangle} \overline{\sum_{i,j=1}^{L} e^{-\beta(E_i+E_j)+it(E_j-E_i)}}\,, \tag{6}$$

where we note that the double-sum is highly oscillatory for long times $t \gg \min_{i,j} |E_j - E_i|$, hence the benefit of averaging over realizations to obtain a smooth function (a process we denote here by an overline).

A smoking-gun indicator of late-time chaos is the dip-ramp-plateau structure of the spectral form factor shown in Fig. 1 of [2]. This structure is characteristic of random matrix theory, and hints at an interpretation of the SYK model as a maximally chaotic system, holographically dual to a black-hole. Using the diagonalized Hamiltonians you have already computed, plot the spectral form factor for various values of N to reveal this dip-ramp-plateau structure, as shown in Fig. 3.

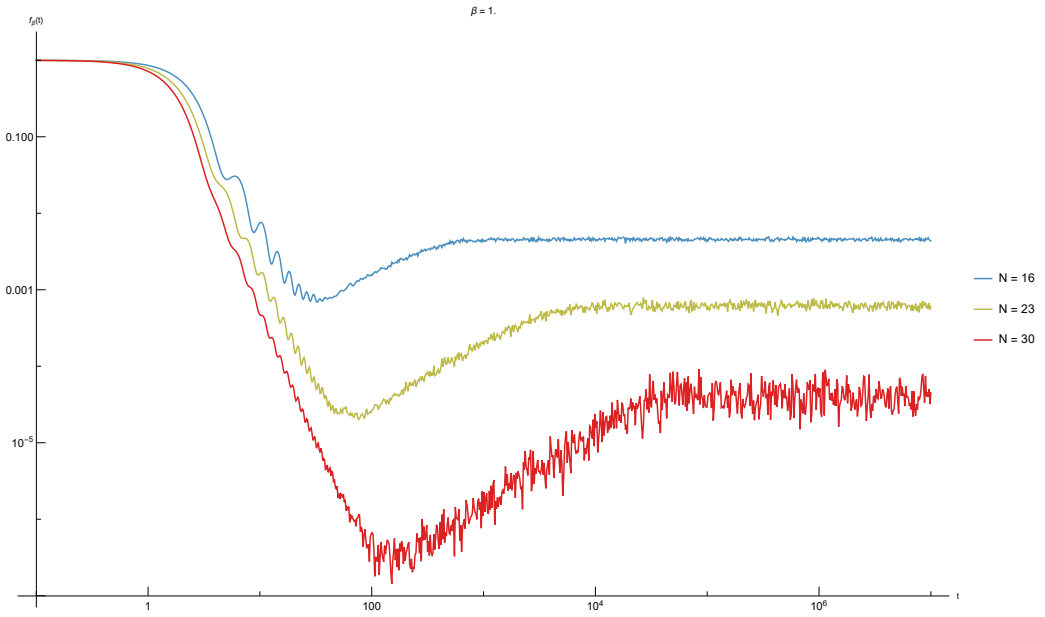

Figure 3: Spectral form factor for $\beta = 1$ and a few values of N.

Details

A recursive construction for the chiral basis of the higher-dimensional Dirac matrices with N even is as follows:

- For $N = 2$, set
$$\Gamma^1_{(2)} = \sigma^1 \qquad \text{and} \qquad \Gamma^2_{(2)} = \sigma^2 \,;$$
- For $N > 2$, set
$$\begin{aligned} \Gamma^i_{(N)} &= \Gamma^i_{(N-2)} \otimes \sigma^3 \qquad \text{for} \qquad i = 1, 2, \ldots, N-2 \,, \\ \Gamma^{N-1}_{(N)} &= -\mathbb{1}_{(N-2)} \otimes \sigma^1 \,, \\ \Gamma^N_{(N)} &= -\mathbb{1}_{(N-2)} \otimes \sigma^2 \,. \end{aligned}$$

Here, the σ^i are Pauli matrices and $\otimes$ is the Kronecker product, with $\mathbb{1}_{(n)}$ the identity matrix of size $2^{n/2}$. For odd $N \geq 3$, we have instead

$$\begin{aligned} \Gamma^i_{(N)} &= \Gamma^i_{(N-1)} \qquad \text{for} \qquad i = 1, 2, \ldots, N-1 \,, \\ \Gamma^N_{(N)} &= i^{\frac{N-1}{2}} \Gamma^1_{(N-1)} \cdots \Gamma^{N-1}_{(N-1)} \,. \end{aligned}$$

When implementing this make sure to check that with these definitions $\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$, and remember that memoïzation is a powerful tool to speed-up calculations where the same ingredients are repeatedly used. Memoïzation in `Mathematica` is as simple as explicitly storing the function value once computed, using the form `f[x_]:=f[x]=...`

Moving on, after constructing each Hamiltonian deflation can be achieved by separating the groups of rows and columns that do not mix with each other. A neat way to do this involves interpreting H_N as an adjacency matrix for a graph of L nodes, where a non-vanishing matrix element $(H_N)_{ij}$ indicates an edge $i \leftrightarrow j$. Then a matrix block corresponds to a connected component in the associated graph (why?), and these can easily be computed by `Mathematica`.[4]

Lastly, we should note that after constructing and diagonalizing a Hamiltonian it is convenient to store the computed energy levels for later reuse. In this way, we accumulate samples and increase our statistics while requiring only minimal storage (for this purpose, it is essential to store all the values in binary format!).

Once the Hamiltonians have been diagonalized, completing the rest of this problem should be straightforward. However, do note that given enough statistics evaluating almost any quantity becomes impracticably slow. One option you may consider to speed-up evaluation-intensive tasks such as plotting is to pre-compile simple algebraic expressions with `Compile`.

Useful functions

You may find it useful to read the `Mathematica` help pages of the following functions:

- **Matrix functions:** `PauliMatrix`, `IdentityMatrix`, `KroneckerProduct`, `SparseArray`, `Eigenvalues`, ...
- **Statistics functions:** `RandomVariate`, `NormalDistribution`, `PDF`, `SmoothKernelDistribution`, `SmoothHistogram`, ...

[4] Because the underlying reason why this is possible for even N is the existence of a residual $\mathbb{Z}_2$ symmetry, the transformation into block-diagonal form cannot depend on the specific realization we consider. Therefore, it need only be computed once for each value of N.

- **Graph functions:** `AdjacencyGraph`, `ConnectedComponents`, ...
- **Other functions:** `Unitize`, `LinearModelFit`, `Compile`, `BinaryReadList`, `BinaryWrite`, ...

References

[1] J. Maldacena and D. Stanford, "Remarks on the Sachdev-Ye-Kitaev model," Phys. Rev. D **94** (2016) no.10, 106002 [arXiv:1604.07818 [hep-th]].

[2] J. S. Cotler *et al.*, "Black Holes and Random Matrices," JHEP **1705** (2017) 118 Erratum: [JHEP **1809** (2018) 002] [arXiv:1611.04650 [hep-th]].