Determination of the Permeability of Free Space using the Biot-Savart Law

Louis Sayer, Lab partner: Saabir Petker
L1 Discovery Labs, Lab Group D (Monday)
Submitted: 20/03/19, Date of Experiment: 18/02/19 and 25/02/19

The Biot-Savart Law describes the magnetic field at a point in space generated by a current-carrying conductor. This equation is evaluated in four different methods to find a value for the permeability of free space. The values from each method are consistent and produce a final value of $(1.28 \pm 0.01) \times 10^{-6} TmA^{-1}$ which is consistent with the literature. Arrangements of a current carrying wire, loop and Helmholtz coils are used.

I. INTRODUCTION

Discovered by Jean-Baptiste Biot and Félix Savart, the Biot-Savart law relates the magnetic field at a point in space caused by the current in a conductor,

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times \hat{r}}{r^2} \tag{1}$$

where $\vec{B}$ is the magnetic field, μ_0 is the permeability of free space and $d\vec{l}$ is the length vector parallel to the current I [1]. The constant μ_0 is determined in this experiment by integrating this law to describe the magnetic field in different scenarios. The accepted value is $(1.26 \times 10^{-6}) TmA^{-1}$ or $(4\pi \times 10^{-7}) TmA^{-1}$ exactly [2].

The integral can be evaluated for a long, straight, current-carrying wire at a distance R and current I to derive the equation

$$B = \frac{\mu_0 I}{2\pi R}, \tag{2}$$

which is used for method A. In methods B and D the form of the equation is more complicated and describes how the magnetic field varies at a distance x from a loop of radius R and current I,

$$B = \frac{\mu_0 I R^2}{2(x^2 + R^2)^{3/2}}. \tag{3}$$

Method B uses a constant distance x and measures the corresponding magnetic field while in method D the current is kept constant and the distance is varied. For method D, x is zero so the equation becomes

$$B = \frac{\mu_0 I}{2R}. \tag{4}$$

Another approach can determine the magnetic field at the centre of a Helmholtz coil (two loops parallel to each other at a distance equal to their radius). For radii R, number of coils n and current I, the magnetic field is equal to

$$B = \frac{8}{5\sqrt{5}} \frac{\mu_0 n I}{R}. \tag{5}$$

This is used in method C. Helmholtz coils can used in experiments requiring the magnetic field in a region to be zero (by creating a magnetic field of equal and opposite strength to the Earth's field) as they have the unique property of having a constant magnetic field in the region enclosed by the loops [3].

II. METHOD

A DC current power supply was used to supply the current to the wire and (two) loops in the experiments. Transverse and axial Gauss meters were alternately used to measure the magnetic fields and were chosen depending on the method. The transverse probe measured the magnetic field perpendicular to the probe while the axial probe measured it parallel to the probe. Both employed the Hall effect. A support rail with a ruler attached was used to determine the distance between the probes and the wires/loops. Connecting cables were used to supply the current.

In method A, the transverse probe was used to measure the magnetic field at a constant distance of 0.7cm while the current was varied. The current was increased from 0 to 20A in increments of 1A and the resulting curve of magnetic field against current was linear. Note that, at currents above 15A the current was disconnected in between readings to minimise damage to the apparatus.

Method B placed the axial probe at the centre of a single loop and measured the magnetic field at increasing currents in the same range. The radius of the coil was determined by measuring the diameter at four separate places and calculating a standard error, which was halved for the radius error. Method C repeated this however the probe was placed at the centre of the Helmholtz coils. Care was taken to ensure the probe measured the field perpendicular to the coils by looking down the probe and aligning it to be pointing in the same plane as the loops.

Method D used the same arrangement as method B however the current was kept constant while the perpendicular distance was increased by increments of 1 cm between 1 and 10cm. Although the other methods plotted magnetic field against current using a weighted least-squares fit, this method plotted magnetic field against $\frac{1}{2(x^2+R^2)^{3/2}}$.

Before all sets of readings, the Gauss meter was zeroed to remove the effects of the earth's magnetic field. Measurements were paused and the Gaussmeter was zeroed if there were sharp spikes in the readings.

III. RESULTS

Table 1 shows the permeability of free space determined by each of the four different methods. These were calculated by inserting the gradients into the equations shown in the introduction. The methods used in the calculations of the errors are described in Appendix I. All of the results are consistent with each other within their errors and the errors are of similar magnitude meaning they can be combined with a weighted mean.

The variation of the magnetic field with current and its variation with $\frac{1}{2(x^2+R^2)^{3/2}}$ are shown in Figs.1 and 2.

TABLE I: Values of the permeability of free space and their associated uncertainties obtained in each of the four methods, and their combined value.

Method	$\mu_0/TmA^{-1}(\times 10^{-6})$
A(Wire)	(1.2 ± 0.2)
B(Loop)	(1.29 ± 0.02)
C(Helmholtz pair)	(1.25 ± 0.02)
D(Loop)	(1.29 ± 0.05)
Combined Value	(1.28 ± 0.01)

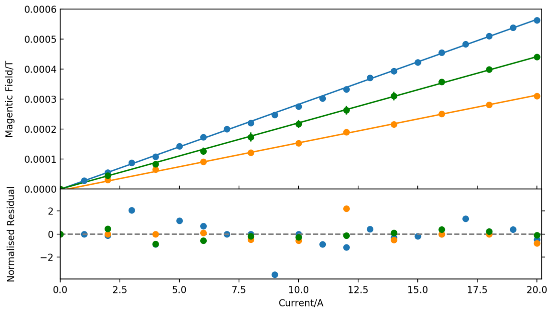

FIG. 1: The magnetic field as a function of the current through a wire, loop and Helmholtz coils. Method A is blue, B is orange and C is green. A linear trend line has been added as well as error bars, the latter are too small to be seen.

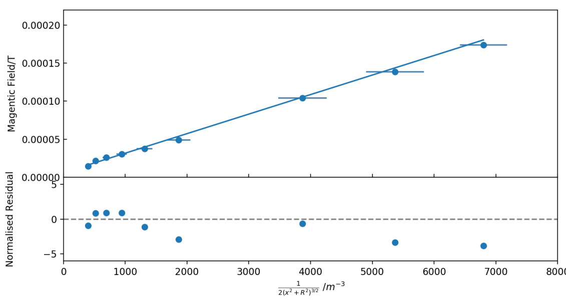

FIG. 2: The magnetic field as a function of $\frac{1}{2(x^2+R^2)^{3/2}}$. A linear trend line and error bars have been added. The 7th data point was considered anomalous by Chauvenet's criterion and therefore omitted.

IV. DISCUSSION

The data support the linear relationships between magnetic field and current as well as magnetic field against $\frac{1}{2(x^2+R^2)^{3/2}}$ as stated by the equations shown in the introduction. They have an expected random distribution in the normalised residuals with no trend immediately visible. The final value of $(1.28{\pm}0.01){\times}10^{-6}\ TmA^{-1}$ is in reasonable agreement with the literature as it is within two standard errors of the accepted value [2]. Table I shows that all values are in agreement with each other up to two standard errors which allowed them to be combined. A weighted mean was used for this to give more credence to the values with lower errors such as method B and C.

The source of dominant error in the methods came from the random error in the magnetic field measurements. This excludes method A where the error in the distance from the probe and the wire was deemed to be ± 0.1 cm which formed a very large percentage error of 14%, the dominant error. In this method, more time should have been devoted to minimising this error by using digital instruments with higher resolutions or increasing the distance to reduce the effective error. A ruler physically aligned in this axis would have allowed for easier measurements as subjective judgement was required to deem when the ruler was in the centre of the loop. This provided a limitation to the method. In methods B,C and D, the magnetic field standard errors formed the dominant errors, particularly in method D where the large errors in the x values gave a gradient of $(2.6 \pm 0.1) \times 10^{-8}$, forming a percentage error of 4%. This was much larger than the errors in the constants and was similar for B and C. Despite this, the variance in the measurements made should have been higher than indicated as the Gauss meter readings were constantly changing, meaning values centred around what appeared as the mean were chosen. The standard deviation was therefore probably higher in all methods and has not been well represented by the final values. This would have made the gradient errors even larger.

The method which produced the most accurate value was method C (Helmholtz pair). It avoided the very large error present in method A and also minimised the error in the position within the coils as the magnetic field is constant within this region. Method B also had low error and produced a good value while method A had large error in the distance which increased the final error. The propagation of errors in the dependent variable of method D gave large errors, which can be seen by the error bars in Fig.2. Observations of higher errors in distance led to the current being favoured as the independent variable in the methods.

V. CONCLUSIONS

In conclusion, the permeability of free space has been determined from measurements of the magnetic field at varying currents and distances. All four methods found the magnetic field proportional to current or $\frac{1}{2(x^2+R^2)^{3/2}}$. This allowed for the calculation of a combined value of $(1.28 \pm 0.01) \times 10^{-6} TmA^{-1}$ which is consistent with the accepted value. The best method appeared to be method B using the Helmholtz pair which was the most accurate and had a low uncertainty. It also had the property of a uniform magnetic field between the coils.

References

[1] H. Young and R. Freedman, University Physics with Modern Physics, 13th edition, Pearson Education, San Francisco, 2012.

[2] P. Mohr, D. Newell, B. Taylor, CODATA Recommended Values of the Fundamental Physical Constants: 2014, Journal of Physical and Chemical Reference Data, page 043102-7.

[3] A.E. Ruark and M. F. Peters, Helmholtz Coils for Producing Uniform Magnetic Fields, J. Opt. Soc. Am., 1926, Vol. 13, Issue 2., pp. 205-212

VI. APPENDIX

All of the values determined in this experiment had an absolute error associated with them. This section describes how these errors were propagated through the experiment to give the final errors in the individual and combined values.

The values for the permeability of free space in Fig.1, measured in methods A, B and C were obtained by measuring the magnetic field at increasing currents and these each had their own uncertainty. For the magnetic field errors we made repeats of our measurements and calculated the standard error α_{B_i} for each i^{th} set:

$$\alpha_{B_i} = \frac{\sigma_{N-1}}{\sqrt{N}} \tag{6}$$

The errors in the current were taken to be one unit on the last digit as there was no noise and the current was able to be set exactly each time, therefore $\alpha_I = 0.01A$.

A method of weighted least-squares fit was used as this allowed for the incorporation of the standard errors of the y values, as the least squares fit neglects this. Using this method meant that the current errors were neglected as using an approach which considers both is more complicated. The standard errors were proportionally higher and therefore these were placed on the y axis as expected. The error in the gradient α_m was calculated as follows:

$$\alpha_m = \sqrt{\frac{\Sigma_i \omega_i}{\Delta'}} \tag{7}$$

Where

$$\Delta' = \Sigma_i \omega_i \Sigma_i \omega_i {x_i}^2 - (\Sigma_i \omega_i x_i)^2 \tag{8}$$

and

$$\omega_i = \alpha_i^{-2} \tag{9}$$

α_i was the standard error of each i^{th} set of the magnetic field values. (These equations, like all other equations mentioned in this appendix, are based on the error analysis formulas given in I.G Hughes and T.P.A Hase, *Measurements and their Uncertainties*, Oxford University Press, Oxford (2010).)

The errors were further propagated in different ways depending on the equation used for determining μ_0. These were all done using the same form of equation as the errors from different sources were either multiplied or divided.

For method A (wire) the error in μ_{0_A} was

$$\alpha_{\mu_{0_A}} = \mu_{0A} \sqrt{(\frac{\alpha_R}{R})^2 + (\frac{\alpha_m}{m})^2} \tag{10}$$

where α_R was the error in distance R and α_m was the error in the gradient. The distance error from the Gauss meter and wire was deemed to be 1mm as it was not clear to precisely measure where the Hall probe was situated in the Gauss meter.The error was therefore increased from the absolute error of 0.5mm.

For method B (loop) the error in μ_{0_B} was

$$\alpha_{\mu_{0_B}} = \mu_{0B} \sqrt{(\frac{\alpha_R}{R})^2 + (\frac{\alpha_m}{m})^2} \tag{11}$$

where α_R was the error in R and α_m was the error in the gradient. The radius error was determined by measuring the diameter at four points and calculating a standard error. This gave a radius of 4.07 ± 0.02 cm.

For method C(Helmholtz coils) the error in μ_{0_C} was

$$\alpha_{\mu_{0_C}} = \mu_{0C} \sqrt{(\frac{\alpha_R}{R})^2 + (\frac{\alpha_m}{m})^2} \tag{12}$$

where α_R was the error in radius R and α_m was the error in the gradient. The error in the current was deemed to be one unit of the last digit as this could be set exactly and there was no noise.

This final incorporation of errors gave the final error for methods A,B and C. For Method D the x errors were much higher than the y errors as can be seen from Fig.2. This meant that the weighted least-squares fit used the x errors rather than y errors. The combined errors in distance and radius of loop were difficult to evaluate at each value and therefore a functional approach was used. For a function Z dependent on A whose errors are α_A, the errors are obtained by

$$\alpha_Z = \mid f(\overline{A} + \alpha_A) - f(\overline{A}) \mid \tag{13}$$

This was evaluated to find the error concerned with the radius and then the error concerned with the distance. These were then combined to give

$$\alpha_X = X \sqrt{(\frac{\alpha_x}{x})^2 + (\frac{\alpha_R}{R})^2} \tag{14}$$

Where X was the x value displayed on the graph, α_x was the distance error and α_R was the error in the radius. The weighted least-squares fit was used with $\omega_i = \alpha_i^{-2}$ using X errors as α_i. The error in the gradient was then used in conjunction with the error in the current (0.01A) and radius (0.02cm) to produce the final error

$$\alpha_{\mu_{0_D}} = \mu_{0D} \sqrt{(\frac{\alpha_I}{I})^2 + (\frac{\alpha_R}{R})^2 + (\frac{\alpha_m}{m})^2} \tag{15}$$

The error in the combined value was calculated using a weighted mean. The combined value error α_{CE} is given by

$$\frac{1}{\alpha_{CE}^2} = \Sigma_i (\frac{1}{\omega_i}) \tag{16}$$

where

$$\omega_i = \frac{1}{{\alpha_i}^2} \tag{17}$$

and α_i are the errors of each value of each method.