Determining the dynamic viscosity of water using the Poiseuille Equation

Louis Sayer, Lab partners: Alex Ogden, Francois Miravalls, Michael Macnaughton
L2 Skills Sessions, Friday
Submitted: 05/12/19, Dates of Experiment: 08/11/19 and 15/11/19

The dynamic viscosity of water is determined using the Poiseuille equation. The mass flow rate of water exiting a reservoir through a capillary tube at a fixed point, is measured for different heights of water. The values from two capillary tubes are consistent and produce a final value of $(1.0 \pm 0.1) mPa\ s$, which is in good agreement with the literature for the given water temperature. The dominant source of error came from the radii of the capillary tubes and a lower error in the final value would have required more homogeneous tube radii measured with higher precision instruments.

I. INTRODUCTION

The dynamic viscosity η of a fluid is a measure of its internal friction [1] and was first defined by Navier in 1823 [2]. It is the proportionality factor between the shear stress τ of two parallel layers in a fluid and the corresponding velocity gradient $\frac{dv}{dr}$, where v is the velocity in the direction of flow and r is the distance from the vessel boundary [3].

$$\tau = \eta \frac{dv}{dr} \tag{1}$$

Poiseuille discovered in the 1840s that the flow rate Q of a fluid when moving through a tube at a pressure P, was proportional to the diameter D^4 and inversely proportional to its length L [2].

$$Q \alpha \frac{PD^4}{L} \tag{2}$$

The proportionality constant was only discovered as dependent on the viscosity of the fluid by Hagenbach in 1860 [2]. Several assumptions are required to arrive at this equation including having a horizontal tube so there is no effect of gravity, having no slippage of the fluid at the tube wall, no turbulent flow and also that the viscosity is constant at all rates of shear. The volumetric flow rate must also be sufficiently small to eliminate the effect of the quadratic term in said variable. It has also been noted that the flow rates in tubes with large radii relative to their lengths deviate from the proportionality with P [2].

In the context of this work, where the mass flow rate through a capillary tube is measured from water exiting a tank, the pressure can be simplified to a function of the height h of the water, its density ρ and the acceleration due to gravity g

$$\frac{dm}{dt} = \frac{\pi \rho^2 g h a^4}{8 \eta L} \tag{3}$$

where m is the mass of water and a is the radius of the capillary tube [2]. Using a capillary tube ensures the length of the tube is large relative to its radius. The fluid is assumed to have laminar flow whereby the layers flow smoothly, parallel to each other and do not mix, unlike in turbulent flow. Water was also assumed to be a Newtonian fluid whereby its viscosity is invariant of pressure and the shear stress is only dependent on the shear rate.

II. METHOD

The mass of water escaping a water reservoir through a capillary tube was measured over time using a mass balance. This allowed the mass flow rate to be measured at different heights of water and was used to calculate the viscosity using Eq.(3). The mass was measured every second over ten seconds and the values collected by a data acquisition device connected to a computer. The apparatus set up is shown below in Fig.(1).

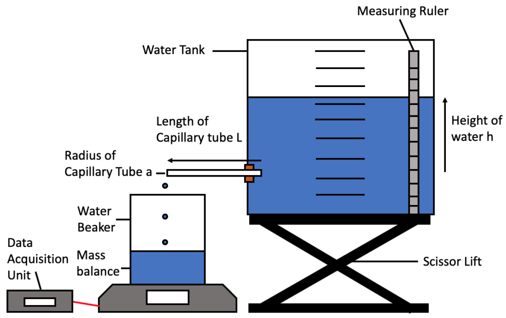

FIG. 1: The configuration of the apparatus used in the experiment. Care was taken to ensure the water tank and capillary tube were parallel to the horizontal.

The starting height of water, which varied between 11 and 19 cm, and the length of the capillary tube were measured with a ruler whose uncertainty was taken to be half a unit of the last digit, which was 0.5 mm. The lengths of the blue and white tubes were (141.0 ± 0.5) mm and (142.0 ± 0.5) mm. Measuring the radii of the capillary tubes required a high degree of accuracy because of its fourth power in Eq.(3). A travelling microscope with a vernier scale was therefore used. Three repeats were taken for each tube from which a standard error was calculated. Two capillary tubes, blue and white, were used as these had high flow rates which appeared more linear when plotted and would allow more readings to be taken. The radii were (0.38 ± 0.02) mm and (0.57 ± 0.02) mm. The measurement period was kept short as it was noticed that the mass flow rate would not remain constant for longer periods as the height of the water decreased. Up to three repeats were made at each height for each tube. The temperature was measured six times and a mean and error were calculated (14.7 ± 0.2)°C.

III. RESULTS

Table I shows the viscosity values determined by the two capillary tubes calculated by using a least-squares fitting of a straight line to find the mass flow rate from the measured values of mass against time. A chi-squared analysis with two parameters was then implemented to find the best-fit values of the mass flow rate against the height of water. This was inserted into Eq.(3) to give the viscosity.

The values are consistent with each other within their errors

and the errors are of similar magnitude meaning they can be combined with a weighted mean as shown.
The variation of mass flow rate with height is consistent with the predicted linear relationship and is shown in Fig.2.

TABLE I: Values of the viscosity of water obtained by the two capillary tubes, and their combined value.

Capillary Tube Type	$\eta(mPa\ s)$
Blue (small radius)	(1.2 ± 0.3)
White (large radius)	(0.9 ± 0.1)
Combined Value	(1.0 ± 0.1)

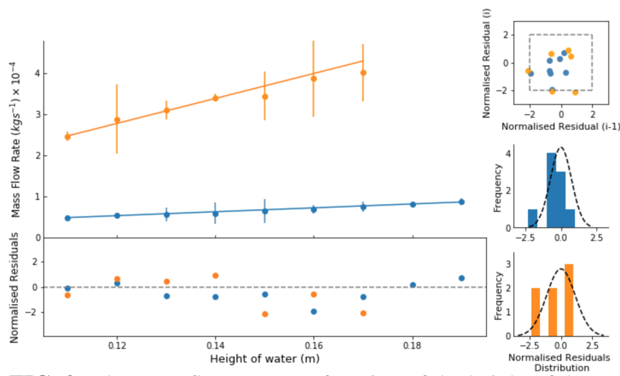

FIG. 2: The mass flow rate as a function of the height of the water. The blue capillary tube is shown in blue and white is shown in orange. A linear trend line has been added as well as error bars, which have been scaled by x5 for both tubes to be more visible. The distribution plots of the normalised residuals and their lag plot have also been included.

IV. DISCUSSION

The data support the linear relationship between mass flow rate and the height of water as stated by Eq.(3). A χ^2_{min} value of 6.3 for the blue tube was well within one standard deviation (0.2 σ) of the number of degrees of freedom of seven. This gave a χ^2_ν of 0.91, which indicates a good fit. The probability of obtaining this value, or higher, $P\left(\chi^2_{min};\nu\right)$, was calculated as 0.50, which indicates a good match between the sample and parent distributions.
The χ^2_{min} value for the white tube was calculated as 10.6 which was within two standard deviations (1.8 σ) of the number of degrees of freedom of 5. This gave a χ^2_ν of 2.1, which was larger than expected. The $P\left(\chi^2_{min};\nu\right)$ was looked at to determine if the null hypothesis that the model gives a good fit should be rejected at the 5% level. The calculated $P\left(\chi^2_{min};\nu\right)$ value was 0.06, meaning the proposition of the model being a good fit was not rejected.
The normalised residuals appear to have a Gaussian distribution as can be seen in Fig.(2), which is to be expected. There is no trend immediately visible, however upon further inspection using the Durbin-Watson statistic [4], there seemed to be a slight correlation in the normalised residuals, which can be seen in the lag plot of Fig.(2). The blue tube had a statistic of 0.9 and the white tube 1.4. This may have been as a result of the height not remaining constant over a measurement period leading to the mass flow rate being lower than expected, although this is difficult to discern with so few data points. Over 65% of both the normalised residuals are within ± 1, which indicates a good fit.
This analysis of the data gave confidence that the model used to determine the viscosity of water was correct despite the use of some assumptions. This was reflected by the excellent agreement of the final combined value of (1.0 ± 0.1) m Pa s with the literature at 14.7 °C as it is within one standard error [3]. Table 1 shows that both values are in agreement with each other to one standard error allowing them to be combined. A weighted mean was calculated thus giving more credence to the value with lower error.
The dominant source of error in the experiment came from the radii of the capillary tubes. These gave errors of 22% and 14% for the blue and white tubes, when scaled to the fourth power, and were the main limitation for the experiment. Errors from other sources such as the $\frac{dm}{dt}/h$ gradient or length were much smaller and of order $\times 10^{-1}$ or less (these can be seen in more detail in the error appendix). The reason for such a large error was the imperfect circular aperture of the capillary tubes, which when measured at six different orientations gave varying values. A possible improvement to the experiment would be to obtain capillary tubes with more consistent radii. Reducing the relative error to one comparable to the other sources of error would have involved measuring the radius of the blue tube to nearly 6 nm, or over x 3000 the precision of that achieved. This would have required more elaborate instruments as the absolute error in the vernier travelling microscope was 0.1mm. Attempting to reduce the other sources of error would have been unproductive unless this were addressed.
Other limitations included an inconstant mass flow rate as discrete drops of water entered the beaker. Efforts were made to model this as continuous, however assumptions such as keeping the height of the water constant required a short measuring period. Increasing the period led to a non-linear mass flow rate as the water pressure decreased as water left the tank. Negative changes in mass were also sometimes measured, which could be attributed to the error in the weighing balance or measuring the mass whilst a drop was entering the beaker (increasing the force exerted). These discrepancies were not removed from the data. The mean temperature of the water (14.7 ± 0.2)°C did not remain constant either during the experiment as it equilibrated with the room temperature, decreasing the viscosity in the process. A water temperatures closer to the room temperature would have made for more accurate readings.

V. CONCLUSIONS

In conclusion, the dynamic viscosity of water has been determined from measurements of the mass flow rate at varying heights of water. Both capillary tubes found a mass flow rate proportional to the height, which allowed for the calculation of a combined value of (1.0 ± 0.1) $(mPa\ s)$. This is consistent with the literature. The dominant error in the final value came from the radii error due to the fourth power in the equation. Ideally more homogeneous capillary tubes should have been used and measured with more precise instruments.

References

[1] H. Young and R. Freedman, University Physics with Modern Physics, 13th ed. , Pearson Education, San Francisco, 2012.
[2] S. Sutera and R.Skalak, The History of Poiseuille's Law, Annu. Rev. Fluid Mech. 1993. 25: 1-19.
[3] G. Kaye and T. Laby, Tables of Physical and Chemical Constants, 16th ed. , Longman, London, 1995.
[4] I. G. Hughes and T. P. A. Hase, Measurements and their Uncertainties, Oxford, University Press, Oxford, 2010.

VI. APPENDIX

The error in the mass flow rate at a given height of water was calculated using a least-squares fitting with two parameters to the experimental distribution of mass against time. This was used as the errors for both variables were homoscedastic and therefore equal weighting could be given to all the points. The gradient was calculated for three repeats (varying number for white tube) and a standard error of the mass flow rate $\alpha_{\frac{dm}{dt}_i}$ at each i^{th} height was determined by :

$$\alpha_{\frac{dm}{dt}} = \frac{\sigma_{N-1}}{\sqrt{N}} \tag{4}$$

Some of the mass flow rates at a given height of water for the white tube did not have repeats and therefore the error in the gradient from the linear least-squares fitting was used. The error in the gradient α_m for these was calculated as follows [4]:

$$\alpha_m = \alpha_{CU}\sqrt{\frac{N}{\Delta}} \tag{5}$$

where

$$\Delta = N\sum_i {x_i}^2 - \left(\sum_i x_i\right)^2 \tag{6}$$

and the common uncertainty α_{CU} is

$$\alpha_{CU} = \sqrt{\frac{1}{N-2}\sum_i (y_i - mx_i - c)^2} \tag{7}$$

These were calculated using polyfit in Python which returns the variance of the gradient and intercept in the diagonal entries of a covariance matrix. The square root of these were the standard deviation for each $\frac{dm}{dt}$ and were used as the errors at different heights of water.
A χ^2 analysis was used to calculate the error for the distribution between $\frac{dm}{dt}$ and height as the $\frac{dm}{dt}$ errors were heteroscedastic and therefore the points with lower error were given more credence. This was done by first minimising the χ^2 statistic giving the best-fit parameters (gradient and intercept). The error of each parameter was then calculated by finding the extremal difference between the best-fit values and their values at the $\Delta\chi^2 + 1$ contour.
Practically, this can be done by increasing one parameter whilst keeping the other constant until it reaches the $\Delta\chi^2 + 1$ contour. Then, the χ^2 is re-minimised and this is repeated until there is no discernible difference in the $\Delta\chi^2 + 1$ calculated. This looks like a series of orthogonal steps between the best-fit parameters to the $\Delta\chi^2 + 1$ contour, touching said contour between the re-minimisation steps.
The calculations for these were computed using Python code and gave a final error for the gradient. The gradient $\frac{dm}{dt}/h$ will be referred to as m with an error α_m, to determine the final error in the viscosity. The error in viscosity α_η was calculated as:

$$\alpha_\eta = \eta\sqrt{\left(\frac{2\alpha_\rho}{\rho}\right)^2 + \left(\frac{\alpha_m}{m}\right)^2 + \left(\frac{4\alpha_a}{a}\right)^2 + \left(\frac{\alpha_l}{l}\right)^2} \tag{8}$$

For the blue tube the gradient and its error were $(4.7460 \pm 0.0003)kg\ s^{-1}\ m^{-1}$ and for the white tube they were $(3.039 \pm 0.001)kg\ s^{-1}\ m^{-1}$.
The density and its error were taken to be $(999.1 \pm 0.1)kg\ m^{-3}$ taking into account the variation in temperature [3]. The error in the length of the tubes was taken to be half one unit of the last digit of the measuring instrument, which was 0.5 mm. Although the acceleration due to gravity appears in Eq.(3), it is an exact value and therefore has no error. The error in the radii of the tubes α_a were found by calculating the standard error of the repeat measurements made. This explicitly was:

$$\alpha_a = \frac{\sigma_{N-1}}{\sqrt{N}} \tag{9}$$

where N was 6. For the blue tube this gave (0.38 ± 0.02) mm and for the white tube (0.57 ± 0.03) mm. The error in the gradient and radii were scaled by x2 and x4 times to account for their indices.
The error in the combined value was calculated using a weighted mean. The combined value error α_{CE} is given by

$$\frac{1}{\alpha_{CE}^2} = \sum_i \omega_i \tag{10}$$

where

$$\omega_i = \frac{1}{\alpha_i^2} \tag{11}$$

and α_i are the errors of the viscosity value for the blue and white tube.