OPTIMAL POLICY

The loss function is $L_0^u = -V_0^u$. Policy problem is:

$$
\min_{\{\hat{Y}_t, \pi_{Ht}, q_t, i_t\}} \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \hat{\beta}^t \Big\{ \Big(s_{y,11} \hat{Y}_t^2 + 2 s_{y,12} \hat{Y}_t \hat{q}_t + s_{y,22} \hat{q}_t^2
$$

$$
+ s_{e11} \hat{Y}_t \hat{a}_t + s_{e12} \hat{Y}_t \hat{C}_t^* + s_{e,22} \hat{q}_t \hat{C}_t^*
$$

$$
+ R_\pi \pi_{Ht}^2 \Big)
$$

$$
+ 2 \lambda_t^\pi \Big[\pi_{Ht} - \hat{\beta} \pi_{Ht+1} - \lambda \kappa_1 (\hat{Y}_t - \bar{Y}_t)
$$

$$
- \lambda \kappa_2 (\hat{q}_t - \bar{q}_t) \Big]
$$

$$
+ 2 \lambda_t^Y \Big[(\hat{Y}_t - \bar{Y}_t) - \omega (\hat{Y}_{t+1} - \bar{Y}_t) + \overset{\mu}{\cancel{\chi}} (i_t - \pi_{Ht+1}) - \epsilon_t \Big]
$$

$$
+ 2 \lambda_t^q \Big[(\hat{q}_t - \bar{q}_t) - (\hat{q}_{t+1} - \bar{q}_{t+1})
$$

$$
+ (1 - \gamma)(i_t - \pi_{Ht+1}) - u_t \Big]
$$

~~Note: I switch ~~xxx~~ notation from $\boxed{\mu \text{ to } \psi}$ in the IS equation~~

OPTIMAL POLICY w/ commitment

FOC:

$$\hat{y}_t : \quad s_{y,11}\hat{y}_t + s_{y,12}\hat{q}_t + s_{e,11}\hat{a}_t + s_{e,12}\hat{c}^*_t - \lambda\kappa_1\lambda^{\pi}_t$$

$$+ \lambda^{y}_t - \varpi\lambda^{y}_{t-1} = 0$$

$$\pi_{Ht} : \quad R_{\pi}\pi_{Ht} + \lambda^{\pi}_t - \beta\lambda^{\pi}_{t-1} - \overset{\mu}{\cancel{\psi}}\lambda^{y}_{t-1} - (1-\gamma)\lambda^{q}_{t-1} = 0$$

$$\hat{q}_t : \quad s_{y,22}\hat{q}_t + s_{y,12}\hat{y}_t + s_{e,22}\hat{c}^*_t + \lambda\kappa_2\lambda^{\pi}_t$$

$$+ \lambda^{q}_t - \lambda^{q}_{t-1} = 0$$

$$\hat{i}_t : \quad \mu\,\cancel{\psi}\lambda^{y}_t + (1-\gamma)\lambda^{q}_t = 0$$

OPTIMAL POLICY WITHOUT COMMITMENT

FOC:

(1) $\hat{Y}_t: \quad s_{y,11}\hat{Y}_t + s_{y,12}\hat{q}_t + s_{e,11}\hat{a}_t + s_{e,12}\hat{C}^*_t + \lambda^Y_t - \lambda K_1 \lambda^\pi_t = 0$

(2) $\pi_{Ht}: \quad R_\pi \pi_{Ht} + \lambda^\pi_t = 0$

(3) $\hat{q}_t: \quad s_{y,22}\hat{q}_t + s_{y,12}\hat{Y}_t + s_{e,22}\hat{C}^*_t + \lambda K_2 \lambda^\pi_t + \lambda^q_t = 0$

(4) $\hat{i}_t: \quad \mu \lambda^Y_t + (1-\gamma)\lambda^q_t = 0$

Simplifying:

(3) $\Rightarrow s_{y22}\hat{q}_t + s_{y12}\hat{Y}_t + s_{e,22}\hat{C}^*_t - \lambda K_2 \underset{(2)}{R_\pi \pi_{Ht}} + \underset{(4)}{\left(\frac{\mu}{1-\gamma}\right)\lambda^Y_t} = 0$

$\Rightarrow s_{y22}\hat{q}_t + s_{y12}\hat{Y}_t + s_{e22}\hat{C}^*_t - \lambda K_2 R_\pi \pi_{H,t}$

$$-\left(\frac{\mu}{1-\gamma}\right)\Big[\underbrace{\underset{(2)}{-\lambda K_1 R_\pi \pi_{Ht}} - s_{y11}\hat{Y}_t - s_{y12}\hat{q}_t - s_{e11}\hat{a}_t - s_{e,12}\hat{C}^*_t}_{(1)}\Big] = 0$$

$x \equiv s$

$$\Rightarrow \overbrace{\left(s_{y,22} + \frac{\overset{\mu}{\psi}}{1-\gamma} s_{y,12}\right)}^{\equiv f_q} \hat{q}_t + \overbrace{\left(s_{y,12} + \frac{\overset{\mu}{\psi}}{1-\gamma} s_{y,11}\right)}^{\equiv f_x} \hat{y}_t$$

$$+ \underbrace{\lambda R_\pi \left(\frac{\cancel{\psi}\,\mu}{1-\gamma} \kappa_1 - \kappa_2\right)}_{\equiv f_\pi} \pi_{H,t} + \underbrace{\frac{\cancel{\psi}\,\mu}{1-\gamma} s_{e,11}}_{\equiv f_a} \hat{a}_t + \underbrace{\frac{\cancel{\psi}\,\mu}{1-\gamma} s_{e,12}}_{\equiv f_{c^*}} \hat{c}_t^* = 0 \quad (\star)$$

An equilibrium under optimal time consistent policy (Markovian) is a stochastic process $\{\hat{q}_t, \pi_{Ht}, \hat{y}_t, i_t\}_{t=0}^{\infty}$ ~~st~~ which is bounded and satisfies: $\underset{AS}{(40)}, \underset{IS}{(41)}, \underset{UIP}{(42)}$, and the optimal policy trade-off $(\star)$.

$$(1) \qquad \tilde{x}_t = \varpi \mathbb{E}_t \tilde{x}_{t+1} - \mu \left[i_t - \mathbb{E}_t\{\pi_{H t+1}\} \right] + \chi \mathbb{E}_t\{\tilde{q}_{t+1}\} + \epsilon_t$$

$$(2) \qquad \pi_{Ht} = \beta \mathbb{E}_t \pi_{H t+1} + \lambda\kappa_1 \tilde{x}_t + \lambda\kappa_2 \tilde{q}_t$$

$$(3) \qquad \tilde{q}_t = \mathbb{E}_t \tilde{q}_{t+1} - (1-\gamma)\left[i_t - \mathbb{E}_t\{\pi_{H\,t+1}\} \right] + u_t$$

$$(4) \qquad f_q \tilde{q}_t + f_x \tilde{x}_t + f_\pi \pi_{Ht} + f_a a_t + f_c c_t^* + f_q \bar{q}_t + f_x \bar{y}_t = 0$$

(2) ← (3):

$$\pi_{Ht} = \beta \mathbb{E}_t \pi_{H t+1} + \lambda\kappa_1 \tilde{x}_t + \lambda\kappa_2 \Big[\mathbb{E}_t \tilde{q}_{t+1} - (1-\gamma) i_t + (1-\gamma)\mathbb{E}_t \pi_{H t+1} + u_t \Big] \qquad (2')$$

(2') ← (1)

$$\begin{aligned}
\pi_{Ht} &= \beta \mathbb{E}_t \pi_{H t+1} + \lambda\kappa_1 \Big[\varpi \mathbb{E}_t \tilde{x}_{t+1} - \mu i_t + \mu \mathbb{E}_t \pi_{H t+1} + \chi \mathbb{E}_t \tilde{q}_{t+1} + \epsilon_t \Big] \\
&\quad + \lambda\kappa_2 \Big[\mathbb{E}_t \tilde{q}_{t+1} - (1-\gamma) i_t + (1-\gamma)\mathbb{E}_t \pi_{H t+1} + u_t \Big] \\
&= \overbrace{\Big[\beta - \lambda\kappa_1 \mu + \lambda\kappa_2 (1-\gamma) \Big]}^{\equiv \rho_\pi} \mathbb{E}_t \pi_{H,t+1} + \overbrace{\lambda\kappa_1 \varpi}^{\equiv \rho_x} \mathbb{E}_t \tilde{x}_{t+1} \\
&\quad + \overbrace{(\lambda\kappa_1 \chi + \lambda\kappa_2)}^{\equiv \rho_q} \mathbb{E}_t \tilde{q}_{t+1} - \overbrace{\Big[\lambda\kappa_1 \mu + \lambda\kappa_2 (1-\gamma) \Big]}^{\equiv \rho_i} i_t \\
&\quad + \lambda\kappa_1 \epsilon_t + \lambda\kappa_2 u_t
\end{aligned} \qquad (2'')$$

Let

$$\rho_\pi := \beta - \lambda\kappa_1\mu + \lambda\kappa_2(1-\gamma)$$

$$\rho_x := \lambda\kappa_1\varpi$$

$$\rho_q := \lambda\kappa_1\chi + \lambda\kappa_2$$

$$\rho_i := \lambda\kappa_1\mu + \lambda\kappa_2(1-\gamma)$$

$(2'')$, (1), (3) $\rightarrow$ (4):

3

$$f_q\left[\mathbb{E}_t\tilde{q}_{t+1} - (1-\gamma)\, i_t + (1-\gamma)\mathbb{E}_t\pi_{H,t+1} + u_t\right]$$

1

$$+ f_x\left[\varpi\mathbb{E}_t\tilde{x}_{t+1} - \mu i_t + \mu\mathbb{E}_t\pi_{H,t+1} + \chi\mathbb{E}_t\tilde{q}_{t+1} + \epsilon_t\right]$$

$$+ f_\pi\left[\rho_\pi\mathbb{E}_t\pi_{H,t+1} + \rho_x\mathbb{E}_t\tilde{x}_{t+1} + \rho_q\mathbb{E}_t\tilde{q}_{t+1} - \rho_i\, i_t + \lambda\kappa_1\epsilon_t + \lambda\kappa_2 u_t\right]$$

$$+ \left[f_a,\ f_{c^*},\ f_q,\ f_x\right]\cdot\begin{pmatrix} a_t \\ c_t^* \\ \bar{i}_t \\ \bar{q}_t \end{pmatrix} = 0$$

Rearranging for i_t:

$$[f_q + f_x \chi + f_\pi \rho_q] \mathbb{E}_t \tilde{q}_{t+1}$$

$$+ [f_q(1-\gamma) + f_x \mu + f_\pi \rho_\pi] \mathbb{E}_t \pi_{H,t+1}$$

$$+ [f_x \varpi + f_\pi \rho_x] \mathbb{E}_t \tilde{x}_{t+1}$$

$$+ [f_q + f_\pi \lambda \kappa_2] u_t + [f_x + f_\pi \lambda \kappa_1] \epsilon_t$$

$$+ [f_a \; f_c^* \; f_{\bar{q}} \; f_{\bar{x}}] \begin{pmatrix} a_t \\ c_t^A \\ \bar{q}_t \\ \bar{q}_t \end{pmatrix}$$

$$= [f_q(1-\gamma) + f_x \mu + f_\pi \rho_i] i_t$$

$$\boxed{\begin{aligned} \Rightarrow \; i_t &= \phi_\pi^* \mathbb{E}_t \pi_{H,t+1} + \phi_x^* \mathbb{E}_t \tilde{x}_{t+1} + \phi_q^* \mathbb{E}_t \tilde{q}_{t+1} \\ &\quad + \text{t.i.p.} \\ \text{where: } \phi_\pi^* &= [f_q(1-\gamma) - f_x \mu + f_\pi \rho_\pi] / [f_q(1-\gamma) + f_x \mu + f_\pi \rho_i] \\ \phi_x^* &= [f_x \varpi + f_\pi \rho_x] / [f_q(1-\gamma) + f_x \mu + f_\pi \rho_i] \\ \phi_q^* &= [f_q + f_x \chi + f_\pi \rho_q] / [f_q(1-\gamma) + f_x \mu + f_\pi \rho_i] \end{aligned}}$$