Behavioral epidemiology: An economic model to evaluate optimal policy in the midst of a pandemic*

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April 15, 2020

Preliminary. Comments welcome.

Let p_t be a probability that a vaccine arrives at t, set $q_t = \sum_{k=t}^{\infty} p_t$.

PREFERENCES:

$$\boxed{\text{gov't}: \quad \overline{Y} + \xi(1 - D) \qquad \text{agents:} \quad Y - C}$$

$$Y = \sum_{t=1}^{\infty} \left\{(1-\delta)\lambda_t \left[S_t + I_t + R_t + RT_t\right] \delta^{t-1} q_t + \left[1 - D_{t+1}\right] \delta^t p_t\right\}$$

$$C = \sum_{t=1}^{\infty} \left\{(1-\delta)(c/2)\left[IT_t + H_t + D_t\right]\delta^{t-1}q_t + (1-\delta)\sum_{k=1}^{t}\left\{(c/2)(1-\alpha_t^k)^2\left[S_t^k + I_t^k + R_t^k\right] + \beta_S\phi^+ S_t^k \hat{I}_t + \beta_S \phi^- I_t^k \hat{S}_t\right\}\delta^{t-1}q_t + (c/2)D_{t+1}\delta^t p_t\right\}$$

$$\overline{Y} = \sum_{t=1}^{\infty} \left\{(1-\bar{\delta})\lambda_t \left[S_t + I_t + R_t + RT_t\right] \bar{\delta}^{t-1} q_t + \left[1 - D_{t+1}\right] \bar{\delta}^t p_t\right\}$$

$$D = \sum_{t=1}^{\infty} D_{t+1} p_t$$

where we used the following notations: $S_t = \sum_{k=1}^{t} S_t^k$, $I_t = \sum_{k=1}^{t} I_t^k$, $R_t = \sum_{k=1}^{t} R_t^k$, $\hat{S}_t = \sum_{k=1}^{t} \alpha_t^k S_t^k$, $\hat{I}_t = \sum_{k=1}^{t} \alpha_t^k I_t^k$.

In order to avoid dealing with α_t^k hitting boundaries, we add a barrier to the agents' costs, thus the agents maximize

$$Y - C + (1-\delta)c\kappa \sum_{t=1}^{\infty} \left[\log \alpha_t^k + \log(1-\alpha_t^k)\right] \delta^{t-1} q_t$$

where κ is sufficiently small.

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AGENTS STATE EQ'S [forward]:

$$S^k_{t+1} = (1-\tau_t\gamma)\left[S^k_t - \beta_w \lambda^2_t S^k_t I_t - \beta_s \alpha^k_t S^k_t \hat{I}_t\right], \qquad k=1,\dots t \tag{$\mathbb{S}^k_t\delta^{t-1}$}$$

$$S^{t+1}_{t+1} = \tau_t\gamma\left[S_t - \beta_w \lambda^2_t S_t I_t - \beta_s \hat{S}_t \hat{I}_t\right] \tag{$\mathbb{S}^{t+1}_t\delta^{t-1}$}$$

$$I^k_{t+1} = (1-\tau_t)\left[(1-1/t_i)I^k_t + \beta_w \lambda^2_t S^k_t I_t + \beta_s \alpha^k_t S^k_t \hat{I}_t\right], \qquad k=1,\dots t \tag{$\mathbb{I}^k_t\delta^{t-1}$}$$

$$IT_{t+1} = (1-1/t_i)IT_t + \tau_t\left[(1-1/t_i)I_t + \beta_w \lambda^2_t S_t I_t + \beta_s \hat{S}_t \hat{I}_t\right] \tag{$\mathbb{IT}_t\delta^{t-1}$}$$

$$R^k_{t+1} = (1-\tau_t\gamma)\left[R^k_t + (1-m_i)/t_i I^k_t\right], \qquad k=1,\dots t \tag{$\mathbb{R}^k_t\delta^{t-1}$}$$

$$RT_{t+1} = RT_t + (1-m_i)/t_i IT_t + \tau_t\gamma\left[R_t + (1-m_i)/t_i I_t\right] + (1-m_h)/t_h H_t \tag{$\mathbb{RT}_t\delta^{t-1}$}$$

$$H_{t+1} = (1-1/t_h)H_t + m_i/t_i(I_t + IT_t) \tag{$\mathbb{H}_t\delta^{t-1}$}$$

$$D_{t+1} = D_t + m_h/t_h H_t$$

with the following initial conditions: $S^1_1 = 1 - e_0$, $I^1_1 = e_0$, $R^1_1 = IT_1 = RT_1 = H_1 = D_1 = 0$. Moreover, $I^{t+1}_{t+1} = R^{t+1}_{t+1} = 0$ at all dates. Here, the red variables are those which are FIXED at the moment the agents make their social distancing decisions.

In what follows we denote the respective dual variables in the GOV'T PROBLEM by $\overline{\mathbb{S}}^k_t$, $\overline{\mathbb{I}}^k_t$, $\overline{\mathbb{R}}^k_t$, $\overline{\mathbb{IT}}_t$, $\overline{\mathbb{RT}}_t$ and $\overline{\mathbb{H}}_t$. Note that the gov't variables are NOT discounted.

STRATEGY

1 solve the agents' problem by choosing a) (S, I, R, IT, RT, H) and b) α treating the red variables fixed. This produces a) the agents' adjoint equations for $(\mathbb{S}, \mathbb{I}, \mathbb{R}, \mathbb{IT}, \mathbb{RT}, \mathbb{H})$ and b) the social distancing FOC.

2 solve the gov't problem by selecting a) (S, I, R, IT, RT, H), b) $(\mathbb{S}, \mathbb{I}, \mathbb{R}, \mathbb{IT}, \mathbb{RT}, \mathbb{H})$, c) α, d) τ and e) λ. This produces a) the gov't adjoint equations for $(\overline{\mathbb{S}}, \overline{\mathbb{I}}, \overline{\mathbb{R}}, \overline{\mathbb{IT}}, \overline{\mathbb{RT}}, \overline{\mathbb{H}})$, b) the set of adjoint equations to the agent's adjoints $(\overline{S}, \overline{I}, \overline{R}, \overline{\mathbb{IT}}, \overline{RT}, \overline{H})$, c) one more FOC for social distancing, d) the FOC for testing and e) the FOC for lockdown.

AGENTS ADJOINT EQ'S [backward]:

$$\begin{aligned}\mathbb{S}^k_t &= \delta\left[(1-\tau_{t+1}\gamma)\mathbb{S}^k_{t+1} + \tau_{t+1}\gamma\mathbb{S}^{t+2}_{t+1}\right] + \delta\left[(1-\tau_{t+1})\mathbb{I}^k_{t+1} + \tau_{t+1}\mathbb{IT}_{t+1} - (1-\tau_{t+1}\gamma)\mathbb{S}^k_{t+1} - \tau_{t+1}\gamma\mathbb{S}^{t+2}_{t+1}\right](\beta_w\lambda^2_{t+1}I_{t+1} + \beta_s\alpha^k_{t+1}\hat{I}_{t+1})\\ &\quad + \delta(1-\delta)\lambda_{t+1}q_{t+1} + \delta p_t + \delta(1-\delta)(c/2)q_{t+1} + \delta(c/2)p_t - \delta(1-\delta)(1-\alpha^k_{t+1})^2(c/2)q_{t+1} - \delta(1-\delta)\alpha^k_{t+1}\hat{I}_{t+1}\phi^+ q_{t+1}, \qquad k=1,\dots t+1\end{aligned} \tag{$\overline{S}^k_{t+1}$}$$

$$\begin{aligned}\mathbb{I}^k_t &= \delta(1-1/t_i)\left[(1-\tau_{t+1})\mathbb{I}^k_{t+1} + \tau_{t+1}\mathbb{IT}_{t+1}\right] + \delta(1-m_i)/t_i\left[(1-\tau_{t+1}\gamma)\mathbb{R}^k_{t+1} + \tau_{t+1}\gamma\mathbb{RT}_{t+1}\right] + \delta m_i/t_i\mathbb{H}_{t+1}\\ &\quad + \delta(1-\delta)\lambda_{t+1}q_{t+1} + \delta p_t + \delta(1-\delta)(c/2)q_{t+1} + \delta(c/2)p_t - \delta(1-\delta)(1-\alpha^k_{t+1})^2(c/2)q_{t+1} - \delta(1-\delta)\alpha^k_{t+1}\hat{S}_{t+1}\phi^- q_{t+1}, \qquad k=1,\dots t\end{aligned} \tag{$\overline{I}^k_{t+1}$}$$

$$\mathbb{IT}_t = \delta\left[(1-1/t_i)\mathbb{IT}_{t+1} + (1-m_i)/t_i\mathbb{RT}_{t+1} + m_i/t_i\mathbb{H}_{t+1}\right] + \delta p_t + \delta(c/2)p_t \tag{$\overline{IT}_{t+1}$}$$

$$\begin{aligned}\mathbb{R}^k_t &= \delta\left[(1-\tau_{t+1}\gamma)\mathbb{R}^k_{t+1} + \tau_{t+1}\gamma\mathbb{RT}_{t+1}\right] + \delta(1-\delta)\lambda_{t+1}q_{t+1} + \delta p_t + \delta(1-\delta)(c/2)q_{t+1} + \delta(c/2)p_t\\ &\quad - \delta(1-\delta)(1-\alpha^k_{t+1})^2(c/2)q_{t+1}, \qquad k=1,\dots t\end{aligned} \tag{$\overline{R}^k_{t+1}$}$$

$$\mathbb{RT}_t = \delta\mathbb{RT}_{t+1} + \delta(1-\delta)\lambda_{t+1}q_{t+1} + \delta p_t + \delta(1-\delta)(c/2)q_{t+1} + \delta(c/2)p_t \tag{$\overline{R}_{t+1}$}$$

$$\mathbb{H}_t = \delta\left[(1-1/t_h)\mathbb{H}_{t+1} + (1-m_h)/t_h\mathbb{RT}_{t+1}\right] + \delta p_t + \delta(c/2)p_t \tag{$\overline{H}_{t+1}$}$$

with the following terminal conditions: $\lim_{T\to\infty}\mathbb{S}^k_T/p_T = \lim_{T\to\infty}\mathbb{I}^k_T/p_T = \lim_{T\to\infty}\mathbb{R}^k_T/p_T = \delta(1+c/2)$ for all k and $\lim_{T\to\infty}\mathbb{IT}_T/p_T = \lim_{T\to\infty}\mathbb{RT}_T/p_T = \lim_{T\to\infty}\mathbb{H}_T/p_T = \delta(1+c/2)$.

SOCIAL DISTANCING:

$$\begin{aligned}0 &= \left\{c(1-\alpha^k_t)\left[S^k_t + I^k_t + R^k_t\right] - \beta_s\phi^+ S^k_t\hat{I}_t - \beta_s\phi^- I^k_t\hat{S}_t\right\}(1-\delta)q_t - \beta_s S^k_t\hat{I}_t\left[(1-\tau_t\gamma)\mathbb{S}^k_t + \tau_t\gamma\mathbb{S}^{t+1}_t - (1-\tau_t)\mathbb{I}^k_t - \tau_t\mathbb{IT}_t\right]\\ &\quad + (1-\delta)c\kappa\left[1/\alpha^k_t - 1/(1-\alpha^k_t)\right]q_t \qquad k=1,\dots t\end{aligned} \tag{η^k_t}$$

Note that for $\kappa = 0$, the unique solution to the FOC is $\alpha_t^k = A_t^k$. The problem is that it might lie outside the unit interval, thus the optimal social distancing must be adjusted: $\alpha_t^k = \max\{0, \min\{1, A_t^k\}\}$.

On the other hand, for $\kappa > 0$, the FOC admits the unique solution in $(0, 1)$. Moreover, as $\kappa \downarrow 0$, it converges to $\max\{0, \min\{1, A_t^k\}\}$.

Next, we provide the exact solution to the FOC with $\kappa > 0$. First of all, note that the equation can be rewritten as

$$\alpha_t^k = A_t^k + B_t^k \left[1/\alpha_t^k - 1/(1-\alpha_t^k)\right]$$

where A_t^k and B_t^k are two constants given by $B_t^k = \kappa/\left[S_t^k + I_t^k + R_t^k\right]$,

$$A_t^k = 1 - (\beta_s/c)\phi^+ \frac{S_t^k \hat{I}_t}{S_t^k + I_t^k + R_t^k} - (\beta_s/c)\phi^- \frac{I_t^k \hat{S}_t}{S_t^k + I_t^k + R_t^k} - \frac{\beta_s/c}{(1-\delta)q_t}\left[(1-\tau_t\gamma)\mathbb{S}_t^k + \tau_t\gamma\mathbb{S}_t^{t+1} - (1-\tau_t)\mathbb{I}_t^k - \tau_t\mathbb{I}\mathbb{T}_t\right]$$

The equation can be rewritten as a polynomial of degree 3. Change variables to $\tilde{\alpha}_t^k = \alpha_t^k - \frac{1+A_t^k}{3}$ to obtain the following equivalent depressed cubic:

$$(\tilde{\alpha}_t^k)^3 + \tilde{A}_t^k \tilde{\alpha}_t^k + \tilde{B}_t^k = 0$$

with $\tilde{A}_t^k = \frac{3(A_t^k - 2B_t^k) - (1+A_t^k)^2}{3}$ and $\tilde{B}_t^k = \frac{9(1+A_t^k)(A_t^k - 2B_t^k) - 2(1+A_t^k)^3 + 27B_t^k}{27}$. Use the Viete's formula and express the desired root using trigonometric functions (it cannot be written in real radicals):

$$\alpha_t^k = \frac{1 + A_t^k}{3} + 2\sqrt{\tilde{A}_t^k/3}\cos\left(\arccos\left(3\tilde{B}_t^k/(2\tilde{A}_t^k)\sqrt{-3/\tilde{A}_t^k}\right)/3 - 2\pi/3\right)$$

It can be verified that α_t^k defined above lies within the unit interval and coincides with $\max\{0, \min\{1, A_t^k\}\}$ as $\kappa \downarrow 0$.

TESTING:

$$X_t = \tau_t \left\{\gamma\left[S_t - \beta_w S_t I_t - \beta_s \hat{S}_t \hat{I}_t\right] + \gamma\left[R_t + (1-m_i)/t_i I_t\right] + (1 - 1/t_i)I_t + \beta_w \lambda_t^2 S_t I_t + \beta_s \hat{S}_t \hat{I}_t\right\} \qquad (\chi_t)$$

As before, we use the following notations: $\overline{S}_t = \sum_{k=1}^t \overline{S}_t^k$, $\overline{I}_t = \sum_{k=1}^t \overline{I}_t^k$, $\overline{R}_t = \sum_{k=1}^t \overline{R}_t^k$, $\hat{\overline{S}}_t = \sum_{k=1}^t \alpha_t^k \overline{S}_t^k$, $\hat{\overline{I}}_t = \sum_{k=1}^t \alpha_t^k \overline{I}_t^k$.

GOV'T ADJOINT EQ'S [backward]:

$$
\begin{aligned}
\overline{\mathbb{S}}_t^k &= (1-\tau_{t+1}\gamma)\overline{\mathbb{S}}_{t+1}^k + \tau_{t+1}\gamma\overline{\mathbb{S}}_{t+1}^{t+2} + \left[(1-\tau_{t+1})\overline{\mathbb{I}}_{t+1}^k + \tau_{t+1}\overline{\mathbb{IT}}_{t+1} - (1-\tau_{t+1}\gamma)\overline{\mathbb{S}}_{t+1}^k - \tau_{t+1}\gamma\overline{\mathbb{S}}_{t+1}^{t+2}\right](\beta_w\lambda_{t+1}^2 I_{t+1} + \beta_s\alpha_{t+1}^k\hat{I}_{t+1}) \\
&\quad - (1-\delta)\delta\phi^-\alpha_{t+1}^k\hat{\bar{I}}_{t+1}q_{t+1} - \chi_{t+1}\tau_{t+1}\left[\gamma + (1-\gamma)\left(\beta_w\lambda_{t+1}^2 I_{t+1} + \beta_s\alpha_{t+1}^k\hat{I}_{t+1}\right)\right] \\
&\quad + (1-\delta)\eta_{t+1}^k\left[c(1-\alpha_{t+1}^k) - \beta_s\phi^+\hat{I}_{t+1}\right]q_{t+1} + \eta_{t+1}^k\beta_s\left[(1-\tau_{t+1})\mathbb{I}_{t+1}^k + \tau_{t+1}\mathbb{IT}_{t+1} - (1-\tau_{t+1}\gamma)\mathbb{S}_{t+1}^k - \tau_t\gamma\mathbb{S}_{t+1}^{t+2}\right]\hat{I}_{t+1} \\
&\quad - (1-\delta)\beta_s\phi^-\sum_{j=1}^{t+1}\eta_{t+1}^j I_{t+1}^j\alpha_{t+1}^k q_{t+1} + (1-\bar{\delta})\bar{\delta}^t\lambda_{t+1}q_{t+1} + (\xi+\bar{\delta}^t)p_t, \qquad k = 1,\dots t+1 \\
\overline{\mathbb{I}}_t^k &= (1-1/t_i)\left[(1-\tau_{t+1})\overline{\mathbb{I}}_{t+1}^k + \tau_{t+1}\overline{\mathbb{IT}}_{t+1}\right] + (1-m_i)/t_i\left[(1-\tau_{t+1}\gamma)\overline{\mathbb{R}}_{t+1}^k + \tau_{t+1}\gamma\overline{\mathbb{RT}}_{t+1}\right] + m_i/t_i\overline{\mathbb{H}}_{t+1} \\
&\quad + \sum_{j=1}^{t}\left[(1-\tau_{t+1})\overline{\mathbb{I}}_{t+1}^j + \tau_{t+1}\overline{\mathbb{IT}}_{t+1} - (1-\tau_{t+1}\gamma)\overline{\mathbb{S}}_{t+1}^j - \tau_{t+1}\gamma\overline{\mathbb{S}}_{t+1}^{t+2}\right](\beta_w\lambda_{t+1}^2 S_{t+1}^j + \beta_s\alpha_{t+1}^k S_{t+1}^j\alpha_{t+1}^k) \\
&\quad + \delta\sum_{j=1}^{t}\left[(1-\tau_{t+1})\mathbb{I}_{t+1}^j + \tau_{t+1}\mathbb{IT}_{t+1} - (1-\tau_{t+1}\gamma)\mathbb{S}_{t+1}^j - \tau_{t+1}\gamma\mathbb{S}_{t+1}^{t+2}\right](\beta_w\lambda_{t+1}^2\overline{S}_{t+1}^j + \beta_s\alpha_{t+1}^k\overline{S}_{t+1}^j\alpha_{t+1}^k) \\
&\quad - (1-\delta)\delta\phi^+\alpha_{t+1}^k\hat{\bar{S}}_{t+1}q_{t+1} - \chi_{t+1}\tau_{t+1}\left[1 - 1/t_i + \gamma(1-m_i)/t_i + (1-\gamma)\left(\beta_w\lambda_{t+1}^2 I_{t+1} + \beta_s\alpha_{t+1}^k\hat{I}_{t+1}\right)\right] \\
&\quad + (1-\delta)\eta_{t+1}^k\left[c(1-\alpha_{t+1}^k) - \beta_s\phi^-\hat{S}_{t+1}\right]q_{t+1} + \beta_s\sum_{j=1}^{t+1}\eta_{t+1}^j\left[(1-\tau_{t+1})\mathbb{I}_{t+1}^j + \tau_{t+1}\mathbb{IT}_{t+1} - (1-\tau_{t+1}\gamma)\mathbb{S}_{t+1}^j - \tau_t\gamma\mathbb{S}_{t+1}^{t+2}\right]S_{t+1}^j\alpha_{t+1}^k \\
&\quad - (1-\delta)\beta_s\phi^+\sum_{j=1}^{t+1}\eta_{t+1}^j S_{t+1}^j\alpha_{t+1}^j\alpha_{t+1}^k q_{t+1} + (1-\bar{\delta})\bar{\delta}^t\lambda_{t+1}q_{t+1} + (\xi+\bar{\delta}^t)p_t, \qquad k = 1,\dots t+1 \\
\overline{\mathbb{IT}}_t &= (1-1/t_i)\overline{\mathbb{IT}}_{t+1} + (1-m_i)/t_i\overline{\mathbb{RT}}_{t+1} + m_i/t_i\overline{\mathbb{H}}_{t+1} + (\xi+\bar{\delta}^t)p_t \\
\overline{\mathbb{R}}_t^k &= (1-\tau_{t+1}\gamma)\overline{\mathbb{R}}_{t+1}^k + \tau_{t+1}\gamma\overline{\mathbb{RT}}_{t+1} + \eta_{t+1}^k c(1-\alpha_{t+1}^k)(1-\delta)q_{t+1} - \chi_{t+1}\tau_{t+1}\gamma + (1-\bar{\delta})\bar{\delta}^t\lambda_{t+1}q_{t+1} + (\xi+\bar{\delta}^t)p_t, \qquad k = 1,\dots t \\
\overline{\mathbb{RT}}_t &= \overline{\mathbb{RT}}_{t+1} + (1-\bar{\delta})\bar{\delta}^t\lambda_{t+1}q_{t+1} + (\xi+\bar{\delta}^t)p_t \\
\overline{\mathbb{H}}_t &= (1-1/t_h)\overline{\mathbb{H}}_{t+1} + (1-m_h)/t_h\overline{\mathbb{RT}}_{t+1} + (\xi+\bar{\delta}^t)p_t
\end{aligned}
$$

with the following terminal conditions: $\lim_{T\to\infty}\overline{\mathbb{S}}_T^k/p_T - \delta^T = \lim_{T\to\infty}\overline{\mathbb{I}}_T^k/p_T - \delta^T = \lim_{T\to\infty}\overline{\mathbb{R}}_T^k/p_T - \delta^T = \xi$ for all k and $\lim_{T\to\infty}\overline{\mathbb{IT}}_T/p_T - \delta^T = \lim_{T\to\infty}\overline{\mathbb{RT}}_T/p_T - \delta^T = \lim_{T\to\infty}\overline{\mathbb{H}}_T/p_T - \delta^T = \xi$.

ADJOINT to AGENTS ADJOINT EQ's [forward]:

$$
\begin{aligned}
\overline{S}_{t+1}^k &= \delta(1-\tau_t\gamma)\left[\overline{S}_t^k - \beta_w\lambda_t^2\overline{S}_t^k I_t - \beta_s\alpha_t^k\overline{S}_t^k\hat{I}_t\right] - \eta_t^k\beta_s S_t^k\hat{I}_t(1-\tau_t\gamma), \qquad k = 1,\dots t \\
\overline{S}_{t+1}^{t+1} &= \delta\tau_t\gamma\left[\overline{S}_t - \beta_w\lambda_t^2\overline{S}_t I_t - \beta_s\hat{\overline{S}}_t\hat{I}_t\right] - \sum_{k=1}^{t}\eta_t^k\beta_s S_t^k\hat{I}_t\tau_t\gamma \\
\overline{I}_{t+1}^k &= \delta(1-\tau_t)\left[(1-1/t_i)\overline{I}_t^k + \beta_w\lambda_t^2\overline{S}_t^k I_t + \beta_s\alpha_t^k\overline{S}_t^k\hat{I}_t\right] + \eta_t^k\beta_s S_t^k\hat{I}_t(1-\tau_t), \qquad k = 1,\dots t \\
\overline{IT}_{t+1} &= \delta\left\{(1-1/t_i)\overline{IT}_t + \tau_t\left[(1-1/t_i)\overline{I}_t + \beta_w\lambda_t^2\overline{S}_t I_t + \beta_s\hat{\overline{S}}_t\hat{I}_t\right]\right\} + \sum_{k=1}^{t}\eta_t^k\beta_s S_t^k\hat{I}_t\tau_t \\
\overline{R}_{t+1}^k &= \delta(1-\tau_t\gamma)\left[\overline{R}_t^k + (1-m_i)/t_i\overline{I}_t^k\right], \qquad k = 1,\dots t \\
\overline{RT}_{t+1} &= \delta\left\{\overline{RT}_t + (1-m_i)/t_i\overline{IT}_t + \tau_t\gamma\left[\overline{R}_t + (1-m_i)/t_i\overline{I}_t\right] + (1-m_h)/t_h\overline{H}_t\right\} \\
\overline{H}_{t+1} &= \delta\left[(1-1/t_h)\overline{H}_t + m_i/t_i(\overline{I}_t + \overline{IT}_t)\right]
\end{aligned}
$$

where the initial conditions are

$$
\begin{aligned}
\overline{S}_2^1 &= -\eta_1^1\beta_s S_1^1\hat{I}_1(1-\tau_1\gamma) \\
\overline{S}_2^2 &= -\eta_1^1\beta_s S_1^1\hat{I}_1\tau_1\gamma \\
\overline{I}_2^1 &= \eta_1^1\beta_s S_1^1\hat{I}_1(1-\tau_1) \\
\overline{IT}_2 &= \eta_1^1\beta_s S_1^1\hat{I}_1\tau_1
\end{aligned}
$$

$\overline{R}_2^1 = \overline{RT}_2 = \overline{H}_2 = 0$. In addition, $\overline{I}_t^t = \overline{R}_t^t = 0$ at all dates.

MULTIPLIER ON TESTING

$$
\begin{aligned}
&\chi_t \left\{\gamma \left[S_t - \beta_w S_t I_t - \beta_s \hat{S}_t \hat{I}_t\right] + \gamma \left[R_t + (1 - m_i)/t_i I_t\right] + (1 - 1/t_i) I_t + \beta_w \lambda_t^2 S_t I_t + \beta_s \hat{S}_t \hat{I}_t\right\} = \\
&+\gamma \overline{\mathbb{S}}_t^{t+1} \left[S_t - \beta_w \lambda_t^2 S_t I_t - \beta_s \hat{S}_t \hat{I}_t\right] + \gamma \overline{\mathbb{R}}_t^{t+1} \left[R_t + (1 - m_i)/t_i I_t\right] + \overline{\mathbb{IT}}_t \left[(1 - 1/t_i) I_t + \beta_w \lambda_t^2 S_t I_t + \beta_s \hat{S}_t \hat{I}_t\right] \\
&\quad - \gamma \sum_{k=1}^{t} \overline{\mathbb{S}}_t^k \left[S_t^k - \beta_w \lambda_t^2 S_t^k I_t - \beta_s \alpha_t^k S_t^k \hat{I}_t\right] - \gamma \sum_{k=1}^{t} \overline{\mathbb{R}}_t^k \left[R_t^k + (1 - m_i)/t_i I_t^k\right] - \sum_{k=1}^{t} \overline{\mathbb{I}}_t^k \left[(1 - 1/t_i) I_t^k + \beta_w \lambda_t^2 S_t^k I_t + \beta_s \alpha_t^k S_t^k \hat{I}_t\right] \\
&+\gamma \mathbb{S}_t^{t+1} \left[\overline{S}_t - \beta_w \lambda_t^2 \overline{S}_t I_t - \beta_s \hat{\overline{S}}_t \hat{I}_t\right] + \gamma \mathbb{R}_t^{t+1} \left[\overline{R}_t + (1 - m_i)/t_i \overline{I}_t\right] + \mathbb{IT}_t \left[(1 - 1/t_i) \overline{I}_t + \beta_w \lambda_t^2 \overline{S}_t I_t + \beta_s \hat{\overline{S}}_t \hat{I}_t\right] \\
&\quad - \gamma \sum_{k=1}^{t} \mathbb{S}_t^k \left[\overline{S}_t^k - \beta_w \lambda_t^2 \overline{S}_t^k I_t - \beta_s \alpha_t^k \overline{S}_t^k \hat{I}_t\right] - \gamma \sum_{k=1}^{t} \mathbb{R}_t^k \left[\overline{R}_t^k + (1 - m_i)/t_i \overline{I}_t^k\right] - \sum_{k=1}^{t} \mathbb{I}_t^k \left[(1 - 1/t_i) \overline{I}_t^k + \beta_w \lambda_t^2 \overline{S}_t^k I_t + \beta_s \alpha_t^k \overline{S}_t^k \hat{I}_t\right]
\end{aligned}
$$

LOCKDOWN

The derivative with respect to λ_t can be written as $a_t - 2\lambda_t b_t$ where a_t and b_t are given by

$$
\begin{aligned}
a_t &= (1 - \bar{\delta}) \bar{\delta}^{t-1} \left[S_t + I_t + R_t + RT_t\right] q_t + (1 - \delta)\delta \left[\overline{S}_t + \overline{I}_t + \overline{R}_t + \overline{RT}_t\right] q_t \\
b_t &= \beta_w \sum_{k=1}^{t} \left[(1 - \tau_t)\overline{\mathbb{I}}_t^k + \tau_t \overline{\mathbb{IT}}_t - (1 - \tau_t \gamma)\overline{\mathbb{S}}_t^k - \tau_t \gamma \overline{\mathbb{S}}_t^{t+1}\right] S_t^k I_t + \delta \beta_w \sum_{k=1}^{t} \left[(1 - \tau_t)\mathbb{I}_t^k + \tau_t \mathbb{IT}_t - (1 - \tau_t \gamma)\mathbb{S}_t^k - \tau_t \gamma \mathbb{S}_t^{t+1}\right] \overline{S}_t^k I_t + (1 - \gamma)\beta_w \tau_t \chi_t S_t I_t
\end{aligned}
$$

The optimal policy is as it follows:

$$
\begin{cases}
\lambda_t = 1 & \text{whenever } b_t \leqslant 0 \\
\lambda_t = \min\{1, \max\{\lambda_{min}, \frac{a_t}{2b_t}\}\} & \text{whenever } b_t > 0
\end{cases}
$$

MULTIPLIER ON SOCIAL DISTANCING

The derivative with respect to α_t^k can be written as $\mathcal{B}_t - \sum_{j=1}^{t} \mathcal{A}_t^{kj} \eta_t^j$ where $\mathcal{A}_t$ and $t \times 1$ vector $\mathcal{B}_t$ are given by

$$
\begin{aligned}
\mathcal{A}_t^{kj} &= (1 - \delta)\beta_s \left[\phi^+ S_t^j I_t^k + \phi^- I_t^j S_t^k\right] q_t + \beta_s \left[(1 - \tau_t)\mathbb{I}_t^j + \tau_t \mathbb{IT}_t - (1 - \tau_t \gamma)\mathbb{S}_t^j - \tau_t \gamma \mathbb{S}_t^{t+1}\right] S_t^j I_t^k \\
&\quad + (1 - \delta) c \left(S_t^k + I_t^k + R_t^k + \kappa \left[1/(\alpha_t^k)^2 + 1/(1 - \alpha_t^k)^2\right]\right) q_t \mathbb{1}(j = k) \\
\mathcal{B}_t^k &= \beta_s \left[(1 - \tau_t)\overline{\mathbb{I}}_t^k + \tau_t \overline{\mathbb{IT}}_t - (1 - \tau_t \gamma)\overline{\mathbb{S}}_t^k - \tau_t \gamma \overline{\mathbb{S}}_t^{t+1}\right] S_t^k \hat{I}_t + \delta \beta_s \left[(1 - \tau_t)\mathbb{I}_t^k + \tau_t \mathbb{IT}_t - (1 - \tau_t \gamma)\mathbb{S}_t^k - \tau_t \gamma \mathbb{S}_t^{t+1}\right] \overline{S}_t^k \hat{I}_t \\
&\quad + \beta_s \sum_{j=1}^{t} \left[(1 - \tau_t)\overline{\mathbb{I}}_t^j + \tau_t \overline{\mathbb{IT}}_t - (1 - \tau_t \gamma)\overline{\mathbb{S}}_t^j - \tau_t \gamma \overline{\mathbb{S}}_t^{t+1}\right] \alpha_t^j S_t^j I_t^k + \delta \beta_s \sum_{j=1}^{t} \left[(1 - \tau_t)\mathbb{I}_t^j + \tau_t \mathbb{IT}_t - (1 - \tau_t \gamma)\mathbb{S}_t^j - \tau_t \gamma \mathbb{S}_t^{t+1}\right] \alpha_t^j \overline{S}_t^j I_t^k \\
&\quad + (1 - \delta)\delta c (1 - \alpha_t^k)(\overline{S}_t^k + \overline{I}_t^k + \overline{R}_t^k) q_t - (1 - \delta)\delta \left(\phi^+ \left[\overline{S}_t^k \hat{I}_t + S_t^k \hat{\overline{I}}_t\right] + \phi^- \left[\overline{I}_t^k \hat{S}_t + I_t^k \hat{\overline{S}}_t\right]\right) q_t - \tau_t \chi_t (1 - \gamma)(S_t^k \hat{I}_t + \hat{S}_t I_t^k)
\end{aligned}
$$

Then, we have the simple formula for η:

$$\eta_t = [\mathcal{A}_t]^{-1} \mathcal{B}_t$$

ALGORITHM

0 fix large T such that $p_T \approx 0$ and choose $(\chi, \eta, \alpha, \lambda)$

1 given $(\chi, \eta, \alpha, \lambda)$, solve both forwards systems (S, I, R, IT, RT, H), $(\overline{S}, \overline{I}, \overline{R}, \overline{IT}, \overline{RT}, \overline{H})$ and find τ

2 given $(\chi, \eta, \alpha, \lambda)$ and the result of step 1, solve both backwards systems $(\mathbb{S}, \mathbb{I}, \mathbb{R}, \mathbb{IT}, \mathbb{RT}, \mathbb{H})$ and $(\overline{\mathbb{S}}, \overline{\mathbb{I}}, \overline{\mathbb{R}}, \overline{\mathbb{IT}}, \overline{\mathbb{RT}}, \overline{\mathbb{H}})$

3 given results of steps 1 and 2, update $(\chi, \eta, \alpha, \lambda)$ to $(\chi', \eta', \alpha', \lambda')$ and repeat until convergence