

Name \_\_\_\_\_

Final Exam  
Intertemporal Choice  
Fall, 2014  
Answers

You are expected to answer all parts of all questions. If you cannot solve part of a question, *do not give up*. The exam is written so that you should be able to answer later parts even if you are stumped by earlier parts.

Write all answers on the exam itself; if you run out of room, use the back of the previous page.

## Part I. Short Questions.

1. **CARA Portfolio Choice (Merton (1969)).** Consider a consumer with Constant Absolute Risk Aversion utility  $u(c) = -\alpha^{-1}e^{-\alpha c}$ , with assets  $a_{T-1}$  who is deciding how much to invest in a risky security that will earn a normally distributed stochastic return  $\mathbf{R}_T \sim \mathcal{N}(\mathbf{R}, \sigma)$  versus a safe asset that will earn return  $\mathbf{R} < \mathbf{R}$ .

Consumption in the last period of life will be the entire amount of resources. If the consumer invests an absolute amount of money  $\$S$  in the risky asset, then

$$\begin{aligned} c_T &= S\mathbf{R}_T + (a_{T-1} - S)\mathbf{R} \\ &= a_{T-1}\mathbf{R} + \underbrace{(\mathbf{R}_T - \mathbf{R})S}_{\equiv \Phi_T} \end{aligned}$$

where  $\Phi_T$  is the equity premium realized in period  $T$ . Given  $S$  and defining the expected equity premium as  $\Phi \equiv \mathbb{E}_{T-1}[\mathbf{R}_T - \mathbf{R}]$ :

- a) Show that the expectation of utility as of time  $T - 1$  is:

$$\mathbb{E}_{T-1}[u(c_T)] = -\alpha^{-1}e^{-\alpha(a_{T-1}\mathbf{R})}e^{-\alpha(S\Phi - \alpha S^2\sigma^2/2)} \quad (1)$$

*Answer:*

See **CARAPortfolio**.

- b) Given this result, show that the FOC imply:

$$\Phi = \alpha S \sigma^2 \quad (2)$$

$$S = \frac{\Phi}{\alpha \sigma^2}. \quad (3)$$

and explain both the intuition and why the result is not very plausible. (Hint: Homer and Montgomery).

*Answer:*

For the derivations, see **CARAPortfolio**.

This result is qualitatively intuitive in the sense that the greater is risk aversion or the greater is the risk, the less the consumer wants to invest in the risky asset, while the greater is the expected excess return, the more the consumer wants to invest. Note, however, that the model implausibly says that the dollar amount invested in the risky asset does not depend on the total dollar amount of resources  $a_{T-1}$ . So, Montgomery Burns and Homer Simpson should have the exact same dollar holdings of the risky asset! If Buffett is richer than Simpson, Burns excess wealth is held in the safe form. Not very plausible. (That is why models with CARA utility are increasingly unfashionable in the economics and finance literatures).

2. **Romer (1986).** Romer (1986) presents a growth model in which private producers engage in perfect competition, but a benevolent social planner can achieve positive economic growth in perpetuity (forever). Explain why this is a difficult combination to achieve and explain how Romer's model solves the problem. (Be sure to mention Euler's theorem in your answer).

*Answer:*

Euler's theorem says that for a function  $F(K, L)$  that is homogeneous of degree one we have the following:

$$F(K, L) = F_K K + F_L L.$$

In words, this says that total production is used up after compensating capital and labor at marginal product prices. This makes endogenous growth difficult, since there are no resources left over to spend on R&D.

Romer overcomes this problem by introducing external effects to individual firms' investment decisions. When a firm invests in its capital stock, it *also* increases the stock of total knowledge available in the economy  $\Xi$ . In essence, investment in capital does double duty as R&D too. This overcomes the problem, because the investment is being done by people who have their own selfish reasons for doing it, and generates a growing technology factor endogenously.

(Note:  $u'_t = \mathbb{E}_t [R_{t+1}\beta u'_{t+1}]$  is the Euler *Equation*, not Euler's theorem.)

## Part II. Medium Questions.

1. **Lucas Asset Pricing with CRRA Utility.** Consider a Lucas (1978) model in which, instead of having logarithmic utility, consumers have CRRA utility:  $u(c) = (1 - \rho)^{-1} c^{1-\rho}$ .

- a) Explain the set of assumptions that Lucas (1978) makes that allow him to use the partial equilibrium consumption Euler equation

$$u'(c_t^i) = \beta \mathbb{E}_t^i [u'(c_{t+1}^i) \mathbf{R}_{t+1}] \quad (4)$$

to obtain a general equilibrium asset-pricing equation of the form:

$$P_t = \beta \mathbb{E}_t \left[ \left( \frac{u'(d_{t+1})}{u'(d_t)} \right) (P_{t+1} + d_{t+1}) \right]. \quad (5)$$

*Answer:*

For all answers, see `LucasAssetPrice`

- b) Define an object

$$\mathbf{M}_{t,t+n} = \beta^n \left( \frac{u'(d_{t+n})}{u'(d_t)} \right) \quad (6)$$

called the ‘stochastic discount factor.’ Analogously, define the cumulative SDF between  $t$  and  $t + 2$  as  $\mathbf{M}_{t,t+2} \equiv \mathbf{M}_{t,t+1} \mathbf{M}_{t+1,t+2}$ , and so on. Using these definitions, explain how to use this object to obtain an equation relating the current asset price to expected future prices via

$$P_t = \mathbb{E}_t [\mathbf{M}_{t,t+1} d_{t+1} + \mathbf{M}_{t,t+2} d_{t+2} + \mathbf{M}_{t,t+3} d_{t+3} + \dots]. \quad (7)$$

*Answer:*

- c) Use (5) to show that if the utility function is of the CRRA form and then the pricing equation reduces to

$$P_t = \beta d_t^\rho \mathbb{E}_t [d_{t+1}^{-\rho} (P_{t+1} + d_{t+1})] \quad (8)$$

*Answer:*

- d) Using (8), show that in the logarithmic utility case where  $\rho = 1$ , the price eventually reduces to

$$\begin{aligned} P_t &= d_t \left( \frac{\beta}{1 - \beta} \right) \\ &= d_t \left( \frac{1}{1/\beta - 1} \right) \end{aligned}$$

$$\begin{aligned}
&= d_t \left( \frac{1}{1 + \vartheta - 1} \right) \\
&= \frac{d_t}{\vartheta}
\end{aligned} \tag{9}$$

*Answer:*

- e) Now assume that under CRRA utility with  $\rho > 1$ ,  $P_t/d_t^\rho$  is a constant, and assume that the dividend process is white noise (that is,  $\mathbb{E}_t[d_{t+1}] = \mathbb{E}_t[d_{t+2}] = \mathbb{E}_{t+1}[d_{t+2}] \equiv \bar{d} \dots$ ). Under the assumption that the white noise process for dividends is  $d_{t+n} \sim \mathcal{N}(0, \sigma^2) \forall n > 0$ , derive the equilibrium value for  $P/d^\rho$  and discuss the time series relationships between  $P$  and  $d$  and explain how they depend on the value of  $\rho$ .

*Answer:*

See the appendix to `LucasAssetPrice`.

2. **Scarcity.** Recent work by Mullainathan and Shafir (2013) can be interpreted as claiming that psychological effects of poverty (perhaps interpretable as “stress”) effectively cause poor people to become more impatient. This question asks you to think about some implications of that hypothesis using a buffer-stock model of saving.

Specifically, suppose that there are two psychological states that a person can be in: Either “relaxed” or “stressed.” In the “relaxed” state the time preference rate is lower than in the “stressed” state. Suppose that, in either state, people do not perceive that there is a possibility that they might make a transition to the other state. That is, a person in the “stressed” state believes he will always stay in the stressed state, and similarly for the relaxed state. However, suppose that in truth, whenever a person’s market resource ratio falls below a certain cutoff point, that person inevitably switches to the stressed state. If the  $m$  ratio rises above the cutoff level (call it  $\hat{m}$ ) the person will make a transition back to the “relaxed” state. The reason “stress” has this effect is supposedly that the brain does not operate as well in a stressed condition as in a relaxed condition.

Answer the following questions under the assumption that a target wealth ratio exists for either a consumer in the “stressed” or in the “relaxed” state.

- Whose target wealth ratio is higher, the “stressed” person’s or the “relaxed” person’s? (Draw the phase diagram showing the target level of wealth for the two kinds of people).

*Answer:*

The stressed person’s higher impatience shifts their  $\Delta c_t^e = 0$  locus to the left in  $m$ - $c$  space; but the budget constraint that yields the  $\Delta m_t^e = 0$  locus is unchanged. Hence the stressed person’s target level of wealth is lower.

- Suppose a “relaxed” person who is at the target level of wealth for a relaxed person suddenly gets hit by a negative shock to wealth (car crash; child’s wedding; medical expenses) that push his wealth below  $\hat{m}$ . Will this person be expected subsequently to recover toward his original target wealth?

*Answer:*

We must assume  $\hat{m}_{stressed} < \hat{m} < \hat{m}_{relaxed}$  for the model to make sense (ie to have two types of consumers in the steady state). Given this, the answer is no. The consumer switches regimes when they cross the  $\hat{m}$  threshold and thereafter tends towards their new steady state level of wealth,  $\hat{m}_{stressed}$ . Since in the absence of shocks they never re-cross the threshold  $\hat{m}$ , they will remain stressed forever and so will remain at the lower level of wealth forever.

- Suppose a “stressed” person who is at the target level of wealth for a stressed person suddenly gets hit by a positive shock (inheritance; lottery winnings)

that push his wealth above  $\hat{m}$ . Will this person be expected to recover toward his original target wealth?

*Answer:*

Again assume  $\hat{m}_{stressed} < \hat{m} < \hat{m}_{relaxed}$ . Given this, the answer is no. The consumer switches regimes when they cross the  $\hat{m}$  threshold and thereafter tends towards their new steady state level of wealth,  $\hat{m}_{relaxed}$ . Since in the absence of shocks they never re-cross the threshold  $\hat{m}$ , they will remain stressed forever and so will remain at the higher level of wealth forever.

- A recent study examined the consequences of giving poor people in a developing country a large lump sum of money. While they were better off for a while, eventually they subsided back into poverty indistinguishable from their initial circumstances. Is this what you would expect from the Mullainathan and Shafir (2013) model?

*Answer:*

There are two possible conclusions to draw from this. The first is that the model is wrong, since if the lump sum of money was big enough it would push the consumer across the threshold and relax them. They would switch regimes and target a new, higher level of wealth  $\hat{m}_{relaxed}$  and never go back. The second possible conclusion to draw is that the lump sum of money simply was not big enough to push the consumers across the threshold of relaxation.

## Part III. Long Question.

### Saving and Growth Redux (Gourinchas and Jeanne (2007)).

Consider a Ramsey/Cass-Koopmans growth model with constant labor-augmenting technological progress at rate  $g = \dot{Z}_t/Z_t$ . The population is constant at  $L_t = 1$  and productivity in period 0 is normalized to  $Z_0 = 1$ . The social planner has a CRRA felicity function  $u(C) = C^{1-\rho}/(1-\rho)$  and solves the optimization problem

$$\max \int_0^\infty u(C_t) e^{-\vartheta t} dt \quad (10)$$

s.t.

$$\dot{K}_t = \underbrace{K_t^\alpha (Z_t)^{1-\alpha}}_{\equiv Y_t} - C_t - \delta K_t \quad (11)$$

1. Rewrite the problem in per-efficiency-unit terms, designating small-letter variables as equal to the capital-letter variable divided by labor productivity, e.g.  $c_t = C_t/Z_t$ , and show that in the steady state  $\dot{K}/K = \dot{C}/C = g$ . (Hint: Begin by rewriting the utility function in terms of consumption per efficiency unit and the growth in productivity since time 0.)

*Answer:*

The utility function is rewritten as

$$\frac{C_t^{1-\rho}}{1-\rho} = \frac{(c_t Z_t)^{1-\rho}}{1-\rho} \quad (12)$$

$$= \frac{c_t^{1-\rho}}{1-\rho} Z_t^{1-\rho} \quad (13)$$

$$= \frac{c_t^{1-\rho}}{1-\rho} (Z_0 e^{gt})^{1-\rho} \quad (14)$$

$$= \frac{c_t^{1-\rho}}{1-\rho} e^{g(1-\rho)t} \quad (15)$$

Rewrite  $\dot{K}$  as

$$\frac{dK}{dt} = \frac{d(k_t Z_t)}{dt} \quad (16)$$

$$= \frac{dk_t}{dt} Z_t + k_t \frac{dZ_t}{dt} \quad (17)$$

$$= (\dot{k}_t + g k_t) Z_t \quad (18)$$

so that equation (11) becomes

$$\begin{aligned} (\dot{k}_t + g k_t) Z_t &= Y_t - C_t - \delta K_t \\ \dot{k}_t + g k_t &= y_t - c_t - \delta k_t \\ \dot{k}_t &= y_t - c_t - (\delta + g) k_t \end{aligned} \quad (19)$$



and the optimization problem can be rewritten as

$$\max \int_0^\infty \left( \frac{c_t^{1-\rho}}{1-\rho} \right) e^{-\hat{\vartheta}t} dt \quad (20)$$

s.t.

$$\dot{k}_t = k_t^\alpha - c_t - (\delta + g)k_t \quad (21)$$

where  $\hat{\vartheta} = \vartheta - g(1 - \rho)$ .

The steady-state of this model will occur at the point where  $\dot{k}_t = \dot{c}_t = 0$ . But since  $k_t = K_t/Z_t$  and  $c_t = C_t/Z_t$ ,  $\dot{k}$  and  $\dot{c}$  can be zero only if  $K$  and  $C$  are growing at the same rate as  $Z$ . Thus we know that in the steady-state,

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = g. \quad (22)$$

2. Use the first order condition for consumption per efficiency unit to show that the steady-state normalized capital stock will be given by

$$k = \left[ \frac{\alpha}{\rho\phi + \vartheta + \delta} \right]^{\frac{1}{1-\alpha}} \quad (23)$$

*Answer:*

The consumption Euler equation tells us that

$$\frac{\dot{c}}{c} = \rho^{-1}(f'(k) - (\delta + g) - \hat{\vartheta}) \quad (24)$$

$$= \rho^{-1}(f'(k) - (\delta + g) - \vartheta + g(1 - \rho)) \quad (25)$$

$$= \rho^{-1}(f'(k) - \delta - \vartheta - \rho g) \quad (26)$$

but by definition  $\dot{c}/c = 0$  in steady-state, implying

$$f'(k) = \vartheta + \delta + \rho g \quad (27)$$

$$\alpha k^{\alpha-1} = \vartheta + \delta + \rho g \quad (28)$$

$$k = \left( \frac{\vartheta + \delta + \rho g}{\alpha} \right)^{1/(\alpha-1)} \quad (29)$$

$$= \left( \frac{\alpha}{\vartheta + \delta + \rho g} \right)^{1/(1-\alpha)} \quad (30)$$

3. Explain why the steady-state ratio of investment to GDP is given by

$$i/y = (g + \delta)k^{1-\alpha} \quad (31)$$

*Answer:*

The level of investment necessary to keep the capital stock constant is

$$i = (g + \delta)k \quad (32)$$

and income is  $y = k^\alpha$ , so

$$i/y = (g + \delta)k/k^\alpha \quad (33)$$

$$= (g + \delta)k^{1-\alpha} \quad (34)$$

4. The prior analysis examined the case where the social planner has no ability to borrow and lend on international capital markets. We now want to examine the opposite extreme case, where international capital markets are perfectly frictionless. Under these circumstances, the social planner can borrow or lend internationally at rate  $r$  and can therefore set the level of domestic capital at any desired value, independent of domestic saving. Show that, regardless of the country's net worth, the social planner will always choose to set the domestic capital stock to the value

$$k_t = \left( \frac{\alpha}{r + \delta} \right)^{1/(1-\alpha)}. \quad (35)$$

(Hint: Think about the social planner's choice of the level of capital as in the HallJorgenson model, but where the price of capital is normalized to 1.)

*Answer:*

The country's wealth is maximized by investing in domestic capital up to the point where the marginal product of an additional unit (net of depreciation) exactly matches the return available from investing abroad:

$$f'(k) - \delta = \bar{r} \quad (36)$$

$$\alpha k^{\alpha-1} = \bar{r} + \delta \quad (37)$$

$$k^{\alpha-1} = \left( \frac{\bar{r} + \delta}{\alpha} \right) \quad (38)$$

$$k = \left( \frac{\bar{r} + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}} \quad (39)$$

$$= \left( \frac{\alpha}{\bar{r} + \delta} \right)^{\frac{1}{1-\alpha}} \quad (40)$$

because if the capital stock exceeds this amount then its marginal product is less than what could be earned from investing abroad while if the capital stock is smaller than this amount the country would be better off borrowing a bit from overseas at interest rate  $\bar{r}$  and investing domestically earning a rate of return greater than  $\bar{r}$ .

5. Suppose now that there is a world productivity growth rate  $\bar{g}$  which is shared by all countries, and world capital markets are perfect and the world interest rate is determined by the optimizing behavior of a world social planner. Explain why the world interest rate will be given by

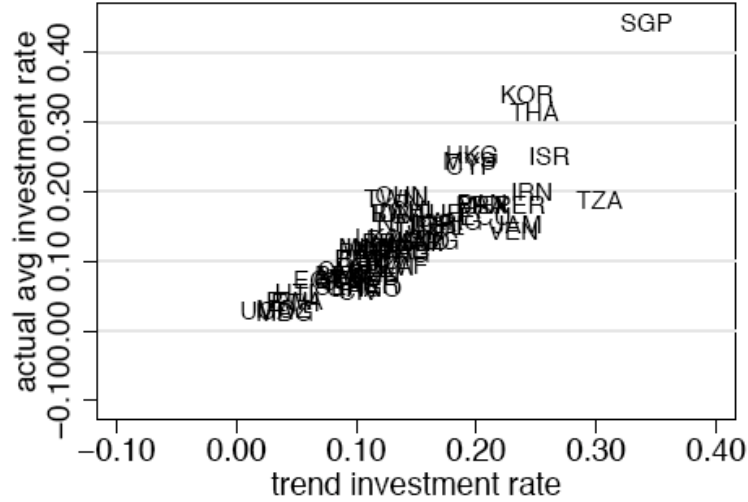
$$\bar{r} = \bar{\vartheta} + \bar{\rho}\bar{g} \quad (41)$$

*Answer:*

Under the circumstances described, the optimization problem of the world social planner is the same as the problem faced by the closed-economy social planner in the first part of the question. In that problem, we showed that the balanced growth equilibrium would occur at the point where  $f'(k) = \delta + \vartheta + \rho g$ . If the world interest rate is given by the marginal product of capital minus depreciation, we have  $\bar{r} = \bar{\vartheta} + \bar{\rho} \bar{g}$

6. Now imagine that countries differ only in the extent to which they tax capital income. The after-tax rate of return on a unit investment in capital is  $r\mathfrak{U}$ , where a country with no taxes has  $\mathfrak{U} = 1$ . Suppose that all countries share in the same rate of world technological progress, and all are in their steady states. Figure 1 (from Gourinchas and Jeanne (2007)) shows the relationship between the “trend” investment rate needed to keep a country’s capital/output ratio constant, and the actual investment rate over the period 1980-2000. Discuss how this model interprets the strong upward slope evident in the figure.

**Figure 1** Investment: Trend Vs Actual



*Answer:*

The amount of investment needed to keep the capital stock constant is  $(\bar{g} + \delta)k^{1-\alpha}$ . Since  $\bar{g}$  and  $\delta$  are assumed to be constant across countries, the only factor that could explain such large differences in investment rates is large differences in  $k$  due to large differences in  $\mathfrak{U}$ . That is, the low-investing countries would be interpreted as those where capital income is taxed at a high rate and therefore it is not worthwhile to own much capital. Small steady-state capital means small steady-state investment requirements.

7. Now suppose there are exogenous permanent differences in  $g$  across countries. What relationship would you expect to see between investment and  $g$ ? Between

saving and  $g$ ? Would you expect the fast-growing countries to be net borrowers, net lenders, or is it unclear?

*Answer:*

There is nothing in this model to diminish the magnitude of the human wealth effect, which implies a strong negative relationship between saving and growth.

The foregoing analysis showed that, *ceteris paribus*, there is a positive relationship between growth and the investment ratio.

So faster growth implies lower saving and higher investment. To pay for this, fast growing countries should borrow from slow growing countries.



## References

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