Linear Difference Equations and Asset Pricing

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1 Solutions (and Uniqueness) of Difference Equations

From the previous lecture notes, pricing a sequence $\{y_{t+j}\}$ of payoffs:

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j}, \text{ at time } t. \text{ (Sequential formulation)} \tag{1}$$

Can be written,

$$P_t = y_t + \beta P_{t+1} \tag{2}$$

1.1 Solving with Guess and Verify

How can we solve a difference equation?

Example: $y_t = \bar{y}$

$$P_t = \bar{y} + \beta P_{t+1} \tag{3}$$

A Guess: $P_t = \bar{P}$, independent of t. Plug in equation (3):

$$\bar{P} = \bar{y} + \beta \bar{P} \tag{4}$$

$$\Rightarrow \bar{P} = \frac{\bar{y}}{1-\beta}, \text{ consistent with } P = \sum_{t=0}^{\infty} \beta^t y_t \tag{5}$$

<u>Role of $|\beta| < 1$:</u>

- Keep from "exploding": stability
- Will have equivalent condition for more complicated difference equations

1.2 Rational Bubbles

Let $y_t = \bar{y}$ for all t.
Fundamental value:

$$P_t = \sum_{j=0}^{\infty} \beta^j \bar{y} \tag{6}$$

$$= \frac{\bar{y}}{1-\beta}, \text{ (unique)} \tag{7}$$

Remember that this solves the recursive problems as well:

$$\frac{\bar{y}}{1-\beta} = \bar{y} + \beta \left(\frac{\bar{y}}{1-\beta} \right) \Rightarrow \text{true!} \tag{8}$$

Is $P_t = \frac{\bar{y}}{1-\beta}$ the unique solution to $P_t = \bar{y} + \beta P_{t+1}$? **No!** Like the undetermined coefficient in differential equations.

Example:

$$P_t = \underbrace{\frac{\bar{y}}{1-\beta}}_{\substack{\text{fundamental} \\ \text{value}}} + \underbrace{c\beta^{-t}}_{\text{bubble term}} \quad \text{for any } c \tag{9}$$

Check: $P_t = \bar{y} + \beta P_{t+1}$

$$\frac{\bar{y}}{1-\beta} + c\beta^{-t} = \bar{y} + \beta\left[\frac{\bar{y}}{1-\beta} + c\beta^{-(t+1)}\right] \tag{10}$$

$$= \bar{y} + \left(\frac{\beta}{1-\beta}\right)\bar{y} + c\beta^{-t} \tag{11}$$

$$= \frac{\bar{y}}{1-\beta} + c\beta^{-t} \tag{12}$$

So it fulfills the difference equation for any c, t, etc. *Rational* as every agent in the economy would agree on the price, no-one needs to be tricked or making a pricing mistake, and there is no arbitrage. An example of a self-fulfilling equilibrium.

1.2.1 Size of the “Rational Bubble"

$$\underbrace{P_0 - P_{fund}}_{\substack{\text{difference from}\\ \text{fundamental}}} = \frac{\bar{y}}{1-\beta} - \frac{\bar{y}}{1-\beta} + c\beta^0 = c \tag{13}$$

Expectations:
- Prices rise because they are expected to rise.
- Self fulfilling. Will depend on coordination of expectations.
- Is Fiat money a bubble?

2 Extending our Asset Pricing Model

We will generalize our results to include systems of equations, with dynamics.

2.1 Recall: Properties

- Dividend stream y_t
- Discount factor β

- Present discounted value = price :
  $P = \sum_{t=0}^{\infty} \beta^t y_t$, and if $y_t = \bar{y}$, $P = \bar{y}(1-\beta)^{-1}$

- How to model the evolution of y_t?
  - Will use systems of linear difference equations in an underlying state x_t

- Example: dividends are a linear function of evolving aggregate and idiosyncratic variables

Recall: Recursive Formulation $\quad P_t = y_t + \beta P_{t+1}$

2.2 Applying to Dynamics

- Let x_t be a n dimensional vector of states.

- Let A, G be matrices.

- Stack first order difference equations, giving another *canonical form*:

$$x_{t+1} = A \cdot x_t, \qquad (A \text{ is } n \times n \text{ matrix, } x \text{ is } n \times 1 \text{ vector}) \tag{14}$$

$$y_t = G \cdot x_t, \qquad (G \text{ is } 1 \times n \text{ vector, } y_t \text{ is a scalar, i.e. } 1 \times 1) \tag{15}$$

- 'A' gives evolution of the state, given x_0

- 'G' gives observation of the state
  "Finding the state is an art"

Example:

- Asset payoff follows difference equation (not first order!):

$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1} \tag{16}$$

- What is the value of this asset at time t?

State

Guess: $x_t \equiv \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$, 2 $\times$ 1 vector.

<u>What is the difference equation for x_t?</u>

$$\underbrace{\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}}_{x_t} \tag{17}$$

and observation:

$$y_t = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}}_{x_t} \tag{18}$$

Therefore, the set of difference equations in our *canonical form* are:

$$x_{t+1} = Ax_t \tag{19}$$

$$y_t = Gx_t \tag{20}$$

Price is:

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} \tag{21}$$

$$= \sum_{j=0}^{\infty} \beta^j G \cdot x_{t+j} \tag{22}$$

If $x_{t+1} = A \cdot x_t$, then $x_{t+2} = A \cdot (Ax_t) = A^2 x_t$, and $x_{t+j} = A^j x_t$

$$\Rightarrow P_t = \sum_{j=0}^{\infty} \beta^j G \cdot A^j \cdot x_t \tag{23}$$

$$= G \cdot \left[\sum_{j=0}^{\infty} (\beta A)^j \right] x_t \tag{24}$$

Remember that if λ is scalar: $\sum_{j=0}^{\infty} (\beta\lambda)^j = (1 - \beta\lambda)^{-1} = \frac{1}{1-\beta\lambda}$.
With matrices and inverses, this is similar: $\sum_{j=0}^{\infty} \beta^j A^j = (I - \beta A)^{-1}$,
where the matrices' dimensions are: $A : n \times n,\ I = n \times n$ identity, $(I - \beta A)^{-1} : n \times n$

$$\boxed{P_t = G\,(I - \beta A)^{-1}\, x_t} \quad (* : \text{very important memorize}) \tag{25}$$

- Asset pricing formula for first-order linear difference equations.
- Summary of sizes:

- $P_t : 1 \times 1$ scalar
- $G : 1 \times n$ vector
- $A : n \times n$ matrix
- $I : n \times n$ identity matrix
- $\beta : 1 \times 1$ scalar
- $x_t : n \times 1$ state vector

2.3 Stability

- Recall in the example with $x_t = \lambda^t$ that $|\beta\lambda| < 1$ for the series to converge.
- For matrix equations, need a similar condition where eigenvalues of βA are all < 1, or $\max|\mathrm{eig}(A)| < \frac{1}{\beta}$
- Can use software to check the eigenvalues.

A Connection to Differential Equations

Difference equations are just differential equations in discrete time.

- Let $y(t)$ be the flow dividends, a function of t.
- Let r be the instantaneous interest rate.

- Let the length of a period be Δ, and take the limit as it goes to 0.
- Dividends over Δ period $\approx \Delta y(t) \equiv y_t(\Delta)$
- Discounting over Δ period $\approx 1 - \Delta r \equiv \beta(\Delta)$

The difference equation is: $P_t = y_t + \beta P_{t+1}$.
Using the above $\Rightarrow$ Let function $p(t)$ be the price of asset:

$$p(t) = \Delta \cdot y(t) + (1 - \Delta r) \cdot p(t + \Delta) \tag{A.1}$$

Rearrange:

$$\Delta r \cdot p(t + \Delta) = \Delta \cdot y(t) + p(t + \Delta) - p(t) \tag{A.2}$$

$$\Rightarrow rp(t + \Delta) = y(t) + \frac{p(t + \Delta) - p(t)}{\Delta} \tag{A.3}$$

Take limit as $\Delta \to 0$, i.e. discrete $\to$ continuous t

$$\boldsymbol{\partial} p(t) = \frac{p(t + \Delta) - p(t)}{\Delta}, \text{ definition of a derivative} \tag{A.4}$$

where $\boldsymbol{\partial} p(t) = \frac{d}{dt} p(t)$

$$\Rightarrow \underbrace{rp(t)}_{\substack{\text{opportunity cost}\\ \text{of buying a unit}\\ \text{of the asset}}} = \underbrace{y(t)}_{\substack{\text{flow}\\ \text{dividends}}} + \underbrace{\boldsymbol{\partial} p(t)}_{\substack{\text{capital}\\ \text{gains}}} \tag{A.5}$$

- Consider this pricing equation and arbitrage:
  What if $rp(t) < y(t) + \boldsymbol{\partial} p(t)$ instead of being an equation?

B Popping Bubbles

- In our discrete time model, keep $y_t = \bar{y}$ deterministic for simplicity:

- Let the bubble term have a chance of popping each period.
- Therefore, prices are a random variable.
- Linear asset pricing if random:

$$P_t = y_t + \beta \mathbb{E}_t\left[P_{t+1}\right] \quad \text{(Expected value of } P_{t+1} \text{ given information at } t\text{)} \tag{B.1}$$

B.1 Bubble Evolution

$$\text{Let } C_{t+1} = \begin{cases} \frac{1}{\lambda} C_t & \text{with prob. } \lambda \in (0,1) \\ 0 & \text{with prob. } 1-\lambda \end{cases} \tag{B.2}$$

i.e., C_t multiplied by $\frac{1}{\lambda}$ each time until bubble breaks. Then $C_t = 0\ \forall t$

Note:

$$\mathbb{E}_t\left[C_{t+1}\right] = \lambda \left(\frac{1}{\lambda} C_t\right) + (1-\lambda) \cdot 0 = C_t \tag{B.3}$$

If $\mathbb{E}_t\left[y_{t+1}\right] = y_t$, then this term is called a *martingale*.

B.2 Price Level

We can check that for any C_0:

$$P_t = \begin{cases} \frac{\bar{y}}{1-\beta} + (\beta\lambda)^{-t} \cdot C_0 & \text{if bubble hasn't popped} \\ \frac{\bar{y}}{1-\beta} & \text{after bubble pops} \end{cases} \tag{B.4}$$

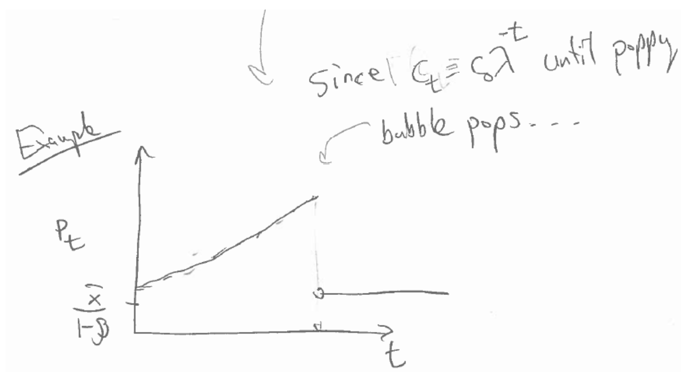

Figure 1: Graphical representation of the price level when the bubble pops