Question 1: (Transitions and Government Objective Functions)

Consider a standard setup of the neoclassical growth model in a competitive equilibrium: A representative consumer orders its welfare by[1]

$$\sum_{t=0}^{\infty} \beta^t \log(c_t)$$

where $0 < \beta < 1$. Or $\beta \equiv \frac{1}{1+\rho}$ for $\rho > 0$. The technology in the economy is,

$$y_t = f(k_t) = z k_t^{\alpha}$$

for $0 < \alpha < 1$ and $z > 0$. Labor of mass 1 is supplied inelastically.

Given exogenous government expenditures of real goods, g_t, the feasibility condition is

$$c_t + k_{t+1} + g_t \leq y_t + (1-\delta)k_t$$

In a competitive equilibrium, the government will finance g_t through taxes on capital or lump-sump taxes, $\{\tau_{kt}, \tau_{ht}\}$. Negative taxes are subsidies.

(a) Find the steady state level of capital and consumption $\left\{\bar{k}, \bar{c}\right\}$ if $g_t = \tau_{kt} = \tau_{ht} = 0$.

(b) Now, assume that while the government will still have $g_t = 0$, they can choose a constant tax ($\bar{\tau}_k > 0$) or subsidize ($\bar{\tau}_k < 0$) the return to capital faced by the consumer. Since they have no need for expenditures, then if $\bar{\tau}_k > 0$ the government simply rebates the revenues to consumers as a lump-sum subsidy ($\bar{\tau}_h < 0$). Similarly, to pay for a capital subsidy the government sets a lump-sum tax. Find the steady state $\left\{\bar{k}, \bar{c}\right\}$ for a given $\bar{\tau}_k$ tax (or subsidy).[2]

(c) The objective of government (A) is to maximize steady state consumption per capita by choosing the $\bar{\tau}_k$. Formulate this as an optimal problem for the government, and solve for its optimal $\bar{\tau}_k$ policy and the corresponding steady state $\left\{\bar{c}, \bar{k}\right\}$. What is the sign of $\bar{\tau}_k$, and why?

(d) Now, a new government (B) comes to power with the objective of maximizing consumer welfare (i.e. our usual objective) by choosing a constant $\bar{\tau}_k$. Find the optimal $\bar{\tau}_k$ policy and the corresponding steady state $\left\{\bar{c}, \bar{k}\right\}$. What is the sign of $\bar{\tau}_k$, and why?

(e) Assuming that government (A) was in power for a long-time and the economy was in a steady state. The new government (B) is elected with no anticipation, and associated new tax policy is immediately changed to the optimal value forever. Draw the dynamics of $\{k_t, c_t\}_{t=0}^{\infty}$ as the economy evolves from the initial steady state of government (A) to the new steady state of government (B).

(f) Compare the steady states of the two governments to discuss whether $\bar{\tau}_k$ was set too high or too low in government (A).[3]

[1] Let c_t, k_t, y_t, and g_t be in per-capita terms.
[2] Hint: $\bar{\tau}_h$ adjusts to balance the government's budget and is non-distorting.
[3] Be explicit on what criteria one should use to make this judgment.

Question 2: (Sequential Formulation of Neoclassical Growth with Interest Rates) A competitive (i.e., price taking) equilibrium exists for households and firms. This is identical to the competitive equilibrium of the deterministic neoclassical growth model we did in class except:

- Consumer's can smooth consumption by investing in capital, but also have access to financial assets (with prices determined in general equilibrium).
- There will not have a set of assets traded at time 0 which provide a claim to consumption at time t (i.e., our old complete set of assets with prices q_t^0 doesn't exist).
- Instead, household's can buy and sell (at each time t rather than time 0) claims to consumption at time $t+1$ (i.e., 1 period bonds). This single asset can be sold to each other(or to the government) on competitive markets that operate at each time period (i.e., spot markets).
- Instead of a lifetime budget constraint, households have sequential budget constraints.
- The price of the consumption good at time t is normalized to 1, so the price system will be $\{r_t, w_t, i_{t+1}\}$ where r_t and k_t are the real rental rates for capital and labor in time t goods, and i_{t+1} is the net interest rate on a bond purchased at time t.
- The gross interest rate of purchasing a unit of the bond is $1 + i_{t+1}$. Consequently, buying a claim to 1 unit of the good at time t costs $\frac{1}{1+i_{t+1}}$.[4]
- The government also has a sequential budget constraint, and smooths in expenditures through the bond market and tax policy (i.e., it can also buy and sell bonds).
- Bond's are in 0 net supply. That is, if consumers hold $B_t > 0$ of the bonds, then the government would need to hold $-B_t < 0$ of the bonds.

To summarize the entire equilibrium for a representative consumer and firm,

Allocation: $\{c_t, k_t, B_t\}_{t=0}^{\infty}$. Bond holdings, B_t, are pieces of paper. k_0 and B_0 are given.

Price System: $\{r_t, w_t, i_{t+1}\}_{t=0}^{\infty}$

Government Policy: $\{\tau_{ct}, g_t, B_t^g\}_{t=0}^{\infty}$. That is, a consumption tax, government expenditures, and government bond holdings (which is negative if they owe the households money). Assume that the Government Policy is given exogenously, though it will need to be budget feasible.

Feasibility: The firm operates the same neoclassical production function (with $f'(\cdot) > 0$ and $f''(\cdot) < 0$ as before, and the bond markets clear,

$$c_t + k_{t+1} - (1-\delta)k_t + g_t = f(k_t) \tag{1}$$

$$B_t^g = -B_t, \text{ i.e. bonds are in 0 net supply} \tag{2}$$

[4] From our interest notes, consider a claim to a unit of consumption delivered at time $t+1$ but priced in time t good, q_{t+1}^t. Then the interest rate on this 1 period claim at time t is defined as $\rho_{t,t+1}$ by $q_{t+1}^t \equiv \frac{1}{1+\rho_{t,t+1}}$. This calculation is doing the same thing for these 1 period claims.

Government Budget: The government policy is given exogenously, but it must "balance" in the long-run (i.e., when government debt is taken into account with the endogenous prices).

$$\underbrace{g_t}_{\text{Expenditures}} + \underbrace{\frac{1}{1+i_{t+1}}}_{\text{With Interest}} \underbrace{B^g_{t+1}}_{\text{New Bonds}} \leq \underbrace{\tau_{ct} c_t}_{\text{Tax Income}} + \underbrace{B^g_t}_{\text{Previous Bonds}} \quad \text{for all } t \geq 0 \tag{3}$$

For example, if $B^g_t = B^g_{t+1}$ then the government is rolling over their bonds and paying (or getting) the interest. There will also be a no-ponzi scheme condition (e.g. $\lim_{t\to\infty} |B^g_t| < \infty$)

Households's Problem: A large number of identical consumers have a typical strictly concave utility function($u'(\cdot) > 0, u''(\cdot) < 0, u'(0) = \infty$), and provide 1 unit of labor inelastically. Taking B_0, k_0, prices, and government polices as given

$$\max_{\{c_t, k_{t+1}, B_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4}$$

$$\text{s.t. } (1+\tau_{ct})c_t + k_{t+1} + \frac{1}{1+i_{t+1}} B_{t+1} \leq B_t + (1-\delta)k_t + r_t k_t + w_t, \quad \text{for } t \geq 0 \tag{5}$$

$$(\text{+ a no-ponzi scheme transversality condition}) \tag{6}$$

Note that the period by period budget constraint is written with the price of the consumption good normalized to 1.

Firm's Problem: A large number of identical firms operate a constant returns to scale (CRS) production function $F(K, N)$ with the usual result that $F(\frac{K}{N}, 1) = \frac{1}{N} f(k)$ when $k \equiv \frac{K}{N}$.[5] Taking prices as given they maximize,

$$\max_{K_t, N_t} \{F(K_t, N_t) - w_t N_t - r_t K_t\} \tag{7}$$

With this complete specification of the equilibrium,

(a) Define a competitive equilibrium

(b) Solve for the first-order-necessary conditions (FONC) of the firms to get expressions for the real rental rate of capital in terms of production function $f(\cdot)$ and the aggregate capital k_t (which is also the aggregate capital to labor ratio here since $N_t = 1$). Why are we able to use a representative firm?

(c) Solve for the FONC of the household's choice of $\{c_t, k_{t+1}, B_{t+1}\}_{t=0}^{\infty}$. There will now be 2 Euler equations: one for capital investment, and another for bond investment. Substitute from the FONC of the firm's problems to express in terms of allocations where possible.

(d) Define the real return on capital, as in the class notes: $R_t \equiv 1 - \delta + f'(k_{t+1})$. What is i_{t+1} in terms of R_t? Interpret this relationship.[6]

[5] Note that if there is a representative firm, then $F(\frac{K}{N}) = f(k)$ and $k = K$.

[6] Hint: stare at the two Euler equations and combine. Is there a way to think of arbitrage between using these two different methods to smooth consumption?

(e) If the government policy has $B_t^g = 0$ for all t, then note that $B_t = 0$ from the feasibility condition. Is there any trading of bonds between consumers, and if not, why not? If so, then why is there still an interest rate?

(f) Let $B_t^g = B_t = 0$ forever. You can either assume that $g_t = \tau_{ct} = 0$, or that these are fixed such that $\bar{\tau}_c, \bar{g}$ balance the government budget. Calculate the steady state $\left\{\bar{k}, \bar{c}, \bar{B}, \bar{r}, \bar{i}\right\}$.

(g) Alternatively, assume that the equilibrium may be away from the steady state. Take (1) the budget constraint of the government at time t and solve for B_t^g, and (2) the budget constraint at time $t+1$ and solve for B_{t+1}^g, and substitute (2) into (1). Repeat this for the $t+2$ constraint, etc.[7] Show that at any point in time, the government debt B_g^t are required for the government to finance the future expenditures $\{g_{t+s}\}_{s=0}^{\infty}$ given a tax sequence $\{\tau_{c,t+s}\}_{s=0}^{\infty}$ and equilibrium allocations [8]

$$-B_t^g = \sum_{s=0}^{\infty} \underbrace{\left(\prod_{\tau=0}^{s} \frac{1}{1+i_{t+\tau}}\right)}_{\text{Discount to time } t+s} \underbrace{\left[\tau_{c,t+s} c_{t+s} - g_{t+s}\right]}_{\text{Deficit at time } t+s} \tag{8}$$

where the allocations c_{t+s} solve the competitive equilibrium, and $i_{t+s} = f'(k_{t+s}) - \delta$. Note that if capital (and hence interest rates) were constant at i, then the discounting is standard: $\prod_{\tau=0}^{s} \frac{1}{1+i_{t+\tau}} = \left(\frac{1}{1+\bar{i}}\right)^s$

(h) Assume the economy is in steady state with no government expenditures at $t = 0$, with $B_0^g = 0$ and $\tau_c = g = 0$. At this point, the government needs to finance a constant stream of $\bar{g} > 0$ forever, and immediately (i.e., unanticipated) policy of $\bar{\tau}_c$ to finance it. Using the results of the previous part: What is the $\bar{\tau}_c$ necessary for the sequence of budget constraints of the government to hold for all t? What is the path of k_t and B_t^g?

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[7] If you are stuck, look at our notes showing the equivalence of the lifetime and the period-by-period budget constraints in the permanent income model and follow the approach.

[8] Hint: The discounting term here is due to the possibility of non-constant interest rates in the economy. For example, if $s = 2$, to discount at rate i_t then i_{t+1} then i_{t+2}, the total discounting $\{k_{t+1+s}, c_{t+s}\}_{s=0}^{\infty}$ is, $\prod_{\tau=0}^{2} \frac{1}{1+i_{t+\tau}} = \frac{1}{1+i_t} \frac{1}{1+i_{t+1}} \frac{1}{1+i_{t+2}}$. The convention is that if $s = 0$, the product is 1.