

# Stochastic Permanent Income Model and Government Fiscal Policy

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## 1 Stochastic Permanent Income

### 1.1 Basic setup.

Linear State Space + Normal Shock:

- Let

$$x_{t+1} = Ax_t + Cw_{t+1} \quad (1)$$

$$y_t = G \cdot x_t \quad (2)$$

where  $A$  is  $n \times n$  matrix,  $x$  is  $n \times 1$  vector,  $C$  is  $n \times m$  matrix,  $w_{t+1} \sim N(0, I_{n \times m})$ , i.i.d. normal shocks;  $G$  is  $1 \times n$  vector,  $y_t$  is a scalar, which means "labor income"

- Consumer's Budget Constraint (assuming  $\beta R = 1$ ):

$$F_{t+1} = \underbrace{\frac{1}{\beta}}_{\text{gross interest rate}} \left( \underbrace{F_t}_{\text{Financial wealth}} + y_t - c_t \right) \quad (3)$$

- Recall if  $\{y_t\}$  is deterministic, and  $R = 1/\beta$ , then for any strictly concave  $u(c)$  they achieved perfect consumption smoothing:

$$c_t = (1 - \beta) \left( \underbrace{F_t}_{\text{Financial wealth}} + \underbrace{\sum_{j=0}^{\infty} \beta^j y_{t+j}}_{\text{PDV of human wealth}} \right) \quad (4)$$

- If  $y_t$  is stochastic, can we just replace the above equation with expected value?:

$$c_t = (1 - \beta)(F_t + \underbrace{\mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]}_{\substack{\text{expected PDV} \\ \text{of human wealth} \\ \text{with information at} \\ \text{time } t}}) \quad (5)$$

**Note:** if  $u'(c)$  is not linear, then this is only an approximation

- Combine (3) and (5):

$$F_{t+1} = \frac{1}{\beta} \left[ F_t + y_t - (1 - \beta) \left( F_t + \mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right) \right] \quad (6)$$

$$= F_{t+1} = \frac{1}{\beta} \left[ \beta F_t + y_t - (1 - \beta) \mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right] \quad (7)$$

$$\Rightarrow F_{t+1} - F_t = \frac{1}{\beta} \left[ y_t - (1 - \beta) \mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right] \quad (8)$$

i.e. agents adds difference between  $y_t$  and permanent income. Now use (5) at  $t$  and  $t + 1$ ,

$$c_{t+1} = (1 - \beta) \left[ F_{t+1} + \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j+1} \right] \right] \quad (9)$$

$$c_t = (1 - \beta) \left[ F_t + \mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right] \quad (10)$$

$$\Rightarrow c_{t+1} - c_t = (1 - \beta)(F_{t+1} - F_t) + (1 - \beta) \left[ \mathbb{E}_{t+1} \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j+1} \right] - \mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right] \quad (11)$$

Use (8) to find (after many steps):

**Proposition:**

$$c_{t+1} - c_t = (1 - \beta) \sum_{j=0}^{\infty} \beta^j \left( \underbrace{\mathbb{E}_{t+1}(y_{t+j+1})}_{\substack{\text{Forecast of } t+1, t+2, \dots \\ \text{with time } t+1 \\ \text{information}}} - \underbrace{\mathbb{E}_t(y_{t+j+1})}_{\substack{\text{With time } t \\ \text{information}}} \right) \quad (12)$$

- Consumption only changes due to “surprise” of new information changing expected value

- Only unanticipated changes in  $y_{t+j}, \dots$  or other information which changes forecasts.
- Could be unanticipated changes in government policy or shock realizations.
- Finally, for a shock between  $t \rightarrow t+1$  with our linear state space model:

$$c_{t+1} - c_t = (1 - \beta) \left[ \sum_{j=0}^{\infty} \beta^j (\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1})) \right] \quad (13)$$

$$= (1 - \beta) \left[ G(I - \beta A)^{-1} x_{t+1} - G(I - \beta A)^{-1} A x_t \right] \quad (14)$$

$$= (1 - \beta) G(I - \beta A)^{-1} \left[ \underbrace{A x_t + C w_{t+1}}_{x_{t+1}} - A x_t \right] \quad (15)$$

### Solution with Linear Gaussian State Space

$$\boxed{c_{t+1} - c_t = \underbrace{(1 - \beta)}_{\text{Propensity to Consume}} \underbrace{G(I - \beta A)^{-1} \cdot C w_{t+1}}_{\text{PDV of impulse response a shock to } x_{t+1}}} \quad (16)$$

i.e PDV of changes to forecasts from the realized shock.

## 1.2 Special case of Quadratic Preferences

- Recall Euler equation for Permanent Income Model:

$$u'(c_t) = \beta(1 + r)u'(c_{t+1}), \forall t = 0, \dots, T - 1 \quad (17)$$

If stochastic consumption and  $\beta = \frac{1}{1+r}$ , just replace with expectation?

$$\underbrace{u'(c_t)}_{\text{Marginal utility this period}} = \underbrace{\mathbb{E}_t[u'(c_{t+1})]}_{\text{Expectation of marginal utility next period}} \quad (18)$$

- Let  $u(c) = \frac{a_1}{2}c^2 + a_2c + a_3 \Rightarrow u'(c) = a_1c + a_2$

In Euler equation:

$$a_1c_t + a_2 = \mathbb{E}_t(a_1c_{t+1} + a_2) \quad (19)$$

$$c_t = \mathbb{E}_t(c_{t+1}) \quad (20)$$

i.e., Euler equation implying perfect consumption smoothing with a deterministic process translates to consumption being a martingale if stochastic!

- Note:
  - In general,  $\mathbb{E}_t(u(c)) \neq u(\mathbb{E}_t(c))$
  - Then, we can use the linear-stochastic state space model for forecasting  $E_t c_{t+1}$  - Due to linearity, it simply forecasts mean.
  - This is a general result called "Certainty Equivalence" of optimizing a quadratic objective subject to a linear-gaussian state space model.
  - The decision is identical in a model with or without the certainty
  - However, the realized sequence contingent on the sequence, and utility, are not the same.

## 2 Examples

### 2.1 Pre-announced Tax Cut

- This will use a shock to knowledge about deterministic income processes, rather than a constant stream of shocks to income.
- Setup:
  - Government announce at  $t = 0$  that at  $t = 1$  it will borrow  $\alpha$  from international markets at interest rate  $(1 + r)$  per period and give it to each consumer.
  - They also announce that to eventually balance the budget, they will pay it back at  $t = 2$  for simplicity by increasing taxation that period.
  - Assume consumers had deterministic  $y_{t+j}$ , which happens to consumption?

- $c_{t+1} - c_t = (1 - \beta) \sum_{j=0}^{\infty} \beta^j [\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1})]$

Define:  $\{\hat{y}_{t+1}\}_{j=0}^{\infty} = \left\{ y_t, \underbrace{y_{t+1} + \alpha, y_{t+2} - \alpha(1+r)\beta}_{\text{Only difference}}, y_{t+3}, \dots, y_{t+j} \dots \right\}$

- Note that from  $t$  to  $t + 1$ , the agent has the news that  $\{y_{t+j}\} \rightarrow \{\hat{y}_{t+j}\}$
- This is a change in expectations:

$$c_1 - c_0 = (1 - \beta) \sum_{j=0}^{\infty} \beta^j (\hat{y}_{j+1} - y_{j+1}) \quad (21)$$

$$\Rightarrow c_1 - c_0 = (1 - \beta) \sum_{j=0}^{\infty} \beta^j (y_{j+1} - y_{j+1}) + (1 - \beta) [\alpha - \beta(1 + r)\alpha] \quad (22)$$

- Notes: If  $\beta = \frac{1}{1+r}$ , then  $c_1 - c_0 = 0$   
 i.e. tax cut has no effect because of anticipated rise in taxes. Later, we will investigate cases why  $\beta = \frac{1}{1+r}$  comes out of general equilibrium.

## 2.2 Timing of Tax Cuts

- Setup:
  - Between time 0 and 1, government announces that it will cut taxes to give  $\alpha$  to each individual at a deterministic time  $T \geq 1$
  - Assume they do not need to pay it back and taxes will not raise to compensate.
  - What happens to consumption at time  $\{0, \dots, T, T+1, \dots\}$ ?
  - Assume  $y_{t+j+1}$  are deterministic.

- Solve:

$$c_1 - c_0 = (1 - \beta) \sum_{j=0}^{\infty} \beta^j [\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1})] \quad (23)$$

$$= (1 - \beta) \sum_{j=0}^{\infty} \beta^j [y_{t+j+1} - y_{t+j+1}] + (1 - \beta) \cdot \beta^{T-1} \cdot \alpha \quad (24)$$

$$= \underbrace{(1 - \beta)}_{\text{MPC out of wealth}} \underbrace{\beta^{T-1} \cdot \alpha}_{\text{Change in permanent income}} \quad (25)$$

- For  $t \geq 1$ :

$$\mathbb{E}_{t+1}(y_{t+j+1}) = \mathbb{E}_t(y_{t+j+1}) \quad (26)$$

$$\Rightarrow c_{t+1} - c_t = 0, \forall t \geq 1 \quad (27)$$

- That is:
  - Changes only happen at announcement, not at tax cut,  $T$ .
  - A similar approach with stochastic income would yield the same result.

Variation: The only reason that  $T$  enters the above is that PDV of the  $\alpha$  delivery is discounted for the  $T$  period. If instead, the government announces they will set aside  $\alpha$ , put it in the bank at  $R$  interest, and then deliver the  $\alpha$  with interest at time  $T$ . In that case, interest compounds for  $T - 1$  period, which means that

$$c_1 - c_0 = (1 - \beta) \beta^{T-1} (R^{T-1} \alpha) = (1 - \beta) \alpha$$

i.e., the tax break (no matter when it is actually implemented) adds  $\alpha$  to the PDV of lifetime earning.

## 2.3 Example from Friedman-Muth

- Setup:

$$y_t = z_t + u_t \quad (28)$$

$$z_{t+1} = z_t + \sigma_1 w_{1t+1} \quad (29)$$

$$u_{t+1} = \sigma_2 w_{2t+1} \quad (30)$$

where  $y_t$  is income,  $z_t$  is the *persistent* or "permanent income",  $u_t$  is transitory changes in income;

- Which one is a martingale (i.e., random walk here)?
- Define the vector of shocks  $w_{t+1} = \begin{pmatrix} w_{1t+1} \\ w_{2t+1} \end{pmatrix} \sim N(0_2, I_{2 \times 2})$ , i.e. iid normal distributed, mean 0, variance 1.

- Setup in linear state space form:

$$\text{Since } x_t = \begin{pmatrix} z_t \\ u_t \end{pmatrix}, \text{ we have:} \quad (31)$$

$$\underbrace{\begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix}}_{x_{t+1}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_A \cdot \underbrace{\begin{pmatrix} z_t \\ u_t \end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}}_C \underbrace{\begin{pmatrix} w_{1t+1} \\ w_{2t+1} \end{pmatrix}}_{w_{t+1}} \quad (32)$$

$$y_t = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_G \cdot \underbrace{\begin{pmatrix} z_t \\ u_t \end{pmatrix}}_{x_t} \quad (33)$$

$$I - \beta A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 - \beta & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \quad (34)$$

$$(I - \beta A)^{-1} = \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix}, \text{ since diagonal matrix, its inverse is just 1 over each element} \quad (35)$$

$$G(I - \beta A)^{-1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \quad (36)$$

- Recall:

$$c_t = (1 - \beta) \left[ F_t + \mathbb{E}_t \left( \sum_{j=0}^{\infty} \beta^j y_{t+j} \right) \right] \quad (37)$$

$$= (1 - \beta) \left[ F_t + G(I - \beta A)^{-1} x_t \right] \quad (38)$$

$$\text{in this example} = (1 - \beta) \left[ F_t + \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \cdot \begin{pmatrix} z_t \\ u_t \end{pmatrix} \right] \quad (39)$$

$$c_t = (1 - \beta) \left[ F_t + \frac{1}{1 - \beta} z_t + u_t \right] \Rightarrow \quad (40)$$

$$c_t = (1 - \beta) F_t + z_t + (1 - \beta) u_t \quad (41)$$

Note: coefficient on  $u_t$  is  $(1 - \beta)$ , the marginal propensity to consumer (MPC) out of transitory income: coefficient of  $z_t$  is 1, which is the MPC out of permanent income. The marginal propensity to consumer out of financial wealth  $F_t$  is the same as before.

- Recall:

$$c_{t+1} - c_t = (1 - \beta) G (1 - \beta A)^{-1} \cdot C \cdot w_{t+1} \quad (42)$$

$$= (1 - \beta) \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \cdot \begin{pmatrix} w_{1t+1} \\ w_{2t+1} \end{pmatrix} \quad (43)$$

$$= \sigma_1 w_{1t+1} + (1 - \beta) \sigma_2 w_{2t+1} \quad (44)$$

i.e. Consumes all of the permanent shock, and the MPC out of the transitory shock.

- What about savings?

Recall:

$$F_{t+1} - F_t = \frac{1}{\beta} \left[ y_t - (1 - \beta) \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \quad (45)$$

$$= \frac{1}{\beta} \left[ G \cdot x_t - (1 - \beta) G(I - \beta A)^{-1} x_t \right] \quad (46)$$

$$= \frac{1}{\beta} G \left[ I - (1 - \beta) G(I - \beta A)^{-1} \right] x_t \quad (47)$$

$$G \cdot I = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad (48)$$

$$G(I - \beta A)^{-1} = \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \text{ from before } \Rightarrow \quad (49)$$

$$F_{t+1} - F_t = \frac{1}{\beta} \left[ \begin{pmatrix} 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 - \beta \end{pmatrix} \right] \begin{pmatrix} z_t \\ u_t \end{pmatrix} \quad (50)$$

$$= \frac{1}{\beta} \begin{pmatrix} 0 & \beta \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix} \quad (51)$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix} \Rightarrow \quad (52)$$

$$\boxed{F_{t+1} - F_t = u_t} \quad (53)$$

i.e. Consumer spends all of  $z_t$ , saves nothing but a fraction of transitory income (Note returns on savings to  $F_{t+1}$ )