Question 1

A price taking consumer has an exogenous endowment $\{y_t\}_{t=0}^{\infty}$. They choose consumption to maximize their welfare given a discount rate $\beta \in (0,1)$, and a concave $u(\cdot)$.

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

$$\text{s.t.} \sum_{t=0}^{\infty} q_t^0 c_t \leq \sum_{t=0}^{\infty} q_t^0 y_t \tag{2}$$

where $\{q_t\}_{t=0}^{\infty}$ are the price of one unit of consumption good delivered at time t measured in units of time 0 consumption (i.e., use a $q_0^0 = 1$ normalization of the price level here).

Assume the utility has the form,

$$u(c) = \begin{cases} \frac{1}{1-\gamma} c^{1-\gamma} & \text{if } \gamma > 0, \gamma \neq 1 \\ \log c & \text{if } \gamma = 1 \end{cases}$$

Note: you will note that the marginal utility, $u'(c) = c^{-\gamma}$ holds for all $\gamma > 0$, including the special log case. This means you will not need to treat it separately.

Assume there is a large number of identical agents in the economy, all with identical processes $y_t = y_0 \delta^t$ for $0 < \delta < 1/\beta$.

Finally, recall that if $q_0^0 = 1$, then r_{0t} is the "yield to maturity on a t-period zero-coupon bond purchased at time 0" through,

$$\frac{q_t^0}{q_0^0} \equiv \frac{1}{(1 + r_{0t})^t}$$

(a) What is the feasibility condition in the economy (i.e. relate c_t and y_t)? (hint: can use a representative agent with a large number of price taking agents).

(b) Solve for q_t^0 in this model, explaining why q_0^0 can be chosen for convenience. Then use this to find r_{0t} from the definition above.

(c) In the special case of $\gamma = 1$, compute q_t^0 and r_{0t}. Compute the special case of $\gamma = 1$ and $\delta = 1$.

(d) Interpret r_{01} if $\gamma = 1$ for the $\delta > 1$ and $\delta < 1$ cases

(e) Interpret r_{01} for $\gamma > 0$ and $\delta = 1$. In particular, discuss any reliance on γ.

Question 2

Consider an economy with a large number of identical, price-taking agents. Endowments, $\{y_t\}_{t=0}^{\infty}$, are stochastic with $y_t \in \{L, H\}$. The evolution of endowments follows a Markov chain between these L and H states with transition matrix

$$P \equiv \begin{bmatrix} \pi_{LL} & \pi_{LH} \\ \pi_{HL} & \pi_{HH} \end{bmatrix}$$

where $\pi_{ij} > 0, \pi_{LL} + \pi_{LH} = 1$, and $\pi_{HL} + \pi_{HH} = 1$

Consider a 2 period model (which is equivalent to comparing two sequential periods of an infinite horizon model). Consumers are able to buy and sell contingent claims for delivery of goods next period. Given a current state of the economy, S, paying $q(L|S)$ delivers 1 unit of the good if L is the state next period, and 0 units of the good if H is the state next period. Similarly, $q(H|S)$ delivers 1 unit of the good next period if the state is H and L otherwise. (Hint: Note that I have put $S \in \{L, H\}$ in the price since the prices may depend on the current state of the economy)

(a) The consumer will maximize utility by choosing consumption today c_0 and consumption tomorrow depending on the state of the world: $c_1(L)$ and $c_1(H)$.

Note that c_0 is deterministic because they know the current state of the world. The endowment today depends on the current state S, i.e. $y_0(S)$ and the endowment tomorrow is stochastic $y_1(L|S)$ or $y_1(H|S)$.

Normalizing the price of a consumption good today to 1 (i.e., $q_0^0 = 1$), and facing prices $q(L|S)$ and $q(H|S)$, write out the optimization problem of the consumer given their endowment.

(b) What is the feasibility condition in this economy (i.e., relate c and y in the various states of the world and time periods. Hint: you can use a representative agent)?

(c) Use your optimization problem for the consumer and the feasibility condition to write out an asset pricing equation (the Euler equation) to relate the price to the probabilities, utilities, and endowments in various states of the world. Solve for $q(S'|S)$ for the various $S, S' \in \{L, H\}$. This is the price of a one-period state contingent bond.

(d) Derive the risk-free price of a consumption good next period using your $q(S'|S)$. This is the price of a bond that delivers one unit of consumption regardless of the realized state of the economy. Denote this as $q(S)$.

(e) Use this to derive the net interest rate on a 1 period risk-free bond, $r_{01}(S)$

$$q(S) \equiv \frac{1}{1 + r_{01}(S)}$$

Interpret $r_{01}(L)$ and $r_{01}(H)$. In particular, explain why bond prices are a function of the current state, S. Consider the role of income growth and consumption smoothing.