Question 1

(Bubbles)

A dividend process follows the deterministic process:

$$x_{t+1} = A \cdot x_t \tag{1}$$

where x_t is an $n \times 1$ vector and A is an $n \times n$ matrix, and

$$y_t = G \cdot x_t \tag{2}$$

where y_t is the dividend(a scalar), and G is an $1 \times n$ vector. Assume future profits are discounted by $\beta \in (0, 1)$, and that $I - \beta A$ is invertible.

(a) What is the stock price of the firm p_t, in terms of A and G if there was no bubble?

(b) What is the stock price of the firm today, p_t, in terms of the price tomorrow, p_{t+1}, and the state today, x_t?

(c) A friend guesses that the stock price should be

$$p_t = H \cdot x_t + c \cdot \lambda^t \tag{3}$$

for some vector $H \in \mathbb{R}^n$ and scalars c, λ.
Get as far as you can in finding formulas for H, c, λ. (**Hint**: use the guess and verify to find the undetermined constants, with the recursive definition of the price from (b)).

(d) Is H unique? How about c and λ?

Question 2

(Practice with State Spaces)

A dividend obeys:

$$y_{t+1} = \lambda_0 + \lambda_1 y_t + \lambda_2 y_{t-1} \tag{4}$$

where y_t is scalar

The stock price obeys:

$$p_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} \tag{5}$$

(a) Find a solution for the price p_t of the form:

$$p_t = a_0 + a_1 y_t + a_2 y_{t-1} \tag{6}$$

for some a_0, a_1, and a_2 in terms of model parameters.[1] (No need to actually invert matrices, etc. to find the solution to the particular a_0, a_1, a_2)

[1] **Hint:** Set it up as a linear state space.

Question 3

(Rationalizing Interest Rates)

A consumer chooses consumption and savings to maximize their welfare subject to a budget constraints.

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{7}$$

$$\text{s.t. } F_{t+1} = R_t \left(F_t + y_t - c_t\right), \quad \text{for all } t \geq 0 \tag{8}$$

$$c_t \geq 0, \quad \text{for all } t \geq 0 \tag{9}$$

$$\lim_{T \to \infty} \beta^T u'(c_T) F_{T+1} = 0 \quad \text{(Transversality Condition)} \tag{10}$$

Where $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, $\beta \in (0,1)$, $\gamma > 0$, $F_0 = 0$, and $\{y_t\}_{t=0}^{\infty}$ is an exogenous, deterministic sequence of labor income with at least some positive y_t. $\{R_t\}_{t=0}^{\infty}$ are the gross interest rate on financial assets, and are an exogenous, deterministic sequence known to the consumer.

(a) Setup the Lagrangian for this problem, being clear on Lagrange Multipliers and equality/inequality constraints.[2]

(b) Show that the $c_t \geq 0$ constraint can never bind, then find the Euler equation for the consumer at all $t \geq 0$.[3]

(c) For some $\delta \geq 0$ and $\phi \geq 0$, let the labor income process be

$$y_t = \begin{cases} y_0 \delta^t & t = 0, \dots T \\ y_0 \delta^t \phi^{t-T} & t = T+1, \dots \infty \end{cases}$$

It just happens that $\{R_t\}_{t=0}^{\infty}$ is such that the consumer optimally sets $c_t = y_t$ and $F_{t+1} = 0$ for all t (i.e., this is a particular sequence of R_t which rationalizes this behavior). Find a formula for $\{R_t\}_{t=0}^{\infty}$ and justify your formula.[4]

(d) Interpret your formula for R_t in terms of (i) the consumer's impatience, and (ii) the consumer's income growth.

[2]Hint: You can be sloppy and skip the multiplier on the Transversality condition, as we have done in class. As always, I strongly suggesting using present-value Lagrange multipliers to simplify algebra.

[3]Hint: The only change from our standard problem is the time varying interest rate. You will need to be careful with the timing when taking first order conditions. To proof that $c_t > 0$, you will need to use the marginal utility and use the fact that there is at least some positive income.

[4]Hint: What does optimality mean? Also, be a little careful around T for the calculation of R_t

Question 4

(Practice with Lagrange Multipliers)

A consumer chooses consumption and savings to maximize their welfare subject to a budget constraints.

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t c_t \tag{11}$$

$$\text{s.t. } F_{t+1} = R\left(F_t - c_t\right), \quad \text{for } t = 0, \dots T \tag{12}$$

$$c_t \geq 0, \quad \text{for } t = 0, \dots T \tag{13}$$

$$F_{T+1} = 0 \tag{14}$$

where $T < \infty, F_0 > 0, R > 0$, and $\beta \in (0,1)$.[5] Note that there is positive initial wealth, but no labor income. They are choosing how to spend their wealth

(a) Check if the utility function is concave.

(b) Setup the Lagrangian and find the first-order necessary conditions.[6]

(c) Using the first-order necessary conditions, find the optimal path of $\{c_t, F_{t+1}\}_{t=0}^{T}$. Is the solution always unique, and if not, why?

(d) Interpret how does the optimal allocation depends on the relationship between R and β.

[5] Hint: This has **not** assumed any relationship between β and R. Consider that $\beta R \lesseqqgtr 1$ as potentially having different behaviors.

[6] Hint: You will have to be very careful with Lagrange multipliers here. The complementarity conditions will be important for the c_t constraints. Also, what do you know about linear objectives and linear constraints?