Assignment 3: Nonlinear Equations and Numerical Optimization

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The goal of this exercise is to introduce basic numerical methods. We use data from a field experiment. Each choice is binary, and the lotteries are denoted ($\mathscr{C}_1$ and $\mathscr{C}_2$). They are composed of a low and high payoffs. Subjects are presented with five gambles entailing binary choices. In each case, the first option has lower expected value and dispersion than the second. At the fifth choice, expected values are similar across lotteries, and the second option still has higher variance. In total, there is 25 decisions, which are stored in dat_choices.

Table 1: Risk Aversion Lotteries: Ordered Lotteries Selection

	$\mathscr{C}_1$				$\mathscr{C}_2$			
	$\mathbb{P}_1$	$\mathbb{P}_2$	EV	SD.	$\mathbb{P}_1$	$\mathbb{P}_2$	EV	SD.
	48	48	48	0	40	64	52	16.97
	40	64	52	16.97	32	80	56	33.94
List 1	32	80	56	33.94	24	96	60	50.91
	24	96	60	50.91	16	112	64	67.88
	16	112	64	67.88	8	120	64	79.2
	48	48	48	0	42	66	54	16.97
	42	66	54	16.97	36	84	60	33.94
List 2	36	84	60	33.94	30	102	66	50.91
	30	102	66	50.91	24	120	72	67.88
	24	120	72	67.88	16	128	72	79.2
	48	48	48	0	38	62	50	16.97
	38	62	50	16.97	28	76	52	33.94
List 3	28	76	52	33.94	18	90	54	50.91
	18	90	54	50.91	8	104	56	67.88
	8	104	56	67.88	0	112	56	79.2
	42	42	42	0	36	60	48	16.97
	36	60	48	16.97	30	78	54	33.94
List 4	30	78	54	33.94	24	96	60	50.91
	24	96	60	50.91	18	114	66	67.88
	18	114	66	67.88	10	122	66	79.2
	54	54	54	0	44	68	56	16.97
	44	68	56	16.97	34	82	58	33.94
List 5	34	82	58	33.94	24	96	60	50.91
	24	96	60	50.91	14	110	62	67.88
	14	110	62	67.88	6	118	62	79.2

Notes: Table lists five lotteries. $\mathscr{C}_1$ and $\mathscr{C}_2$ refer to the choices of the lottery, while $\mathbb{P}_1$ and $\mathbb{P}_2$ refers to the payoffs. $\mathbb{E}V$ is for expected value, while SD is the standard deviation across payoffs. Payoffs are in Canadian \$. Source: SRDC-CIRANO Field Experiment on Education Financing.

Exercise 1 Part 1

Let individuals be endowed with a CRRA utility function. The utility of an individual with risk aversion θ over choice c

$$u(c,\theta)=\begin{cases}\dfrac{c^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1\\ \log(c), & \text{if } \theta = 1\end{cases} \tag{1}$$

- Write a function crra that as input c and θ and return utility.
- Illustrate the shape of the utility function, for $c = 50$, and risk aversion θ ranging from -5 to 5.

Exercise 2 Part 2

- Create a matrix that records all choices in Table 1.
- Using bisection techniques, find the level of risk aversion that makes an individual indifferent between each of the 25 choices.
- Use turning points to construct identified sets for each set of lotteries.
- Use these identified sets along with dat_choice to construct the distribution of risk aversion in the population for each list of questions.

Exercise 3 Part 3

Consider an individual i endowed with an initial wealth w_i which is set to 20. Individuals have CRRA utility function. Consider the choice between $\mathscr{C}_{1l} = (c_l^1, c_l^2)$ and $\mathscr{C}_{2l} = (c_2^1, c_2^2)$. The utility of an agent i for lottery $\mathscr{C}_{1l}$ is:

$$V_i^r(w_i + \mathscr{C}_{1l}, \theta_i) = \overline{V}_{il}^{1r} + \xi_{il}^r = u(w_i + c_{1l}^1, \theta_i) + u(w_i + c_{1l}^2, \theta_i) + \xi_{il}^r \tag{2}$$

The associated probability of choosing $\mathscr{C}_{11}$ over $\mathscr{C}_{2l}$ is:

$$\Pr\left[V_i^r(w_i + \mathscr{C}_1, \theta_i) > V_i^r(w_i + \mathscr{C}_2, \theta_i)\right] \quad = \Pr\left[\xi_{il}^{2r} - \xi_{il}^{1r} < \overline{V}_{il}^{1r} - \overline{V}_{il}^{2r}\right] \tag{3}$$

1. Write the likelihood associated to least risky lottery under the assumption that $\xi_{il}^r \sim EV_1$ (extreme value type 1 = gumbel)
2. One parameter optimization: Grid search: Simulate a grid of potential values for θ, evaluate the likelihood for each of value of the grid and select the maximum. Apply this for individuals 900 and 115.

Exercise 4 Part 4

Consider the Beale function

$$f(x, y) = (1.5 - x + xy)^2 + (2.25 - x + xy^2)^2 + (2.625 - x + xy^3)^2 \tag{4}$$

We are interested in the minimum of this function.

1. Brute force. Create a grid of values for (x,y) from -5 to 5 with a step size of 0.01 and evaluate $f(x, y)$ for all those points. Find the minimum of the function on this grid and let it be denoted (x_0, y_0)
2. Using the approximation of the gradient seen in the lecture, compute $df(x_0, y_0)$

Exercise 5 Part 5

Consider the Rosenbrock Banana function

$$f(x, y) = (1 - x)^2 + 5(y - x^2)^2 \tag{5}$$

We are interested in the minimum of this function.

1. Program the steepest descent algorithm to find the minimum of this function.

Exercise 6 Part 6

In this part, we are interested in the empirical question: Do Workplace Smoking Bans Reduce Smoking? We will use the dataset "SmokeBan" from the AER package, which contains 10,000 observations on 7 variables.

- smoker factor. Is the individual a current smoker?
- ban factor. Is there a work area smoking ban?
- age. age in years.
- education. factor indicating highest education level attained: high school (hs) drop out, high school graduate, some college, college graduate, master's degree (or higher).
- afam factor. Is the individual African-American?
- hispanic factor. Is the individual Hispanic?
- gender factor indicating gender.

1. Estimate the effect of all variables on the probability of smoking using a probit model. For estimation purpose, use the pre-programmed packages in optim.

2. Using Fisher Information, derive the standard error of the estimates.

Exercise 7 Part 7

Consider an individual i endowed with an initial wealth w_i which is set to 20. Individuals have CRRA utility function. Consider the choice between $\mathscr{C}_{1l} = (c_l^1, c_l^2)$ and $\mathscr{C}_{2l} = (c_2^1, c_2^2)$. The utility of an agent i for lottery $\mathscr{C}_{1l}$ is:

$$V_i^r(w_i + \mathscr{C}_{1l}, \theta_i) = \overline{V}_{il}^{1r} + \xi_{il}^r = u(w_i + c_{1l}^1, \theta_i) + u(w_i + c_{1l}^2, \theta_i) + \xi_{il}^r \tag{6}$$

The associated probability of choosing $\mathscr{C}_{11}$ over $\mathscr{C}_{2l}$ is:

$$\Pr\left[V_i^r(w_i + \mathscr{C}_1, \theta_i) > V_i^r(w_i + \mathscr{C}_2, \theta_i)\right] \quad = \Pr\left[\xi_{il}^{2r} - \xi_{il}^{1r} < \overline{V}_{il}^{1r} - \overline{V}_{il}^{2r}\right] \tag{7}$$

1. Write the likelihood associated to the choice problem under the assumption that $\xi_{il}^r \sim \mathscr{N}(0, \sigma)$.
2. Using dat_choices, estimate risk aversion and sigma for individual 5.
   - nloptr has the largest collection of optimization tools in R.
   - Install the nloptr packages.
   - Using starting values of 2.1 and 0.8 for risk aversion and σ respectively, test the following algorithms
     - ISRES (Improved Stochastic Ranking Evolution Strategy)
     - LBFGS
     - BOBYQA
     - Nelder-Mead Simplex
   - Select the algorithm that gives you the best performance, and estimate the pair risk aversion/$sigma$ for everyone in the sample.
   - Represent the distribution of estimated risk aversion.

Exercise 8 Part 8

At question l, an individual is faced with the choice between $\mathscr{C}_{1,l} = (c_{11,l}, c_{12,l})$, and $\mathscr{C}_{2,l} = (c_{21,l}, c_{22,l})$, with the probability of the first reward being p_l. Assuming that individual background consumption is denoted by w_i, and the value of each alternative is given by $V_{il1}(w_i + \mathscr{C}_{1,l})$ and $V_{il2}(w_i + \mathscr{C}_{2,l})$:

$$V_{1i,l}(w_i + \mathscr{C}_{1,l}, \theta_{il}, p_l) \quad = \overline{V}_{1i,l} + \epsilon_{1i,l}^d \tag{8}$$

where

$\overline{V}_{il}^{1r} = 0.5u(w_i + c_{11,l}, \theta_{il}) + 0.5u(w_i + c_{12,l}, \theta_{il})$

Letting $\epsilon_{il}^{d} \equiv \epsilon_{1i,l}^{d} - \epsilon_{2i,1}^{d} \sim \mathcal{N}(0, \sigma_u)$. The choice probabilities are given:

$$\Pr(y_{il} = 1) = F_{\epsilon}\left(\bar{V}_{il1}(w_i + \mathscr{C}_{1l}, x_{il} + y_{1l}, p_l) - \bar{V}_{il2}(w_i + \mathscr{C}_{2l}, x_{il} + y_{2l}, p_l)\right) \tag{9}$$

1. Assume the scale parameter is known ($\sigma = 0.5$), For a grid of risk aversion parameters (0.1 to 2), report the relationship between the choice probability and the risk aversion assuming that the error terms follow a normal distribution.

2. Conduct a monte carlo study to assess whether the risk aversion parameter is identified using multiple questions. A monte carlo analysis is as follows:

   (a) Set a risk aversion parameter r

   (b) Given r, simulate some data under the specific model and choice structure. Denote this as y^{sim}.

   (c) Given y^{sim}, estimate the model and record the estimate.

   (d) Redo steps b-c, 100 times, and construct confidence intervals.

3. Do the monte carlo study this grid of parameters

grid	r	σ
1	0.5	0.1
2	0.5	0.9
3	0.8	0.1
4	0.8	0.9
5	1.5	0.1
6	1.5	0.9
7	2.8	0.1
8	2.8	0.9

Table 2: Time Preferences Lotteries

	$\mathbb{P}_1$	$Time_1$	$\mathbb{P}_2$	$Time_2$
List 1	75	1	75.31	31
	75	1	75.63	31
	75	1	76.25	31
	75	1	78.13	31
	75	1	81.25	31
	75	1	87.50	31
List 2	75	8	75.31	31
	75	8	75.63	31
	75	8	76.25	31
	75	8	78.13	31
	75	8	81.25	31
	75	8	87.50	31
List 3	75	31	75.31	61
	75	31	75.63	61
	75	31	76.25	61
	75	31	78.13	61
	75	31	81.25	61
	75	31	87.50	61
List 4	75	91	75.31	121
	75	91	75.63	121
	75	91	76.25	121
	75	91	78.13	121
	75	91	81.25	121
	75	91	87.50	121
List 5	75	1	75.31	361
	75	1	75.63	361
	75	1	76.25	361
	75	1	78.13	361
	75	1	81.25	361
	75	1	87.50	361
List 6	75	8	75.31	368
	75	8	75.63	368
	75	8	76.25	368
	75	8	78.13	368
	75	8	81.25	368
	75	8	87.50	368
List 7	75	31	75.31	391
	75	31	75.63	391
	75	31	76.25	391
	75	31	78.13	391
	75	31	81.25	391
	75	31	87.50	391
List 8	75	91	75.31	451
	75	91	75.63	451
	75	91	76.25	451
	75	91	78.13	451
	75	91	81.25	451
	75	91	87.50	451

Exercise 9   Part 9

We consider time preferences decisions given by Table 2. Consider the following framework. At time 0, when the experiment takes place, the intertemporal utility of consumption stream $\{c_0, c_1, ...c_\tau..\}$ is equal to

$$V_0 = U(c_0) + \beta \sum_{t=1}^{T} \delta^t U(c_t),$$

where $U(.)$ is the per-period utility function; β is the present-bias parameter; and δ measures the classical discount factor. We assume that the per-period utility, denoted $U(.)$, belongs to the Constant-Relative-Risk Aversion (CRRA) family; that is:

$$U(c_t;\theta) = \frac{c_t^{1-\theta}}{1-\theta}.$$

Assume that each individual is endowed with a background (reference) level of consumption denoted w that represents the optimal level of consumption in the absence of any windfall cash payment.

Put generally, the per-period utility in period 0 of choosing any amount a to be paid at time t is equal to

$$\begin{cases} U(w+a), & \text{if } t=0 \\ \beta\delta^t U(w+a), & \text{if } t=1,2,\dots,T. \end{cases}$$

Each potential choice q consists of two mutually exclusive cash payments—c_q at time t and d_q at time $t+\tau$. In the absence of consumption smoothing induced by the cash payments, choosing between c_q at time t and d_q at time $t+\tau$ boils down to comparing two distinct sums of utilities. To introduce noisy measurements of preference parameters, we assume an additive idiosyncratic error, which is i.i.d. across questions and across individuals, and denoted by ε_{iq}.

Setting the indicator y_q to 1 when option 1 (c_q) is chosen and to 0 when option 2 (d_q) is chosen, the expression for the probabilities of choosing the earlier payment are given by the following expressions:

$$\Pr\{y_q = 1\} = \begin{cases} \Pr\left\{\varepsilon_q > (U(w) - U(w+c_q) + \beta\delta^\tau[U(w+d_q) - U(w)\right\}, & \text{no front-end delay} \\ \Pr\left\{\varepsilon_q > \beta\delta^t[U(w) - U(w+c_q] + \beta\delta^{t+\tau}[U(w+d_q) - U(w)]\right\}, & \text{front-end delay,} \end{cases} \tag{10}$$

- Under the assumption that $\varepsilon_i \sim N(0, \sigma_i)$, write the likelihood of the choice problem.
- Using estimates from risk aversion derived in part 7, and dat_time, estimate β, δ and σ for all individuals in the sample. Derive the standard deviation of these estimates.
- Evaluate the correlation between θ (risk aversion), and β and δ