Computational Methods in Economics

Numeric Integration

March 31, 2020

Motivation

Numerical integration, which is also called quadrature, has a history extending back to the invention of calculus and before. The fact that integrals of elementary functions could not, in general, be computed analytically, while derivatives could be, served to give the field a certain panache, and to set it a cut above the arithmetic drudgery of numerical analysis during the whole of the 18th and 19th centuries.

Introduction (1)

Second Fundamental Theorem of Calculus

$$\int_a^b F'(x)dx = F(b) - F(a) \tag{1}$$

Fundamental!!!, but consider

$$\int_a^b \exp(-x^2)dx \tag{2}$$

Even though $\exp(-x^2)$ is **Riemann** Integrable.

Introduction (2)

- Numerical integration methods are based, in one way or another, on the obvious device of adding up the value of the integrand at a sequence of abscissas within the range of integration.
- The goal is to obtain the integral as accurately as possible with the smallest number of function evaluations of the integrand.

General Statement

- Consider a sequence of points $x_0; x_1; \ldots; x_{N-1}; x_N$, that are spaced apart by a constant step h,

$$x_i = x_0 + ih \quad i = 0, 1, \ldots, N \tag{3}$$

- A function $f(x)$ has known values at the x_i's,

$$f(x_i) = f_i \tag{4}$$

- The goal is to integrate the function $f(x)$ between a lower limit a and an upper limit b.

Graphical Analysis

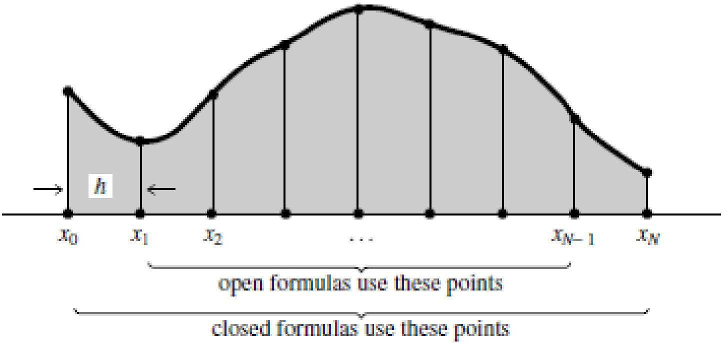


Figure 4.1.1. Quadrature formulas with equally spaced abscissas compute the integral of a function between x_0 and x_N. Closed formulas evaluate the function on the boundary points, while open formulas refrain from doing so (useful if the evaluation algorithm breaks down on the boundary points).

Early days: Kepler/Simpson's Approximation

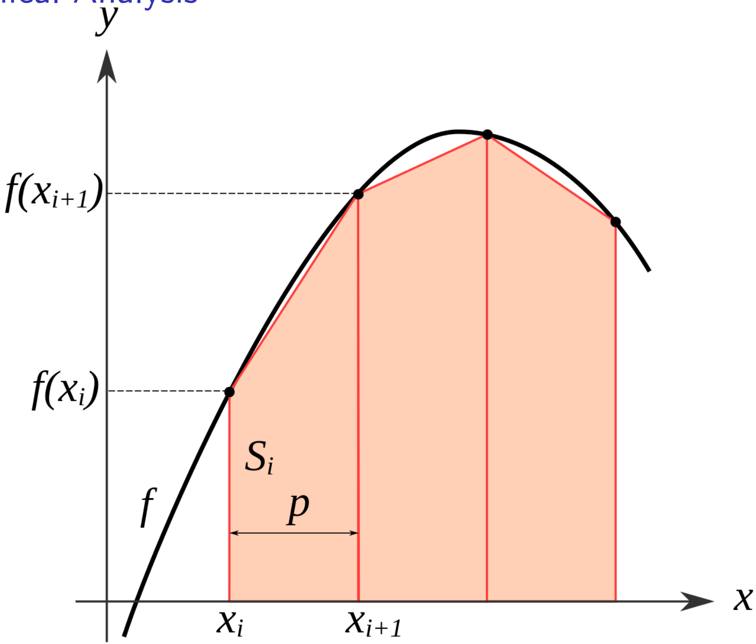
$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \tag{5}$$

Trapezoidal Integration Rule

$$\int_a^b f(x)dx = h\left[\frac{f(a)}{2} + f(a+h) + \ldots + f(a+(n-1)*h) + \frac{f(b)}{2}\right] \tag{6}$$

where $h = (b-a)/n$,

Graphical Analysis



Gaussian Quadrature

- Until now, the integral of a function was approximated by the sum of its values at a set of equally spaced points, multiplied by some weighting coefficients.
- The idea of Gaussian quadratures is to not only choose the weighting coefficients, but also the location of the abscissas at which the function is to be evaluated.
- Thus, we will have twice the number of degrees of freedom, and it turns out we can achieve Gaussian quadrature formulas whose order is, essentially, twice that of the Newton-Cotes formula with the same number of function evaluations.
- The catch is the function needs to be very smooth.

Gaussian Quadrature (2)

- The key reason why quadrature methods are used - we can arrange the choice of weights and abscissas to make the integral exact for a class of integrands polynomials times some known function $W(x)$
- For example, given $W(x)$, in other words, and given an integer N, we can find a set of weights w_j and abscissas x_j such that the approximation

$$\int_a^b W(x) f(x) = \sum_{j=0}^{N-1} w_j f(x_j) \tag{7}$$

is exact if $f(x)$ is a polynomial.

Quadrature

$$\int_a^b w(x)f(x) = \sum_i^{n+1} w_i f(x_i) \tag{8}$$

Interval	Weight	Name
$[-1, 1]$	1	Legendre
$[-1, 1]$	$(1 - x^2)^{-1/2}$	Tschebyscheff, 1st
$[-1, 1]$	$(1 - x^2)^{1/2}$	Tschebyscheff, 2nd
$[0, \infty)$	$\exp(-x)$	Laguerre
(∞, ∞)	$\exp(-x^2)$	Hermite

Monte Carlo Integration

- When all fails!
- General formula; consider $F = int_a^b f(x)dx$. Given a set of N uniform random variables (pdf is $\frac{1}{b-a}$) $X_i \in [a, b]$. A monte carlo estimator is given by

$$f \approx (b-a)\frac{1}{N-1}\sum_{i=0}^{N} f(X_i) \tag{9}$$

In practice, we usually do

$$X_i = a + \xi(b-a) \quad \text{with}\, \xi \in [0, 1] \tag{10}$$

Coding

- Well established software suites to do numerical integration.
- GSL is my favourite (can be ported to various languages).