Multiple OLS Regression: Estimation

Introduction to Econometrics,Fall 2017

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Review the last lecture

Properties OLS estimators in multiple regression model

If the four least squares assumptions in the multiple regression model hold:

- The OLS estimators $\hat{\beta}_0, \hat{\beta}_1 ... \hat{\beta}_k$ are unbiased.

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- The OLS estimators $\hat{\beta}_0, \hat{\beta}_1 ... \hat{\beta}_k$ are normally distributed in large samples.
- the formal proof need use the knowledge of linear algebra and matrix.We will prove the ubiasedness in a simple case.

Partitioned regression

Partitioned regression: OLS estimator in multiple regression

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- Regress $X_{1,i}$ on other regressors

$$X_{1,i} = \hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + ... + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}$$

where $\tilde{X}_{1,i}$ is the fitted OLS residual(just a variation of u_i)

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- Then we could prove that

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2}$$

Proof of Partitioned regression result(1)

- we know $Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_{1,i} + \hat{\beta}_2 X_{2,i} + ... + \hat{\beta}_k X_{k,i} + \hat{u}_i$ where $\sum \hat{u}_i = \sum \hat{u}_i X_{ji} = 0, j = 1, 2, ..., k$

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- Now

$$
\begin{aligned}
\frac{\sum_{i=1}^n \tilde{X}_{1,i} Y_i}{\sum_{i=1}^n \tilde{X}_{1,i}^2} &= \frac{\sum \tilde{X}_{1,i}(\hat{\beta}_0 + \hat{\beta}_1 X_{1,i} + \hat{\beta}_2 X_{2,i} + ... + \hat{\beta}_k X_{k,i} + \hat{u}_i)}{\sum \tilde{X}_{1,i}^2} \\
&= \hat{\beta}_0 \frac{\sum_{i=1}^n \tilde{X}_{1,i}}{\sum_{i=1}^n \tilde{X}_{1,i}^2} + \hat{\beta}_1 \frac{\sum_{i=1}^n \tilde{X}_{1,i} X_{1,i}}{\sum_{i=1}^n \tilde{X}_{1,i}^2} + ... \\
&+ \hat{\beta}_k \frac{\sum_{i=1}^n \tilde{X}_{1,i} X_{k,i}}{\sum_{i=1}^n \tilde{X}_{1,i}^2} + \frac{\sum_{i=1}^n \tilde{X}_{1,i} \hat{u}_i}{\sum_{i=1}^n \tilde{X}_{1,i}^2}
\end{aligned}
$$

Proof of Partitioned regression result(2)

- $\tilde{X}_{1,i}$ is the fitted OLS residual for the regression

$$X_{1,i} = \hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + ... + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}$$

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$$X_{1,i} = \hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + ... + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}$$

- so it is a variation of u_i, then we have

$$\sum_{i=1}^{n} \tilde{X}_{1,i} = 0 \; and \; \sum_{i=1}^{n} \tilde{X}_{1,i} X_{j,i} = 0 \;, j = 2, 3, ..., k$$

Proof of Partitioned regression result(3)

- We also have

$$
\begin{aligned}
&\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i} \\
&= \sum_{i=1}^{n} \tilde{X}_{1,i} (\hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + ... + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}) \\
&= \hat{\gamma}_0 \cdot 0 + \hat{\gamma}_2 \cdot 0 + ... + \hat{\gamma}_k \cdot 0 + \sum \tilde{X}_{1,i}^2 \\
&= \sum \tilde{X}_{1,i}^2
\end{aligned}
$$

Proof of Partitioned regression result(4)

- Recall: $\hat{u}_i$ are the fitted residuals from the regression of Y against all X, then

$$\sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} \hat{u}_i X_{j,i} = 0 \ , j = 1, 2, 3, ..., k$$

Proof of Partitioned regression result(5)

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- We also have

$$\begin{aligned}
&\sum_{i=1}^{n} \tilde{X}_{1,i} \hat{u}_i \\
&= \sum_{i=1}^{n} (X_{1,i} - \hat{\gamma}_0 - \hat{\gamma}_2 X_{2,i} - ... - \hat{\gamma}_k X_{k,i}) \hat{u}_i \\
&= 0 - \hat{\gamma}_0 \cdot 0 - \hat{\gamma}_2 \cdot 0 - ... - \hat{\gamma}_k \cdot 0 \\
&= 0
\end{aligned}$$

wrap up so far

- OLS Regression

$$\sum_{i=1}^{n}\tilde{X}_{1,i}=\sum_{i=1}^{n}\tilde{X}_{1,i}X_{j,i}=0\ ,j=2,3,...,k$$

$$\sum_{i=1}^{n}\tilde{X}_{1,i}X_{1,i}=\sum\tilde{X}_{1,i}^{2}$$

$$\sum_{i=1}^{n}\tilde{X}_{1,i}\hat{u}_{i}=0$$

wrap up so far

- OLS Regression
- and $\tilde{X}_{1,i}$ is the fitted OLS residual for the regression

$$X_{1,i} = \hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + ... + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}$$

$$\sum_{i=1}^{n} \tilde{X}_{1,i} = \sum_{i=1}^{n} \tilde{X}_{1,i} X_{j,i} = 0 \ , j = 2, 3, ..., k$$

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- we obtained

$$\sum_{i=1}^{n} \tilde{X}_{1,i} = \sum_{i=1}^{n} \tilde{X}_{1,i} X_{j,i} = 0 \ , j = 2, 3, ..., k$$

$$\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i} = \sum \tilde{X}_{1,i}^2$$

$$\sum_{i=1}^{n} \tilde{X}_{1,i} \hat{u}_i = 0$$

Proof of Partitioned regression result(6)

- we have shown that

$$\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}Y_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} = \hat{\beta}_0\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} + \hat{\beta}_1\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}X_{1,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} + ...$$
$$+ \hat{\beta}_k\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}X_{k,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} + \frac{\sum_{i=1}^{n}\tilde{X}_{1,i}\hat{u}_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2}$$

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$$+\hat{\beta}_k\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}X_{k,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2}+\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}\hat{u}_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2}$$

- then

$$\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}Y_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2}=\hat{\beta}_1$$

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- then

$$\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}Y_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} = \hat{\beta}_1$$

.

- Identical argument works for $j = 2, 3, ..., k$, thus

$$\hat{\beta}_j = \frac{\sum_{i=1}^{n}\tilde{X}_{j,i}Y_i}{\sum_{i=1}^{n}\tilde{X}_{j,i}^2}$$

The intuition of Partitioned regression : "Partialling Out"

- First, we regress X_j against the rest of the regressors (and a constant) and keep $\tilde{X}_j$ which is the "part" of X_j that is **uncorrelated**

The intuition of Partitioned regression : “Partialling Out”

- First, we regress X_j against the rest of the regressors (and a constant) and keep $\tilde{X}_j$ which is the “part” of X_j that is **uncorrelated**
- Then, to obtain $\hat{\beta}_j$, we regress Y against $\tilde{X}_j$ which is *“clean”* from correlation with other regressors.

The intuition of Partitioned regression : "Partialling Out"

- First, we regress X_j against the rest of the regressors (and a constant) and keep $\tilde{X}_j$ which is the "part" of X_j that is **uncorrelated**
- Then, to obtain $\hat{\beta}_j$, we regress Y against $\tilde{X}_j$ which is *"clean"* from correlation with other regressors.
- $\hat{\beta}_j$measures the effect of X_1 after the effects of $X_2, ..., X_k$ have been *partialled out or netted out*.

Example: Test scores and Student Teacher Ratios

```
tilde.str <- residuals(lm(str ~ el_pct+avginc, data=ca))
mean(tilde.str) # should be zero
```

```
## [1] 1.305121e-17
```

```
sum(tilde.str)
```

```
## [1] 5.412337e-15
```

```
cov(tilde.str,ca$avginc)# should be zero too
```

```
## [1] 3.650126e-16
```

Example: Test scores and Student Teacher Ratios(2)

```
tilde.str_str <- tilde.str*ca$str
tilde.strstr <- tilde.str^2
sum(tilde.str_str)
```

```
## [1] 1396.348
```

```
sum(tilde.strstr)# should be equal the result above.
```

```
## [1] 1396.348
```

Example: Test scores and Student Teacher Ratios(3)

```
sum(tilde.str*ca$testscr)/sum(tilde.str^2)
```

```
## [1] -0.06877552
```

```
summary(lm(ca$testscr~tilde.str))
```

```
##
## Call:
## lm(formula = ca$testscr ~ tilde.str)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -48.50 -14.16   0.39  12.57  52.57
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
```

Proof that OLS is unbiased(1)

- Use partitioned regression formula

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2}$$

Proof that OLS is unbiased(1)

- Use partitioned regression formula

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2}$$

- Substitute
  $Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + ... + \beta_k X_{k,i} + u_i, i = 1, ..., n$,then

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\sum \tilde{X}_{1,i}(\beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + ... + \beta_k X_{k,i} + u_i)}{\sum \tilde{X}_{1,i}^2} \\
&= \beta_0 \frac{\sum_{i=1}^{n} \tilde{X}_{1,i}}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} + \beta_1 \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i}}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} + ... \\
&+ \beta_k \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} X_{k,i}}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} + \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} u_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2}
\end{aligned}$$

Proof that OLS is unbiased(2)

- Because

$$\sum_{i=1}^{n} \tilde{X}_{1,i} = \sum_{i=1}^{n} \tilde{X}_{1,i} X_{j,i} = 0 \ , j = 2, 3, ..., k$$

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- Therefore

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} u_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2}$$

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- we have that

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- Take expectations of $\hat{\beta}_1$ and based on **Assumption 1** again

$$\begin{aligned} E[\hat{\beta}_1] &= E\left[E[\hat{\beta}_1|X]\right] \\ &= \beta_1 + 0 \end{aligned}$$

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- Identical argument works for $j = 2, 3, ..., k$