Problem set 3

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Question 2 - Required

Consider a simple version of the stochastic growth model in which a planner solves

$$\max_{\{c_t, k_{t+1}\}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \log(c_t) \right] \tag{1}$$

subject to

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha}, \quad \text{with } k_0 \text{ given} \tag{2}$$

and where $z_{t+1} = \rho z_t + \sigma \varepsilon_{t+1}$, with z_0 given. Set $\beta = 0.99, \alpha = 0.33, \rho = 0.95,$ and $\sigma = 0.01$. Solve the model three ways: (1) analytically (2) log linearizing it yourself using the method of undetermined coefficients or the implicit function theorem (or, even better, use both and show they are equivalent), and (3) a quadratic approximation using Dynare. Compare the policy functions and Euler errors. Starting from the steady state, draw a path of z_t 's for 200 periods (assuming $\varepsilon_t \sim \mathcal{N}(0,1)$) and compare the simulated paths for k_t (use the same draws for all three simulations). Maybe also compute a log linearization using Dynare and compare with the results from your log linearization.

Analytical solution

Solution. To solve this model analytically, we start by writing the Lagrangian associated with (1)-(2)

$$\mathcal{L} = \mathbb{E}_0 \left[\max_{\{c_t, k_{t+1}\}} \left[\sum_{t=0}^{\infty} \beta^t \left(\log(c_t) + \lambda_t \left(e^{z_t} k_t^{\alpha} - c_t - k_{t+1} \right) \right) \right] \right]. \tag{3}$$

The next step is to take the FOC[1] of (3) w.r.t. c_t, k_{t+1} and λ_t:

$$\begin{aligned} c_t: \quad \frac{\partial \mathcal{L}}{\partial c_t} &= \frac{\partial}{\partial c_t} \mathbb{E}_t \left[\beta^t \left\{ \log(c_t) + \lambda_t \left(e^{z_t} k_t^{\alpha} - c_t - k_{t+1} \right) \right\} \right] \\ &= \mathbb{E}_t \left[\beta^t \left\{ \frac{1}{c_t} - \lambda_t \right\} \right] = 0 \Longrightarrow \boxed{\lambda_t = c_t^{-1}} \end{aligned} \tag{4}$$

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Homework for the Advanced Topics in Macro I Course, Block 2. **Instructors:** dr. M. Z. Pedroni (UvA) and dr. E. Proehl (UvA).

[1] Assume that the usual regularity conditions in which the derivative and integral sign can swap places apply.

$$
\begin{aligned}
\lambda_t: \quad \frac{\partial \mathcal{L}}{\partial \lambda_t} &= \frac{\partial}{\partial \lambda_t} \mathbb{E}_t \left[\beta^t \left\{ \log(c_t) + \lambda_t \left(e^{z_t} k_t^{\alpha} - c_t - k_{t+1} \right) \right\} \right] \\
&= \mathbb{E}_t \left[\beta^t \left\{ e^{z_t} k_t^{\alpha} - c_t - k_{t+1} \right\} \right] \\
&= \beta^t \left[e^{z_t} k_t^{\alpha} - c_t - \mathbb{E}_t \left[k_{t+1} \right] \right] = 0 \\
&\implies \boxed{c_t = e^{z_t} k_t^{\alpha} - \mathbb{E}_t \left[k_{t+1} \right]}
\end{aligned}
\tag{5}
$$

$$
k_{t+1}: \quad \frac{\partial}{\partial k_{t+1}} \mathbb{E}_t \left[\beta^t \left\{ \log(c_t) + \lambda_t \left(e^{z_t} k_t^{\alpha} - c_t - k_{t+1} \right) \right\} + \beta^{t+1} \left\{ \log(c_{t+1}) + \lambda_{t+1} \left(e^{z_{t+1}} k_{t+1}^{\alpha} - c_{t+1} - k_{t+2} \right) \right\} \right]
$$

$$
= -\beta^t \lambda_t + \mathbb{E}_t \left[\beta^{t+1} \lambda_{t+1} \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} \right] = 0 \implies \boxed{\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} e^{z_{t+1}} \alpha k_{t+1}^{\alpha-1} \right]}. \tag{6}
$$

We can summarize the equilibrium conditions in our economy as:

- The Euler equation, obtained by fowarding Equation (4) in (6):

$$
c_t^{-1} = \beta \mathbb{E}_t \left[c_{t+1}^{-1} e^{z_{t+1}} \alpha k_{t+1}^{\alpha-1} \right] \tag{7}
$$

- The Aggregate Production constraint, which represents the intra-temporal substitution between consumption and investment, obtained from (5):

$$
c_t = e^{z_t} k_t^{\alpha} - \mathbb{E}_t \left[k_{t+1} \right] \tag{8}
$$

In the steady state (SS), we have $c_t = \bar{c}$ and $k_t = \bar{k} \quad \forall\, t$. In addition, the shocks z_t will be equal to their mean of 0 and there is no uncertainty. Replacing this in the Euler Equation (7) gives:

$$
\begin{aligned}
\bar{c}^{-1} &= \beta \left[\bar{c}^{-1} e^0 \alpha \bar{k}^{\alpha-1} \right] \\
1 &= \beta \alpha \bar{k}^{\alpha-1} \\
(\beta\alpha)^{-1} &= \bar{k}^{\alpha-1} \\
\beta\alpha &= \bar{k}^{1-\alpha} \\
\therefore \bar{k} &= (\beta\alpha)^{\frac{1}{1-\alpha}},
\end{aligned}
\tag{9}
$$

while working with (8) will result in

$$
\begin{aligned}
\bar{c} &= e^0 \bar{k}^{\alpha} - \bar{k} \\
&= (\beta\alpha)^{\frac{\alpha}{1-\alpha}} - (\beta\alpha)^{\frac{1}{1-\alpha}}.
\end{aligned}
\tag{10}
$$

Plugging the parameter values given in the exercise in (9) and (10) results in:

$$
\begin{aligned}
\bar{k} &= (0.99 \cdot 0.33)^{\frac{1}{1-0.33}} \approx 0.1883, \\
\bar{c} &= (0.99 \cdot 0.33)^{\frac{0.33}{1-0.33}} - (0.99 \cdot 0.33)^{\frac{1}{1-0.33}} \approx 0.3880.
\end{aligned}
$$

To get the policy functions analytically, we use a first guess for the policy function for k, namely $\pi(k, z) = B_0 + B_1 e^z k^{\alpha}$ (and use $k = k_t$ and $k' = k_{t+1}$):

$$
\begin{aligned}
c &= e^z k^{\alpha} - \pi_k(k, z) \\
\pi_c(k, z) &= e^z k^{\alpha} - B_0 - B_1 e^z k^{\alpha} = (1 - B_1) e^z k^{\alpha} - B_0
\end{aligned}
$$

Such that the Euler equation looks like:

$$
\begin{aligned}
\frac{1}{\pi_c(k,z)} &= \beta\mathbb{E}_t\left[\frac{e^{z'}\alpha k'^{(\alpha-1)}}{\pi_c(k',z')}\right] \\
\frac{1}{(1-B_1)e^z k^\alpha - B_0} &= \beta\mathbb{E}_t\left[\frac{e^{z'}\alpha\pi_k(k,z)^{(\alpha-1)}}{(1-B_1)e^{z'}\pi_k(k,z)^\alpha - B_0}\right] \\
\frac{1}{(1-B_1)e^z k^\alpha - B_0} &= \beta\mathbb{E}_t\left[\frac{e^{z'}\alpha\left[B_0 + B_1 e^z k^\alpha\right]^{(\alpha-1)}}{(1-B_1)e^{z'}\left[B_0 + B_1 e^z k^\alpha\right]^\alpha - B_0}\right]
\end{aligned}
$$

We have to impose $B_0 = 0$ to satisfy this equation, such that:

$$
\begin{aligned}
\frac{1}{(1-B_1)e^z k^\alpha} &= \beta\mathbb{E}_t\left[\frac{e^{z'}\alpha\left[B_1 e^z k^\alpha\right]^{(\alpha-1)}}{(1-B_1)e^{z'}\left[B_1 e^z k^\alpha\right]^\alpha}\right] \\
&= \beta\mathbb{E}_t\left[\frac{\alpha\left[B_1 e^z k^\alpha\right]^{(-1)}}{(1-B_1)}\right] \\
&\longrightarrow B_1 = \alpha\beta
\end{aligned}
$$

With these values, we can conclude that the policy functions are the following:

$$
\begin{aligned}
\pi_c(k,z) &= (1-\alpha\beta)e^z k^\alpha \\
\pi_k(k,z) &= \alpha\beta e^z k^\alpha
\end{aligned}
$$

□

Log-linearization

Solution. The strategy to log-linearize is:

1. Substitute all variables by the product of the variable in the SS times the exponential of the log deviation, i.e., $x_t = \bar{x}e^{\hat{x}_t}$;
2. Take a first order approximation in $\bar{x}_t$ around $\hat{x}_t = 0$. We can also use the following relations: $e^{\hat{x}} \approx 1 + \hat{x}$, $x_t y_t \approx \bar{x}\bar{y}e^{\hat{x}_t\hat{y}_t}$, $(1+\hat{x}_t)^a \approx (1 + a\hat{x}_t)$ and $\hat{x}_t\hat{y}_t \approx 0$.

Start from the Euler equation (7), by convenience reproduced below:

$$
c_t^{-1} = \beta\mathbb{E}_t\left[c_{t+1}^{-1}e^{z_{t+1}}\alpha k_{t+1}^{\alpha-1}\right].
$$

Define $a_t \equiv e^{z_t}$. Observe that the log deviation this variable, denoted by $\hat{a}_t$, will be exactly z_t (using the fact that $\bar{z} = 0$) since

$$
\hat{a}_t = \log(a_t) - \log(\bar{a}) = \log(e^{z_t}) - \underbrace{\log(\overbrace{e^{\bar{z}}}^{1})}_{0} = z_t.
$$

Now rewrite the Euler equation as:

$$
\begin{aligned}
\left(\cancel{\bar{c}}e^{\hat{c}_t}\right)^{-1} &= \beta\mathbb{E}_t\left[\left(\cancel{\bar{c}}e^{\hat{c}_{t+1}}\right)^{-1}\left(\bar{a}e^{\hat{a}_t}\right)\alpha\left(\bar{k}e^{\hat{k}_{t+1}}\right)^{\alpha-1}\right] \\
(1-\hat{c}_t) &= \underbrace{\beta\alpha\bar{k}^{\alpha-1}}_{=1}\mathbb{E}_t\left[\left(1-\hat{c}_{t+1}\right)\left(1+\hat{a}_{t+1}\right)\left(1+(\alpha-1)\hat{k}_{t+1}\right)\right] \\
(1-\hat{c}_t) &= \mathbb{E}_t\left[1-\hat{c}_{t+1}+\hat{a}_{t+1}+(\alpha-1)\hat{k}_{t+1}\right] \\
\hat{c}_t &= \mathbb{E}_t\left[\hat{c}_{t+1}+(1-\alpha)\hat{k}_{t+1}-z_{t+1}\right] \qquad (11)
\end{aligned}
$$

The aggregate production constraint (Equation 8) can be rewritten as follows:

$$\begin{aligned}
\bar{c}e^{\hat{c}_t} &= \bar{a}e^{\hat{a}_t}\left(\bar{k}e^{\hat{k}_t}\right)^{\alpha} - \mathbb{E}_t\left[\bar{k}e^{\hat{k}_{t+1}}\right] \\
\bar{c}(1+\hat{c}_t) &= \bar{a}(1+\hat{a}_t)\bar{k}^{\alpha}(1+\alpha\hat{k}_t) - \mathbb{E}_t\left[\bar{k}(1+\hat{k}_{t+1})\right] \\
\bar{c}(1+\hat{c}_t) &= \bar{a}(1+\hat{a}_t+\alpha\hat{k}_t)\bar{k}^{\alpha} - \bar{k}\mathbb{E}_t\left[(1+\hat{k}_{t+1})\right]
\end{aligned}$$

Deducting the steady-state (see equation 10), we obtain:

$$\begin{aligned}
\bar{c}\hat{c}_t &= \hat{a}_t\bar{k}^{\alpha} + \alpha\hat{k}_t\bar{k}^{\alpha} - \bar{k}\mathbb{E}_t\left[\hat{k}_{t+1}\right] \\
\bar{c}\hat{c}_t &= (z_t+\alpha\hat{k}_t)\bar{k}^{\alpha} - \bar{k}\mathbb{E}_t\left[\hat{k}_{t+1}\right] \qquad (12)
\end{aligned}$$

And the model is complete by recalling that the shock equation to technology is:

$$z_{t+1} = \rho z_t + \sigma\varepsilon_{t+1}, \quad |\rho|<1, \quad z_t \sim N\left(0, \frac{\sigma^2}{1-\rho^2}\right). \qquad (13)$$

□

In order to use the method of *undetermined coefficients* to find the coefficients for the policy function, we need the two equilibrium conditions that comes from (11)-(12):

$$\begin{aligned}
0 &= (z_t+\alpha\hat{k}_t)\bar{k}^{\alpha} - \bar{k}\mathbb{E}_t\left[\hat{k}_{t+1}\right] - \bar{c}\hat{c}_t \quad \text{and} \\
0 &= \mathbb{E}_t\left[\hat{c}_{t+1} + (1-\alpha)\,\hat{k}_{t+1} - z_{t+1}\right] - \hat{c}_t.
\end{aligned}$$

Following the lecture slides, we can write this in matrix form as follows:

$$\begin{bmatrix}0\\0\end{bmatrix} = \mathbb{E}\left(\begin{bmatrix}-\bar{c} & \alpha\bar{k}^{\alpha}\\ -1 & 0\end{bmatrix}\begin{bmatrix}\hat{c}_t\\ \hat{k}_t\end{bmatrix} + \begin{bmatrix}0 & -\bar{k}\\ 1 & (1-\alpha)\end{bmatrix}\begin{bmatrix}\hat{c}_{t+1}\\ \hat{k}_{t+1}\end{bmatrix} + \begin{bmatrix}\bar{k}^{\alpha}\\ 0\end{bmatrix}z_t + \begin{bmatrix}0\\ -1\end{bmatrix}z_{t+1}\right) \qquad (14)$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \mathbb{E}\left(\mathbf{A}\begin{bmatrix}\hat{c}_t\\ \hat{k}_t\end{bmatrix} + \mathbf{B}\begin{bmatrix}\hat{c}_{t+1}\\ \hat{k}_{t+1}\end{bmatrix} + \mathbf{C}z_t + \mathbf{D}z_{t+1}\right), \qquad (15)$$

and complete the system with equation 13.

The policy functions can be set up as:

$$\begin{bmatrix}\hat{c}_{t+1}\\ \hat{k}_{t+1}\end{bmatrix} = \mathbf{H}\begin{bmatrix}\hat{c}_t\\ \hat{k}_t\end{bmatrix} + \mathbf{G}\ z_t \qquad (16)$$

When substituting these back into 14, we obtain:

$$\begin{bmatrix}0\\0\end{bmatrix} = \mathbb{E}\left(\mathbf{A}\begin{bmatrix}\hat{c}_t\\ \hat{k}_t\end{bmatrix} + \mathbf{B}\left(\mathbf{H}\begin{bmatrix}\hat{c}_t\\ \hat{k}_t\end{bmatrix} + \mathbf{G}\ z_t\right) + \mathbf{C}z_t + \mathbf{D}z_{t+1}\right)$$

Using (13) and the specification that $\varepsilon_t \sim N(0,1)$, we can group similar terms to obtain:

$$\begin{bmatrix}0\\0\end{bmatrix} = \mathbb{E}\left((\mathbf{A}+\mathbf{BH})\begin{bmatrix}\hat{c}_t\\ \hat{k}_t\end{bmatrix} + (\mathbf{BG}+\mathbf{C}+\rho\mathbf{D})\,z_t\right)$$

This must be satisfied for any $\hat{c}_t, \hat{k}_t, z_t$, such that:

$$\begin{aligned}
0 &= \mathbf{A}+\mathbf{BH} \\
0 &= \mathbf{BG}+\mathbf{C}+\rho\mathbf{D}
\end{aligned}$$

Plugging in the values for the matrices:

$$0 = \begin{bmatrix} -\bar{c} & \alpha\bar{k}^{\alpha} \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\bar{k} \\ 1 & (1-\alpha) \end{bmatrix} \mathbf{H}$$

$$0 = \begin{bmatrix} 0 & -\bar{k} \\ 1 & (1-\alpha) \end{bmatrix} \mathbf{G} + \begin{bmatrix} \bar{k}^{\alpha} \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \bar{k}^{\alpha} \\ -\rho \end{bmatrix} + \begin{bmatrix} 0 & -\bar{k} \\ 1 & (1-\alpha) \end{bmatrix} \mathbf{G}$$

As these equations are of first order, they can be directly written as:

$$\mathbf{H} = \begin{bmatrix} 0 & -\bar{k} \\ 1 & (1-\alpha) \end{bmatrix}^{-1} \begin{bmatrix} \bar{c} & -\alpha\bar{k}^{\alpha} \\ 1 & 0 \end{bmatrix} \tag{17}$$

$$\mathbf{G} = \begin{bmatrix} 0 & -\bar{k} \\ 1 & (1-\alpha) \end{bmatrix}^{-1} \begin{bmatrix} -\bar{k}^{\alpha} \\ \rho \end{bmatrix} \tag{18}$$

These matrices G and H are the solutions to the policy functions 16.

As discussed in class, the solution will be stable if all eigenvalues of the matrix $\mathbf{H}$ lie in the unit circle. We can do a manual check as follows:

1. Compute $\mathbf{B}^{-1}$

$$\mathbf{B}^{-1} = \begin{bmatrix} 0 & -\bar{k} \\ 1 & (1-\alpha) \end{bmatrix}^{-1} = \frac{1}{\bar{k}} \begin{bmatrix} (1-\alpha) & \bar{k} \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1-\alpha}{\bar{k}} & 1 \\ -\frac{1}{\bar{k}} & 0 \end{bmatrix}.$$

2. Multiply by $-\mathbf{A}$

$$-\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} \frac{1-\alpha}{\bar{k}} & 1 \\ -\frac{1}{\bar{k}} & 0 \end{bmatrix} \begin{bmatrix} \bar{c} & -\alpha\bar{k}^{\alpha} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \bar{c}\frac{1-\alpha}{\bar{k}} + 1 & -\alpha\bar{k}^{(\alpha-1)}(1-\alpha) \\ -\frac{\bar{c}}{\bar{k}} & \alpha\bar{k}^{(\alpha-1)} \end{bmatrix}$$

3. From here we could compute the trace and determinant of $-\mathbf{B}^{-1}\mathbf{A}$ to find the roots of the characteristic equation. In the interest of time, we just plugged the values in Matlab:

```
cstar = 0.3880;
kstar = 0.1883;
alpha = 0.33;

a = c_ss*(1-alpha)/k_ss + 1;
b = -(1-alpha)*alpha*k_ss^(alpha-1);
c = -c_ss/k_ss;
d = alpha*k_ss^(alpha-1);

H = [a b ;c d];

[v,d] = eig(H)
```

Using the above procedure, we found out that the eigenvalues of H are 3.0609 and 0.3300. Since one Eigenvalue is outside the unit circle, we could expect an explosive behavior of the system (in fact this will be the case in our simulations in the last item).

To calculate $\mathbf{G}$, we can follow the same procedure:

1. Compute $\mathbf{B}^{-1}$ as before

2. Multiply by $\begin{bmatrix} -\bar{k}^{\alpha} \\ \rho \end{bmatrix}$

$$\mathbf{B}^{-1}\begin{bmatrix} -\bar{k}^{\alpha} \\ \rho \end{bmatrix} = \begin{bmatrix} \frac{1-\alpha}{\bar{k}} & 1 \\ -\frac{1}{\bar{k}} & 0 \end{bmatrix}\begin{bmatrix} -\bar{k}^{\alpha} \\ \rho \end{bmatrix} = \begin{bmatrix} -\bar{k}^{(\alpha-1)} + \rho \\ \bar{k}^{(\alpha-1)} \end{bmatrix}$$

Dynare

Solution. We have in our <Dynare mod-file> three different specifications:

1. The log linerized model, in which we specify the steady states as parameters and use Equations (11)-(13);

2. The non-linearized model, in which we specified the steady states as initial values and used Equations (7)-(8);

3. A version of model in item (2), but replacing all variables by their exponential values in the model. This would allows us to compare if our log-linearization version is correct.

□

Using the model described in (2), Dynares gives us the graphs of the shocked variables (when there is an increase of 1 in ε, which can be seen in Figure (1). We can see that in this model both c and k instantaneously increases (although consumption increases more than capital), and takes more than 40 periods to return to the steady state. The magnitude of the shock is not that high: consumption increases around 0.004, which compared to the steady state of 0.3880 doesn't seem to be much. A similar result applies to k.

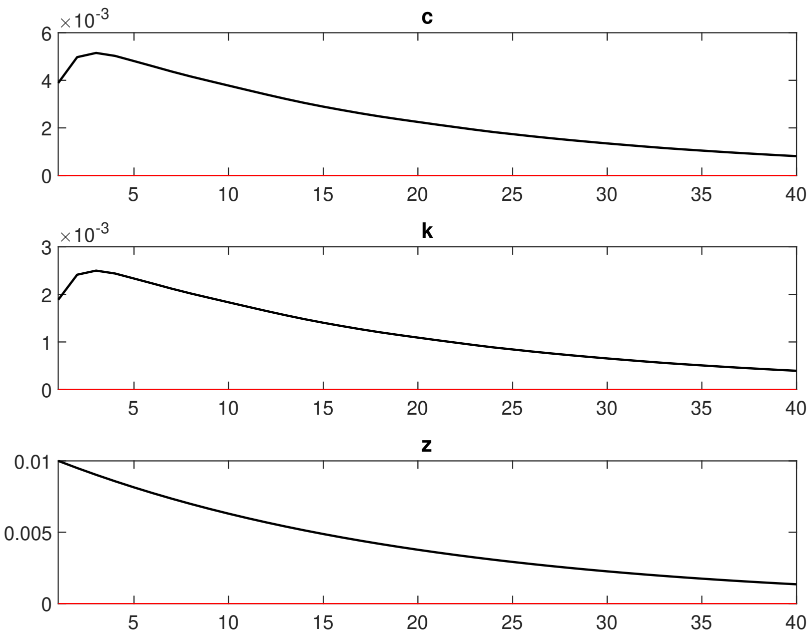

Figure 1 – Response from c, k and z to a shock of one unit in ε.

The resulting policy function[2] is in Table 1.

[2] Actually we got a bigger table from this, but we don't know how to interpret the other lines.

Table 1 – Policy and transition function produced by Dynare

	c	k	z
Constant	0.3881	0.1883	0
k(-1)	0.6801	0.3300	0
z(-1)	0.3687	0.1789	0.95

The first line of Table 1 gives us simply the steady states for c, k and z, which are compatible with our previous computations. As for the second line, it gives us the coefficient of k_t in the equations of c and k_{t+1} (we used k as predetermined variable in Dynare). The same for the third line, it represents the coefficient of $z_(t-1)$ in the equations for the other variable - note how the last cell of the table corresponds exactly to the value of ρ chosen. From this is possible to recover the coefficients to draw the trajectory for k, which is done in the next item.

Comparison of the methods

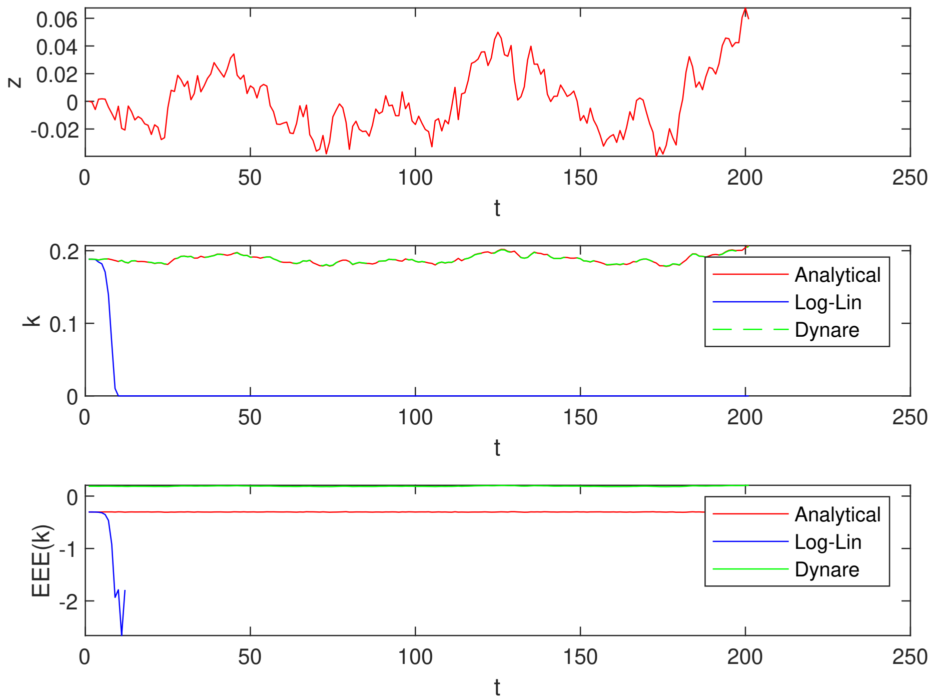

Figure 2 – z_t, trajectory of k using different methods and Euler errors, for a simulation with 200 periods and assuming $\varepsilon_t \sim \mathcal{N}(0,1)$.

In the middle graph of Figure 2 we observe that the analytical solution (red line) matches the trajectory for k produced using the quadratic approximation from Dynare (dashed green line, overlapping with the red one). The results obtained with Dynare and the analytical solution are matching our computed steady state of approximately 0.1883. The only difference between these two procedures can be observed in the bottom graph, where the Euler errors are actually different: the values obtained from the model using Dynare are positive while the ones from the analytical solution are negative, but both close to zero.

With respect to the log-linearization, as we expected, the solution is explosive (which is consistent with the eigenvalues from our H matrix). Therefore, the steady-state deviations for capital, namely $\hat{k}$ are exploding in the direction of $-\infty$, such that k reaches zero rather quickly, whereas consumption becomes infinite. Hence, after some periods, the Euler equation errors can't be computed anymore. We were unable to locate the error in our computations such that we could not fix it on time.