105 / 100

Problem set 5

Aishameriane Schmidt*, Antonia Kurz†.

1st December, 2020

Consider the Aiyagari model with aggregate risk discussed in class. We introduce one modification: unemployment insurance. This is modeled by an unemployment benefit for the unemployed agents which is financed by taxes imposed on the employed agents, i.e., the government solely implements this unemployment insurance scheme and always runs a balanced budget.

The exact details and parameters of this model are described in Wouter J. Den Haan, Kenneth L. Judd, and Michel Juillard. Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics and Control*, 34(1):1–3, 2010.

a. *Solve this model with the Krusell-Smith algorithm using the log-linear forecasting rule with the mean only and stochastic simulation. [required, 50 points]*

Solution. In general lines, each one of the agents in this economy has the values for interest and wage rate in its optimisation scheme, and those are functions of the aggregate capital stock. Because the latter is a function of the individual decisions, its distribution is a state *function*[1] in the system. The method from Krusell & Smith (1998) solves this problem by approximating the distribution via its moments, which then is a finite and discrete set. The overall steps for solving the model are represented in Figure 1. Our programming solution follows the same scheme from the lecture, in which we have two aggregate states of the economy (good and bad) and two idiosyncratic shocks (making agents unemployed vs. employed).

As indicated in Figure 1, the procedure (after an initial guess) starts with the household optimization problem, which can be solved with the methods that we have already been using (VFI or PFI). We used a variation of the grid-based Euler-equation algorithm with a grid improvement proposed by Maliar, Maliar & Valli (2010) to take into account the lack of uniform concentration of the capital along the distribution (better explained later in this item). Then we simulate the economy, i.e., by using the individual policy rules, we can obtain the overall distribution of capital, taking into account the aggregate and idiosyncratic shocks. This step, which has a (*) in Figure 1, can be done either using stochastic or deterministic methods. In question (a), we show results for the first one, whereas in item (b), we describe a way to use the non-stochastic simulation. Finally, by using OLS, it is possible to update the initial guess for the capital distribution and start the routines again until reaching convergence.

When simulating the shocks, we use the proposed transition probabilities of the states (aggregate and idiosyncratic) by Haan, Judd & Juillard (2010) to have a) the transition probabilities of the aggregate

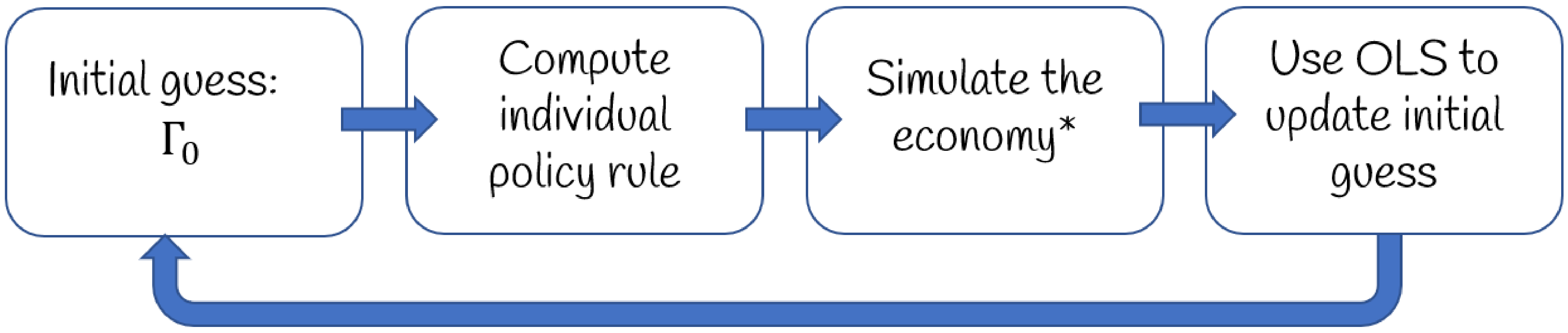
*Contact: aishameriane.schmidt@tinbergen.nl

†Contact: a.kurz@tinbergen.nl

Homework for the Advanced Topics in Macro I Course, Block 2. **Instructors:** dr. M. Z. Pedroni (UvA) and dr. E. Proehl (UvA).

[1] This is due to the fact that we have infinitely many households, which makes impossible to use the density in the individual policy rules.

Figure 1 – Main steps in solving the Ayagari Model with Aggregate shocks using the Krusell-Smith algorithm



Notes: The Krusell-Smith algorithm is based on recursive equilibria where from an initial guess for the law of motion of capital is used to solve the households' optimization problem which is required to obtain the aggregate distribution. In particular, KS showed that this cross-sectional distribution can be approximated by using a finite set of moments. From this simulation step, the aggregate distribution that will serve as new initial guess is obtained by a simple OLS estimation procedure. In question (a), we simulate the economy from step 3 using a stochastic method while in question (b) we explain a non-stochastic algorithm.

Source – Own elaboration based on Algan et al. (2014).

states, and with those b) the transition probabilities of the idiosyncratic states conditional on the aggregate state. Hence, we can simulate the states by drawing random numbers combined with the calculated probabilities. In particular, we start from the bad aggregate state and a random idiosyncratic state for every individual, but due to discarding the burn-in periods, this starting value do not influence the simulated periods significantly our presented results.

Before we can solve the model, we need to set up a grid for all possible asset levels (capital levels) for the households (firms). However, in this heterogeneous agent model, the behaviour of capital holdings in the tails (in particular the lower tail) of its distribution is not the same as it is in the rest of the points. In general lines, Maliar, Maliar & Valli (2010) explains that the low levels of capital are underrepresented. To correct for this, they propose using a simple polynomial rule for setting up the grid:

$$g_j = \left(\frac{j}{J}\right)^{\theta} a_{max}$$

where g_j is the j^{th} gridpoint for capital/assets, $J+1$ the number of points in the grid and θ the parameter shifting more mass of gridpoints to lower values of assets when increasing (we use $\theta = 7$ as they in their calibration).

We set our initial level of capital to 90% of every value on this grid to start the model solving algorithm. For the stochastic simulation, we set the initial level of capital for all agents equal to the steady state k^{SS}, namely $k^{SS} = \left(\frac{\frac{1}{\beta}-(1-\delta)}{\alpha}\right)^{\frac{1}{\alpha-1}}$, obtained by solving the problems of households and firms as below.

The (individual) infinite horizon household problem (for household i) that we need to solve is given by:

$$\begin{aligned} \max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}} \; & \mathrm{E} \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^i\right)^{1-\gamma} - 1}{1-\gamma} \\ \text{s.t. } & c_t^i + a_{t+1}^i = r_t a_t^i + \left((1-\tau_t)\,\bar{l}\varepsilon_t^i + \mu\left(1-\varepsilon_t^i\right)\right) w_t + (1-\delta) a_t^i \\ & a_{t+1}^i \geq 0 \end{aligned}$$

in which $\bar{l}$ is time endowment, $\varepsilon = 1$ when employed and 0 when unemployed and μ is the unemployment benefit as a share of wage, financed by tax τ on labour income (it is an assumption of this setup that the government has a balanced budget at each point of time).

In equilibrium, the aggregate level of households' assets, A, must equal the capital used on the firm side, K. The firms' demand for capital comes from the profit maximisation problem with a Cobb-Douglas production function, such that their optimality conditions are:

$$w_t = (1-\alpha) z_t \left(\frac{K_t}{\bar{l} L_t}\right)^{\alpha} \tag{1}$$

$$r_t = \alpha z_t \left(\frac{K_t}{\bar{l} L_t}\right)^{\alpha - 1}, \tag{2}$$

where the variable z_t is the aggregate productivity (good or bad state).

To forecast the capital distribution using the Krusell-Smith (KS) algorithm, we rely on the first moment of the distribution for assets, so it is also necessary in the initialisation to set up a grid for the mean asset level of households. What we can obtain is a four-dimensional grid for assets (where they vary across time), namely for: individual level of capital, mean level of capital in the economy, the aggregate shock and the idiosyncratic shock.

Krusell & Smith proposes in their algorithm to run the regression from equation (3), also known as *log-linear forecasting rule*:

$$\log \bar{a}' = B_z^0 + B_z^1 \log \bar{a} \tag{3}$$

where the parameters B are depending on the aggregate productivity state z. We update them according to $B' = (1-\rho) B_{old} + \rho B_{new}$ with $\rho = 0.3$. For every regression, we also calculate the associated R^2. For every iteration (until the parameters converge to a steady level), a new distribution of capital, using the new estimated parameters, replaces the initial one - and hence, being inserted in the algorithm all over again until reaching the tolerance limits[2].

Before estimating the regression, we need to make a simulation of the asset levels (individual and aggregate/mean). In item (a), we implement a stochastic version and in (b), we implement a non-stochastic procedure. This is done right after solving the individual (household) optimisation to obtain the households' choices for c and k' (so basically, we use an algorithm giving us the households' policy function).

Note that the households' policy functions for consumption and next period capital are determined every iteration (since the household uses wage and interest rates, which depends on distribution of capital). We set the mean of capital for the next period, $\bar{a}'$, according to our parameters from 3 (in the first period, we use $B^0 = 0$ and $B^1 = 1$ for both states). The household then takes expectations of future prices, such that wages and interest rate of the next period are determined with equations 1 and 2, $\bar{a}'$, and all other variables of the previous iterations. The policy function iteration (PFI) algorithm uses prices and levels to find the most appropriate policy function for $a''(a', \bar{a}')$ by cubic interpolation of the previously found $a' = a(a, \bar{a})$ in the new calculated levels a' and $\bar{a}'$. Using policy function $a''(a', \bar{a}')$ and the budget constraint, the consumption level is calculated - if it occurs to be negative, it is replaced with the value 10^{-10}. These two policy functions are calculated for each of the combinations of the aggregate and idiosyncratic states. The goal of the algorithm is to find the overall policy functions such that c results from the sum of Euler functions for each state combination weighted with its probability. From c, we can infer a'. The algorithm stops when the differences across the iterations are small enough.

With the households' choices for c and a', we start into the stochastic simulation for T periods to obtain the new distribution of assets (and the respective cross-sectional mean): first, we get the mean of the current cross-sectional asset levels, but restrict the values to the bounded grid we set before. We can then obtain a policy function $a'(a, \bar{a}, z, \varepsilon)$ by interpolating in a, all possible idiosyncratic shocks, and the known aggregate shocks z from the simulation and cross-sectional means $\bar{a}$. With the known values, this reduces to $a'(a, \varepsilon)$ such that we can interpolate it again in a, but now also in the simulated idiosyncratic shocks ε. These possible values for a' need to be restricted to be within the set bounds such that we can use them in the next period's iteration as a.

By running this simulation, we have obtained the distribution and mean for asset levels over all periods. This is presented in item (b) (see Figure 2), together with the same results obtained using the non-stochastic method. □

[2] We used a smaller value for the difference between the estimated coefficients at each step because of computational time. Instead using a large value of 10^{10} like Maliar, Maliar & Valli, we used 2^5.

b. *Modify your algorithm to use non-stochastic simulation and compare the precision in terms of Euler equation errors. [required, 35 points]*

Solution. When using Krusell & Smith (1998), one of the steps involves finding the aggregate policy function as described in Figure 1. For item (b), we will explain the method used in Maliar, Maliar & Valli (2010). This is different from the histogram based method from Young (2010) that was seen in the lecture.

The stochastic simulation implemented in part (a) has the drawback of introducing sampling noise, a consequence of the fact that the policy function depends on the current (random) draw. Moreover, there is undersampling of values in the tails since it is more likely to get draws of values that are close to the mean. This justifies the use of a non-stochastic procedure of the KS algorithm. In the lecture, we discussed the method proposed by Young (2010) and in here, we are using the routines from Maliar, Maliar & Valli (2010) that are based on a simulation on a grid of pre-determined points[3], which is described below (following the description of Algan et al. (2014)).

As said before, the procedure needs a pre-determined (fine) grid for the capital stock, which will be used to approximate the distribution with a linear spline. If each point of this grid corresponds to the optimal choice for a', then one can obtain the initial capital value, a by inverting the policy function. Hence, it is possible to compute the probability that the initial capital value is less or equal to a. The latter is used to compute the distribution of a' (but this only works if the policy function presents monotonicity).

More specifically, to characterise the initial capital distribution (or capital holdings distribution, to use the same expression as Algan et al. (2014)), one needs the percentage of agents that are employed/unemployed and have zero capital holdings and the ones who have positive capital holdings. These same quantities are what we need to obtain for the next period, which will take into account the individual optimisation decisions and the aggregate shock realisation. Equations (41)-(43) from Algan et al. (2014) show how the cumulative distribution of the end of the period can be computed by inverting the policy function. Finally, given the possible shocks, the next begin of the period distribution can be obtained by computing a weighted average of the distributions for the unemployed and employed, where the weights are a combination of the current and future employment status and the aggregate state of the economy.

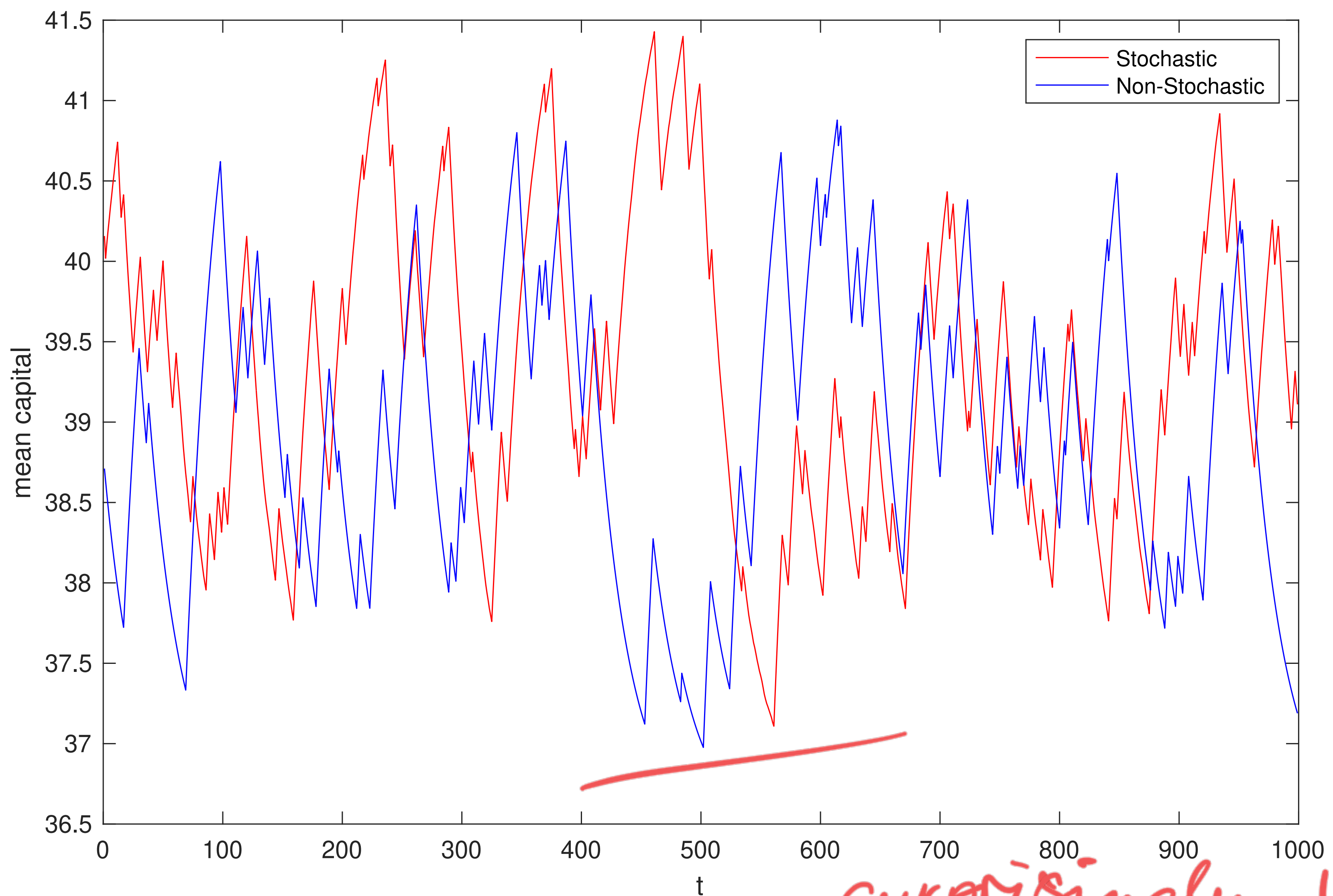
Figure 2 has the time series of the aggregate capital distribution (first moment) considering both methods (stochastic - red line) and non-stochastic (blue line). It is hard to draw conclusions from the graph, since both distributions alternate in being higher or lower than the other. In general, they follow a similar pattern in terms of increasing and decreasing movements, except for the periods t between 450 and 650: in this region the non-stochastic series is in its lower values while the stochastic procedure returns the highest values for the average capital. Nevertheless, looking at the scale, we see that these differences do not exceed 2 or 3 in the y axis.

The Euler Equation errors for both methods are represented in Figure (3). We see that in the beginning, the series seem to be close to each other, but the distance is widened for points $t > 300$. Although the series change positions, the figure indicates that the non-stochastic method is, in general, associated with a higher Euler Equation error value.

□

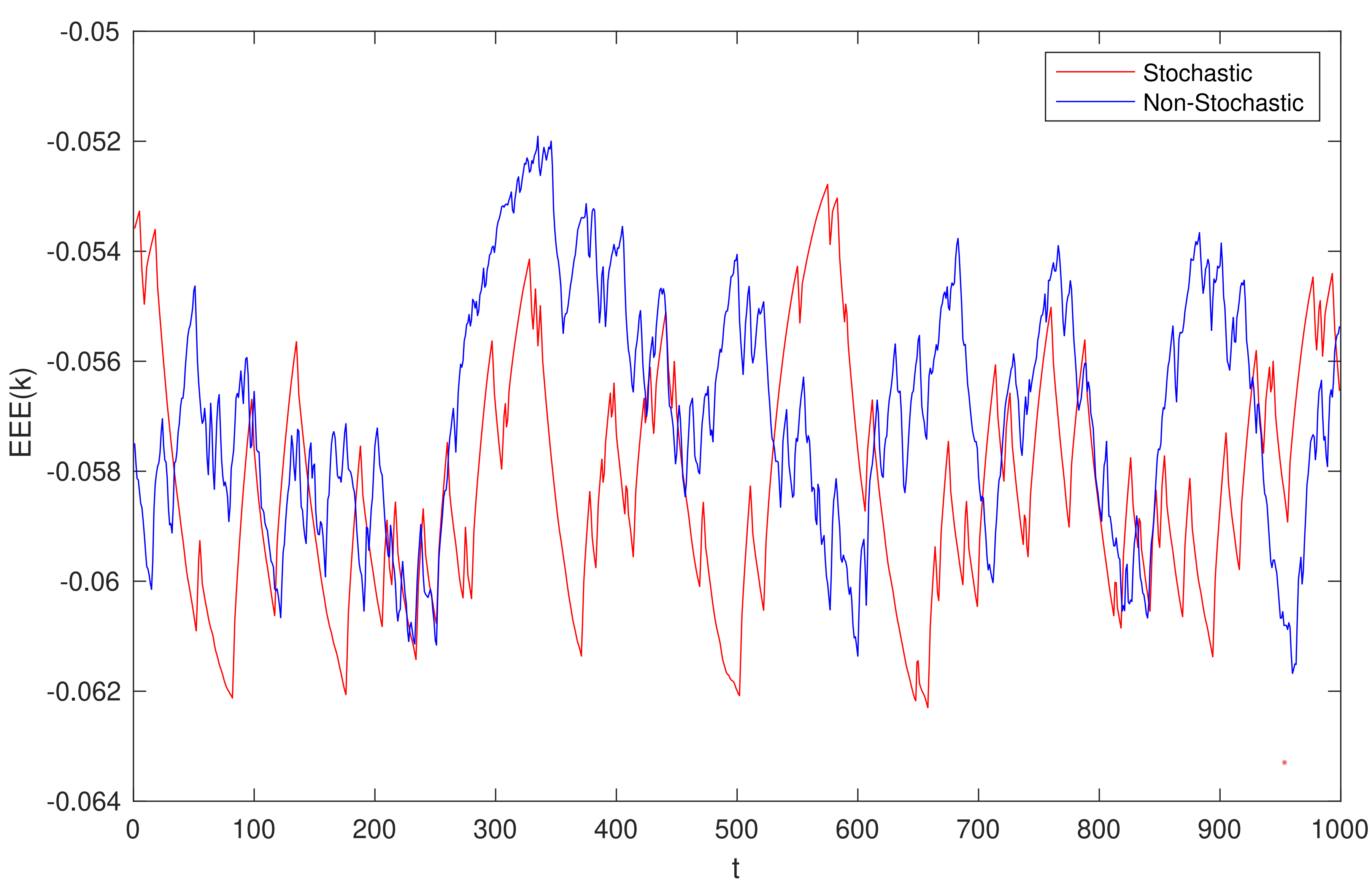
50/50

[3] The biggest difference between both methods is that Young (2010) assumes a continuum of households, which would be better to deal with the probability in the tails.



surprisingly large diff.

Figure 2 – Mean (aggregate) capital values over time with both simulation methods



35/35

Figure 3 – Euler Equation Errors for both simulation methods over time

How, do you average?

c. *Compute and plot the expected ergodic savings distribution.[required, 15 points]*

Solution. The savings distribution is obtained through the accounting relation that investment is equal to savings and investment in our model is just the capital holdings for next period. When using the stochastic simulation method, it is possible to obtain the distribution for both groups of agents (unemployed and employed), which is represented in Figure . We can see that the distributions present more noise in the tails, which was expected as discussed in the previous items. And coherent with what is observed in the real data, the savings distribution of the employed agents is shifted to the right in comparison to the distribution of savings from the unemployed - meaning that employed people have higher capital holdings.

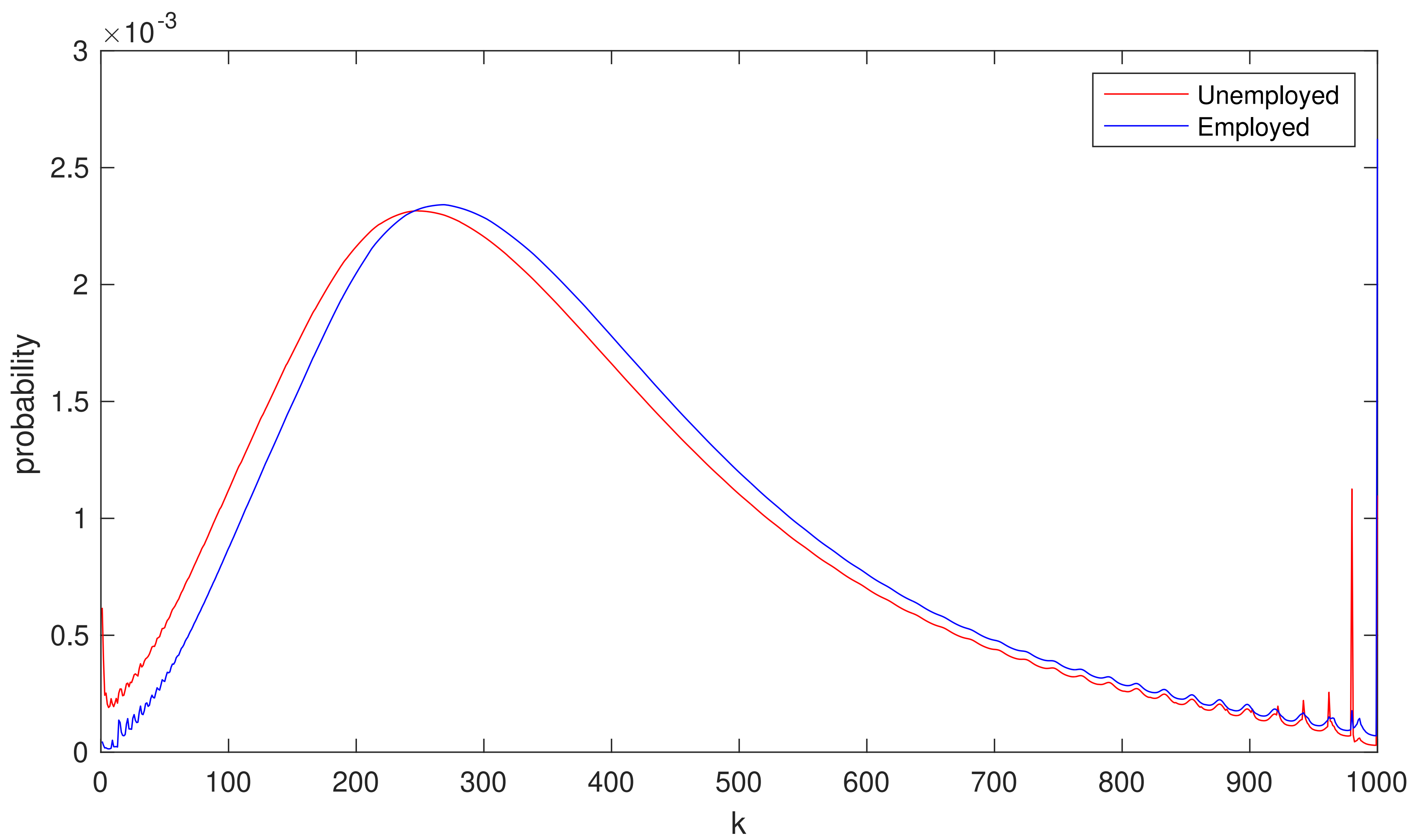
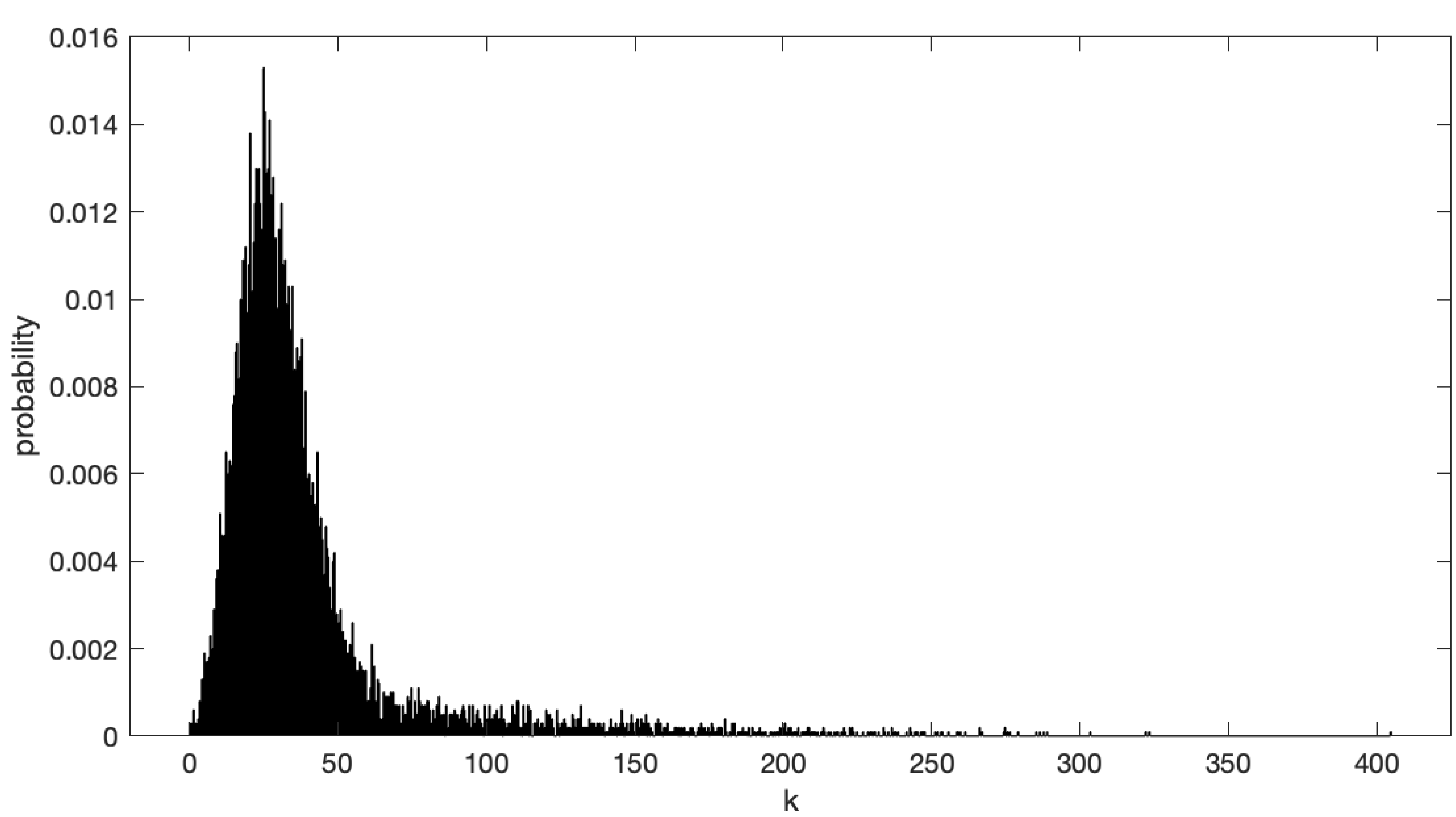


Figure 4 – Distribution of Savings for both employment status, using non-stochastic method

□

Figure (5) has the histogram of the savings distribution for the simulated 10.000 individuals using the non-stochastic simulation method. We can observe that this distribution is skewed towards the right tail, consistent with having few individuals who hold large quantities of assets.



10 / 15

Figure 5 – Histogram of Savings given 10.000 individuals sampled (stochastic method)

d. *Set the unemployment benefit rate to 0.65 and recompute the model solution. Choose the simulation method with higher precision in (b). Plot the expected ergodic distribution and compare to the base case. [optional, 10 bonus points]*

Solution. For this item, we increased the number of grid points from 1.000 to 5.000. After a *while*, we got the results that are represented in Figures 6 and 7. As before, we have the stochastic simulation series (same as in the previous questions), but now the non-stochastic one with higher precision and a higher unemployment benefit. The mean capital of the non-stochastic one behaves differently now - the mean of the mean seems to be still around 39 despite the higher unemployment benefit.

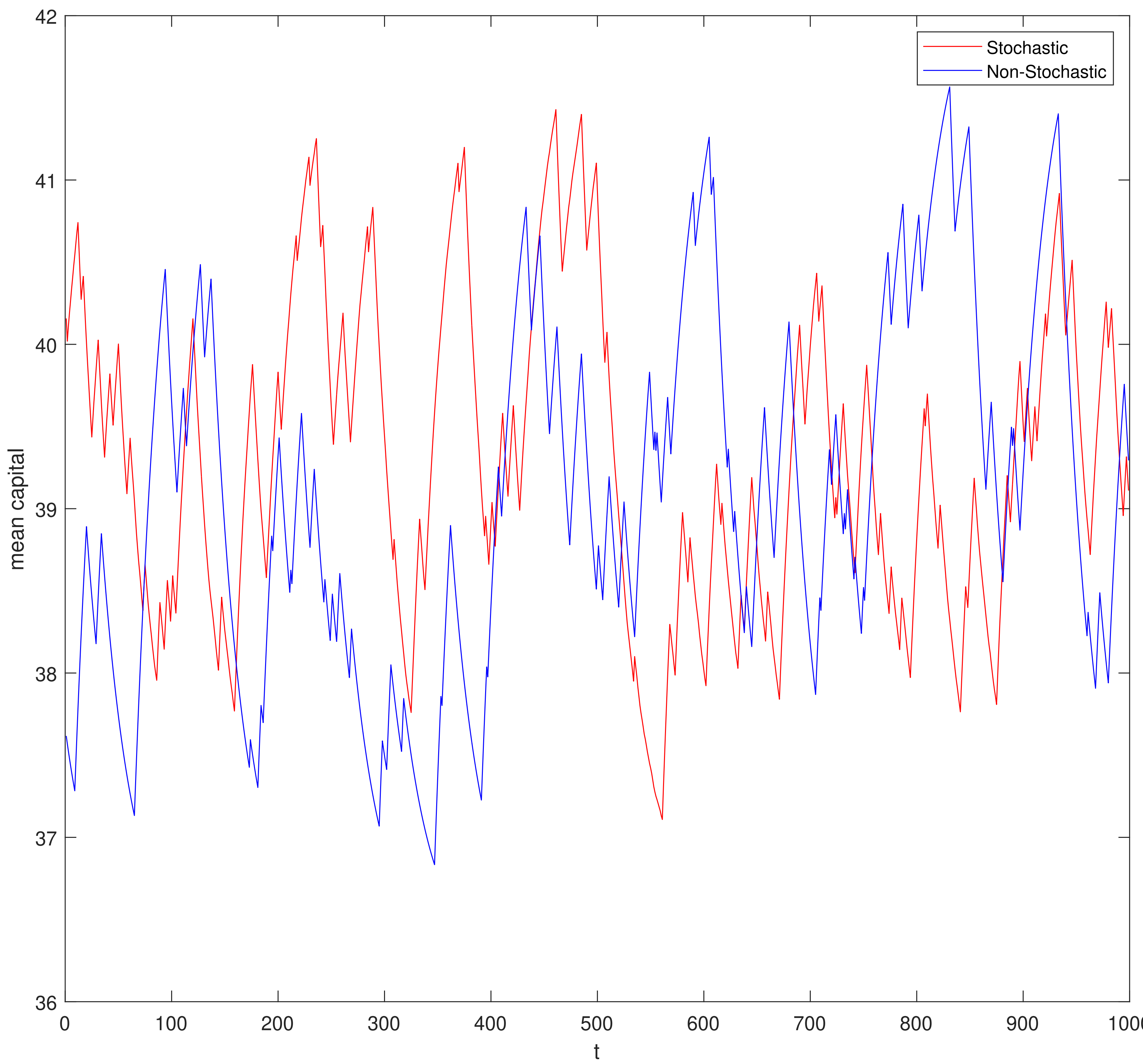


Figure 6 – Mean (aggregate) capital values over time with both simulation methods using a finer grid for the non-stochastic method and $\mu = 0.65$

The Euler Equation errors for both methods are represented in Figure (3). We see that again the series seem to be close to each other in the beginning, but the distance is widened for points $t > 300$. The non stochastic method still seems to have higher values for the EEE than the stochastic counterpart. This seems unexpected given the higher definition of the grid.

Finally, we can compare the ergodic savings distribution for both cases. We have the previous plot, from item (b) and the new plot displayed in Figure 8. It looks like the distribution became more spread and there is a huge amount of noise in the right tail. Given the unemployment benefits changed and its resulting increase in taxes, labour supply by households will go down. This induces firms to substitute

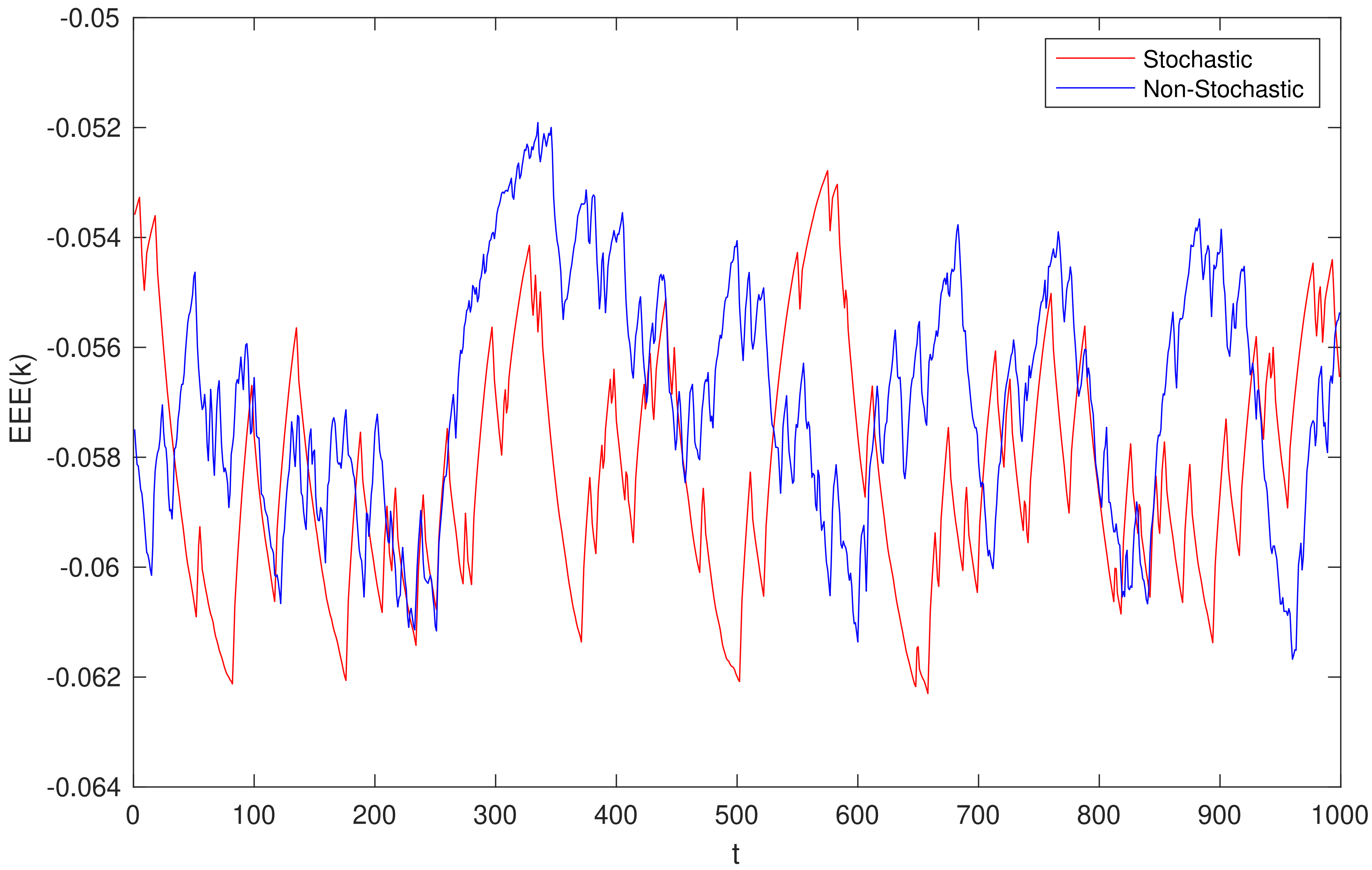


Figure 7 – Euler Equation Errors for both simulation methods over time

the now more expensive labour with capital such that their capital demand goes up and households are more encouraged to save. The shift tho higher levels of savings in combination with a finer grid, it seems reasonable that the most affected households are the ones in the top income savings level - there is more noise and more mass of probability.

□

Bibliography

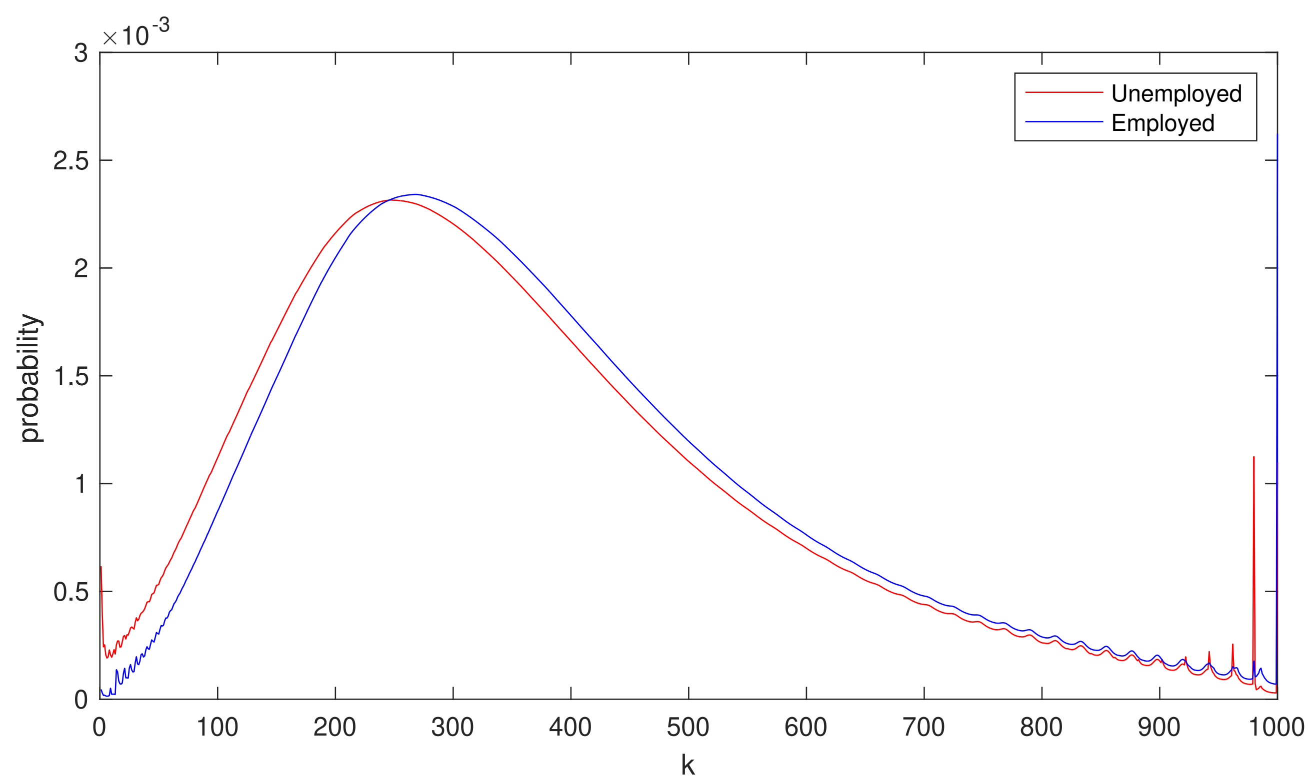
ALGAN, Y. et al. Solving and simulating models with heterogeneous agents and aggregate uncertainty. In: *Handbook of computational economics*. [S.l.]: Elsevier, 2014. v. 3, p. 277–324. Cited 2 times on pages 2 and 4.

HAAN, W. J. D.; JUDD, K. L.; JUILLARD, M. Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics and Control*, Elsevier, v. 34, n. 1, p. 1–3, 2010. Cited on page 1.

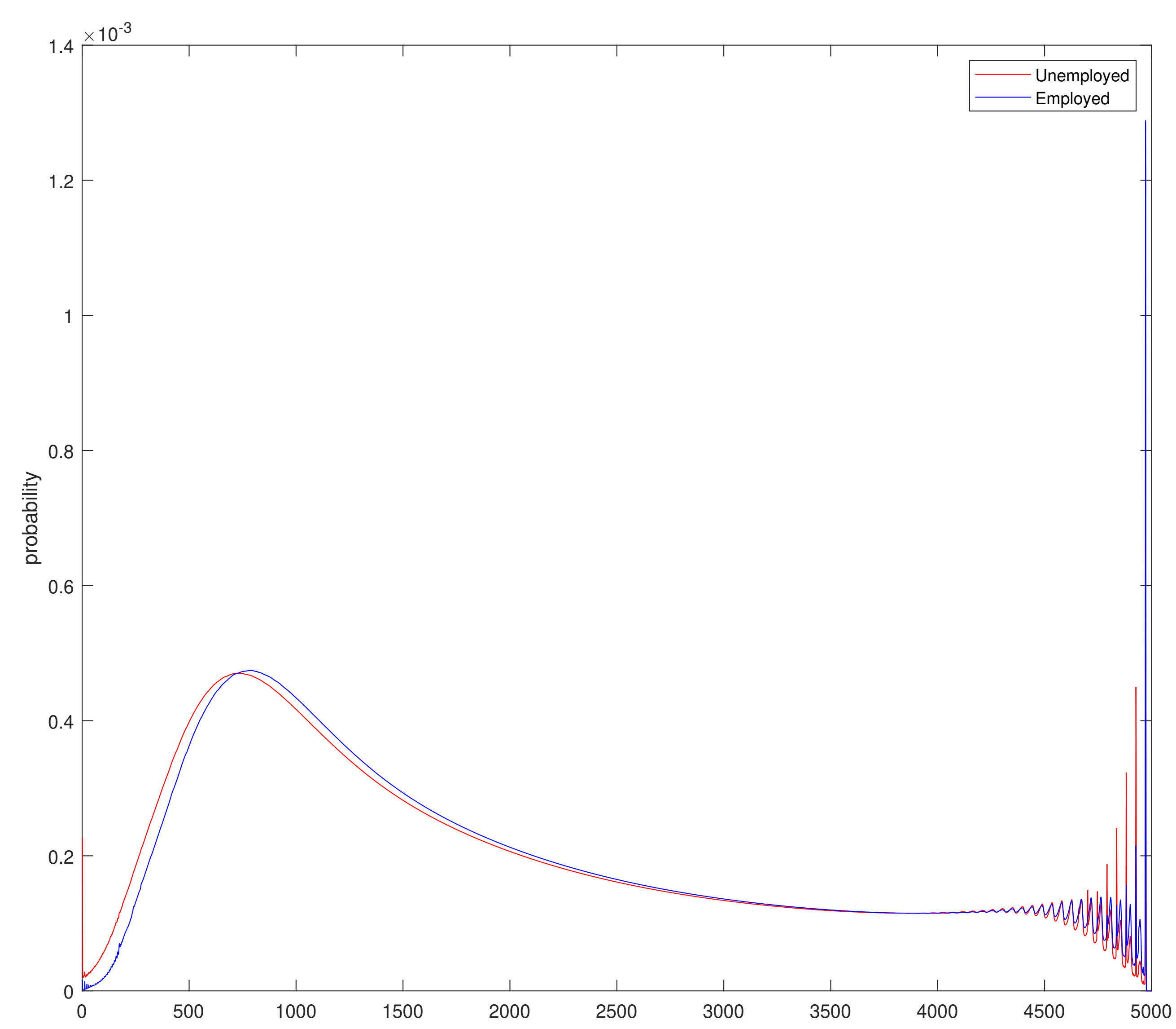
KRUSELL, P.; SMITH, A. A. Income and wealth heterogeneity in the macroeconomy. *Journal of political Economy*, The University of Chicago Press, v. 106, n. 5, p. 867–896, 1998. Cited 3 times on pages 1, 3, and 4.

MALIAR, L.; MALIAR, S.; VALLI, F. Solving the incomplete markets model with aggregate uncertainty using the krusell–smith algorithm. *Journal of Economic Dynamics and Control*, Elsevier, v. 34, n. 1, p. 42–49, 2010. Cited 4 times on pages 1, 2, 3, and 4.

YOUNG, E. R. Solving the incomplete markets model with aggregate uncertainty using the krusell–smith algorithm and non-stochastic simulations. *Journal of Economic Dynamics and Control*, Elsevier, v. 34, n. 1, p. 36–41, 2010. Cited on page 4.



Grid with 1.000 poins and $\mu = 0.15$



10/10

Grid with 5.000 poins and $\mu = 0.65$.

Figure 8 – Histogram of savings for each group using stochastic simulation.