Problem set 2

Aishameriane Schmidt*, Antonia Kurz†.

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Question 1

Consider the AR(1) process

$$z_{t+1} = \rho z_t + \left(1 - \rho^2\right) \epsilon_{t+1} \tag{1}$$

where ϵ_{t+1} is a standard normal.

(a)

Compute (analytically) the stationary distribution of this Markov process for $|\rho| < 1$.

Solution. We can rewrite (1) using a lag operator[1] as follows:

$$\begin{aligned} z_{t+1} &= \rho z_t + \left(1 - \rho^2\right) \epsilon_{t+1} \\ z_{t+1} - \rho z_t &= \left(1 - \rho^2\right) \epsilon_{t+1} \\ z_{t+1}(1 - \rho L) &= \left(1 - \rho^2\right) \epsilon_{t+1}. \end{aligned}$$

Since $|\rho| < 1$, we can invert the process to find:

$$z_{t+1} = (1 - \rho L)^{-1} \left(1 - \rho^2\right) \epsilon_{t+1} \tag{2}$$

$$= \left(1 - \rho^2\right) \sum_{i=1}^{\infty} \rho^i \epsilon_{t+1-i}. \tag{3}$$

From (2) it is clear that z_{t+1} will inherit the distribution from the error term, since everything else in the equation is deterministic. More specifically, in this model we know that $\epsilon_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1) \forall\ i$ and because linear combinations of a normal random variable is also normal, we know that $z_t \sim \mathcal{N}(\mu, \sigma^2)$. To find the parameters, we use to apply the unconditional expectation and variance and recall that the errors are iid:

$$\mathbb{E}[z_{t+1}] = \mathbb{E}\left[\left(1 - \rho^2\right) \sum_{i=1}^{\infty} \rho^i \epsilon_{t+1-i}\right] = \left(1 - \rho^2\right) \sum_{i=1}^{\infty} \rho^i \mathbb{E}[\varepsilon_{t+1-i}] = \left(1 - \rho^2\right) \sum_{i=1}^{\infty} 0 = 0, \tag{4}$$

*Contact: aishameriane.schmidt@tinbergen.nl

†Contact: a.kurz@tinbergen.nl

Homework for the Advanced Topics in Macro I Course, Block 2. **Instructors:** dr. M. Z. Pedroni (UvA) and dr. E. Proehl (UvA).

[1] A lag operator L is such that $\rho L X_t = \rho X_{t-1}$.

and

$$\begin{aligned}
\mathbb{V}\text{ar}[z_{t+1}] &= \mathbb{V}\text{ar}\left[\left(1-\rho^2\right)\sum_{i=1}^{\infty}\rho^i\epsilon_{t+1-i}\right] \\
&= \left(1-\rho^2\right)^2\sum_{i=1}^{\infty}\rho^{2i}\underbrace{\mathbb{V}\text{ar}\left[\epsilon_{t+1-i}\right]}_{1} \\
&= \left(1-\rho^2\right)^2\sum_{i=1}^{\infty}\rho^{2i} \\
&= \frac{\left(1-\rho^2\right)^2}{1-\rho^2} \\
&= 1-\rho^2. \qquad (5)
\end{aligned}$$

This means that our stationary distribution is $\mathcal{N}(0, 1-\rho^2)$. □

(b)

Assume $\rho = 0.7$. Implement Tauchen's method with equidistant grid points with $m = 3$, $N = 5$ and importance sampling with $N = 5$. Simulate the process for $T = 2000$ periods and throw away the first 500 periods to limit the impact of your initial value. Compare the mean, standard deviation and autocorrelation resulting from the simulated data with the true values of the stationary distribution for both the equidistant grid and importance sampling.

(c)

Assume $\rho = 0.7$. Implement Rouwenhorst's method with $N = 5$. Simulate the process for $T = 2000$ periods and throw away the first 500 periods to limit the impact of your initial value. Compare the mean, standard deviation and autocorrelation resulting from the simulated data with your results from (c). ***Hint:*** *Make this code generic and reuse it in Problem 2.*

(d)

Repeat (b) and (c) for $\rho = 0.99$. Interpret your results.

Solution. **Obs:** *we are answering items (b-d) from Question 1 together. Please note that we also implemented importance sampling for Tauchen and are reporting the results in here (which accounts for the optional question 3).*

As seen in class, when dealing with a stochastic model where the Markov process associated is continuous requires a discretization of the process. The methods studied return a finite grid of possible points and a probability transition matrix.

The first method is from Tauchen in 1986 and the approximation is based on a grid with equally spaced values (or bins) for the process. The procedure to find the probability transition matrix and the grid is described in Algorithm 1. This method will give us n points over a grid that will have its range depending on the unconditional variance of the process we are approximating. From there, we use the underlying distribution (in our case, a normal CDF) to find the probabilities to put in the transition matrix.

A modification of Algorithm 1 can be done by assembling a grid in which each interval has the same probability (therefore not the same distance). This variation is called *Tauchen with importance sampling* and in practice means that states that are more likely to occur in the original process will also have a higher chance to be drawn from the approximation.

Lastly, we have Rouwenhorst method, which can model more precisely process, in particular when the autocorrelation is high. The algorithm is described in Algorithm 3.

Finally, after having the transitions matrices for each point, we still need to simulate a series. This is done, in our code, using the function 'SimulTS', which is detailed in Algorithm 4. Note that this algorithm

Algorithm 1: Tauchen's method with equidistant grid

Input: ρ_z, μ_ϵ, σ_ϵ, m, N.
Output: $\{z_i\}_i^n$, P
begin

1. Find the unconditional mean and variance of the process using (4) and (5);
2. Build a grid with n points;
   - 2.1 The last gridpoint is equal to $m \cdot \mathbb{V}\text{ar}[z_{t_t}] \equiv \sigma_z$ - call this point z_N;
   - 2.2 For a symmetric distribution we use the first gridpoint equal to minus the last one: $z_1 \equiv -z_N$;
   - 2.3 The space d between the grid points is computed using $d = \frac{z_n - z_1}{N-1}$ and the points will be of the form $z_{i+1} = z_i + d$, $i \in \{1, \ldots, n-1\}$.
3. Find the transition matrix $P_{n \times n}$:

for $i \in \{1, \ldots, N\}$ **do**
  for $j \in \{1, \ldots, N\}$ **do**
    if $j = 1$ **then**
      $P[i,j] = \Phi\left(\frac{z_1 + d/2 - \rho z_i}{\sigma_\epsilon}\right)$;
    else if $j = N$ **then**
      $P[i,j] = 1 - \Phi\left(\frac{z_N - d/2 - \rho z_i}{\sigma_\epsilon}\right)$;
    else
      $j \in \{2, \ldots N-1\}$
      $P[i,j] = \Phi\left(\frac{z_j + d/2 - \rho z_i}{\sigma_\epsilon}\right) - \Phi\left(\frac{z_j - d/2 - \rho z_i}{\sigma_\epsilon}\right)$;
  end
end

4. Check whether the rows of the matrix P sum up to one;

end

Algorithm 2: Tauchen's method with importance sampling

Input: ρ_z, μ_ϵ, σ_ϵ, m, N, t, t_{burn}.
Output: $\{z_i\}_i^n$, P
begin

1. Find the unconditional mean and variance of the process using (4) and (5);
2. Find $N-1$ bin points, B_i, and N grid corresponding grid points, G_i:

for $i \in \{1, \ldots, N-1\}$ **do**
  $B_i = \Phi^{-1}\left(i/n\right) \cdot \sigma_z$;
  $G_i = \Phi^{-1}\left((i-0.5)/n\right) \cdot \sigma_z$;
end
$G_N = \Phi^{-1}\left((n-0.5)/n\right) \cdot \sigma_z$;

3. Find the transition matrix $P_{n \times n}$ in the same way described in Algorithm 1;
4. Check whether the rows of the matrix P sum up to one;

end

Algorithm 3: Rouwenhorst's method

Input: ρ_z, σ_ϵ, N.
Output: Pi (stationary dist.), P_N (transition matrix), s (prob of success)
begin

1. Compute the degrees of freedom p, q, ψ:
   - $p = q = 0.5(1 + \rho_z)$
   - $\psi = \sigma_z\sqrt{N-1}$, where σ_z is computed using Equation (5);

2. Compute P_N recursively as follows:

$$P_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

for $i \in \{3, \ldots, N\}$ **do**

$$\begin{aligned} P_N = & p \begin{bmatrix} P_{N-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{pmatrix} 1 & -p \end{pmatrix} \begin{bmatrix} 0 & P_{N-1} \\ 0 & 0 \end{bmatrix} \\ & + \begin{pmatrix} 1-q \end{pmatrix} \begin{bmatrix} 0 & 0 \\ P_{N-1} & 0 \end{bmatrix} + q \begin{bmatrix} 0 & 0 \\ 0 & P_{N-1} \end{bmatrix} \end{aligned}$$

end

3. Compute $s = \frac{1-p}{2-p-q}$
4. Find Pi as follows:

for $i \in \{1, \ldots, N\}$ **do**

$$Pi_i = \begin{pmatrix} N-1 \\ i-1 \end{pmatrix} s^{i-1}(1-s)^{N-1}$$

end

5. Build an equidistant grid with N where the end point is equal to $z_N = \psi$ and the first point is $z_1 = -z_N$.

end

produces the positions in the grid for our series, so we still need to apply this later on in the corresponding grid to get the actual time series values.

Using the specification given in the problem together with Algorithm 4 we simulated for each method a series with size 1500. Just as comparison, we also generated values using the true DGP to see how the actual values would behave. The results are in Figure (1). We can see that for a small N we get only a few possible values for the process, which translates in the left graphs to have a 'polygonal' shape. For a small value of ρ we can observe changes in the states from one to the other, however when the persistence increases (right panel) we see that Tauchen's method performs very poorly - in fact the method with equidistant grid doesn't change state at all. As expected, Rouwenhorst method is able to capture the dependence of the data better.

When comparing the moments of the simulated processes (Table 1) we also see that Rouwenhorst outperforms both versions of Tauchen's method, and is able to better approximate both mean and standard deviation. However for a higher value of ρ, the distance of the simulated series' moments and the actual values is higher.

Figures (2) and (3) have the autocorrelation plots considering $\rho = 0.7$ and $\rho = 0.9$, respectively.

Algorithm 4: Simulating a time series from a transition probability matrix and an initial value

Input: z_0 (initial value), P (transition matrix), t (number of periods), t_{burn} (burn-in).
Output: $\{z_i\}_i^{t-t_{\text{burn}}}$
begin

1. Compute the cumulative distribution of the rows of P and store in a new matrix;
2. Get $t-1$ random values from a Uniform$(0,1)$ and store in the vector u;
3. Create a vector z with dimension $t \times 1$ whose first element is z_0

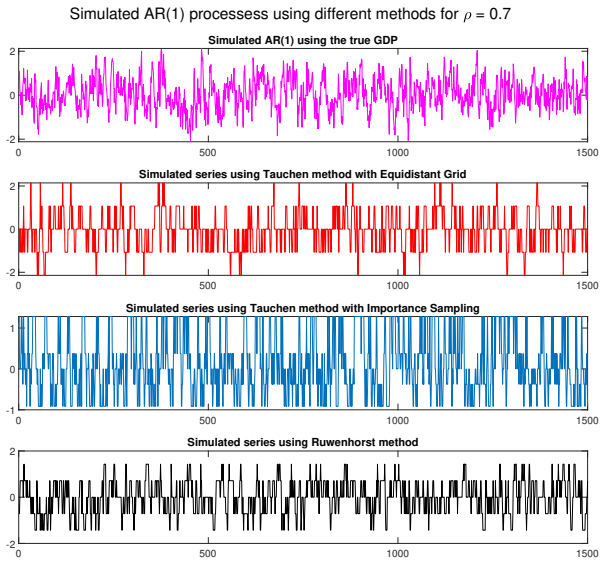
for $i \in \{2, \ldots, t\}$ **do**

- Compare u_{i-1} with the cumulative distribution of the point z_{i-1}, store the corresponding position in z_i

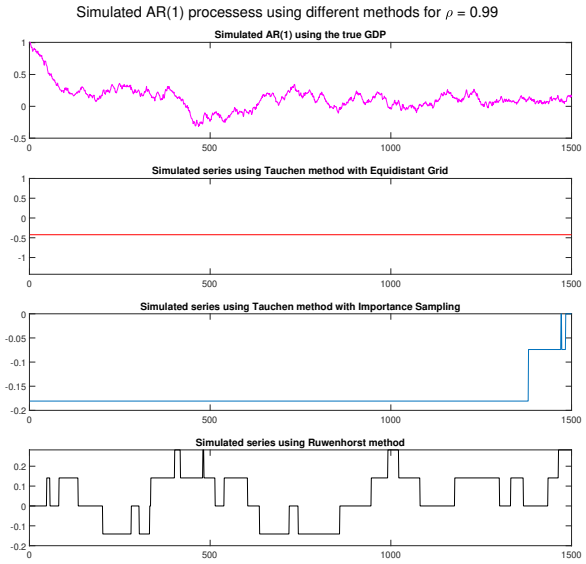
end

4. Discard the first t_{burn} observations of z

end



(a) Plot of 1500 realizations of the simulated AR(1), for $\rho = 0.7, N = 5$, burn-in= 500 and $m = 3$ (for Tauchen w/ equidistant grid).



(b) Plot of 1500 realizations of the simulated AR(1), for $\rho = 0.99, N = 5$, burn-in= 500 and $m = 3$ (for Tauchen w/ equidistant grid).

Figure 1 – Comparison between Tauchen (both versions) and Rouwenhorst methods for generating the AR(1) process defined in (1).

Table 1 – Comparison of the three different methods with the true process moments for different values of ρ in Question 1

	$\rho = 0.7$		$\rho = 0.99$	
	Mean	Standard Dev.	Mean	Standard Dev.
Tauchen w/ Equid. Grid	-0.052847	0.81999	-0.42320	$7.9963 \times e^{-15}$
Tauchen w/ IS	0.136430	0.76249	-0.17130	0.03306
Rouwenhorst	-0.004761	0.72289	0.041944	0.12059
True Process	0	0.71474	0	0.14107

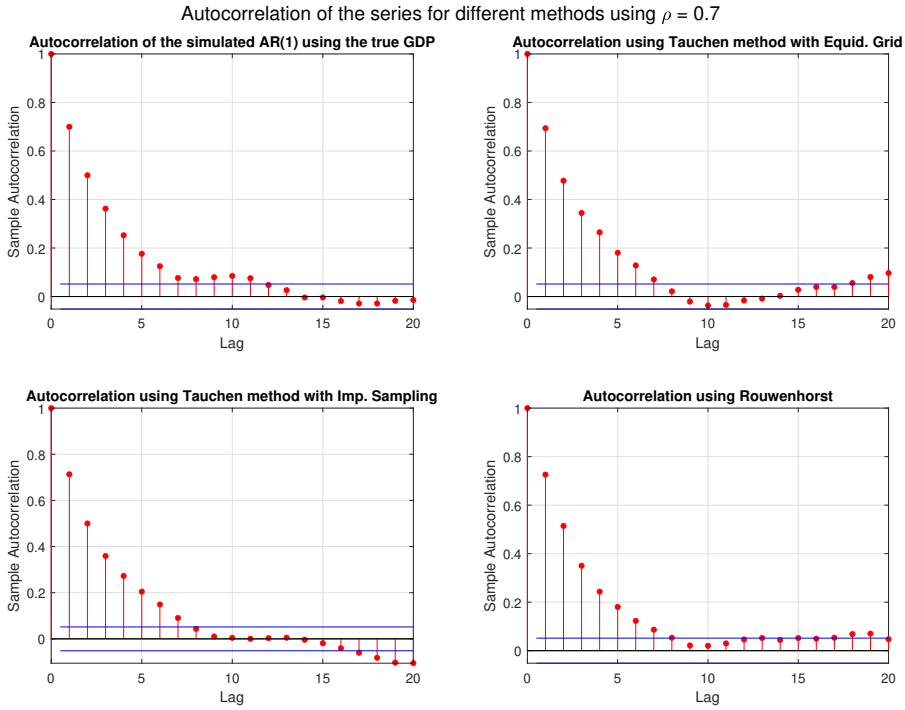


Figure 2 – Comparison the autocorrelation function of the approximated AR(1) series, defined in (1), when generated by Tauchen (both versions) and Rouwenhorst methods using $\rho = 0.7$. The first graph corresponds to simulated values using the true DGP.

For the first case we see that the three methods mimic well what would be expected from the true process. Tauchen's with Equidistant Grid for the case with $\rho = 0.9$ has a constant autocorrelation since the series didn't change state at all. □

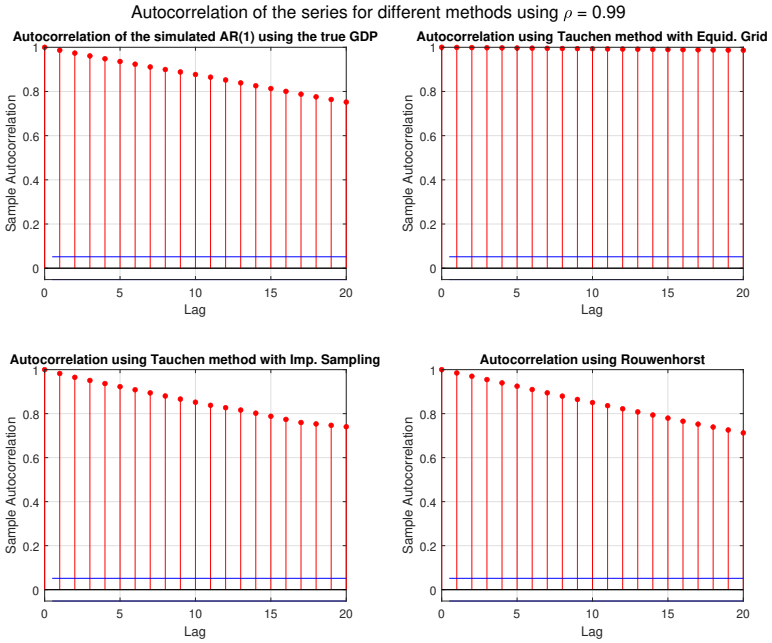


Figure 3 – Comparison of the autocorrelation function of the approximated AR(1) series, defined in (1), when generated by Tauchen (both versions) and Rouwenhorst methods using $\rho = 0.99$. The first graph corresponds to simulated values using the true DGP.

Problem 2

Consider the stochastic growth model described in class with $u(c) = \log(c)$ and $f(k) = k^\alpha$, where the process z is defined by

$$\begin{aligned} z_t &= \exp(y_t) \\ y_t &= \rho y_{t-1} + \epsilon_t \\ \epsilon_t &\sim \mathcal{N}\left(0, \sigma^2\right) \end{aligned}$$

(a)

Use Rouwenhorst's method with $N = 3$ to discretize the Markov process.

Solution. We used the same specifications as in problem 1, such that the DGP of y is the following AR(1) process:

$$y_{t+1} = \rho y_t + \epsilon_{t+1} \tag{6}$$

where ϵ_{t+1} is a standard normal and $\rho = 0.7$. As in problem 1, we get N=3 gridpoints for y and we translate them into 3 gridpoints for z according to $z_t = \exp(y_t)$. For more details on the procedure, we refer to algorithm 3, which was adapted for this different process. We use these values of y and the corresponding transition matrix P_N in part (b). □

(b)

Implement value function and policy function iteration and compare their performance in terms of speed and Euler equation errors (for a given number of grid points).

Solution. We had to make a new code in comparison to week 1 in order to incorporate the stochastic process in the model. Again, we worked with 1000 gridpoints for capital k and z denotes the TFP affecting production. The latter gets discretized by using three different states of TFP, namely $z = 0.1380$, $z = 1$ and $z = 7.2449$. What is fundamentally different in this procedure than in week 1's is the maximisation to find optimal K': for every current state of z, there is an optimal K', such that our value function is increased in a third dimension that has 3 components, one for each z. The same happens to the corresponding policy function. Furthermore, the future discounted value function (as an element of today's optimisation) is now a weighted average according to the transition matrix P for the states of z, obtained with Rouwenhorst's method in part (a). Hence, every iteration produces $N = 3$ new guesses for the value function.

As one can see in figure 4 and 5, a lower value for TFP results in lower values of the value function and lower optimal K' (as efficiency of production is lower, less can be invested in capital stock). As TFP is a convex function of a normally distributed variable, the dynamics above and below $z = 1$ are not symmetric.

The computational time of this method is $96.042s$. The Euler Equation Error is low for the lowest value of z, but is very high the higher z and K. It is concavely increasing in K for $z = 7.224$. The plots of the EEE for each z are in Figure (6).

We could not implement the policy function algorithm: since we did not manage to do this in week 1 and we have not received the feedback for that assignment, we faced the same difficulties now and due to time constraints our algorithm is incomplete and does not produces reliable results.

What we tried to do is to follow the suggestion given of using the information in slide 37, but incorporating the stochastic process z and the transition matrix P_N. For sake of completeness, Figure (7) has the values obtained for the policy function of capital for the three different states. We can see that this is not the right solution since it is not increasing in K, but we could not find the error in our code to fix it on time.

□

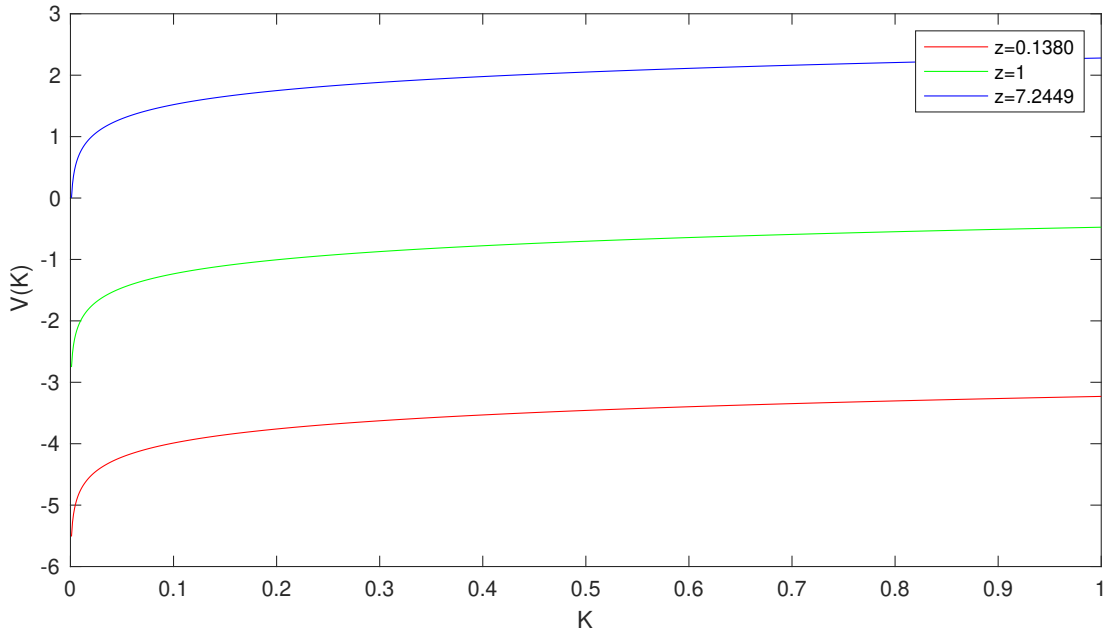


Figure 4 – Value Function Iteration with Rouwenhorst's method using $\rho = 0.7$: Value Function given current capital K for different states of z.

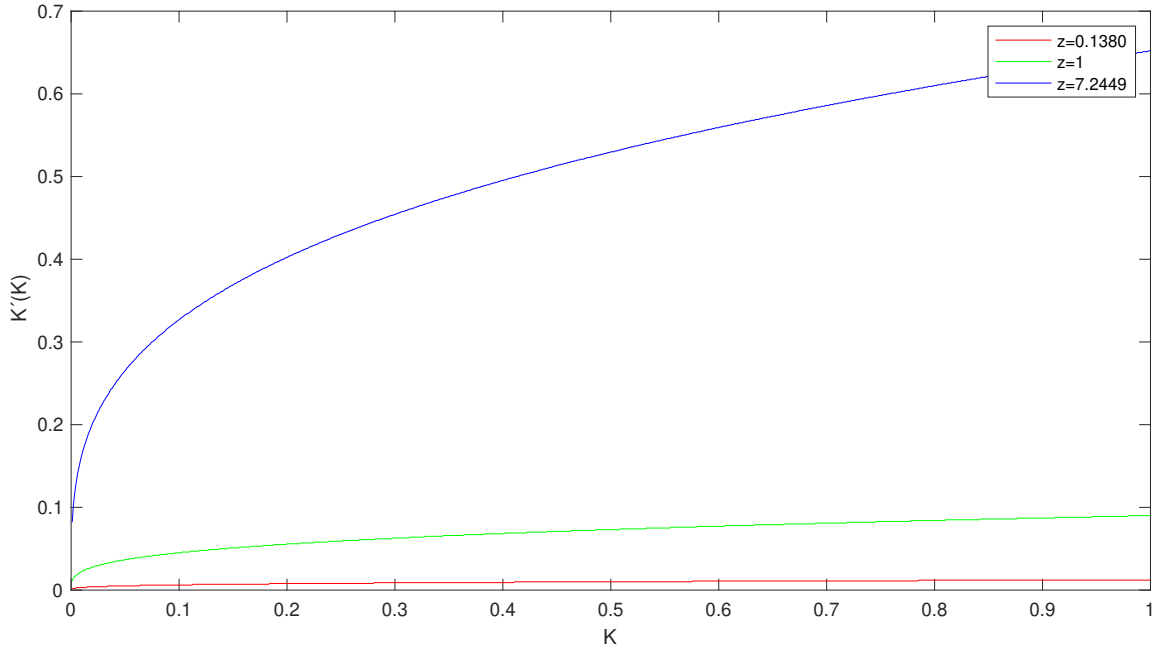


Figure 5 – Value Function Iteration with Rouwenhorst's method using $\rho = 0.7$: Optimal capital K' given current capital K for different states of z.

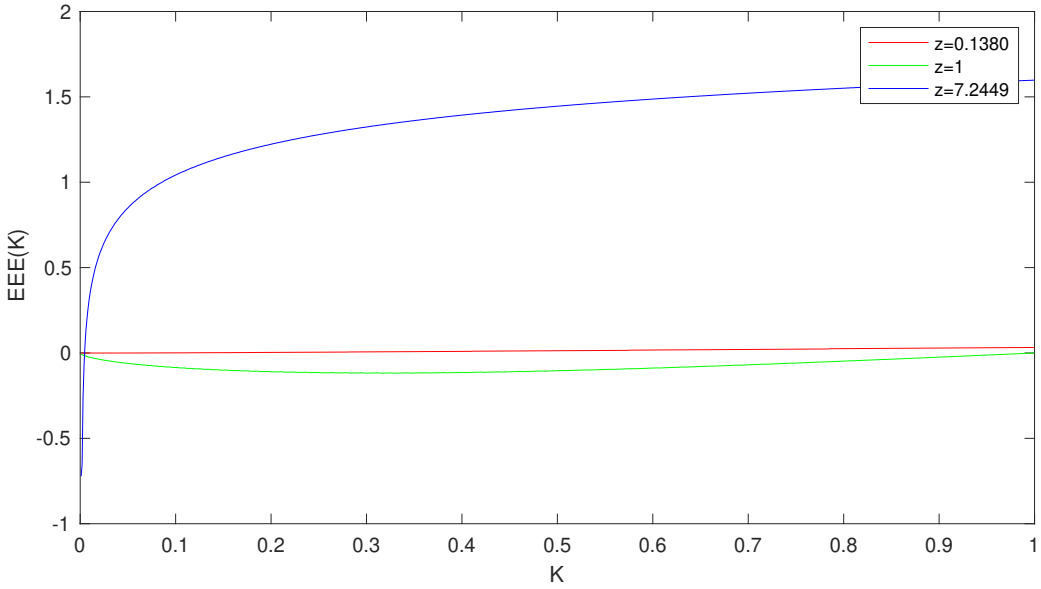


Figure 6 – Value Function Iteration with Rouwenhorst's method using $\rho = 0.7$: Euler Equation Error given current capital K for different states of z.

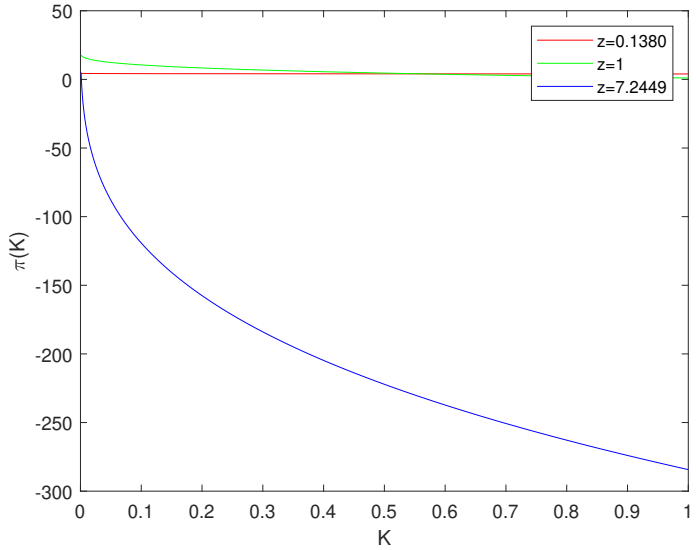


Figure 7 – Policy Function Iteration with Rouwenhorst's method using $\rho = 0.7$: Policy function for k for different states of z.