Existence and Uniqueness of Equilibrium Asset Prices over Infinite Horizons

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Reminders

Standard pricing theory:

$$P_t = \mathbb{E}_t M_{t+1} G_{t+1} \tag{1}$$

where

- G_{t+1} is a random payoff
- M_{t+1} is a **stochastic discount factor**
- P_t is current price

M_{t+1} is also called the **state price deflator**

This holds for **any** admissible payoff G_{t+1}

Now consider pricing **dividend stream** $\{D_t\}$

Purchasing the asset today gives (ex-dividend)

- claim to $D_{t+1}, D_{t+1}, \ldots$
- D_{t+1} and the right to sell tomorrow, at P_{t+1}

Hence payoff is $G_{t+1} = D_{t+1} + P_{t+1}$

Hence

$$P_t = \mathbb{E}_t M_{t+1}(D_{t+1} + P_{t+1}) \tag{2}$$

A recursion in prices...

Predictions

A (representative) investor has (unobservable) SDF process $\{M_t\}$

1. Take **any** dividend stream $\{D_t\}$
2. Plug the joint distribution of $\{D_t\}$ into

$$P_t = \mathbb{E}_t M_{t+1}(D_{t+1} + P_{t+1}) \tag{3}$$

3. Recover $\{P_t\}$ and compare with data

Example. How does $\ln(P_{t+1}/P_t)$ compare to observed returns?

A highly falsifiable theory!

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Testing, in practice

1. Calibrate
2. Log-linearize
3. Compare generated sample moments with observed sample moments

Questions:

- Does the model actually have a solution?
- Is it unique?
- When you log linearize, how big are your errors?

  (If you don't know, doesn't that worry you?)

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So, the technical challenge is to

$$\textbf{solve } P_t = \mathbb{E}_t M_{t+1}(D_{t+1} + P_{t+1}) \textbf{ for prices}$$

- Even for highly complex models
  - SDFs generated by recursive pref models / habit formation / etc.
  - Dividend processes with multiple factors, stochastic volatility
- Avoid log linearization

But this agenda raises important questions:

- When does a solution exist?
- When is it unique?
- How can we compute it?

These are the questions we tackle in this study

Problem Statement

If we regard

$$P_t = \mathbb{E}_t \, M_{t+1}(P_{t+1} + D_{t+1}) \quad \text{for all } t \geqslant 0, \tag{4}$$

as a map

- from the joint distribution of $\{D_t, M_t\}$
- to a price process $\{P_t\}$ satisfying (4)

then when—and under what circumstances—is this mapping well defined?

Also, what properties does the solution have when it exists?

Note: We ignore bubbles! (seek only stationary solutions)

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Background: Contractions

Let $\mathcal{H}$ be a Banach space and let $T\colon \mathcal{H} \to \mathcal{H}$

When does T have a unique fixed point in $\mathcal{H}$?

Go-to theory for scientists: when $\exists\, \alpha < 1$ such that

$$\|Tg - Th\| \leqslant \alpha \|g - h\| \quad \text{for all } g, h \in \mathcal{H}$$

Implies existence of a unique $h^* \in \mathcal{H}$ such that $Th^* = h^*$ and

$$T^n h \to h^* \quad \text{as } n \to \infty, \qquad \text{for all } h \in \mathcal{H}$$

Let's apply this to the risk neutral case, where $M_{t+1} = \beta$

$$P_t = \beta \, \mathbb{E}_t \left(P_{t+1} + D_{t+1} \right) \tag{5}$$

Let $D_t = d(X_t)$ where

- $\{X_t\}$ is Markov on $\mathbb{X}$ with

$$\Pi(x, B) := \mathbb{P}\{X_{t+1} \in B \mid X_t = x\}$$

- $d \colon \mathbb{X} \to \mathbb{R}$

Here

- $\mathbb{X}$ called the **state space**
- Dividends are stationary (can easily modify)

Now, to solve $P_t = \beta \mathbb{E}_t (P_{t+1} + D_{t+1})$, conjecture that

$$P_t = p^*(X_t) \text{ for some fixed function } p^* \colon \mathbb{X} \to \mathbb{R}$$

Such an **equilibrium price function** must satisfy

$$p^*(x) = \beta \int (p^*(y) + d(y)) \Pi(x, \mathrm{d}y) \qquad \text{(all } x \in \mathbb{X}\text{)}$$

Corresponding **equilibrium price operator** is

$$Tp(x) = \beta \int (p(y) + d(y)) \Pi(x, \mathrm{d}y)$$

Easy to show that T is a contraction of modulus β

Example. Suppse d is bounded and let $\|\cdot\|$ be the sup norm

$$
\begin{aligned}
|Tg(x) - Th(x)| &= \beta \left| \int (g(y) - h(y)) \Pi(x, \mathrm{d}y) \right| \\
&\leqslant \beta \int |g(y) - h(y)| \, \Pi(x, \mathrm{d}y) \\
&\leqslant \beta \|g - h\|
\end{aligned}
$$

$$
\therefore \quad \|Tg - Th\| \leqslant \beta \|g - h\|
$$

But agent's aren't risk neutral!

In general, $M_{t+1} \neq \beta$

That would be fine if

$$\exists \, \alpha < 1 \text{ such that } \mathbb{P}\{M_{t+1} \leqslant \alpha\} = 1$$

But usually this isn't true:

$$M_{t+1} > 1 \text{ with positive probability}$$

We like future payouts in bad states!

So how about the **Lucas case**

$$P_t = \mathbb{E}_t \, \beta \, \frac{u'(C_{t+1})}{u'(C_t)} (P_{t+1} + D_{t+1}) \tag{6}$$

Lucas' clever trick: solve for $Y_t = P_t u'(C_t)$ instead

From (6),

$$Y_t = \beta \mathbb{E}_t (Y_{t+1} + u'(C_{t+1}) D_{t+1}) \tag{7}$$

The rest of the story is similar (contraction of mod β)

Lucas' trick requires that SDF can be **factored** as follows:

$$M_{t+1} = \beta \frac{\Phi_{t+1}}{\Phi_t} \quad \text{where } \{\Phi_t\} \text{ stationary and } \beta < 1$$

But many popular SDFs lack such structure (EZ, etc.)

Also, empirical literature suggests that such SDFs are misspecified

- Borovička, Hansen and Scheinkman (2016, JoF)

We need a more general method!

A General Formulation

Consider generalization

$$Y_t = \mathbb{E}_t \left[\Phi_{t+1} (Y_{t+1} + G_{t+1}) \right] \quad \text{for all } t \geqslant 0, \tag{8}$$

where

- $\{\Phi_t\}$ and $\{G_t\}$ are given stochastic processes
- $\{Y_t\}$ is endogenous

Example. $\{Y_t\}$ is price, $\{G_t\}$ is cash flow and $\{\Phi_t\}$ is the SDF

Example. $Y_t = P_t / D_t$, $G_t \equiv 1$ and $\Phi_t = M_{t+1} D_{t+1} / D_t$

To repeat,

$$Y_t = \mathbb{E}_t \left[\Phi_{t+1} (Y_{t+1} + G_{t+1}) \right]$$

Assume that

$$\Phi_{t+1} = \phi(X_t, X_{t+1}, \eta_{t+1})$$

and

$$G_{t+1} = g(X_t, X_{t+1}, \eta_{t+1})$$

Here

- $\{\eta_t\}$ is an innovation sequence
- ϕ and g are positive Borel measurable maps

Any Markov solution h^* must satisfy

$$h^*(x) = \int \int \phi(x, x', \eta) \left[h^*(x') + g(x, x', \eta) \right] \nu(\mathrm{d}\eta) \Pi(x, \mathrm{d}x')$$

In other words, h^* is a fixed point of the **equilibrium price operator** T defined by

$$Th = Vh + \hat{g}$$

where

$$Vh(x) := \int h(x') \left[\int \phi(x, x', \eta) \nu(\mathrm{d}\eta) \right] \Pi(x, \mathrm{d}x')$$

and

$$\hat{g}(x) := \int \int \phi(x, x', \eta) g(x, x', \eta) \nu(\mathrm{d}\eta) \Pi(x, \mathrm{d}x')$$

One Step Contractions

Hey, I know, let's impose conditions that make T a contraction

Then we'll have a unique equilibrium price function

This is easy to do...

And then our problem will be solved... right??

Example. Suppose g bounded and let $\| \cdot \|$ be the sup norm

Suppose also that

$$\alpha := \sup_{x \in \mathbb{X}} \int \int \phi(x, x', \eta) \Pi(x, \mathrm{d}x) \nu(\mathrm{d}\eta) < 1$$

Then

$$|Tf(x) - Th(x)| = \left| \int (f(y) - h(y)) \left[\int \phi(x, y, \eta) \nu(\mathrm{d}\eta) \right] \Pi(x, \mathrm{d}y) \right|$$

$$\leqslant \int |f(y) - h(y)| \left[\int \phi(x, y, \eta) \nu(\mathrm{d}\eta) \right] \Pi(x, \mathrm{d}y)$$

$$\leqslant \alpha \|f - h\|$$

$$\therefore \quad \|Tf - Th\| \leqslant \alpha \|f - h\|$$

Let's test this out on **Bansal and Yaron (2004)**

Utility is given by

$$V_t = \left[(1-\beta)C_t^{1-1/\psi} + \beta \left\{\mathcal{R}_t\left(V_{t+1}\right)\right\}^{1-1/\psi}\right]^{1/(1-1/\psi)}$$

where $\{C_t\}$ is the consumption path extending on from time t and

$$\mathcal{R}_t(V_{t+1}) := \left(\mathbb{E}_t V_{t+1}^{1-\gamma}\right)^{1/(1-\gamma)}$$

- $\beta \in (0,1)$ is a time discount factor
- γ governs risk aversion and ψ is the EIS

Dividends and consumption grow according to

$$\ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_t \, \eta_{c,t+1}$$

$$\ln(D_{t+1}/D_t) = \mu_d + a z_t + \phi_d \, \sigma_t \, \eta_{d,t+1}$$

$$z_{t+1} = \rho z_t + \phi_z \, \sigma_t \, \eta_{z,t+1}$$

$$\sigma^2_{t+1} = \max \left\{ v \, \sigma^2_t + d + \phi_\sigma \, \eta_{\sigma,t+1}, \, 0 \right\}$$

Here

- $\{\eta_{i,t}\}$ are IID and standard normal
- The state X_t can be represented as $X_t = (z_t, \sigma_t)$

The SDF is given by

$$M_{t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{W_{t+1}}{W_t - 1} \right)^{\theta - 1}$$

where

- W_t is the value of aggregate wealth
- $\theta := (1 - \gamma)/(1 - 1/\psi)$

The growth adjusted SDF is

$$\Phi_{t+1} := M_{t+1} \frac{D_{t+1}}{D_t} = \beta^{\theta} \frac{D_{t+1}}{D_t} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{W_{t+1}}{W_t - 1} \right)^{\theta - 1}$$

The wealth value process W_t can be represented as $W_t = w(X_t)$, where

$$\beta^{\theta}\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{w(X_{t+1})}{w(X_t)-1}\right)^{\theta}\right] = 1$$

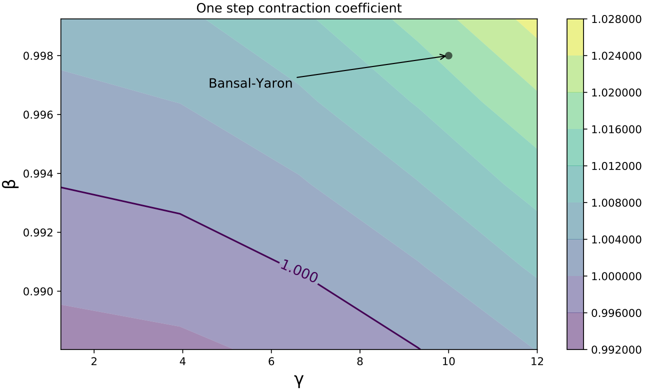
Strategy:

1. solve this functional equation for w
2. compute

$$\alpha := \sup_x \int\int \phi(x, x', \eta)\Pi(x, \mathrm{d}x)\nu(\mathrm{d}\eta)$$

3. test whether $\alpha < 1$

The results are concerning...


One step contraction coefficient
Bansal-Yaron
1.000
β
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0.998
0.996
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2
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1.028000
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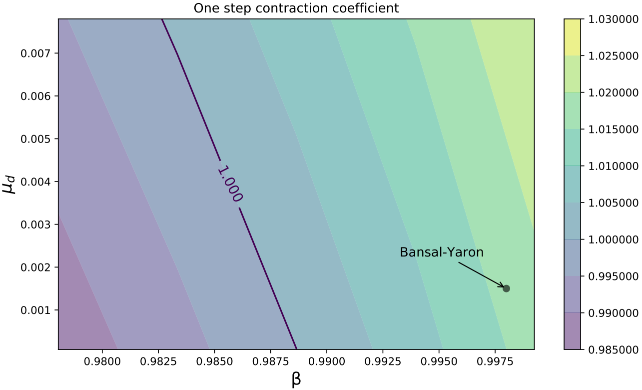
One step contraction coefficient
0.007
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1.005000
1.000000
0.995000
0.990000
0.985000

So Bansal and Yaron's model **fails** this test

But what does this mean?

The condition is only sufficient

When it fails we get no information...

Similar scenarios arise for other models

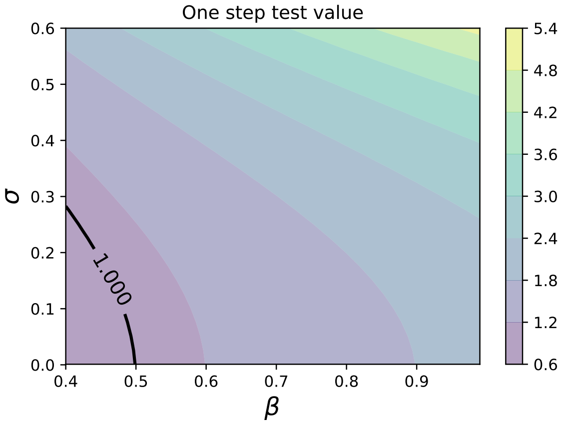
For example, **Calin et al.** (2005, ECMA) give a one-step contraction condition for the **habit formation** case

$$M_{t+1}\frac{D_{t+1}}{D_t} = k_0 \exp((1-\gamma)(\rho-\alpha)X_t)$$

where

- $k_0 := \beta \exp(b(1-\gamma) + \sigma^2(\gamma-1)^2/2)$
- α, γ, β are preference parameters
- the state process obeys

$$X_{t+1} = \rho X_t + b + \sigma\eta_{t+1} \quad \text{with} \quad \{\eta_t\} \overset{\text{IID}}{\sim} N(0,1)$$


One step test value
0.6
0.5
0.4
0.3
0.2
0.1
0.0
σ
1.000
0.4
0.5
0.6
0.7
0.8
0.9
β
5.4
4.8
4.2
3.6
3.0
2.4
1.8
1.2
0.6

Back to the Drawing Board

So, to sum up so far

- Equilib price function is fixed point of $Th = Vh + \hat{g}$
- Requiring contractivity of T is **too strict**

Fortunately, we can do better!

The reason is: V is **linear**

By the **Neumann series lemma**, T has a unique fixed point in $\mathcal{H}$ whenever

$$r(V) < 1, \quad \text{where} \quad r(V) := \text{spectral radius of } V$$

How tight is this condition?

Very close to necessary!

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Very close to necessary!

Theorem. If V is compact, then the following statements are **equivalent**:

1. $r(V) < 1$
2. There exists an $n \in \mathbb{N}$ such that T^n is a contraction on $\mathcal{H}$
3. There exists a unique equilibrium price function h^* in $\mathcal{H}_+$ and $\lim_{n \to \infty} T^n h = h^*$ for every $h \in \mathcal{H}_+$

(Necessity follows from compactness, positivity of V and the Krein–Rutman theorem)

Stable Case

Theorem. If $r(V) < 1$, then

1. The unique equilibrium price function h^* is equal to $\sum_{n \geqslant 0} V^n \hat{g}$

2. The process $\{Y_t^*\}$ defined by $Y_t^* := h^*(X_t)$ for all t is a solution to the forward looking recursion

3. The forward projection

$$Y_t^F := \mathbb{E}_t \sum_{n=1}^{\infty} \prod_{i=1}^{n} \Phi_{t+i} \, G_{t+n} \tag{9}$$

is finite and equal to Y_t^* with probability one

Unstable Case

Theorem. Let $\mathcal{H}$ be such that all strictly positive elements of $\mathcal{H}$ are quasi-interior to $\mathcal{H}_+$

If, in addition, V is compact and $r(V) > 1$, then no equilibrium price function exists in $\mathcal{H}_+$

Intuition

- For any $h > 0$, the dynamics $V^n h$ are similar to the dynamics of $V^n e$, where e is the principal eigenfunction
- If $r(V) > 1$, then $V^n e$ diverges
- Hence $V^n h$ diverges
- Hence $T^n h = V^n h + \sum_{i=1}^n V^i \hat{g}$ diverges

The L_1 Case

Let's return to $P_t = \mathbb{E}_t M_{t+1}(D_{t+1} + P_{t+1})$

Theorem. When $\mathcal{H} = L_1(\pi)$ with $\pi \stackrel{\mathscr{D}}{=} X_t$, we have

$$r(V) = \lim_{n \to \infty} \left\{ \mathbb{E} \prod_{t=1}^{n} M_t \right\}^{1/n}$$

Example. (Sanity check) In the risk-neutral case, where $M_t \equiv \beta$,

$$r(V) = \lim_{n \to \infty} \left\{ \mathbb{E} \prod_{t=1}^{n} M_t \right\}^{1/n} = \beta$$

Hence unique equilibrium price function exists whenever $\beta < 1$

Since

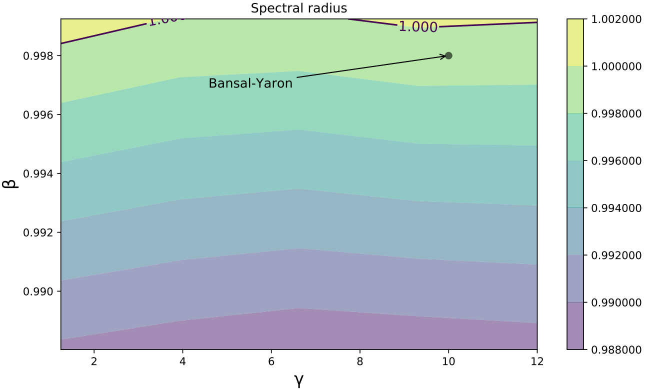
$$r(V) = \lim_{n \to \infty} \left\{ \mathbb{E} \prod_{t=1}^{n} M_t \right\}^{1/n}$$

we have, intuitively,

$$r(V) < 1 \quad \iff \quad \text{“eventual contraction, on average”}$$

Weaker than contraction in one step (essentially necessary)

Let's compute the coefficient $r(V)$ in some applications and see what we get



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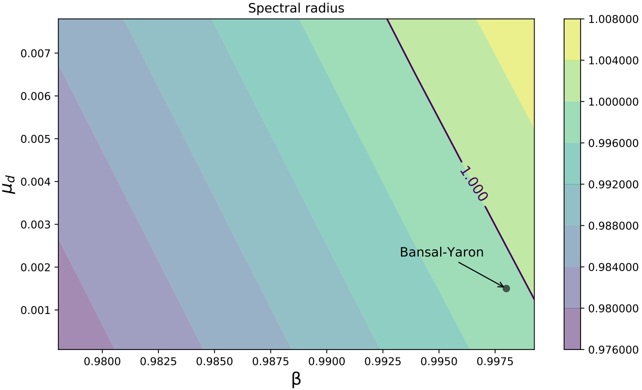
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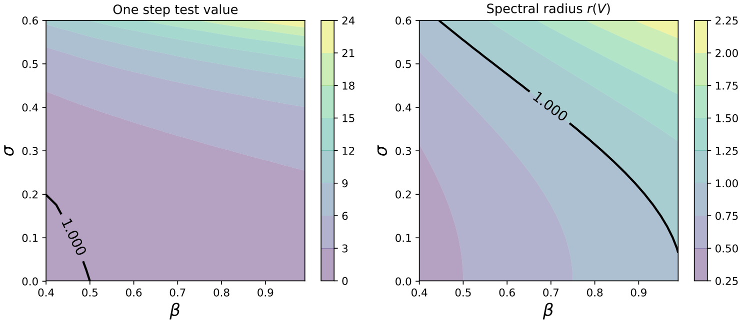
How does this work for the model studied in Calin et al?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0.0 0.1 0.2 0.3 0.4 0.5 0.6
0 3 6 9 12 15 18 21 24
0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00 2.25

Conclusion

$$E = mc^2$$