Behavioral Learning Equilibria in the New Keynesian Model

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Abstract

We generalize the concept of behavioral learning equilibrium (BLE) to a high dimensional linear system and apply it to the standard New Keynesian model. For each endogenous variable in the economy, boundedly rational agents use a simple, but optimal AR(1) forecasting rule with parameters consistent with the observed sample mean and autocorrelation of past data. Agents do not fully recognize the complex structure of the economy, but learn to use an optimal parsimonious AR(1) rule, which satisfies the orthogonality condition for RE. We find that BLE exists, under general stationarity conditions, typically with near unit root autocorrelation parameters. BLE thus exhibits a novel feature, persistence amplification: the persistence in inflation and output gap is much higher than the persistence in exogenous shocks. We provide a general framework estimate a BLE. We illustrate our approach on U.S. data and show that the standard New Keynesian model fits aggregate data better under BLE than RE. We analyze optimal monetary policy under BLE and we find that the transmission channel of monetary policy is stronger under BLE at the estimated parameter values.

JEL classification: C11; E62; E03; D83; D84

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1 Introduction

Rational Expectations Equilibrium (REE) requires that economic agents' subjective probability distributions coincide with the objective distribution that is determined, in part, by their subjective beliefs. There is a vast literature that studies the drawbacks of REE. Some of these drawbacks include the fact that REE requires an unrealistic degree of computational power and perfect information on the part of agents. Alternatively, the adaptive learning literature (see, e.g., Evans and Honkapohja (2001, 2013) and Bullard (2006) for extensive surveys and references) replaces Rational Expectations with beliefs that come from an econometric forecasting model with parameters updated using observed time series. A large part of this literature involves studying under which conditions learning will converge to the rational expectations equilibrium. When the perceived law of motion (PLM) of agents is correctly specified, convergence of adaptive learning to an REE can occur. However, in general the PLM may be misspecified. As shown in White (1994), an economic model or a probability model is only a more or less crude approximation to whatever might be the true relationships among the observed data. Consequently it is necessary to view economic and/or probability models as misspecified to some greater or lesser degree. Whenever agents have *misspecified* PLMs, a reasonable learning process may settle down to a misspecification equilibrium. In the literature, different types of misspecification equilibria have been proposed, e.g. Restricted Perceptions Equilibrium (RPE) where the forecasting model is underparameterized (Sargent, 1991; Evans and Honkapohja, 2001; Adam, 2003; Branch and Evans, 2010) and Stochastic Consistent Expectations Equilibrium (SCEE) (Hommes and Sorger, 1998; Hommes et al., 2013), where agents learn the optimal parameters of a simple, parsimonious AR(1) rule.[1]

A SCEE is a very natural misspecification equilibrium, where agents in the economy do not know the actual law of motion or even recognize all relevant explanatory variables, but rather prefer a parsimonious forecasting model. The economy is too complex to fully understand and therefore, as a first-order approximation, agents forecast the state of the economy by simple autoregressive models (e.g. Fuster et al., 2010). In the simplest model applying this idea, agents run a univariate AR(1) regression to generate out-of-sample forecasts of the state of the economy. Hommes and Zhu (2014) provide the first-order SCEE with an *intuitive behavioral* interpretation and refer to them as *Behavioral Learn-*

[1] Branch (2006) provides a stimulating survey discussing the connection between these types of misspecification equilibria.

ing Equilibria (BLE). Although it is possible for some agents to use more sophisticated models, one may argue that these practices are neither straightforward nor widespread. A simple, parsimonious BLE seems a more plausible outcome of the coordination process of individual expectations in large complex socio-economic systems (Grandmont, 1998).

Hommes and Zhu (2014) formalize the concept of BLE in the simplest class of models one can think of: a one-dimensional linear stochastic model driven by an exogenous linear stochastic AR(1) process. Agents do not recognize, however, that the economy is driven by an exogenous AR(1) process, but simply forecast the state of the economy using a univariate AR(1) rule. The parameters of the AR(1) forecasting rule are not free, but fixed (or learned over time) according to the observed sample average and first-order sample autocorrelation. Within this simple but general class of models, Hommes and Zhu (2014) fully characterize the existence and multiplicity of BLE and provide stability conditions under a simple adaptive learning scheme –Sample Autocorrelation Learning (SAC-learning). Although this class of models is simple, it contains two important standard applications: an asset pricing model driven by autocorrelated dividends and the New Keynesian Phillips curve with inflation driven by autocorrelated output gap (or marginal costs). As shown in Fuhrer (2009), however, the skeleton model of the New Keynesian Phillips curve with AR(1) driving variable leaves implicit the determination of real output and the role of monetary policy in influencing output and inflation.

In this paper we extend the BLE concept to a general n-dimensional linear stochastic framework and provide a method to estimate BLE in this framework. As an application we consider the standard 3-equation dynamic stochastic general equilibrium (DSGE) model-the New Keynesian (NK) model-, study the empirical fit of the model and the role of monetary policy under BLE. Agents' perceived law of motion (PLM) is a simple univariate AR(1) process for each variable to be forecasted. Two consistency requirements are imposed upon BLE to pin down the parameters of the forecasting model: for each endogenous variable observed sample averages and first-order sample autocorrelations match the corresponding parameters of the forecasting rule. Agents thus learn the optimal AR(1) forecasting rule for each endogenous variable in the economy.

Numerous empirical studies show that overly parsimonious models with little parameter uncertainty can provide better forecasts than models consistent with the actual data-generating complex process (e.g. Nelson, 1972; Stock and Watson, 2007; Clark and West, 2007; Enders, 2010). In a similar vein (but without analytical results) Slobodyan and Wouters (2012) study a New Keynesian DSGE model with agents using a constant gain

AR(2) forecasting rule. Chung and Xiao (2014) and Xiao and Xu (2014) study learning and predictions with an AR(1) or VAR(1) model in a two dimensional New Keynesian model with limited information and show, based on simulations, that the simple AR(1) model is more likely to prevail in reality when they make predictions. Laboratory experiments in the NK framework also show that simple forecasting rules such as AR(1) describe individual forecasting behavior surprisingly well (Assenza et al., 2014; Pfajfar and Zakelj, 2016).

Our paper bears clear resemblance to the seminal work of Krusell and Smith (1998), where the behavior of macroeconomic aggregates can be almost perfectly described by the mean of the wealth distribution and the aggregate productivity shock. In our BLE agents are boundedly rational and forecast macro variables by learning the mean and the first-order autocorrelations of all observable endogenous macro variables. Such irrationality is hard to detect however, as agents learn to forecast correctly the mean and the first-order autocorrelation of each variable in the economy. In fact, for each endogenous variable the optimal univariate rule satisfies the orthogonality condition for rational expectations.

The main contributions of our paper are fourfold: (1) we derive existence and stability conditions of BLE in a general linear framework, (2) we provide a simple and general method for Bayesian likelihood estimation of BLE, (3) we estimate the baseline NK model based on U.S. data and show that the relative model fit is better under BLE than REE, (4) we analyze the optimal monetary policy under BLE and compare it with REE.

Many models of learning lead to excess volatility, where the volatility under learning is typically higher than under REE. Our BLE model exhibits another novel feature, *persistence amplification*: the persistence of inflation and output gap under BLE is significantly higher than under REE. In fact, even when autocorrelations of the exogenous shocks to fundamentals are small, inflation and output gap along BLE are typically near unit root processes. As a consequence, when we estimate the NK model under BLE, we find important differences in parameter estimates compared with the REE. We further analyze optimal monetary policy under BLE and find finite optimal Taylor rule coefficients under a wide range of calibrations. Further, we find that transmission channel of monetary policy is stronger under BLE at the estimated parameters.

Related literature

The issue of persistence has been of great interest to macroeconomists and policymakers. A number of models with frictions have been proposed to replicate persistence, such as habit formation in consumption, indexation to lagged inflation in price-setting, rule-of-thumb behavior, or various adjustment costs (Phelps, 1968; Taylor, 1980; Fuhrer and Moore, 1992, 1995; Christiano et al., 2005; Smets and Wouters, 2003, 2005; Boivin and Giannoni, 2006; Giannoni and Woodford, 2003). These models essentially improve the empirical fit by adding lags in the model equations. Estimating these rich models with frictions under the assumption of rational expectations, one typically finds that substantial degrees of persistence are supported by the data. Therefore these additional sources of persistence appear necessary to match the inertia of macroeconomic variables. Estimation of these models typically also involve highly persistent structural shocks. Our BLE model is applied to a New Keynesian framework without habit formation or indexation, but nevertheless exhibits strong persistence. Learning causes persistence amplification: small autocorrelations of exogenous shocks are strongly amplified as agents learn to coordinate on a simple AR(1) forecasting rule with near unit root parameters consistent with observed sample average and sample autocorrelations. The high persistence of inflation and output thus arises from a self-fulfilling mistake (Grandmont, 1998).

Our BLE concept fits with the literature employing adaptive learning to analyze the evolution of U.S. inflation and monetary policy. Adaptive learning can help in understanding some particular historical episodes, such as high inflation in the 1980s, which are often harder to explain under rational expectations. For example, Orphanides and Williams (2003) consider a form of imperfect knowledge in which economic agents rely on adaptive learning to form expectations. This form of learning represents a relatively modest deviation from rational expectations that nests it as a limiting case. They find that policies that would be efficient under rational expectations can perform poorly when knowledge is imperfect. Milani (2005, 2007) also assumes that agents form expectations through adaptive learning using correctly specified economic models and updating the parameters through constant-gain learning (CGL) based on historical data. He shows empirically that when learning replaces rational expectations, the estimated degrees of habits and indexation drop closer to zero, suggesting that persistence arises in the model economy mainly from expectations and learning. Eusepi and Preston (2011) study expectations-driven business cycles based on learning, and find that learning dynamics generate forecast errors similar to the Survey of Professional Forecasters. Estrella and Fuhrer (2002) study the shortcomings of REE models with a focus on inertia and shock propagation structure. Fuhrer (2009) provides a good survey on inflation persistence. He examines a number of empirical measures of reduced form persistence including the first-order autocorrelation and the autocorrelation function of the inflation series. He also investigates the sources of persistence, including learning of agents in a rational- expectation setting.

Our behavioral learning equilibrium concept is closely related to the Exuberance Equilibria (EE) in Bullard et al. (2008), where agents' perceived law of motion is misspecified. However, because of difficulty of computation, in Bullard et al. (2008) there are only numerical results on the exuberance equilibria, while here we analytically show the existence and stability of BLE in a general linear framework with an application to the NK model, as well as empirically validate BLE based on U.S. data. Another related misspecification equilibrium is Limited Information Learning Equilibrium (LILE) defined in Chung and Xiao (2014), which is defined by the least-squares projection of variables on the past information of the actual law of motion equal to that in the perceived law of motion. Different from the LILE, our general Behavioral Learning Equilibrium is defined by the conditions that sample means and first-order autocorrelations of each variable of the actual law of motion are consistent with those corresponding to the perceived law of motion. We further study the effects of monetary policy under the more plausible BLE. The concept of natural expectations in Fuster et al. (2010) and Fuster et al. (2011, 2012) is another misspecification concept, where agents use simple, misspecified models, e.g., linear autoregressive models. Natural expectations, however, do not pin down the parameters of the forecasting model through consistency requirements as for a restricted perceptions equilibrium nor do they allow the agents to learn an optimal misspecified model through empirical observations. Cho and Kasa (2015) study model validation in an environment where agents are aware of misspecification and try to detect it through adaptive learning. In our BLE misspecification is self-fulfilling and the outcome of the SAC-learning process.

The paper is organized as follows. Section 2 introduces the main concepts of BLE in a general n-dimensional setup, the theoretical results on existence and stability of BLE in a linear framework, and the empirical estimation methodology. Section 3 applies BLE to the 3-equation New Keynesian model and studies the existence, stability and estimation results under BLE. Section 4 studies optimal monetary policy and how policy can mitigate persistence and volatility amplification under BLE. Finally, Section 5 concludes.

2 BLE in a Multivariate Framework

Hommes and Zhu (2014) introduced BLE in the simplest setting, a one-dimensional linear stochastic model driven by an exogenous linear stochastic AR(1) process. In this paper we generalize BLE to n-dimensional (linear) stochastic models driven by exogenous linear stochastic AR(1) processes of multiple shocks. To ease the exposition we initially follow the presentation in Hommes and Zhu (2014), but generalize their 1-dimensional model to an n-dimensional framework. In addition, most macroeconomic models include lagged state variables through features such as interest rate smoothing, habit formation in consumption or indexation in prices and wages. Therefore, we further extend our model with lagged state variables.

Let the law of motion of an economic system be given by the stochastic difference equation

$$\boldsymbol{x}_t \quad = \quad \boldsymbol{F}(\boldsymbol{x}^e_{t+1},\ \boldsymbol{x}_{t-1},\ \boldsymbol{u}_t,\ \boldsymbol{v}_t), \tag{2.1}$$

where $\boldsymbol{x}_t$ is an $n \times 1$ vector of endogenous variables denoted by $[x_{1t}, x_{2t}, \cdots, x_{nt}]'$ and $\boldsymbol{x}^e_{t+1}$ is the expected value of $\boldsymbol{x}$ at date $t+1$. This notation highlights that expectations may not be rational. Here $\boldsymbol{F}$ is a continuous n-dimensional vector function, $\boldsymbol{u}_t$ is a vector of exogenous stationary variables and $\boldsymbol{v}_t$ is a vector of white noise disturbances.

Agents are boundedly rational and do not know the exact form of the actual law of motion (2.1). They only use a simple, parsimonious forecasting model where agents' perceived law of motion (PLM) is a simple univariate AR(1) process for each variable to be forecasted. As shown in Enders (2010, p.84-85), coefficient uncertainty increases as the model becomes more complex, and hence it could be that an estimated AR(1) model forecasts a real ARMA(2,1) process better than an estimated ARMA(2,1) model. Numerous empirical studies also show that overly parsimonious models with little parameter uncertainty can provide better forecasts than models consistent with the actual data-generating complex process (e.g. Nelson, 1972; Stock and Watson, 2007; Clark and West, 2007). Thus agents' perceived law of motion (PLM) is assumed to be the simplest VAR model with minimum parameters, i.e. a restricted VAR(1) process

$$\boldsymbol{x}_t = \boldsymbol{\alpha} + \boldsymbol{\beta}(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}) + \boldsymbol{\delta}_t, \tag{2.2}$$

where $\boldsymbol{\alpha}$ is a vector denoted by $[\alpha_1, \alpha_2, \cdots, \alpha_n]'$, $\boldsymbol{\beta}$ is a diagonal matrix[2] denoted by

[2] Chung and Xiao (2014) also argue using simulations that the simple AR(1) model is more likely

$\begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & \beta_n \end{bmatrix}$ with $\beta_i \in (-1, 1)$ and $\{\boldsymbol{\delta}_t\}$ is a white noise process; $\boldsymbol{\alpha}$ is the unconditional mean of $\boldsymbol{x}_t$ and β_i is the first-order autocorrelation coefficient of variable x_i. Given the perceived law of motion (2.2), the 2-period ahead forecasting rule for $\boldsymbol{x}_{t+1}$ that minimizes the mean-squared forecasting error is

$$\boldsymbol{x}^e_{t+1} = \boldsymbol{\alpha} + \boldsymbol{\beta^2(x_{t-1} - \alpha)}. \tag{2.3}$$

Combining the expectations (2.3) and the law of motion of the economy (2.1), we obtain the implied actual law of motion (ALM)

$$\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{\alpha} + \boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}),\ \boldsymbol{x}_{t-1},\ \boldsymbol{u}_t,\ \boldsymbol{v}_t). \tag{2.4}$$

In the case that the ALM (2.4) is stationary, let the variance-covariance matrix $\boldsymbol{\Gamma}(0) := E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})']$ and the first order autocovariance matrix $\boldsymbol{\Gamma}(1) := E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{x}_{t+1} - \overline{\boldsymbol{x}})']$, where $\overline{\boldsymbol{x}}$ is the mean of $\boldsymbol{x}_t$. Let $\boldsymbol{\Omega}$ be the diagonal matrix in which the ith diagonal element is the variance of the ith process, that is $\boldsymbol{\Omega} = \text{diag}[\gamma_{11}(0), \gamma_{22}(0), \cdots, \gamma_{nn}(0)]$, where $\gamma_{ii}(0)$ is the ith diagonal entry of $\boldsymbol{\Gamma}(0)$. Let $\boldsymbol{L}$ be the diagonal matrix in which the ith diagonal element is the first-order autocovariance of the ith process, that is $\boldsymbol{L} = \text{diag}[\gamma_{11}(1), \gamma_{22}(1), \cdots, \gamma_{nn}(1)]$, where $\gamma_{ii}(1)$ is the ith diagonal entry of $\boldsymbol{\Gamma}(1)$. Let $\boldsymbol{G}$ denote the diagonal matrix in which the ith diagonal element is the first-order autocorrelation coefficient of the ith process $x_{i,t}$. Hence

$$\boldsymbol{G} = \boldsymbol{L}\boldsymbol{\Omega}^{-1}. \tag{2.5}$$

Behavioral Learning Equilibrium (BLE)

Extending Hommes and Zhu (2014), the concept of BLE is generalized as follows.

Definition 2.1 *A vector $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where μ is a probability measure, $\boldsymbol{\alpha}$ is a vector and $\boldsymbol{\beta}$ is a diagonal matrix with $\beta_i \in (-1, 1)$ $(i = 1, 2, \cdots, n)$, is called a behavioral learning equilibrium (BLE) if the following three conditions are satisfied:*

to prevail in reality because of limited information restrictions when they model predictions in a two dimensional New Keynesian model. In addition, as far as prediction is concerned, based on our numerous empirical analyses, the short-term forecasts based on an AR(1) model are better than more general VAR models in most cases, because in more general VAR models too many parameters need to be estimated and hence coefficient uncertainty increases.

S1 The probability measure μ is a nondegenerate invariant measure for the stochastic difference equation (2.4);

S2 The stationary stochastic process defined by (2.4) with the invariant measure μ has unconditional mean $\boldsymbol{\alpha}$, that is, the unconditional mean of x_i is α_i, $(i = 1, 2, \cdots, n)$;

S3 Each element x_i for the stationary stochastic process of $\boldsymbol{x}$ defined by (2.4) with the invariant measure μ has unconditional first-order autocorrelation coefficient β_i, $(i = 1, 2, \cdots, n)$, that is, $\boldsymbol{G} = \boldsymbol{\beta}$.

In other words, a BLE is characterized by two natural observable consistency requirements: the unconditional means and the unconditional first-order autocorrelation coefficients generated by the actual (unknown) stochastic process (2.4) coincide with the corresponding statistics for the perceived linear VAR(1) process (2.2), as given by the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This means that in a BLE agents correctly perceive the two simplest and most important statistics: the mean and first-order autocorrelation (i.e., persistence) of each relevant variable of the economy, without fully understanding its structure and recognizing all explanatory variables and cross correlations. A BLE is *parameter free*, as along a BLE the two parameters of each linear forecasting rule are pinned down by simple and observable statistics. Hence, agents do not fully understand the linear structure of the stochastic economy, e.g. they do not take the cross-correlation of state variables into account, but rather use a parsimonious univariate AR(1) forecasting rule for each state variable. A simple BLE may be a plausible outcome of the coordination process of expectations of a large population. Laboratory experiments within the New Keynesian framework also provide empirical evidence of the use of simple univariate AR(1) forecasting rules to forecast inflation and output gap (Adam, 2007; Pfajfar and Zakelj, 2016; Assenza et al., 2014).

Furthermore, we note that along a BLE the orthogonality condition

$$
\begin{aligned}
&E[x_{i,t} - \alpha_i - \beta_i(x_{i,t-1} - \alpha_i)] = 0,\\
&E\{[x_{i,t} - \alpha_i - \beta_i(x_{i,t-1} - \alpha_i)]x_{i,t-1}\} = E\{[x_{i,t} - \alpha_i - \beta_i(x_{i,t-1} - \alpha_i)](x_{i,t-1} - \alpha_i)\} = 0
\end{aligned}
$$

is satisfied. That is, the forecast $\alpha_i + \beta_i(x_{i,t-1} - \alpha_i)$ is the linear projection of $x_{i,t}$ on the vector $(1, x_{i,t-1})'$. For each variable, agents cannot detect the correlation between the forecasting error $x_{i,t} - \alpha_i - \beta_i(x_{i,t-1} - \alpha_i)$ and the vector $(1, x_{i,t-1})'$ in the forecast model. The linear projection produces the smallest mean squared error among the class of linear forecasting rules (e.g., Hamilton (1994). Therefore, for each variable agents use the optimal forecast within their class of univariate AR(1) forecasting rules (Branch, 2006).

Sample autocorrelation learning

In the above definition of BLE, agents' beliefs are described by the linear forecasting rule (2.3) with fixed parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. However, the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are usually unknown to agents. In the adaptive learning literature, it is common to assume that agents behave like econometricians using time series observations to estimate the parameters as new observations become available. Following Hommes and Sorger (1998), we assume that agents use sample autocorrelation learning (SAC-learning) to learn the parameters α_i and β_i, $i = 1, 2, \cdots, n$. That is, for any finite set of observations $\{x_{i,0}, x_{i,1}, \cdots, x_{i,t}\}$, the sample average is given by

$$\alpha_{i,t} = \frac{1}{t+1} \sum_{k=0}^{t} x_{i,k}, \tag{2.6}$$

and the first-order sample autocorrelation coefficient is given by

$$\beta_{i,t} = \frac{\sum_{k=0}^{t-1}(x_{i,k} - \alpha_{i,t})(x_{i,k+1} - \alpha_{i,t})}{\sum_{k=0}^{t}(x_{i,k} - \alpha_{i,t})^2}. \tag{2.7}$$

Hence $\alpha_{i,t}$ and $\beta_{i,t}$ are updated over time as new information arrives. It is easy to check that, independently of the choice of the initial values $(x_{i,0}, \alpha_{i,0}, \beta_{i,0})$, it always holds that $\beta_{i,1} = -\frac{1}{2}$, and that the first-order sample autocorrelation $\beta_{i,t} \in [-1, 1]$ for all $t \geq 1$.

As shown in Hommes and Zhu (2014), define

$$R_{i,t} = \frac{1}{t+1} \sum_{k=0}^{t} (x_{i,k} - \alpha_{i,t})^2.$$

Then the SAC-learning is equivalent to the following recursive dynamical system[3]

[3]The system in (2.8) is a decreasing gain algorithm, where all observations receive equal weight and therefore the weight on the latest observation decreases as the sample size grows. There is also a constant gain correspondence of SAC-learning, where past observations are discounted at a geometric rate. This can be obtained by replacing the weights $\frac{1}{t+1}$ by some positive constant κ, see the online appendix to Hommes & Zhu (2014) for further details.

$$
\begin{cases}
\alpha_{i,t} = \alpha_{i,t-1} + \dfrac{1}{t+1}(x_{i,t} - \alpha_{i,t-1}), \\
\beta_{i,t} = \beta_{i,t-1} + \dfrac{1}{t+1} R_{i,t}^{-1} \Big[(x_{i,t} - \alpha_{i,t-1}) \big(x_{i,t-1} + \dfrac{x_{i,0}}{t+1} - \dfrac{t^2+3t+1}{(t+1)^2} \alpha_{i,t-1} - \dfrac{1}{(t+1)^2} x_{i,t} \big) \\
\qquad - \dfrac{t}{t+1} \beta_{i,t-1} (x_{i,t} - \alpha_{i,t-1})^2 \Big], \\
R_{i,t} = R_{i,t-1} + \dfrac{1}{t+1} \Big[\dfrac{t}{t+1} (x_{i,t} - \alpha_{i,t-1})^2 - R_{i,t-1} \Big].
\end{cases}
\tag{2.8}
$$

The actual law of motion under SAC-learning is therefore given by

$$
\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}^2 (\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}_{t-1}),\ \boldsymbol{x}_{t-1},\ \boldsymbol{u}_t,\ \boldsymbol{v}_t), \tag{2.9}
$$

with $\alpha_{i,t}$, $\beta_{i,t}$ as in (2.8).

In Hommes and Zhu (2014), F is a one-dimensional linear function. In this paper $\boldsymbol{F}$ may be an n-dimensional linear vector function and includes the lagged term $\boldsymbol{x}_{t-1}$.

2.1 Main results in a multivariate linear framework

Assume that a reduced form model is an n-dimensional linear stochastic process $\boldsymbol{x}_t$, driven by an exogenous VAR(1) process $\boldsymbol{u}_t$. More precisely, the actual law of motion of the economy is given by

$$
\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{x}_{t+1}^e,\ \boldsymbol{u}_t,\ \boldsymbol{v}_t) = \boldsymbol{b}_0 + \boldsymbol{b}_1 \boldsymbol{x}_{t+1}^e + \boldsymbol{b}_2 \boldsymbol{x}_{t-1} + \boldsymbol{b}_3 \boldsymbol{u}_t + \boldsymbol{b}_4 \boldsymbol{v}_t, \tag{2.10}
$$

$$
\boldsymbol{u}_t = \boldsymbol{a} + \boldsymbol{\rho} \boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{2.11}
$$

where $\boldsymbol{x}_t$ is an $n \times 1$ vector of endogenous variables, $\boldsymbol{b}_0$ and $\boldsymbol{a}$ are vectors of constants, $\boldsymbol{b}_1, \boldsymbol{b}_2$ and $\boldsymbol{b}_4$ are $n \times n$ matrices of coefficients, $\boldsymbol{b}_3$ is an $n \times m$ matrix, $\boldsymbol{\rho}$ is an $m \times m$ matrix, $\boldsymbol{u}_t$ is an $m \times 1$ vector of exogenous variables which is assumed to follow a stationary VAR(1) as in (2.11), and $\boldsymbol{v}_t$ is an $n \times 1$ vector of i.i.d. stochastic disturbance terms with mean zero and finite absolute moments, with variance-covariance matrix $\boldsymbol{\Sigma_v}$. Hence all of the eigenvalues of $\boldsymbol{\rho}$ are assumed to be inside the unit circle. In addition, $\boldsymbol{\varepsilon}_t$ is assumed to be an $m \times 1$ vector of i.i.d. stochastic disturbance terms with mean zero and finite absolute moments, with variance-covariance matrix $\boldsymbol{\Sigma_\varepsilon}$ and is independent of $\boldsymbol{v}_t$.

Rational expectations equilibrium

Assume that agents are rational. The perceived law of motion (PLM) corresponding to the minimum state variable REE of the model is:

$$\boldsymbol{x}_t^* = \boldsymbol{c}_0 + \boldsymbol{c}_1\boldsymbol{x}_{t-1}^* + \boldsymbol{c}_2\boldsymbol{u}_t + \boldsymbol{c}_3\boldsymbol{v}_t. \tag{2.12}$$

Assuming that shocks $\boldsymbol{u}_t$ are observable when forecasting $\boldsymbol{x}_{t+1}$, the one-step ahead forecast is:

$$E_t\boldsymbol{x}_{t+1}^* = \boldsymbol{c}_0 + \boldsymbol{c}_2\boldsymbol{a} + \boldsymbol{c}_1\boldsymbol{x}_t^* + \boldsymbol{c}_2\boldsymbol{\rho}\boldsymbol{u}_t, \tag{2.13}$$

and the corresponding actual law of motion is:

$$\boldsymbol{x}_t^* = \boldsymbol{b}_0 + \boldsymbol{b}_1(\boldsymbol{c}_0 + \boldsymbol{c}_2\boldsymbol{a} + \boldsymbol{c}_1\boldsymbol{x}_t^* + \boldsymbol{c}_2\boldsymbol{\rho}\boldsymbol{u}_t) + \boldsymbol{b}_2\boldsymbol{x}_{t-1} + \boldsymbol{b}_3\boldsymbol{u}_t + \boldsymbol{b}_4\boldsymbol{v}_t. \tag{2.14}$$

The rational expectations equilibrium (REE) is the fixed point of

$$\begin{aligned}
\boldsymbol{c}_0 - \boldsymbol{b}_1\boldsymbol{c}_1\boldsymbol{c}_0 - \boldsymbol{b}_1\boldsymbol{c}_0 &= \boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{c}_2\boldsymbol{a}, && (2.15)\\
\boldsymbol{c}_1 - \boldsymbol{b}_1\boldsymbol{c}_1^2 &= \boldsymbol{b}_2, && (2.16)\\
\boldsymbol{c}_2 - \boldsymbol{b}_1\boldsymbol{c}_1\boldsymbol{c}_2 - \boldsymbol{b}_1\boldsymbol{c}_2\boldsymbol{\rho} &= \boldsymbol{b}_3, && (2.17)\\
\boldsymbol{c}_3 - \boldsymbol{b}_1\boldsymbol{c}_1\boldsymbol{c}_3 &= \boldsymbol{b}_4. && (2.18)
\end{aligned}$$

A straightforward computation (see Appendix A) shows that the mean of the REE $\overline{\boldsymbol{x}^*}$ satisfies

$$\overline{\boldsymbol{x}^*} = (\boldsymbol{I} - \boldsymbol{b}_1 - \boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_3\boldsymbol{(I - \rho)^{-1}}\boldsymbol{a}], \tag{2.19}$$

where $\boldsymbol{I}$ denotes a comfortable identity matrix throughout the paper. In the special case with $\boldsymbol{\rho} = \rho\boldsymbol{I}$ [4] and $\boldsymbol{b}_2 = \boldsymbol{0}$, the rational expectations equilibrium $\boldsymbol{x}_t^*$ satisfies

$$\boldsymbol{x}_t^* = (\boldsymbol{I} - \boldsymbol{b}_1)^{-1}\boldsymbol{b}_0 + (\boldsymbol{I} - \boldsymbol{b}_1)^{-1}\boldsymbol{b}_1(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3\boldsymbol{a} + (\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3\boldsymbol{u}_t + \boldsymbol{b}_4\boldsymbol{v}_t. \tag{2.20}$$

Thus its unconditional mean is:

$$\overline{\boldsymbol{x}^*} = E(\boldsymbol{x}_t^*) = (1-\rho)^{-1}(\boldsymbol{I} - \boldsymbol{b}_1)^{-1}[\boldsymbol{b}_0(1-\rho) + \boldsymbol{b}_3\boldsymbol{a}]. \tag{2.21}$$

Its variance-covariance matrix is:

$$\boldsymbol{\Sigma}_{\boldsymbol{x}^*} = E[(\boldsymbol{x}_t^* - \overline{\boldsymbol{x}^*})(\boldsymbol{x}_t^* - \overline{\boldsymbol{x}^*})'] = (1-\rho^2)^{-1}(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}[(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3]' + \boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4' \tag{2.22}$$

[4] Note that $\boldsymbol{\rho}$ is a matrix while ρ is a scalar number throughout the paper.

Furthermore, the first-order autocovariance is

$$\boldsymbol{\Sigma}_{\boldsymbol{x}^*\boldsymbol{x}^*_{-1}} = E[(\boldsymbol{x}^*_t - \overline{\boldsymbol{x}^*})(\boldsymbol{x}^*_{t-1} - \overline{\boldsymbol{x}^*})'] = \rho(1-\rho^2)^{-1}(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}[(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_3]'. \quad (2.23)$$

The first-order autocorrelation of the i-element x^*_i of $\boldsymbol{x}^*$ is the i-th diagonal element of matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}^*\boldsymbol{x}^*_{-1}}$ divided by the corresponding i-th diagonal element of matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}^*}$. Furthermore, if $\boldsymbol{\Sigma}_{\boldsymbol{v}} = \boldsymbol{0}$, then the first-order autocorrelation of the i-element u_i of $\boldsymbol{u}$ is equal to ρ. In this case the persistence of the i-th variable x^*_i in the REE coincides exactly with the persistence of the exogenous driving force $u_{i,t}$. That is, in this case the persistence in the REE only inherits the persistence of the exogenous driving force.

Existence of BLE

Now assume that agents are boundedly rational and do not believe or recognize that the economy is driven by an exogenous VAR(1) process $\boldsymbol{u}_t$, but use a simple univariate linear rule to forecast the state $\boldsymbol{x}_t$ of the economy. Given that agents' perceived law of motion is a restricted VAR(1) process as in (2.2), the actual law of motion becomes

$$\boldsymbol{x}_t = \boldsymbol{b}_0 + \boldsymbol{b}_1[\boldsymbol{\alpha} + \boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha})] + \boldsymbol{b}_2\boldsymbol{x}_{t-1} + \boldsymbol{b}_3\boldsymbol{u}_t + \boldsymbol{b}_4\boldsymbol{v}_t, \quad (2.24)$$

with $\boldsymbol{u}_t$ given in (2.11). If all eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2$, for each $\beta_i \in [-1,1], 1 \leq i \leq n$, lie inside the unit circle, then the system (2.24) of $\boldsymbol{x}_t$ is stationary and hence its mean $\overline{\boldsymbol{x}}$ and first-order autocorrelation $\boldsymbol{G}$ exist.

The mean of $\boldsymbol{x}_t$ in (2.24) is computed as

$$\overline{\boldsymbol{x}} = (\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^2 - \boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{\alpha} - \boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\alpha} + \boldsymbol{b}_3(\boldsymbol{I} - \boldsymbol{\rho})^{-1}\boldsymbol{a}]. \quad (2.25)$$

Imposing the first consistency requirement of a BLE on the mean, i.e. $\overline{\boldsymbol{x}} = \boldsymbol{\alpha}$, and solving for $\boldsymbol{\alpha}$ yields

$$\boldsymbol{\alpha}^* = (\boldsymbol{I} - \boldsymbol{b}_1 - \boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_3(\boldsymbol{I} - \boldsymbol{\rho})^{-1}\boldsymbol{a}]. \quad (2.26)$$

Comparing with (2.19), we conclude that in a BLE the unconditional mean $\boldsymbol{\alpha}^*$ coincides with the REE mean. That is to say, in a BLE the state of the economy $\boldsymbol{x}_t$ fluctuates on average around its RE fundamental value $\boldsymbol{x}^*$.

Consider the second consistency requirement of a BLE on the first-order autocorrelation coefficient matrix $\boldsymbol{\beta}$ of the PLM. The second consistency requirement yields

$$\boldsymbol{G}(\boldsymbol{\beta}) = \boldsymbol{\beta}. \quad (2.27)$$

Recall from Section 2, both $\boldsymbol{G}$ and $\boldsymbol{\beta}$ are diagonal matrices. For convenience let G_i denote the i-th diagonal element of the matrix $\boldsymbol{G}$ in (2.5). Under the assumption that all of the eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2$ for each $\beta_i \in [-1,1] (i = 1, 2, \cdots, n)$ lie inside the unit circle, from the theory of stationary linear time series, $G_i(\beta_1, \beta_2, \cdots, \beta_n) \in [-1, 1]$ and is a smooth function with respect to $(\beta_1, \beta_2, \cdots, \beta_n)$ and other model parameters, see Appendix B[5]. Based on Brouwer's fixed-point theorem for $(G_1, G_2, \cdots, G_n)$, there exists $\boldsymbol{\beta}^* = (\beta_1^*, \beta_2^2, \cdots, \beta_n^*)$ with each $\beta_i^* \in [-1, 1]$, such that $\boldsymbol{G}(\boldsymbol{\beta^*}) = \boldsymbol{\beta^*}$. We conclude:

Proposition 1 *If all eigenvalues of $\boldsymbol{\rho}$ and $\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2$, for each $\beta_i \in [-1, 1]$, are inside the unit circle[6], there exists at least one behavioral learning equilibrium $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ for the economic system (2.24) with $\boldsymbol{\alpha}^* = (\boldsymbol{I} - \boldsymbol{b}_1 - \boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_3(\boldsymbol{I} - \boldsymbol{\rho})^{-1}\boldsymbol{a}] = \overline{\boldsymbol{x}^*}$.*

Stability under SAC-learning

In this subsection we study the stability of BLE under SAC-learning. The ALM of the economy under SAC-learning is given by

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{b}_0 + \boldsymbol{b}_1[\boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}_{t-1})] + \boldsymbol{b}_2\boldsymbol{x}_{t-1} + \boldsymbol{b}_3\boldsymbol{u}_t + \boldsymbol{b}_4\boldsymbol{v}_t, \\ \boldsymbol{u}_t = \boldsymbol{a} + \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t. \end{cases} \tag{2.28}$$

with $\boldsymbol{\alpha}_t$, $\boldsymbol{\beta}_t$ updated based on realized sample average and sample autocorrelation as in (2.8). Appendix C shows that the E-stability principle applies and that stability under SAC-learning is determined by the associated ordinary differential equation (ODE)[7]

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \boldsymbol{\alpha} = (\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^2 - \boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{\alpha} - \boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\alpha} + \boldsymbol{b}_3(\boldsymbol{I} - \boldsymbol{\rho})^{-1}\boldsymbol{a}] - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{G}(\boldsymbol{\beta}) - \boldsymbol{\beta}, \end{cases} \tag{2.29}$$

where $\overline{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the mean given by (2.25) and $\boldsymbol{G}(\boldsymbol{\beta})$ is the diagonal first-order autocorrelation matrix. A BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ corresponds to a fixed point of the ODE (2.29). Moreover,

[5]For example, refer to the expression (3.9) in Hommes and Zhu (2014) for the special 1-dimensional case $n = 1$ and $\boldsymbol{b}_2 = \boldsymbol{0}$. In Section 3 we consider the New Keynesian model with two forward-looking variables and compute the (complicated) expressions of $G_1(\beta_1, \beta_2)$ and $G_2(\beta_1, \beta_2)$ explicitly.

[6]The Schur-Cohn criterion theorem provides necessary and sufficient conditions for all eigenvalues to lie inside the unit circle, see Elaydi (1999). For specific models, one may find sufficient conditions to guarantee that all eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2 + \boldsymbol{b}_2$, for each $\beta_i \in [-1, 1]$, are inside the unit circle. For example, in the case of the NK model, the Taylor principle is a sufficient condition to ensure that all eigenvalues lie inside the unit circle for all $\beta_i \in [-1, 1]$; see Corollary 2 and Appendix E.

[7]See Evans and Honkapohja (2001) for a discussion and mathematical treatment of E-stability.

a BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is locally stable under SAC-learning if it is a stable fixed point of the ODE (2.29). Therefore, we have the following property of SAC-learning stability.

Proposition 2 *A BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is locally stable (E-stable) under SAC-learning if*

(i) all eigenvalues of $(\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^{*2} - \boldsymbol{b}_2)^{-1}(\boldsymbol{b}_1 + \boldsymbol{b}_2 - \boldsymbol{I})$ have negative real parts[8]*, and*

(ii) all eigenvalues of $\boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*)$ have real parts less than 1, where $\boldsymbol{DG}_{\boldsymbol{\beta}}$ is the Jacobian matrix with the (i, j)-th entry $\frac{\partial G_i}{\partial \beta_j}$.

Proof. See Appendix C.

Recall from Subsection 2.1 that $G_i(\beta_1, \beta_2, \cdots, \beta_n) \in (-1, 1)$ so that at least one BLE exists. The proposition above implies that the BLE may also be E-stable under SAC-learning.

2.2 Estimation of BLE

As our application to the NK model in the next section will illustrate, finding an analytical expression for a BLE is in general not possible. Therefore we provide a general iteration method to estimate a BLE of the linear system (2.10) and (2.11). The main challenge here is to estimate the structural parameters together with the BLE belief parameters $\boldsymbol{\beta}^*$ that satisfy a highly non-linear consistency (fixed point) constraint $\boldsymbol{\beta}^* = G(\boldsymbol{\beta}^*)$. We first use the notion of iterative E-stability to find an approximate BLE. An advantage of this method is that when it converges, the BLE is always stable under adaptive learning. We next propose an iterative estimation procedure, which is closely linked to the notion of iterative E-stability and which is a recursion of Bayesian estimation of linear models.

We first re-write the system by augmenting $\boldsymbol{x}_t$ with $\boldsymbol{u}_t$ to obtain

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{b}_3 \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{u}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_0 \\ \boldsymbol{a} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\rho} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{u}_{t-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^e_{t+1} \\ \boldsymbol{u}^e_{t+1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_4 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_t \\ \boldsymbol{\epsilon}_t \end{bmatrix}. \quad (2.30)$$

Defining[9]

$$\begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{u}_t \end{bmatrix} = S_t, \begin{bmatrix} \boldsymbol{v}_t \\ \boldsymbol{\epsilon}_t \end{bmatrix} = \eta_t, \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{b}_3 \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} = \tilde{\gamma}, \tilde{\gamma}^{-1} \begin{bmatrix} \boldsymbol{b}_0 \\ \boldsymbol{a} \end{bmatrix} = \bar{\gamma}, \quad \tilde{\gamma}^{-1} \begin{bmatrix} \boldsymbol{b}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\rho} \end{bmatrix} = \gamma_1,$$

[8] The Routh-Hurwitz criterion theorem provides sufficient and necessary conditions for all the n eigenvalues having negative real parts, see Brock and Malliaris (1989).

[9] We assume the invertibility conditions of the corresponding matrices are satisfied throughout the paper.

$$\tilde{\gamma}^{-1}\begin{bmatrix}\boldsymbol{b}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \gamma_2,\ \ \tilde{\gamma}^{-1}\begin{bmatrix}\boldsymbol{b}_4 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}\end{bmatrix} = \gamma_3.$$

We can re-write the law of motion as

$$S_t = \bar{\gamma} + \gamma_1 S_{t-1} + \gamma_2 S^e_{t+1} + \gamma_3 \eta_t. \tag{2.31}$$

Denote the learning vector for the mean parameters $\boldsymbol{\alpha}$, and the diagonal learning matrix for the first-order autocorrelation parameters $\boldsymbol{\beta}$, with the corresponding learning parameters by α_x and β_x for each endogenous variable x_t as before[10]. Then the agent's PLM, the corresponding one-step ahead expectations and the implied ALM are given as

$$\begin{cases} S_t = \boldsymbol{\alpha} + \boldsymbol{\beta}(S_{t-1} - \boldsymbol{\alpha}) + \boldsymbol{\delta_t}, \\ E_t S_{t+1} = \boldsymbol{\alpha} + \boldsymbol{\beta}^2(S_{t-1} - \boldsymbol{\alpha}), \\ S_t = (\bar{\gamma} + \boldsymbol{\alpha} - \boldsymbol{\beta}^2\boldsymbol{\alpha}) + \gamma_1 S_{t-1} + \gamma_2 \boldsymbol{\beta}^2 S_{t-1} + \gamma_3 \eta_t. \end{cases} \tag{2.32}$$

Our main focus in this section is to estimate log-linearized DSGE models, where the mean $\boldsymbol{\alpha^*}$ is available based on (2.26). Without loss of generality, we focus on the case where $\boldsymbol{\alpha^*} = 0$. Denoting by $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(-1)$ the variance-covariance and first-order covariance matrices as before, one can show that[11]

$$\begin{cases} Vec(\boldsymbol{\Gamma}(0)) = [I - M(\boldsymbol{\beta^*}) \otimes M(\boldsymbol{\beta^*})]^{-1}(\gamma_3 \otimes \gamma_3) Vec(\boldsymbol{\Sigma_\eta}), \\ Vec(\boldsymbol{\Gamma}(-1)) = [I \otimes M(\boldsymbol{\beta^*})] Vec(\boldsymbol{\Gamma}(0)), \end{cases} \tag{2.33}$$

where $M(\boldsymbol{\beta}^*) = \gamma_1 + \gamma_2 \boldsymbol{\beta^*}^2$, and $\boldsymbol{\Sigma_\eta}$ is the variance-covariance matrix of i.i.d disturbances η_t. This implies that $\beta_j{}^* = \frac{Vec(\boldsymbol{\Gamma}(-1))_{N(j-1)+j}}{Vec(\boldsymbol{\Gamma}(0))_{N(j-1)+j}} = G_j(\boldsymbol{\beta}^*, \theta)$, $1 \le j \le N$, where θ represents the set of structural parameters in γ_1, γ_2 and γ_3. Then every BLE satisfies

$$\begin{cases} S_t = \gamma_1 S_{t-1} + \gamma_2 \boldsymbol{\beta^*}^2 S_{t-1} + \gamma_3 \eta_t \\ \beta^*_j = \frac{Vec(\boldsymbol{\Gamma_{-1}})_{N(j-1)+j}}{Vec(\boldsymbol{\Gamma_0})_{N(j-1)+j}} = G_j(\boldsymbol{\beta}^*, \theta), 1 \le j \le N. \end{cases} \tag{2.34}$$

The E-stability conditions of Proposition 2 are easily simplified to the case with zero mean. Accordingly, a BLE $(\mathbf{0},\ \boldsymbol{\beta}^*)$ is locally stable if all eigenvalues of $(I - \gamma_1 - \gamma_2 \boldsymbol{\beta^*}^2)(\gamma_1 + \gamma_2 - I)$ have negative real parts and all eigenvalues of $DG_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*)$ have real parts less than one. Note

[10]Without loss of generality, we assume the first N variables in S_t are the forward-looking variables and we introduce zeros for the remaining state variables and exogenous shocks.

[11]See Appendix B.

that the first condition governs the stability of mean coefficients, while the second condition relates to stability of first-order autocorrelation coefficients *independent* of $\boldsymbol{\alpha^*}$.

Iterative E-stability and Estimation of BLE

The first-order autocorrelation coefficients $\boldsymbol{\beta}^*$ in (2.34) are functions in terms of the structural parameters θ, which satisfy the highly nonlinear equilibrium conditions $G_j(\boldsymbol{\beta}^*, \theta) = \boldsymbol{\beta}^*$ and cannot be computed analytically. In order to find a BLE for a given θ, we use a simple fixed-point iteration, which is formalized below in Algorithm I.

Algorithm I: Approximation of a BLE using Iterative E-stability

---

Denote by θ the set of structural parameters, and by $G(\boldsymbol{\beta^{(k)}}, \theta)$ the first-order autocorrelation function for a given θ.

- Step (0): Initialize the vector of learning parameters at $\boldsymbol{\beta^{(0)}}$.
- Step (I): At each iteration k, using the first-order autocorrelation functions, update the vector of learning parameters as

$$\boldsymbol{\beta^{(k)}} = G(\boldsymbol{\beta^{(k-1)}}, \theta), \tag{2.35}$$

  where $G(\boldsymbol{\beta^{(k-1)}}, \theta)$ is known at iteration $k-1$.
- Step (II): Terminate if $||\boldsymbol{\beta^{(k)}} - \boldsymbol{\beta^{(k-1)}}||_p < \epsilon$, for a small scalar $\epsilon > 0$[12] and a suitable norm distance $||.||_p$, otherwise repeat Step (I).

A BLE $(\mathbf{0}, \boldsymbol{\beta^*})$ is locally stable under (2.35) if all eigenvalues of $DG_{\boldsymbol{\beta}}(\boldsymbol{\beta^*})$ lie inside the unit circle. Then the equilibrium is said to be *iteratively E-stable*. When Algorithm I terminates for some K at a pre-specified ϵ, we say that it has converged to $\boldsymbol{\beta^{(K)}}$. First note that, if Algorithm I converges, it converges to an approximate BLE since

$$||\boldsymbol{\beta^{(K+1)}} - \boldsymbol{\beta^{(K)}}|| < c \Rightarrow ||\boldsymbol{G}(\boldsymbol{\beta^{(K)}}) - \boldsymbol{\beta^{(K)}}|| < c \Rightarrow \boldsymbol{G}(\boldsymbol{\beta^{(K)}}) \approx \boldsymbol{\beta^{(K)}}.$$

Further note that, if we assume the learning dynamics for the mean $\boldsymbol{\alpha^*}$ are E-stable[13], there

[12] Throughout the remainder of this paper, we use the common L^1-Norm as our norm distance, i.e. $||\boldsymbol{\beta^{(k)}} - \boldsymbol{\beta^{(k-1)}}||_p = \sum_{j=1}^{N} |\beta_j^{(k)} - \beta_j^{(k-1)}|$.

[13] Using Proposition 2, the E-stability condition for $\boldsymbol{\alpha^*} = 0$ is satisfied when $\xi((I - (\gamma_1 + \gamma_2 \boldsymbol{\beta^{*}}^2))(\gamma_1 + \gamma_2 - I)) < 0$, where $\xi(G(\boldsymbol{\beta^*})) = max\{Re(\lambda) : \lambda \in \mathbb{C}$ an eigenvalue of $DG(\boldsymbol{\beta^*})\}$.

is a simple connection between Iterative E-stability and E-stability of $\boldsymbol{\beta^*}$: for E-stability, the real parts of all eigenvalues of $DG(\boldsymbol{\beta^*})$ must be less than one, while iterative E-stability requires the eigenvalues to lie inside the unit circle. This immediately implies that iterative E-stability is a stronger condition than E-stability, which gives us the following corollary.

Corollary 1 *Iterative E-stability of $\boldsymbol{\beta^*}$ implies E-stability of $\boldsymbol{\beta^*}$. Therefore if Algorithm I converges, it converges to an E-stable approximate BLE.*

The iteration function in (2.35) plays an important role for the above corollary, where our choice of the function $G(.)$ reduces Algorithm I to the simplest fixed-point iteration known as *Iterative E-stability* in the adaptive learning literature (Evans & Honkapohja, 2001). Iterations of this type have been used as a learning algorithm in the earlier literature, see e.g. DeCanio (1979), Bray (1982) and Evans (1985). In this paper, we use it as our approximation method, which allows us to eliminate E-unstable BLE without additional steps. As an alternative, one could also consider a Quasi-Newton iteration of the following form:

$$\boldsymbol{\beta^{(k)}} = \boldsymbol{\beta^{(k-1)}} - DF(\boldsymbol{\beta^{(k-1)}}, \theta)^{-1} F(\boldsymbol{\beta^{(k-1)}}, \theta). \tag{2.36}$$

where $F(\boldsymbol{\beta}, \theta) = \boldsymbol{\beta} - G(\boldsymbol{\beta}, \theta)$ and $DF_{\boldsymbol{\beta}}(\boldsymbol{\beta}, \theta)$ denotes the Jacobian of $F(\boldsymbol{\beta}, \theta)$[14]. This latter algorithm has been used in e.g. Farmer et. al. (2009) to compute MSV-solutions in Markov-switching models. However, a downside of the Quasi-Newton iteration in our context is that both E-stable and E-unstable BLE are locally stable under (2.36), which means this iteration method is not informative about E-stability of BLE[15]. Therefore we use the notion of *Iterative E-stability* in our estimations.

The discussion up to this point is based on finding E-stable BLE for a given set of structural model parameters θ. In the following, we provide a straightforward extension of Algorithm I to accommodate the joint estimation of the structural parameters and the BLE parameters. In order to estimate the model, we add a set of measurement equations to the law of motion in (2.34) as follows:

$$y_t = \psi_0(\theta) + \psi_1(\theta) S_t + h_t, \tag{2.37}$$

[14]At each iteration k, we approximate the Jacobian using $\frac{\partial F_i(\boldsymbol{\beta^{(k)}})}{\partial \beta_j^{(k)}} \approx \frac{F_i(\boldsymbol{\beta^{(k)}} + h\vec{e_j}) - F_i(\boldsymbol{\beta^{(k)}})}{h}$, $1 \leq i, j \leq N$, where $\vec{e_j}$ denotes a suitable unit vector.

[15]See Appendix D for a formal treatment of this and our online appendix for an example.

where y_t denotes a vector of observable variables, h_t is a vector of measurement errors, $\psi_0(\theta)$ and $\psi_1(\theta)$ are matrices of the structural parameters that relate the state variables S_t and the observable variables y_t. Together with (2.34), (2.37) yields the state-space representation of the DSGE model under BLE. The model is *linear* in the state variables S_t, but the BLE learning parameters $\boldsymbol{\beta}^*$ are a highly non-linear function in terms of the structural parameters θ to be estimated. On the one hand, given the non-linear function $\boldsymbol{\beta}^* = G(\boldsymbol{\beta}^*, \theta)$, we can find $\boldsymbol{\beta}^*$ using iterative E-stability for a given θ. On the other hand, whenever $\boldsymbol{\beta}$ is temporarily fixed at $\boldsymbol{\beta}^{(k)}$ at the k^{th} iteration, the model reduces to a linear state-space model that can be estimated using standard Bayesian methods. Based on this, we consider an iterative routine where the structural parameters θ and belief parameters $\boldsymbol{\beta}$ are estimated sequentially. The estimation is summarized in Algorithm II.

[16]For a detailed textbook derivation of the likelihood function and the posterior distribution, see e.g. Greenberg (2012) or Herbst & Schorfheide (2015). In this paper, we make use of the routines available in Dynare to estimate the model at each step for a given set of fixed learning parameters.

Algorithm II: Bayesian Estimation of an (iteratively) E-stable BLE

Denote by $Y_{1:T} = \{Y_1, \cdots, Y_T\}$ the matrix of the observable variables up to period T, and by $p(\theta)$ the prior distributions for the structural parameters θ that appear in matrices γ_1, γ_2 and γ_3. Consider the system characterized by (2.34) and (2.37):

$$\begin{cases} S_t = \gamma_1(\theta)S_{t-1} + \gamma_2(\theta)\boldsymbol{\beta}^{*2}S_{t-1} + \gamma_3(\theta)\eta_t \\ \beta_j^* = G_j(\boldsymbol{\beta}^*, \theta), 1 \leq j \leq N \\ y_t = \psi_0(\theta) + \psi_1(\theta)S_t + h_t \\ \end{cases} \tag{2.38}$$

- **Step (0)** Initialize a set of learning parameters $\boldsymbol{\beta^{(0)}}$. At the (temporarily) fixed $\boldsymbol{\beta^{(0)}}$, the system (2.38) reduces to a standard state-space representation for the linearized DSGE model.

- **Step (I-a)** At each iteration k, one can obtain the likelihood function using the Kalman filter and the corresponding posterior distribution conditional on $\beta^{(k-1)}$ as follows[16]:

$$p(Y_{1:T}|\theta, \boldsymbol{\beta}^{(k-1)}) = \sum_{t=1}^{T} p(y_t|Y_{1:T-1}, \theta, \boldsymbol{\beta}^{(k-1)}); \quad p(\theta|Y_{1:T}, \boldsymbol{\beta}^{(k-1)}) = \frac{p(Y_{1:T}|\theta, \boldsymbol{\beta}^{(k-1)})p(\theta)}{p(Y_{1:T}, \boldsymbol{\beta}^{(k-1)})}, \tag{2.39}$$

  where $\boldsymbol{\beta^{(k-1)}}$ is obtained from iteration $k-1$, and $p(Y_{1:T}, \boldsymbol{\beta^{(k-1)}})$ denotes the marginal likelihood function. Denote by $\hat{\theta}^{(k)}$ the conditional posterior mode obtained from

$$\hat{\theta}^{(k)} = \underset{\theta}{\operatorname{argmax}} \;\; p(\theta|Y_{1:T}, \boldsymbol{\beta^{(k-1)}}). \tag{2.40}$$

- **Step (I-b)** Using $\hat{\theta}^{(k)}$, update the matrix of learning parameters:

$$\beta_j^{(k)} = G_j(\boldsymbol{\beta^{(k-1)}}, \hat{\theta}^{(k)}), 1 \leq j \leq N. \tag{2.41}$$

- **Step(II)** Proceed to Step (III) if $||\boldsymbol{\beta^{(k)}} - \boldsymbol{\beta^{(k-1)}}|| < \epsilon$ and $||\hat{\theta}^{(k)} - \hat{\theta}^{(k-1)}|| < \epsilon$ for a given scalar $\epsilon > 0$, otherwise repeat Step (I).

- **Step(III)** Use the Metropolis-Hastings algorithm to construct the posterior distribution *conditional on the BLE* at the posterior mode.

A BLE $(\mathbf{0}, \boldsymbol{\beta^*})$ satisfying $G(\boldsymbol{\beta^*}, \theta^*) = \boldsymbol{\beta^*}$ and $\theta^* = \underset{\theta}{\mathrm{argmax}} p(\theta|Y_{1:T}, \boldsymbol{\beta^*})$, is locally stable under (2.40) and (2.41) if all eigenvalues of $DG(\boldsymbol{\beta}^*, \theta^*)$ lie inside the unit circle[17,18].

The estimation routine described as above corresponds to a straightforward extension of Algorithm I, where we allow the structural parameters θ (and therefore the matrices γ_1, γ_2 and γ_3) to be re-estimated at each step of the fixed-point iteration in (2.35). Our approach is similar to e.g. the computation of initial beliefs in Slobodyan & Wouters (2012), where the belief coefficients in $\boldsymbol{\beta}$ are treated as separate parameters and estimated along with θ. The main difference here is that we compute the equilibrium beliefs consistent with the underlying BLE, such that the first-order autocorrelations in the PLM coincide with the ALM at the estimated posterior mode. In other words, the belief parameters are consistent with the actual realizations and there is no adaptive learning involved. Our estimation approach is fast and easy to implement, because it allows us to approximate and estimate a BLE at the posterior mode of a sequence of linear models. Since the beliefs in $\boldsymbol{\beta}^{(k)}$ are updated at each step k based on the first-order autocorrelations of the state variables, the estimated parameters $\hat{\theta}^{(k)}$ tend to lead $\boldsymbol{\beta}^{(k)}$ towards the empirically relevant region. In turn, this allows the system to rapidly converge to the underlying BLE as we illustrate in the next section. Once we find a BLE along with estimated structural parameters under Algorithm II, we check for (iterative) E-stability and multiplicity of stable equilibria by simulating the system under (2.8) and (2.35) with randomized initial values.

As an alternative to this algorithm that directly estimates a BLE, we also consider an estimation routine with SAC-learning based on the Kalman filter output. Since iterative E-stability guarantees convergence under SAC-learning, allowing the agents learn simultaneously with the Kalman filter recursions serves as an indirect approach to estimate a BLE, as well as a robustness check for the empirical fit of a BLE. The model under SAC-learning is conditionally linear for a given set of belief coefficients and therefore one can use the standard Kalman filter to obtain the likelihood function, where the beliefs are updated in each step using the Kalman filter output. Similar approaches have been

[17] In order to formally rule out explosive outcomes, one can augment the algorithm with a projection facility, where the next iteration is projected to a point inside the unit cube if the iteration $G(\boldsymbol{\beta}^{k-1})$ leads to $|\beta_i^{(k)}| > 1$ for some $1 \leq i \leq N$. We do not observe such outcomes in the NK model considered in this paper and therefore do not use a projection facility.

[18] The eigenvalue condition under (2.40) and (2.41) is slightly different than the eigenvalue condition under (2.35), since the second argument θ^* of $G(\boldsymbol{\beta^*}, \theta^*)$ also depends on $\boldsymbol{\beta^*}$, see Appendix D for more details.

used in estimating constant gain least squares and Kalman gain adaptive learning models in Milani (2005, 2007) and Slobodyan & Wouters (2012) respectively. In this paper we focus on the decreasing-gain SAC-learning algorithm since our primary interest is the estimation of the underlying fixed-point BLE, rather than the time-variation in beliefs. See Appendix D for a more detailed description of this approach with SAC-learning.

3 Application: a New Keynesian model

3.1 A baseline model

In this section we apply these results within the framework of a standard New Keynesian model along the lines of Woodford (2003) and Galí (2008). Consider a simple version, linearized around the zero inflation steady state, given by

$$\begin{cases} y_t = y^e_{t+1} - \varphi(r_t - \pi^e_{t+1}) + u_{y,t}, \\ \pi_t = \lambda \pi^e_{t+1} + \gamma y_t + u_{\pi,t}, \end{cases} \tag{3.1}$$

where y_t is the output gap, π_t is the inflation rate, y^e_{t+1} and π^e_{t+1} are expected output gap and expected inflation. Following Bullard and Mitra (2002) and Bullard et al. (2008) we study the NK-model (3.1) with adaptive learning. The terms $u_{y,t}, u_{\pi,t}$ are stochastic shocks and are assumed to follow AR(1) processes

$$u_{y,t} = \rho_y u_{y,t-1} + \varepsilon_{y,t}, \tag{3.2}$$

$$u_{\pi,t} = \rho_\pi u_{\pi,t-1} + \varepsilon_{\pi,t}, \tag{3.3}$$

where $\rho_i \in [0,1)$ and $\{\varepsilon_{i,t}\}$ $(i = y, \pi)$ are two uncorrelated i.i.d. stochastic processes with zero mean and finite absolute moments with corresponding variances σ_i^2.

The first equation in (3.1) is an IS curve that describes the demand side of the economy. In an economy of rational or boundedly rational agents, it is a linear approximation to a representative agent's Euler equation. The parameter $\varphi > 0$ is related to the elasticity of intertemporal substitution in consumption of a representative household, and its inverse can be interpreted as a risk aversion coefficient. The second equation in (3.1) is the New Keynesian Phillips curve which describes the aggregate supply relation. This is obtained by averaging all firms' pricing decisions.The parameter γ is related to the degree of price stickiness in the economy and the parameter $\lambda \in [0,1)$ is the discount factor of a representative household.

We supplement the equations in (3.1) with a standard Taylor-type policy rule, which represents the behavior of the monetary authority in setting the nominal interest rate:

$$r_t = \phi_\pi \pi_t + \phi_y y_t, \tag{3.4}$$

where i_t is the deviation of the nominal interest rate from the value that is consistent with inflation at target and output at potential. The parameters ϕ_π, ϕ_y, measuring the response of i_t to the deviation of inflation and output from long run steady states, are assumed to be non-negative[19].

Substituting the Taylor-type policy rule in (3.4) into (3.1) and writing the model in matrix form gives

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{B}\boldsymbol{x}^e_{t+1} + \boldsymbol{C}\boldsymbol{u}_t, \\ \boldsymbol{u}_t = \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t, \end{cases} \tag{3.5}$$

where $\boldsymbol{x}_t = [y_t, \pi_t]', \boldsymbol{u}_t = [u_{y,t}, u_{\pi,t}]', \boldsymbol{\varepsilon}_t = [\varepsilon_{y,t}, \varepsilon_{\pi,t}]', \boldsymbol{B} = \frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} \begin{bmatrix} 1 & \varphi(1-\lambda\phi_\pi) \\ \gamma & \gamma\varphi + \lambda(1+\varphi\phi_y) \end{bmatrix}$,

$\boldsymbol{C} = \frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} \begin{bmatrix} 1 & -\varphi\phi_\pi \\ \gamma & 1+\varphi\phi_y \end{bmatrix}, \boldsymbol{\rho} = \begin{bmatrix} \rho_y & 0 \\ 0 & \rho_\pi \end{bmatrix}$.

Before turning to BLE, we first consider the Rational Expectations Equilibrium.

3.2 Theoretical results

Comparing the NK model (3.5) with the general framework (2.10), we note that $\boldsymbol{a} = \mathbf{0}$, $\boldsymbol{b}_0 = \mathbf{0}$ and $\boldsymbol{b}_2 = \mathbf{0}$. The Rational Expectation Equilibrium (REE) fixed point in (2.15-2.18) then simplifies to

$$(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{\xi} = \mathbf{0} \tag{3.6}$$

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta}\boldsymbol{\rho} + \mathbf{C}. \tag{3.7}$$

Bullard and Mitra (2002) show that the REE is unique (determinate) if and only if $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$. The REE is then the stable stationary process with mean

$$\overline{\mathbf{x}^*} = \mathbf{0}. \tag{3.8}$$

In the symmetric case $\rho_i = \rho$ for $i = 1, 2, \cdots, n$, the REE $\mathbf{x}^*_t$ satisfies

$$\mathbf{x}^*_t = (\mathbf{I} - \rho\mathbf{B})^{-\mathbf{1}}\mathbf{C}\mathbf{u_t}. \tag{3.9}$$

[19] In our online appendix we also discuss *lagged* and *forward-looking* Taylor rules, responding to lagged and expected future values of y_t and π_t respectively.

Thus its covariance is

$$\mathbf{\Sigma}_{\mathbf{x}^*} = \mathbf{E}(\mathbf{x_t^*} - \overline{\mathbf{x}^*})(\mathbf{x_t^*} - \overline{\mathbf{x}^*})' \quad = \quad (1-\rho^2)^{-1}(\mathbf{I} - \rho\mathbf{B})^{-1}\mathbf{C}\mathbf{\Sigma}_{\boldsymbol{\varepsilon}}[(\mathbf{I} - \rho\mathbf{B})^{-1}\mathbf{C}]'. \quad (3.10)$$

Furthermore, the first-order autocorrelation of the i-element x_i of $\mathbf{x}$ is equal to ρ. That is, in this case the persistence of the REE coincides exactly with the persistence of the exogenous driving force $\mathbf{u}_t$ and the first-order autocorrelations of output gap and inflation are the same, i.e. symmetric, equal to the autocorrelation in the driving force. Under RE, Inflation and output gap only inherit the persistence of the shocks.

Behavioral learning equilibria

Bullard and Mitra (2002) study adaptive learning in this NK setting. They consider a PLM which coincides with the minimum state variable solution (MSV) of the form

$$\boldsymbol{x}_t = \widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{E}}\boldsymbol{x}^e_{t+1} + \widetilde{\boldsymbol{F}}\boldsymbol{u}_t, \quad (3.11)$$

where $\widetilde{\boldsymbol{D}}$, $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{F}}$ are conformable matrices. We will consider learning with *misspecification*. As in the general setup in Section 2, we assume that agents are boundedly rational and use simple univariate linear rules to forecast the output gap y_t and inflation π_t of the economy. Therefore we deviate from Bullard and Mitra (2002) in two important ways: (i) our agents cannot observe or do not use the exogenous shocks $\boldsymbol{u}_t$, and (ii) agents do not fully understand the linear stochastic structure and do not take into account the cross-correlation between inflation and output. Rather our agents learn simple univariate AR(1) forecasting rules for inflation and output gap, as in (2.2). However these AR(1) rules indirectly, in a boundedly rational way, take exogenous shocks and cross-correlations of endogenous variables into account as agents learn the two parameters of each AR(1) rule consistent with the observable sample averages and first-order autocorrelations of the state variables inflation and output gap. The use of simple AR(1) rule is supported by evidence from the learning-to-forecast laboratory experiments in the NK framework in Adam (2007), Assenza et al. (2014) and Pfajfar and Zakelj (2016). The actual law of motion (3.5) becomes

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{B}[\boldsymbol{\alpha} + \boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha})] + \boldsymbol{C}\boldsymbol{u}_t, \\ \boldsymbol{u}_t = \boldsymbol{\rho}u_{t-1} + \boldsymbol{\varepsilon}_t. \end{cases} \quad (3.12)$$

For the actual law of motion (ALM) (3.12), the REE determinacy condition $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$ implies that the ALM is stationary, see Appendix E. Thus the means and first-order autocorrelations are

$$\begin{aligned}
\overline{\boldsymbol{x}} &= (\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}\boldsymbol{\alpha} - \boldsymbol{B}\boldsymbol{\beta}^2\boldsymbol{\alpha}), \\
\boldsymbol{G}(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \begin{bmatrix} G_1(\beta_1, \beta_2) & 0 \\ 0 & G_2(\beta_1, \beta_2) \end{bmatrix} = \begin{bmatrix} \mathrm{corr}(y_t, y_{t-1}) & 0 \\ 0 & \mathrm{corr}(\pi_t, \pi_{t-1})) \end{bmatrix}.
\end{aligned}$$

In order to obtain analytical expressions for $G_1(\beta_1, \beta_2)$ and $G_2(\beta_1, \beta_2)$ we focus on the symmetric case with $\rho_y = \rho_\pi = \rho$. The first-order autocorrelations of output gap and inflation can be expressed in terms of the structural parameters through very complicated calculations (see Appendix F[20])

$$G_1(\beta_1, \beta_2) = \frac{\widetilde{f}_1}{\widetilde{g}_1} \tag{3.13}$$

$$G_2(\beta_1, \beta_2) = \frac{\widetilde{f}_2}{\widetilde{g}_2} \tag{3.14}$$

where

$$\begin{aligned}
\widetilde{f}_1 = {} & \sigma_\pi^2 \Big\{ (\rho + \lambda_1 + \lambda_2 - \lambda\beta_2^2)[1 - \lambda\beta_2^2(\rho + \lambda_1 + \lambda_2)] + [\lambda\beta_2^2(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) - \\
& \rho\lambda_1\lambda_2][(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) - \lambda\beta_2^2\rho\lambda_1\lambda_2] \Big\} + \sigma_y^2 \Big\{ (\varphi\phi_\pi(\rho + \lambda_1 + \lambda_2) - \varphi\beta_2^2)) \\
& [\varphi\phi_\pi - \varphi\beta_2^2(\rho + \lambda_1 + \lambda_2)] + [\varphi\beta_2^2(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) - \varphi\phi_\pi\rho\lambda_1\lambda_2] \\
& [\varphi\phi_\pi(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) - \varphi\beta_2^2\rho\lambda_1\lambda_2] \Big\}, \\
\widetilde{g}_1 = {} & \sigma_\pi^2 \Big\{ [(1 + \lambda^2\beta_2^4) - 2\lambda\beta_2^2(\rho + \lambda_1 + \lambda_2) + (1 + \lambda^2\beta_2^4)(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2)] \\
& -\rho\lambda_1\lambda_2[(1 + \lambda^2\beta_2^4)(\rho + \lambda_1 + \lambda_2) - 2\lambda\beta_2^2(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) + (1 + \lambda^2\beta_2^4)\rho\lambda_1\lambda_2] \Big\} \\
& +\sigma_\pi^2 \Big\{ [((\varphi\phi_\pi)^2 + \varphi^2\beta_2^4) - 2\varphi\phi_\pi\varphi\beta_2^2(\rho + \lambda_1 + \lambda_2) + ((\varphi\phi_\pi)^2 + \varphi^2\beta_2^4)(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2)] \\
& -\rho\lambda_1\lambda_2[((\varphi\phi_\pi)^2 + \varphi^2\beta_2^4)(\rho + \lambda_1 + \lambda_2) - 2\varphi\phi_\pi\varphi\beta_2^2(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) \\
& +((\varphi\phi_\pi)^2 + \varphi^2\beta_2^4)\rho\lambda_1\lambda_2] \Big\},
\end{aligned} \tag{3.15}$$

[20] Appendix F employs the VARMA$(1, \infty)$ representation of the model. Although it is possible to obtain the expressions of $\boldsymbol{G}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ using the direct method in Appendix B, the analytical expressions are much more complicated. Numerical computations based on the two methods are consistent and also coincide with the simple numerical simulation of the first-order autocorrelation coefficients of output gap and inflation obtained from simulated time series generated by the system (3.12), confirming the complicated expressions (3.13-3.14).

$$
\begin{aligned}
\widetilde{f}_2 &= \sigma_y^2\Big\{\gamma^2[(\rho+\lambda_1+\lambda_2)-\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\Big\}+\sigma_\pi^2\Big\{[(1+\varphi\phi_y)(\rho+\lambda_1+\lambda_2)-\beta_1^2]\cdot \\
&\quad [(1+\varphi\phi_y)-\beta_1^2(\rho+\lambda_1+\lambda_2)]+[\beta_1^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-(1+\varphi\phi_y)\rho\lambda_1\lambda_2]\cdot \\
&\quad [(1+\varphi\phi_y)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\beta_1^2\rho\lambda_1\lambda_2]\Big\}, \\
\widetilde{g}_2 &= \sigma_y^2\Big\{\gamma^2[1+\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2-\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)-(\rho\lambda_1\lambda_2)^2]\Big\} \\
&\quad +\sigma_\pi^2\Big\{[((1+\varphi\phi_y)^2+\beta_1^4)-2(1+\varphi\phi_y)\beta_1^2(\rho+\lambda_1+\lambda_2)+((1+\varphi\phi_y)^2+\beta_1^4) \\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]-\rho\lambda_1\lambda_2[((1+\varphi\phi_y)^2+\beta_1^4)(\rho+\lambda_1+\lambda_2)-2(1+\varphi\phi_y)\beta_1^2\cdot \\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+((1+\varphi\phi_y)^2+\beta_1^4)\rho\lambda_1\lambda_2]\Big\}, \qquad (3.16)
\end{aligned}
$$

$$
\lambda_1+\lambda_2 = \frac{\beta_1^2+(\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}, \qquad (3.17)
$$

$$
\lambda_1\lambda_2 = \frac{\lambda\beta_1^2\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}. \qquad (3.18)
$$

From these expressions, it is easy to see that $G_1(\beta_1,\beta_2)$ and $G_2(\beta_1,\beta_2)$ are analytic functions with respect to β_1 and β_2, *independent* of $\boldsymbol{\alpha}$.

The actual law of motion (3.5) depends on eight parameters φ, λ, γ, ϕ_y, ϕ_π, ρ, σ_π^2 and σ_y^2. Only the ratio σ_π^2/σ_y^2 of noise terms matters for the persistence $G_i(\beta_1,\beta_2)$ in (3.13) and (3.14). Hence, the existence of BLE $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ depends on seven structural parameters φ, λ, γ, ρ, ϕ_y, ϕ_π and σ_π^2/σ_y^2 of the NK-model.

Using Proposition 1 and Proposition 2 we have the following properties for the New Keynesian model:

Corollary 2 *Under the contemporaneous Taylor rule, if $\gamma(\phi_\pi-1)+(1-\lambda)\phi_y>0$, then there exists at least one BLE $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$, where $\boldsymbol{\alpha}^*=\mathbf{0}=\overline{\boldsymbol{x}^*}$.*

Corollary 3 *Under the contemporaneous interest rate rule and the condition $\gamma(\phi_\pi-1)+(1-\lambda)\phi_y>0$, a BLE $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ is locally stable under SAC-learning if all eigenvalues of $\boldsymbol{DG_\beta}(\boldsymbol{\beta}^*)=\left(\frac{\partial G_i}{\partial\beta_j}\right)_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}$ have real parts less than 1.*

Proof. See Appendix G.

It is useful to discuss the special case in which shocks are not persistent, that is, $\rho=0$ (no autocorrelation in the shocks). It is easy to see that

$$
G_1(0,0)\Big|_{\rho=0}=0, \qquad G_2(0,0)\Big|_{\rho=0}=0.
$$

That is to say $(\mathbf{0},\mathbf{0})$ is a BLE for $\rho=0$. Hence, when there is no persistence in the exogenous shocks, the BLE coincides with the rational expectation equilibrium.

It is also useful to briefly discuss the non-stationary case, that is, when the coefficient matrix $\boldsymbol{B}$ for expectations $\boldsymbol{x}^e_{t+1}$ in (3.5) has at least one eigenvalue outside the unit circle. In that case, SAC-learning of an AR(1) rule typically leads to explosive dynamics with $\alpha_t \to \pm\infty$ and $\beta_t \to 1$. In the non-stationary case, learning of BLE thus typically leads to explosive time paths of inflation and output.

Persistence amplification

To illustrate the typical output-inflation dynamics, we present a calibration exercise for empirically plausible parameter values. As in the Clarida et al. (1999) calibration we fix $\varphi = 1, \lambda = 0.99$. We fix $\gamma = 0.04$, which lies between the calibrations $\gamma = 0.3$ in Clarida et al. (1999) and $\gamma = 0.024$ in Woodford (2003). For the exogenous shocks, we set the ratio of shocks $\frac{\sigma_\pi}{\sigma_y} = 0.5$, which is within the possible range suggested in Fuhrer (2006). We consider the symmetric case $\rho_1 = \rho_2 = \rho = 0.5$, with weak persistence in the shocks. The baseline parameters on the policy response to inflation deviation and output gap follow a broad literature, $\phi_\pi = 1.5,\ \phi_y = 0.5$, see for example Fuhrer (2006, 2009). At these parameter values, the two eigenvalues of the Jacobian matrix $\boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*)$ are $0.5012 \pm 0.7348i$ (with real parts less than 1), which implies that the BLE is E-stable under SAC-learning based on our theoretical results. The numerical results shown below are robust across a range of plausible parameter values.

Figure 1 illustrates the existence of a unique E-stable BLE $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$[21]. In order to obtain (β_1^*, β_2^*), we numerically compute the corresponding fixed point $\beta_2^*(\beta_1)$ satisfying $G_2(\beta_1, \beta_2^*) = \beta_2^*$ for each β_1 and the corresponding fixed point $\beta_1^*(\beta_2)$ satisfying $G_1(\beta_1^*, \beta_2) = \beta_1^*$ for each β_2 as illustrated in Figure 1. Hence their intersection point (β_1^*, β_2^*) satisfies $G_1(\beta_1^*, \beta_2^*) = \beta_1^*$ and $G_2(\beta_1^*, \beta_2^*) = \beta_2^*$.

A striking and typical feature of the BLE is that the first-order autocorrelation coefficients of output gap and inflation $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$ are substantially higher than those at the REE, that is, the persistence is much higher than the persistence $\rho (= 0.5)$ of the exogenous shocks. We refer to this phenomenon as *persistence amplification*. Agents fail to recognize the exact linear structure and cross-correlations of the economy, but rather learn to coordinate on simple univariate AR(1) rules consistent with simple observable statistics, the mean and the first-order autocorrelations of inflation and output gap. As a result of this *self-fulfilling mistake*, shocks to the economy are strongly amplified.

[21]Note that $(\alpha_1^*, \alpha_2^*) = (0, 0)$.

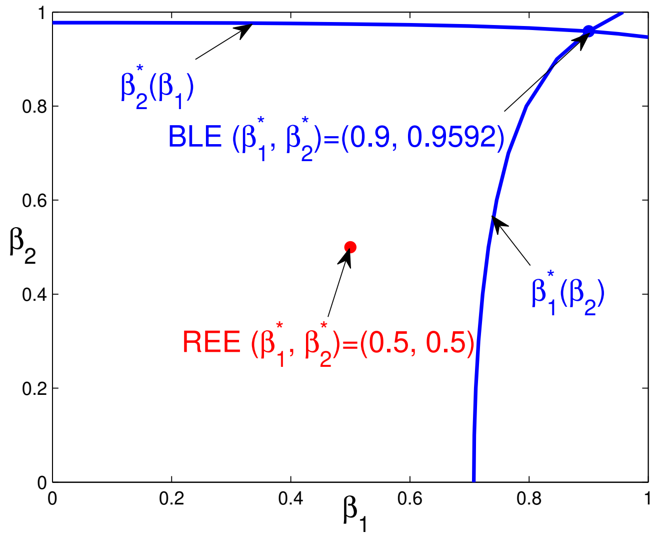
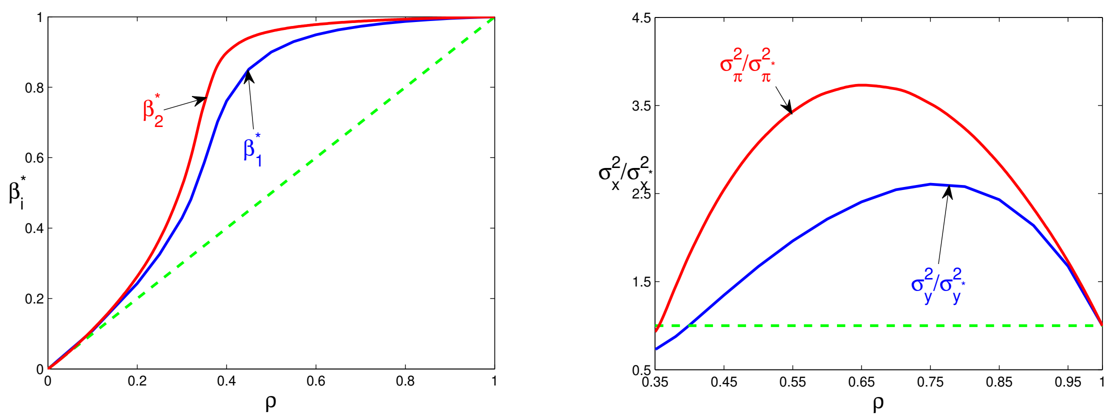


Figure 1: A unique BLE $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$ obtained as the intersection point of the fixed point curves $\beta_2^*(\beta_1)$ and $\beta_1^*(\beta_2)$. The BLE exhibits strong persistence amplification compared to REE (red dot, with $\rho = 0.5$). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_\pi}{\sigma_y} = 0.5$.



(a) (b)

Figure 2: BLE (β_1^*, β_2^*) as a function of the persistence ρ of the exogenous shocks. (a) $\beta_i^*(i = 1, 2)$ with respect to ρ; (b) the ratio of variances $(\sigma_y^2/\sigma_{y^*}^2,\ \sigma_\pi^2/\sigma_{\pi^*}^2)$ of the BLE (β_1^*, β_2^*) w.r.t. the REE. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_\pi}{\sigma_y} = 0.5$.

Figure 2 illustrates how these results depend on the persistence ρ of the exogenous shocks. The figure shows the BLE, i.e. the first-order autocorrelations β_1^* of output gap and β_2^* of inflation, as a function of the parameter ρ. This figure clearly shows the *persistence amplification* along BLE, with much higher ACF than under RE, for all values of $0 < \rho < 1$. Especially for $\rho \geq 0.5$ we have $\beta_1^*, \beta_2^* \geq 0.9$, implying that output gap and inflation have significantly higher persistence than the exogenous driving forces. Figure 2 (right plot) also illustrates the *volatility amplification* under BLE compared to REE. For output gap the ratio of variances $\sigma_y^2/\sigma_{y^*}^2$ reaches a peak of about 2.5 for $\rho \approx 0.75$, while for inflation the ratio of variances $\sigma_\pi^2/\sigma_{\pi^*}^2$ reaches its peak of about 3.5 for $\rho \approx 0.65$. These results suggest that, given the same parameter values, the moments of inflation and output gap implied by BLE and REE are substantially different due to persistence and volatility amplification under BLE. Therefore if the model is estimated on the same dataset under BLE and REE, one might expect important differences in the resulting parameter estimates and the resulting shock propagation mechanism of the model. We explore this implication in the next section by estimating the model under BLE and REE based on U.S. data.

3.3 Estimation of The Baseline Model

Sample Period and Prior Distributions

In this section, we compare the empirical fit of the 3-equation New Keynesian model under REE, BLE and SAC-learning. We augment the Taylor rule with an i.i.d monetary policy shock and an interest rate smoothing parameter to allow the model to match the inertia of the historical interest rate:

$$r_t = \rho_r r_{t-1} + (1-\rho_r)(\phi_\pi \pi_t + \phi_x y_t) + \epsilon_{r,t}. \tag{3.19}$$

We estimate the small-scale system of (3.1) and (3.19) for the U.S economy over the period 1966:I-2016:IV using quarterly macroeconomic data. We also investigate whether our results are sensitive to structural breaks such as the large volatility reduction for most macroeconomic time series during the mid-80s, often referred to as the Great Moderation, or the near-zero level of nominal interest rates that followed the 2007-08 crisis period. We use the following measurement equations for output gap, inflation and interest rate[22]

[22]See Appendix H for more details on the observable variables.

without measurement errors:

$$\begin{cases} log(y_t^{obs}) = \bar{\gamma} + y_t \\ log(\pi_t^{obs}) = \bar{\pi} + \pi_t \\ log(r_t^{obs}) = \bar{r} + r_t, \end{cases} \tag{3.20}$$

where we use the cycle component of HP-filtered quarterly output as our measure of output gap[23] y_t^{obs}, while π_t^{obs} and r_t^{obs} denote the quarterly historical inflation and interest rate series respectively. $\bar{\gamma}, \bar{\pi}$ and $\bar{r}$ correspond to the historical mean levels of output gap, inflation and interest rate. The model is estimated using the same prior distributions under all three specifications, which guarantees that any differences that arise between the estimations is due to the difference in the expectation formation rule. The prior distributions are kept close to those commonly assumed in the literature: the risk aversion coefficient $\tau = \frac{1}{\varphi}$ is assigned a gamma distribution centered at 2 with a standard deviation of 0.5, covering values of approximately up to four. The slope of the Phillips curve γ is assigned a Beta distribution with mean 0.3 and standard deviation 0.15 which falls somewhere between the prior in An & Schorfheide (2007) and Smets & Wouters (2007), covering both flat and steep cases for the Phillips curve[24]. The policy response parameters for output gap and inflation are assigned beta distributions centered around 0.5 and 1.5, which are standard values associated with the Taylor Rule in the literature. The autocorrelation coefficients have a Beta distribution centered at 0.5, and the standard deviations for the shock processes are assumed to follow an Inverted Gamma distribution with a mean of 0.1 and standard deviation of 2, same as in Smets & Wouters (2007). The priors for the steady-state inflation rate, output growth and interest rate are normal distributions centered at their pre-sample means of 0.47, -0.2 and 0.72 respectively, where the pre-sample period covers data from 1954:I to 1965:IV. Finally, we fix the HH discount rate λ at 0.99, which is a standard assumption in most empirical studies.

[23]As shown in the next section, our main results are not sensitive to the choice of output gap measure.

[24]In particular if we denote the nominal price stickiness as ω, its relation with γ is given as $\gamma = \frac{(1-\lambda\omega)(1-\omega)}{\omega}$ (Gali, 2008). Smets & Wouters (2007) assume a prior for ω with mean 0.5.

Convergence Diagnostics

Table 3.3 presents the posterior estimation results for the BLE, RE and SAC-learning models.[25] Before moving onto the estimation results, we first briefly discuss the convergence diagnostics of BLE and SAC-learning. Under BLE, initializing both β_y and β_π[26] at fairly low values of 0.5 and using a convergence criterion $\epsilon = 10^{-5}$, our estimation algorithm takes only 5 steps to converge[27]. The resulting BLE is $(\beta_y^*, \beta_\pi^*) = (0.88, 0.89)$ at the final step. This is fairly close to the sample-autocorrelation moments of the data over this period, which is $(0.87, 0.89)$. The left panel of Figure 3 shows the norm distances between two consecutive sets of $\boldsymbol{\beta}^{(k)}$ and $\theta^{(k)}$ at each step k, both of which rapidly converge towards 0. The largest eigenvalue of the Jacobian matrix $DG(\boldsymbol{\beta}^{(k)}, \theta^{(k)})$ remains strictly inside the unit circle during the estimation, and stabilizes after the second step. The right panel of the same figure shows the convergence of Algorithm I towards $\boldsymbol{\beta}^*$ at the estimated posterior mode with randomized initial values, suggesting that the estimated equilibrium is the unique iteratively E-stable BLE. Similar results emerge when we examine the Monte Carlo simulations of the model under SAC-learning in Figure 4: the left panel shows the histograms of β_y and β_π over 1000 simulations under decreasing gain learning, while the right panel shows the constant gain equivalent with a small gain value of 0.001. It is readily seen that none of the distributions show signs of multiplicity of equilibria. To formally check this, we provide the approximate distributions of β_y and β_π by smoothing the histograms and applying Hartigan's Dip Test of Unimodality[28]. The dip test does not reject the null hypothesis of unimodality, suggesting that there is no evidence of multiple BLE at the estimated structural parameter values based on the simulations. We also observe a small bias in the simulations for both cases, where the

[25] BLE and REE models are estimated using the Dynare toolbox (Adjemian et. al, 2011), while our own toolbox is used for the SAC-learning estimations since it requires an additional learning step in the filter recursions. The posterior distributions are constructed using the Metropolis-Hastings algorithm with 250000 draws, using the first 50000 as the burn-in sample. The step size for the scale parameter of the jumping distribution's covariance matrix is adjusted in both models to obtain a rejection rate of 70% in both models, which is in the commonly assumed appropriate range for the MH algorithm.

[26] In this section we switch to the notation β_y and β_π instead of β_1 and β_2 to make the exposition more clear.

[27] The results are robust to initial values of β_y and β_π.

[28] Hartigan's Dip Test is based on checking for multimodality by using the maximum difference between the empirical distribution, and the theoretical unimodal distribution function that minimizes the maximum difference. The null hypothesis of the test is unimodality, see Hartigan & Hartigan (1985) for more details.

peak of the distributions slightly deviates from the underlying BLE denoted by the dotted line. These Monte Carlo simulations illustrate the advantage of SAC-learning over the standard recursive least squares learning approach: although the autocorrelation coefficients are fairly close to unity for both y_t and π_t, the time series never becomes explosive in our simulations. This is due to the natural projection facility of SAC-learning, with autocorrelations always satisfying $-1 \leq \beta \leq 1$. This makes explosive time paths less likely as discussed in Section 2.

Figure 5 shows the mean and persistence coefficients along with the filtered variables of inflation and output gap under the SAC-learning estimation. For this specification, following the recommendation in Galimberti & Jacqueson (2017), we use a training-sample based initialization as follows: we use the unconditional moment of 0 for the intercept coefficients, and the diffuse moment 0 for the estimated variance of each variable R_t[29]. We use a period of five years over 1961:I-1965:IV as the transient period for the belief coefficients, and compute the likelihood from 1966:I onwards. It is readily seen that the mean coefficients do not substantially deviate from the unconditional mean of 0 and the persistence coefficients indeed converge over the estimation sample, with final values of $\boldsymbol{\beta}^* = (0.86, 0.89)$: this is fairly close to the equilibrium resulting under the BLE estimation.

Posterior Estimation Results

Next we move onto the discussion of our structural parameter estimates. Starting with a comparison of the BLE and REE results, we observe several important differences: both persistence parameters for the inflation and output shocks, ρ_π and ρ_y, are substantially lower under BLE, namely with 0.31 and 0.42 respectively, while they are 0.88 and 0.87 under REE. This is a direct consequence of the difference in the expectation formation rule, and the estimation results confirm the persistence amplification property of a BLE. This backward-looking expectation rule endogenously generates additional inertia for inflation and output gap, which in turn leads to much smaller persistence in the exogenous shocks. The low autocorrelations in u_π and u_y under BLE immediately imply higher estimated standard deviations for the i.i.d shocks of these AR(1) processes at 0.29 and 0.73, while

[29]The initial choice of first-order autocorrelations does not matter since $\beta_{1,t} = -\frac{1}{2}$ regardless of what $\beta_{0,t}$ is.

these are 0.04 and 0.16 under REE[30]. Since interest-rate is not forward-looking, this result does not extend to the interest-rate smoothing ρ_r, which is estimated at 0.85 and 0.80 under BLE and REE respectively, while the volatility of monetary policy shocks is the same at 0.29 under both specifications. The steady-state parameters turn out fairly similar under both estimations, since they relate to sample means of the observable variables and are not affected by the expectation rule. The estimates of monetary policy parameters, ϕ_π and ϕ_y also turn out similar under both estimations, with 1.36 and 0.48 under BLE, and 1.39 and 0.46 under REE. Finally turning to the two structural parameters that determine the contemporaneous relation between the state variables, both the risk aversion coefficient $\frac{1}{\varphi}$ and the slope of the NKPC γ turn out fairly different: these are estimated at 3.02 and 0.035 under BLE, while they are 4.27 and 0.007 under REE respectively. These differences arise due to altered cross-restrictions under learning: the additional inertia that we introduce under learning comes at the cost of a weaker contemporaneous relation between the state variables and shocks. While the state variables are only related to the shocks through $\frac{1}{\varphi}$ and γ under BLE, they are also indirecly related through expectations under REE. As a result, the risk aversion coefficient turns out lower under learning, implying a larger direct impact from the ex-ante risk premium on output gap under BLE. Similarly, γ turns out higher under BLE, implying a stronger direct effect from output gap on inflation. It is also interesting to note that confidence intervals for γ are almost mutually exclusive under these two specifications, with a lower bound of 0.015 under BLE and an upper bound of 0.017 under REE[31].

Overall, our results suggest important differences in the estimated parameters and the propagation of shocks under BLE. These changes lead to a substantial improvement in the empirical fit, evident from the (log marginal) likelihood of −337 under BLE compared with −348 under REE, which yields a Bayes' Factor 4.78 in favour of BLE[32]. Comparing the results to SAC-learning, it is readily seen that there are no substantial differences with

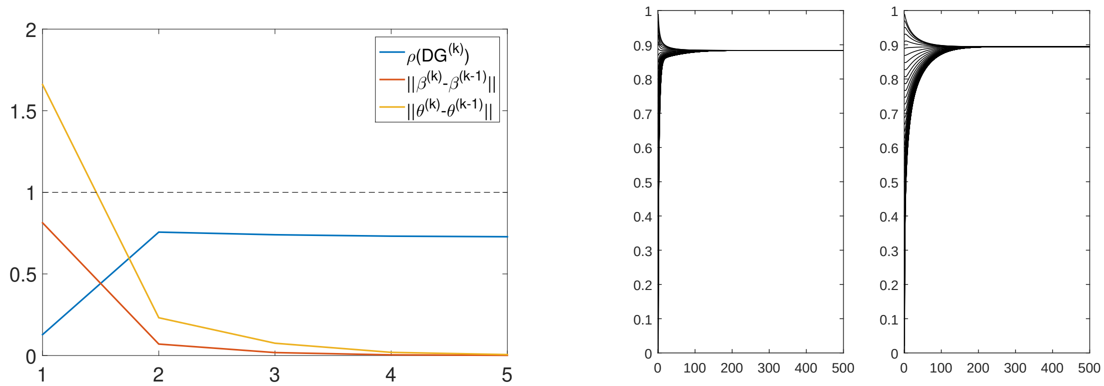
[30]Note that for an AR(1) process $x_t = \rho_x x_{t-1} + u_t, u_t \sim iid(0, \sigma_u)$, the unconditional variance is given by $var(x) = \frac{\sigma_u}{1-\rho_x^2}$. This implies, as ρ_x increases, $var(x)$ also increases.

[31]As noted in the previous section, the first-order coefficients are computed based on the posterior mode values. Therefore for completeness, the discussion and comparison of these results (both in this section and in the upcoming ones) are based on the posterior mode. It is worth noting that, given the small differences across the posterior means and modes, a similar discussion can be easily extended to the posterior mean.

[32]Based on Jeffrey's Guidelines (Greenberg, 2012), if the Bayes' Factor is in favour of a model is larger than 2, then this provides *decisive support* for the model under consideration.

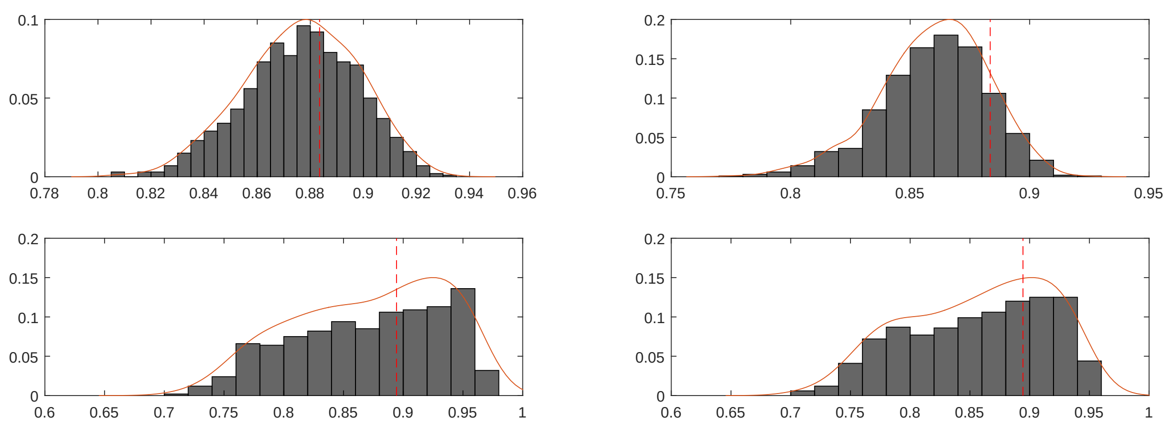
				REE				BLE				SAC			
			Laplace	-348				-337				-341			
			MHM	-348				-337				-341			
			Bayes Factor	1				4.78				3.04			
	Prior			Post.				Post.				Post.			
Para.	Dist.	Mean	St. Dev	Mode	Mean	5 %	95 %	Mode	Mean	5 %	95 %	Mode	Mean	5 %	95 %
σ_y	Inv. Gamma	0.1	2	0.16	0.17	0.12	0.22	0.73	0.74	0.68	0.8	0.75	0.76	0.68	0.83
σ_π	Inv. Gamma	0.1	2	0.04	0.04	0.03	0.05	0.29	0.3	0.27	0.32	0.3	0.3	0.28	0.33
σ_r	Inv. Gamma	0.1	2	0.29	0.3	0.28	0.33	0.29	0.29	0.27	0.32	0.29	0.29	0.27	0.32
$\bar{y}$	Normal	-0.2	0.25	-0.15	-0.49	0.18	-0.12	-0.12	-0.4	0.17	-0.17	-0.21	-0.19	-0.6	0.22
$\bar{\pi}$	Normal	0.47	0.25	0.7	0.71	0.53	0.9	0.79	0.79	0.67	0.91	0.41	0.45	0.18	0.75
$\bar{r}$	Normal	0.72	0.25	0.98	0.98	0.71	1.26	1.1	1.09	0.85	1.32	0.65	0.7	0.33	1.06
γ	Beta	0.3	0.15	0.007	0.01	0.002	0.017	0.035	0.037	0.015	0.065	0.048	0.048	0.019	0.08
$\frac{1}{\varphi}$	Gamma	2	0.5	4.27	4.35	3.35	5.37	3.02	3.15	2.33	3.98	2.86	2.98	2.04	3.94
ϕ_π	Gamma	1.5	0.25	1.38	1.4	1.16	1.65	1.36	1.38	1.1	1.67	1.32	1.35	1.03	1.69
ϕ_y	Gamma	0.5	0.25	0.48	0.51	0.34	0.67	0.49	0.52	0.31	0.73	0.48	0.51	0.29	0.76
ρ_y	Beta	0.5	0.2	0.87	0.86	0.83	0.92	0.43	0.43	0.32	0.53	0.57	0.57	0.46	0.68
ρ_π	Beta	0.5	0.2	0.88	0.87	0.82	0.91	0.32	0.32	0.22	0.43	0.25	0.26	0.14	0.38
ρ_r	Beta	0.5	0.2	0.8	0.8	0.76	0.84	0.85	0.86	0.82	0.91	0.85	0.86	0.81	0.81
β_y^*				-	-	-	-	0.88	-	-	-	0.86	-	-	-
β_π^*				-	-	-	-	0.89	-	-	-	0.88	-	-	-

Table 1: Posterior results under REE, BLE and SAC-learning over the estimation period 1966:I-2016:IV. Lapl. refers to the Laplace approximation of likelihood based on the posterior mode, while MHM refers to the Modified Harmonic Mean estimate of likelihood based on the posterior distribution.



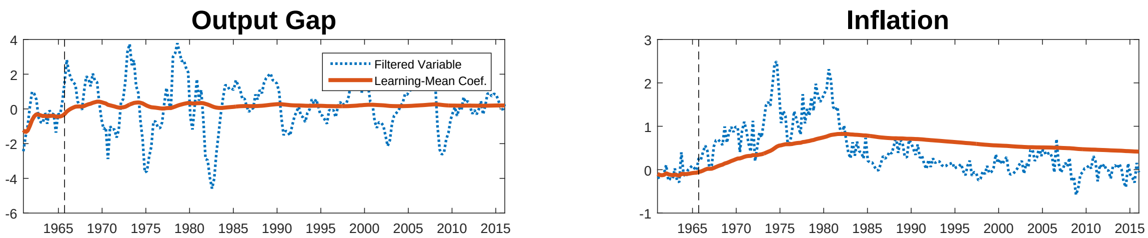
(a) Convergence towards $\boldsymbol{\beta^*}$ (b) Iterative E-stability of the unique $\boldsymbol{\beta}^*$

Figure 3: The estimated BLE is $(\beta_y^*, \beta_\pi^*) = (0.88, 0.89)$. The left panel shows the largest eigenvalue, and the norm distances between consecutive values of $\boldsymbol{\beta^{(k)}}$, as well as $\theta^{(k)}$ across iterations. The second panel shows convergence towards the unique (iteratively) E-stable BLE with randomized initial values.

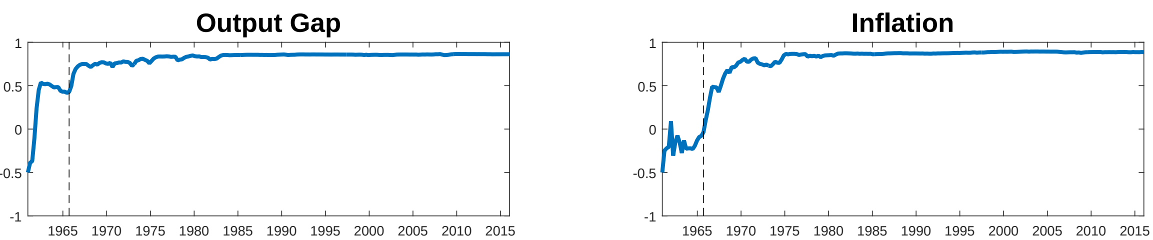


(a) β_y^* and β_π^* under constant gain learning. (b) β_y^* and β_π^* under decreasing gain learning.

Figure 4: Monte Carlo Simulations: frequency distributions and unimodality test for β_y and β_π resulting from 1000 simulations. Hartigan's unimodality test p-values are 0.98 and 0.94 for β_y and β_π under constant gain simulations, and 0.99 and 0.99 for the decreasing gain simulations. Hence the null hypothesis of unimodality is not rejected for any of the distributions.



(a) Filtered Variables and mean parameters



(b) Persistence parameters

Figure 5: Filtered variables and learning parameters over the estimation sample under SAC-learning, where Kalman filter output is used to update the belief coefficients. Converged values of first-order autocorrelations are 0.86 and 0.89 for output gap and inflation respectively.

the BLE specification. Relative to the REE estimations, the exogenous shocks have lower persistence and larger standard deviations, the risk aversion coefficient is lower, Phillips curve slope is higher and monetary policy coefficients are similar. The only difference with the BLE model arises in the steady-state values of inflation and interest-rate, which turn out lower under the SAC-learning estimation. This difference arises since the learning coefficients are time-varying under real-time SAC-learning and Figure 5 suggests that they are slightly above zero on average, which drives down the estimates of the steady-state parameters[33]. Other than these small differences, all parameter estimates are fairly close under BLE and SAC-learning, with implied HPD-intervals well within the range of each other. The likelihood turns out to be -341 under SAC-learning, which is still better than the fit of REE, with a Bayes' Factor of 3.08 in favour of SAC-learning, but worse than BLE, with a Bayes' Factor of 1.7 in favour of BLE[34]. This result suggests that transitory dynamics and the resulting time-variation in the learning parameters do not improve

[33] Shutting off the learning dynamics about mean coefficients and letting the agents learn only about the first-order autocorrelations indeed yields steady-state values similar to BLE.

[34] Based on Jeffrey's Guidelines, this provides *very strong* evidence in favour of BLE.

the model fit in our decreasing-gain learning setup. Overall, these results also allow us to illustrate the advantage of using Algorithm II to estimate a BLE: the estimation under SAC-learning requires a relatively large burn-in sample for the convergence of first-order autocorrelation coefficients, which might become an issue since macroeconomic time series are typically not very long. Furthermore, as we have already seen in Figure 4, the simulations under learning have a relatively large Monte-Carlo variance. In other words, while the simulations converge to the underlying BLE on average, there might be relatively large deviations from the underlying fixed-point for any given simulation. Hence what comes out of the estimation in a real-time learning setup is similar to a single simulation of the model under learning, which in general may not accurately reflect the underlying fixed-point. In the following, we check whether these results are robust to different measures of output gap and different subsamples for the BLE and REE specifications.

Subsample Estimations

We first chek whether our main results hold across different sample periods. To this end, we consider three periods: 1966:I-1979:II, the period before Great Moderation; 1966:I-2008:IV, the period before Great Recession and the zero lower bound episode; and 1984:I-2008:IV, the Great Moderation period.

Table 2 reports the posterior mode and the corresponding Laplace approximation for all periods under BLE and REE. It is readily seen that the difference between parameter estimates are preserved across all three periods: BLE is characterized by lower persistence but larger standard deviation estimates in shocks, a steeper Phillips curve characterized by larger γ, and a smaller risk aversion coefficient. The estimation under BLE provides a better model fit under all three subsamples, with Bayes' Factors of 1.67, 3.93 and 2.44 in favor of the BLE model.

Alternative Definitions of Output Gap

Next we check whether our results are sensitive to which measure of output gap is considered by using two alternative definitions: output gap based on de-trended output, and output gap based on CBO's measure of potential output. The estimations over our main sample are reported in Table 3, which yield the same conclusions as before: BLE is characterized by lower persistence but larger standard deviation estimates in shocks, a steeper Phillips curve characterized by larger γ, and a smaller risk aversion coefficient.

Period	66:I-79:II		66:I-08:IV		84:I-08:IV	
	BLE	REE	BLE	REE	BLE	REE
Laplace	-118.63	-122.47	-313.16	-322.10	-34.32	-39.93
Bayes Factor	1.67		3.93		2.44	
Parameter	Mode		Mode		Mode	
η_y	0.94	0.24	0.76	0.17	0.53	0.09
η_π	0.38	0.06	0.3	0.04	0.18	0.08
η_r	0.20	0.20	0.31	0.32	0.15	0.15
$\bar{y}$	-0.22	-0.14	-0.09	-0.14	0.05	-0.14
$\bar{\pi}$	0.93	0.75	0.84	0.71	0.61	0.58
$\bar{r}$	1.03	0.86	1.25	1.08	1.14	0.97
γ	0.054	0.01	0.033	0.006	0.046	0.007
$\frac{1}{\psi}$	2.44	2.77	4.03	3.78	2.82	3.15
ϕ_π	1.03	1.07	1.29	1.31	1.55	1.57
ϕ_y	0.39	0.36	0.45	0.42	0.48	0.54
ρ_y	0.5	0.87	0.42	0.88	0.44	0.94
ρ_π	0.31	0.85	0.32	0.87	0.24	0.55
ρ_r	0.72	0.68	0.83	0.77	0.9	0.85

Table 2: Comparison of the sub-sample estimations under BLE and REE.

The likelihood is also better under BLE for both definitions, with Bayes' Factors of 11.49 and 10.41 respectively. As a final check, we compare the likelihoods across the three sub-samples with the alternative measures of output gap, which is reported in Table 4: it is readily seen that the likelihood under BLE is better for both measures under all sub-samples, although the difference for the Great Moderation period 1985-I:2008:IV is much smaller. These results are also consistent with our previous findings.

	BLE det.	REE det.	BLE CBO.	REE CBO.
Laplace	-360.96	-387.42	-342.9	-366.88
Bayes Factor	11.49		10.41	
	Post.		Post.	
Parameter	Mode	Mode	Mode	Mode
η_y	0.78	0.08	0.74	0.11
η_π	0.3	0.04	0.29	0.04
η_r	0.3	0.31	0.29	0.3
$\bar{y}$	-0.13	-0.19	-0.54	-0.42
$\bar{\pi}$	0.79	0.66	0.81	0.59
$\bar{r}$	1.09	0.92	1.15	1.05
γ	0.017	0.004	0.024	0.006
$\frac{1}{\psi}$	2.92	4.86	2.65	4.57
ϕ_π	1.44	1.45	1.41	1.43
ϕ_y	0.22	0.16	0.36	0.27
ρ_y	0.42	0.88	0.42	0.89
ρ_π	0.31	0.94	0.31	0.92
ρ_r	0.88	0.79	0.88	0.8

Table 3: Alternative estimations of the 3-equation NKPC model: we compare the results under BLE and REE with two alternative specifications of output gap. In the first case output gap is defined as the deviation of output from a quadratic trend, while in the latter we take the output gap based on CBO's measure of potential output.

	66:I-79:II		66:I-08:IV		84:I-08:IV	
	BLE	REE	BLE	REE	BLE	REE
CBO's estimate	-126.07	-127.6	-321.9	-344.4	-36.7	-46.9
Bayes Factor	0.66		9.77		4.43	
Detrended output	-129.9	-133.1	-333.1	-353.2	-48.4	-66.9
Bayes Factor	1.39		8.73		8.03	

Table 4: Sub-sample estimations with alternative definitions of output gap.

4 Optimal Monetary Policy

Our results so far show that BLE generally provides a better model fit than REE for the 3-equation New Keynesian model, with important differences in the estimated structural parameters and the propagation mechanism under BLE and REE. This leaves an important question for the optimal Taylor rule at BLE: how do the optimal values of Taylor rule parameters differ under BLE and REE? In this section we answer this question by considering optimal monetary policy under both calibrated and estimated parameter values. We assume that the central bank wishes to minimize an expected loss function $E[L]$ in terms of the discounted sum of weighted squared inflation, output gap and interest rate

$$E[L] = (1-\vartheta)E\Big[\Sigma_{t=0}^{\infty}\vartheta^t[\omega_\pi\pi_t^2 + \omega_y y_t^2 + \omega_r r_t^2]\Big] = \omega_\pi\sigma_\pi^2 + \omega_y\sigma_y^2 + \omega_r\sigma_r^2, \qquad (4.1)$$

where $\omega_i,\ i \in \{\pi, y, r\}$ is the relative weight that the central bank places on inflation, output gap and interest rate respectively. The stabilization objective for inflation and output gap is a standard assumption in the literature, see e.g. Boehm and House (2014), Evans and Honkapohja (2003) and Woodford (2003). The weight on interest rate variance can be motivated by different interpretations: Woodford (1999) and Giannoni (2014) suggest that it can proxy for welfare costs of transactions and/or an approximation on the zero lower bound on nominal interest rates, while Caplin and Leahy (1996) suggest that it can represent a gradual learning process for the central bank that is uncertain about the consequences of interest rate fluctuations.

Based on our calculations, the unconditional moments under BLE are given as

$$\sigma_y^2 = \frac{\widetilde{g}_1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\rho\lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}, \tag{4.2}$$

$$\sigma_\pi^2 = \frac{\widetilde{g}_2}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\rho\lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}, \tag{4.3}$$

$$\sigma_r^2 = \phi_y^2\sigma_y^2+\phi_\pi^2\sigma_\pi^2+2\phi_y\phi_\pi E(y_t\pi_t), \tag{4.4}$$

where $\widetilde{g}_1$, $\widetilde{g}_2$, λ_1, λ_2 are given by the equations (3.15), (3.16), (3.17) and (3.18), and

$$\begin{aligned} E(y_t\pi_t) = {} & \Big(-\sigma_y^2\gamma\big(-(1+\gamma\varphi\phi_\pi+\varphi\phi_y)(1+\gamma\varphi\phi_\pi+\varphi\phi_y+\beta_1^2\rho)+\beta_2^4\lambda(1+\gamma\varphi\phi_\pi \\ & +\varphi\phi_y+\beta_1^2\rho)(\lambda+\gamma\varphi+\lambda\varphi\phi_y)+\beta_2^2\rho[\beta_1^4\lambda+\beta_1^2\lambda\rho(1+\gamma\varphi\phi_\pi+\varphi\phi_y)+\gamma\varphi \\ & (-1+\lambda\phi_\pi)(1+\gamma\varphi\phi_\pi+\varphi\phi_y)]-\beta_1^2\beta_2^6\lambda^2\rho(\beta_1^2\lambda+\rho(\lambda+\gamma\varphi+\lambda\varphi\phi_y))\big)+\sigma_\pi^2\varphi \\ & \big(-\phi_\pi(1+\gamma\varphi\phi_\pi+\varphi\phi_y)[-\beta_1^4-\beta_1^2\gamma\rho\varphi\phi_\pi+(1+\varphi\phi_y)(1+\gamma\varphi\phi_\pi+\varphi\phi_y)]+\beta_1^2\beta_2^6\lambda\rho \\ & [-\gamma\varphi(-\beta_1^2+\rho+\rho\varphi\phi_y)+\lambda\rho(\beta_1^4-(1+\varphi\phi_y)^2)]+\beta_2^4\big(\gamma\varphi(1-\beta_1^2\rho+\varphi\phi_y)(1+\gamma\varphi\phi_\pi \\ & +\varphi\phi_y)+\lambda(-1+\beta_1^2\rho-\varphi(\gamma\phi_\pi+\phi_y))(\beta_1^4-(1+\varphi\phi_y)^2)+\beta_1^2\lambda^2\rho\phi_\pi(-\beta_1^4+ \\ & (1+\varphi\phi_y)^2)\big)+\beta_2^2\rho[-\beta_1^6\lambda\rho\phi_\pi+\beta_1^2\lambda\rho\phi_\pi(1+\varphi\phi_y)(1+\gamma\varphi\phi_\pi+\varphi\phi_y)-(-1+\lambda\phi_\pi) \\ & (1+\varphi\phi_y)^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)+\beta_1^4(-1-\varphi(\gamma\phi_\pi+\varphi_y)+\lambda(\phi_\pi+\varphi\phi_\pi\phi_y))]\big)\Big)\Big/ \\ & \Big((-1+\rho^2)(-1+\beta_1^2\beta_2^2\lambda-\varphi(\gamma\phi_\pi+\phi_y))(1+\beta_1^2\rho(-1+\beta_2^2\lambda\rho)+\gamma\varphi\phi_\pi+\varphi\phi_y \\ & -\beta_2^2\rho(\lambda+\gamma\varphi+\lambda\varphi\phi_y))\big(\beta_1^4(-1+\beta_2^4\lambda^2)+2\beta_1^2\beta_2^2\gamma\varphi(-1+\lambda\phi_\pi)+(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2 \\ & -\beta_2^4(\lambda+\gamma\varphi+\lambda\varphi\phi_y)^2\big)\Big). \end{aligned}$$

In the following we study the optimal values (ϕ_y^*, ϕ_π^*) that minimize the central bank's loss function (4.1) at the BLE, where β_1^* and β_2^* are at the BLE (β_1^*, β_2^*).

We first examine monetary policy under BLE and REE at calibrated parameter values: as before in our calibration exercise, we consider the parameters $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_\pi}{\sigma_y} = 0.5$ for both BLE and REE. This ensures that all structural parameters are the same under both specifications, hence the data generating process only differs in terms of the expectation formation rule. The analysis for calibrated parameters provides some insights of how optimal monetary policy under BLE depends on model parameters, particularly the persistence of shocks. As we saw in previous sections, there are important differences in the moments of endogenous variables due to volatility and persistence amplification at BLE. In particular, the implied variances of inflation, output gap and interest rate are different under BLE and REE at these parameter values. Following Woodford (1999) and Giannoni (2014), we normalize the weight on inflation to $\omega_\pi = 1$.

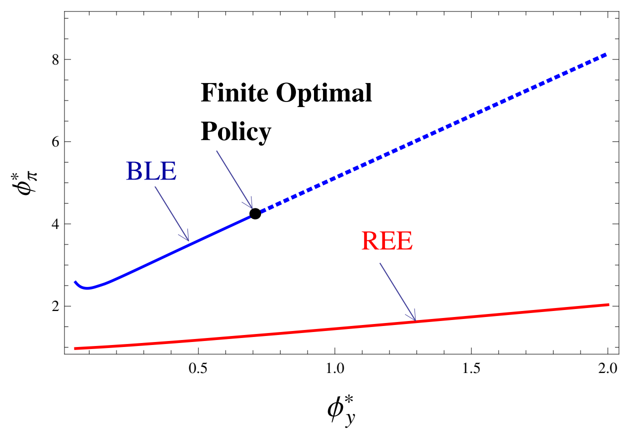
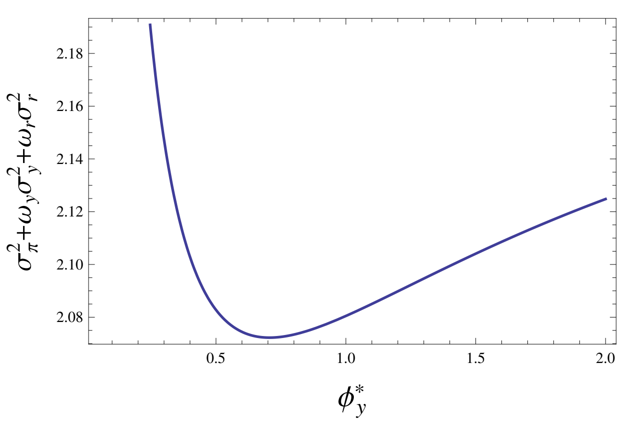
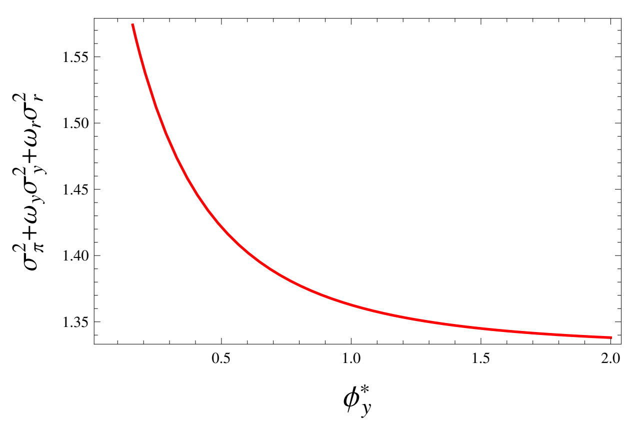


Figure 6: Optimal policies at the BLE and at the REE. Parameters are: $\lambda = 0.99, \varphi = 1, \rho = 0.5, \gamma = 0.04, \sigma_y = 1, \sigma_\pi = 0.5$ and $\omega_y = 0.1, \omega_r = 0.05$.



(a) At the BLE



(b) At the REE

Figure 7: Loss function along the optimal paths (ϕ_y^*, ϕ_π^*) in Figure 6 at the BLE (a) and REE (b). Parameters are: $\lambda = 0.99, \varphi = 1, \rho = 0.5, \gamma = 0.04, \sigma_y = 1, \sigma_\pi = 0.5$ and $\omega_y = 0.1, \omega_r = 0.05$.

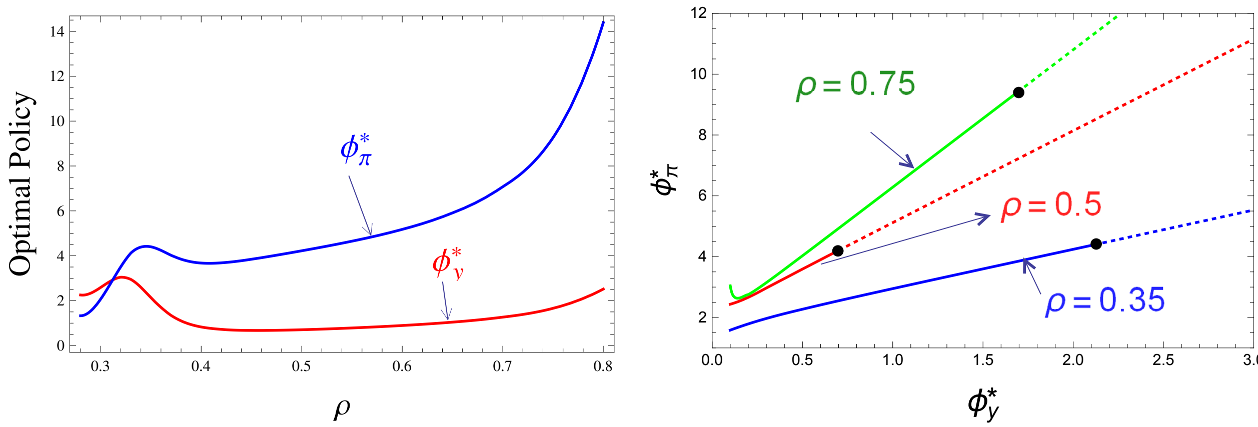
We first focus on a special case with $\omega_y = 0.1$ and $\omega_r = 0.05$, that is, the central bank places a relatively large weight on inflation and a small one on interest rates. The small weight on interest rates allows us to first focus on the trade-off between inflation and output gap stabilization. We leave the discussion of how optimal policy depends on these policy weight for the more empirically relevant case of estimated parameters.

Interestingly, we find that the optimal Taylor rule coefficients (ϕ_y^*, ϕ_π^*) are finite under BLE in this case[35]. As shown in Figure 6, the corresponding optimal policy is $(\phi_y^*, \phi_\pi^*) = (0.9069, 4.8822)$. This is different from REE, where there is no finite optimal policy except when measurement errors are considered, as shown in Boehm and House (2014)[36]. In fact, from Figure 6 it can be seen that in the case ϕ_y^* is small enough (i.e. < 0.9069) the coefficients ϕ_y^* and ϕ_π^* lie on a manifold and the loss function (4.1) decreases gradually along the manifold within this region, which is similar to REE but with higher ϕ_π^*. However, for $\phi_y^* > 0.9069$, the loss function (4.1) starts to increase, while in the REE the loss function (4.1) still decreases as shown in Figure 7. That is to say, there exist finite optimal Taylor rule coefficients at the BLE, but not at the REE. This is mainly because at the BLE the actual law of motion has higher variance than at the REE for most variables (especially inflation) and minimizing the loss function, i.e. minimizing the weighted variances of output gap and inflation, requires balancing the different responses in terms of policy parameters (ϕ_y, ϕ_π).

Next we investigate how optimal monetary policy changes as the persistence of the underlying shocks is varied. At the REE with measurement error the finite coefficients ϕ_y^* and ϕ_π^* increase as the persistence of shocks grows within some range, see Boehm and House (2014). At the BLE, in addition to this, we find that when the persistence of exogenous shocks becomes sufficiently small with $\rho < 0.4$, the finite coefficients ϕ_y^*

[35] We first select a policy parameter domain (e.g. $[0, 100] \times [1, 100]$) and define a lattice with some small step (*e.g.* 0.01). Then for each lattice point (ϕ_y, ϕ_π), we find the BLE $(\beta_1^*(\phi_y, \phi_\pi), \beta_2^*(\phi_y, \phi_\pi))$ and the corresponding central bank's expected loss function $E[L]$ at the BLE. Finally we interpolate the loss function with respect to (ϕ_y, ϕ_π) to find the finite optimal values. It is easy to get analytic expressions of the variances and optimal policy parameters under REE. In contrast, it is impossible to obtain analytic expressions of the optimal policy parameters under BLE and therefore we have to rely on numerical approximations. We find consistent results using different ways to calculate the variances (i.e. based on (4.2) and (4.3) or computing the variances as in Appendix B).

[36] The loss function considered in Boehm and House (2014) does not include interest rate variance. Including interest rates in our loss function here does not affect the result that optimal policy at REE does not exit as long as ω_r remains sufficiently small. Optimal policy becomes finite when ω_r is sufficiently large, these cases are discussed under the estimated parameter values.



(a) Optimal policy

(b) Optimal manifold

Figure 8: Optimal policies at the BLE with respect to ρ (a) and corresponding optimal manifolds for three different ρ (connection points of solid and dotted curves corresponding to finite optimal policies) (b). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \sigma_y = 1, \sigma_\pi = 0.5$ and $\omega_y = 0.1, \omega_r = 0.05$.

and ϕ_π^* increase in a small region as shown in Figure 8a, before they start decreasing again. Furthermore, Figure 8b suggests that the optimal manifold always moves up as the persistence of shocks ρ grows. The finite optimal policy lies at the point in the optimal manifold connecting the solid and dotted lines in Figure 8b. The location of the optimal point corresponding to finite optimal policies depends on the relative values of variances of inflation, output gap and interest rate. In the case ρ is large enough, the loss function is mainly dominated by the variance of inflation and hence the optimal policy ϕ_π^* grows quickly converging to ∞ and the slope of $\frac{\phi_\pi^*}{\phi_y^*}$ converging to a relatively large constant. As ρ becomes smaller, the variance of interest rate starts to play a more dominant role, which leads to smaller values for both ϕ_π^* and ϕ_y^*. The region with small values of ρ is the most relevant for U.S. data since both shock persistence parameters are low at the BLE in our estimation exercises.

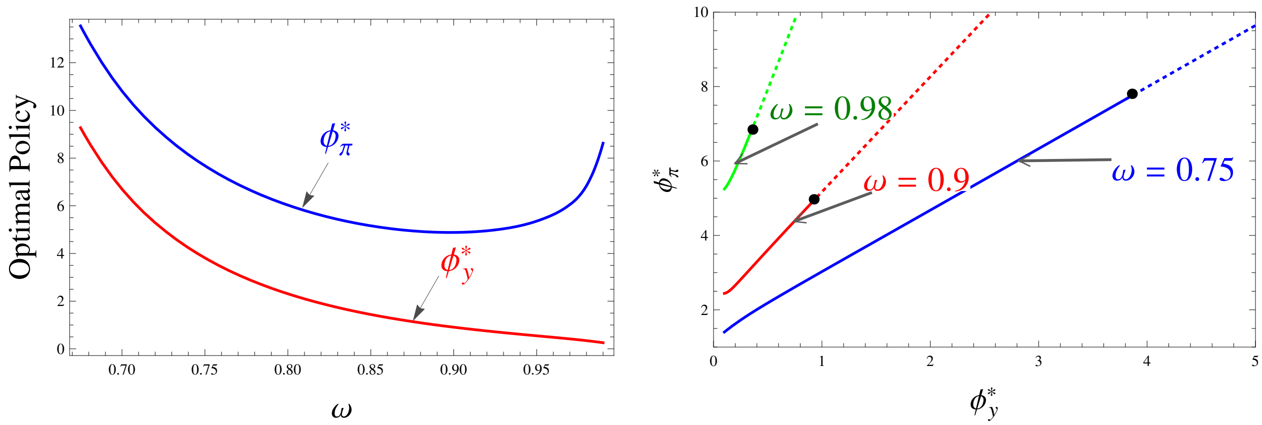
Our analysis so far shows that optimal policy exists under BLE for the calibrated parameters, and the optimal rule (ϕ_y^*, ϕ_π^*) is sensitive to the persistence of exogenous shocks. We next investigate optimal monetary policy at the more empirically relevant case of estimated parameters. In this case the underlying structural parameters are different at BLE and REE, but the variances of inflation, output gap and interest rate are close under these two specifications. In particular, given the estimated parameter values and

ω_π	ω_y	ω_r	BLE		REE		BLE					REE				
			ϕ_y^*	ϕ_π^*	ϕ_y^*	ϕ_π^*	$Var(\pi_t)^*$	$Var(y_t)^*$	$Var(r_t)^*$	$E[L^*]$	$\Delta E[L^*]$	$Var(\pi_t)^*$	$Var(y_t)^*$	$Var(r_t)^*$	$E[L^*]$	$\Delta E[L^*]$
1	0.1	0.05	3.94	3.49	2.74	4.43	0.14	0.49	1.06	0.24	58 %	0.31	0.19	1.41	0.40	27 %
1	0.048	0.236	1.67	1.56	0.94	0.75	0.16	0.82	0.61	0.34	32 %	0.32	1.40	0.71	0.55	1 %
1	0.25	0	15	13.33	4.35	15	0.13	0.27	2.8	0.18	82 %	0.32	0.04	1.8	0.33	60 %
1	0.20	0.05	6.67	3.94	2.64	8.51	0.14	0.35	1.45	0.28	68 %	0.32	0.07	1.61	0.41	46 %
1	0.15	0.09	3.33	2.27	1.64	3.80	0.15	0.53	0.89	0.31	58 %	0.32	0.2	1.31	0.48	31 %
1	0.10	0.15	2.42	1.82	1.25	1.96	0.15	0.64	0.73	0.32	48 %	0.32	0.46	1.05	0.52	15 %
1	0.05	0.2	1.82	1.67	1.06	0.87	0.16	0.77	0.64	0.32	35 %	0.32	1.19	0.77	0.53	2 %
1	0	0.25	1.21	1.67	1.07	0.01	0.16	1.05	0.58	0.31	16 %	0.30	7.91	0.45	0.41	16 %

$\rho_r = 0$

ω_π	ω_y	ω_r	BLE		REE		BLE					REE				
			ϕ_y^*	ϕ_π^*	ϕ_y^*	ϕ_π^*	$Var(\pi_t)^*$	$Var(y_t)^*$	$Var(r_t)^*$	$E[L^*]$	$\Delta E[L^*]$	$Var(\pi_t)^*$	$Var(y_t)^*$	$Var(r_t)^*$	$E[L^*]$	$\Delta E[L^*]$
1	0.1	0.05	1.21	1.67	2.17	4.91	0.14	0.47	0.97	0.24	59 %	0.32	0.09	2.01	0.43	21 %
1	0.048	0.236	0.46	0.91	0.38	0.49	0.16	1.03	0.41	0.30	40 %	0.35	2.34	0.78	0.64	-16 %
1	0.25	0	15	8.79	3.99	15	0.13	0.051	4.65	0.15	85 %	0.32	0.02	2.25	0.33	61 %
1	0.20	0.05	2.12	1.67	2.31	10.36	0.14	0.28	1.46	0.27	69 %	0.32	0.023	2.18	0.43	43 %
1	0.15	0.09	1.06	1.21	1.01	3.75	0.15	0.52	0.79	0.30	60 %	0.33	0.12	1.93	0.54	22 %
1	0.10	0.15	0.76	1.06	0.62	1.62	0.15	0.68	0.59	0.31	59 %	0.33	0.47	1.53	0.61	1 %
1	0.05	0.20	0.46	1.06	0.45	0.60	0.15	1.02	0.44	0.29	41 %	0.35	1.84	0.92	0.62	-15 %
1	0	0.25	0.30	1.06	0.99	0.01	0.16	1.41	0.38	0.25	31 %	0.35	7.93	0.43	0.45	3 %

Table 5: Optimal Taylor rules and some key statistics at BLE and REE: top panel shows the results under the estimated parameter values in each case, while the bottom panel shows the same results without interest rate smoothing. $E[L^*]$ shows the loss function value at the optimal plan, while $\Delta E[L^*]$ shows the percentage improvement at the optimal plan relative to the loss function at the estimated parameters.



(a) Optimal policy

(b) Optimal manifold

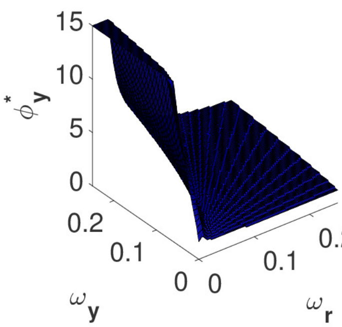
Figure 9: Optimal policies at the BLE with respect to ω (a) and corresponding optimal manifolds for three different ω (connection points of solid and dotted curves corresponding to finite optimal policies) (b) with the contemporaneous interest rate rule. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \frac{\sigma_y}{\sigma_\pi} = 0.5$ and $\rho = 0.5$.

our benchmark policy weights of $(\omega_y, \omega_r) = (0.1, 0.05)$, the expected loss function $E[L]$ is 0.57 at BLE and 0.55 at REE[37]. This ensures that the starting point for the loss functions is similar at BLE and REE, unlike the previous case with calibrated parameters.

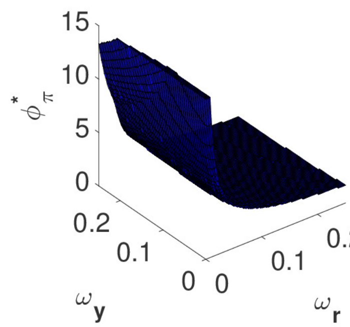
For this exercise, we limit our attention to Taylor parameters ϕ_y and ϕ_π over the range $[0, 15]$. Figure 10 shows the optimal rules at BLE and REE for values between $[0, 0.25]$ for ω_y and ω_r, with ω_π fixed at 1. The first result that stands out is, in this case the optimal monetary policy exists not only at BLE, but also at REE for a wide range of policy weights. This difference at REE with estimated parameters arises from the highly persistent shocks and flatter Phillips curve compared with the calibrated parameters, which introduces a larger trade-off between inflation and output gap stabilization, thereby leading to a finite optimal policy for a wide range of policy weights. This result allows us to compare how the optimal rules differ under BLE and REE.

Table 10 shows the optimal Taylor rules for several pairs of (ω_y, ω_r), along with some

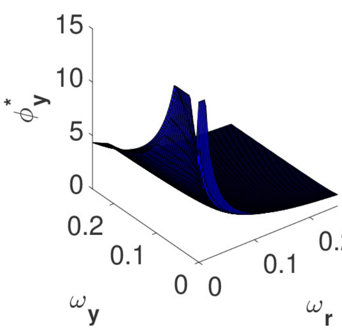
[37]Our analysis with calibrated parameter values is based on the expressions (4.3)-(4.4). These expressions are no longer applicable for the estimated version of the model since it also includes an interest rate smoothing parameter, and the assumption of same persistence for the exogenous shocks is relaxed. Rather, we use the (different) estimated shock persistence parameters. Therefore we proceed by computing the BLE and the associated variances for each value of policy parameters using the Iterative E-stability algorithm.



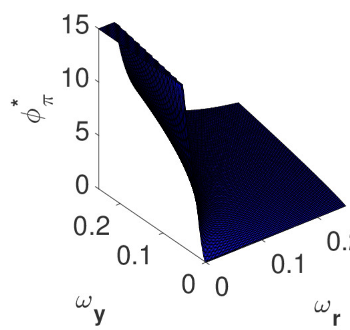
(a) ϕ_y^* at BLE.



(b) ϕ_π^* at BLE.



(c) ϕ_y^* at REE.

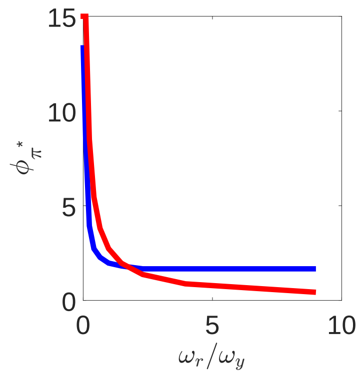


(d) ϕ_π^* at REE.

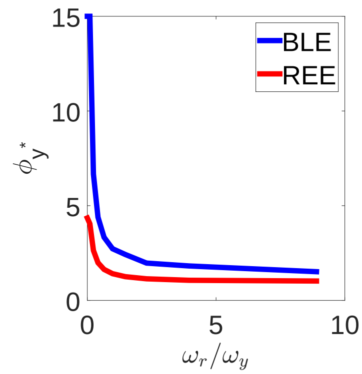
Figure 10: Optimal policies at BLE and REE as a function of ω_y and ω_r over the range $[0, 0.25]$. The upper and lower boundaries are 0 and 15 for the optimal parameters.

key statistics at BLE and REE. The first two rows show our benchmark calibration of $(0.1, 0.05)$, while the second row includes a calibration of $(0.048, 0.236)$ as in Woodford (1999) and Giannoni (2014). The remaining columns start with the corner case of $(0.25, 0)$ with no weight on interest rates, and gradually move to the other corner case of $(0, 0.25)$ with no weight on output gap. Several observations stand out from the table: both at BLE and REE, when $\omega_r = 0$, there is no optimal policy over the range that we consider[38]. As ω_r increases, the optimal rules at both BLE and REE decrease to plausible values. An immediate result that stands out is that, ϕ_π^* is less sensitive to the policy weights at BLE, while ϕ_y^* is less sensitive at REE. In other words, optimal monetary policy at REE

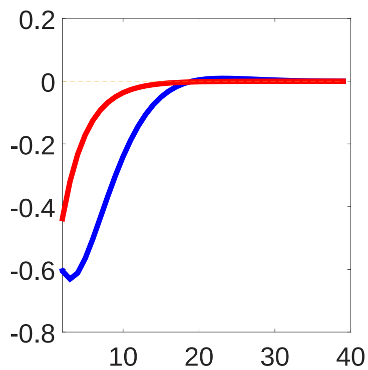
[38] An optimal policy with larger parameters exists at both BLE and REE if we relax the upper bound of 15. We omit these cases from our discussion here.



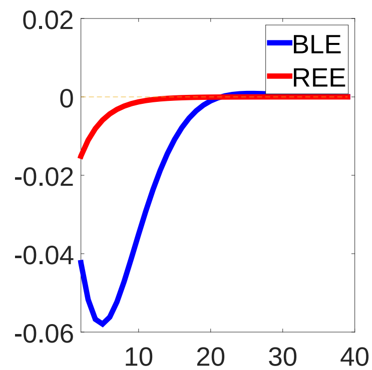
(a) ϕ_π^* as a function of $\frac{\omega_r}{\omega_y}$.



(b) ϕ_y^* as a function of $\frac{\omega_r}{\omega_y}$.



(c) IRF of output gap.



(d) IRF of inflation.

Figure 11: Top panel: changes in optimal parameters as a ratio of weights on interest rates and output gap. Bottom panel: impulse response functions of output gap and inflation to a one standard deviation monetary policy shock.

responds to changes in policy weights mainly through ϕ_π^*, while optimal monetary policy at BLE responds to these changes through ϕ_y^*. A similar effect can also be seen in Figure 11a and 11b, which shows the optimal parameters as a function of the ratio of policy weights $\frac{\omega_r}{\omega_y}$. This result is driven by the fact that both estimates of $\frac{1}{\varphi}$ and γ are larger under BLE, which leads to a stronger transmission channel of monetary policy. This in turn allows interest rate changes to have a larger impact on output gap, and a stronger feedback from output gap to inflation at BLE. The stronger transmission channel at BLE is also evident from Figure 11c and 11d, which shows the impulse responses of inflation and output gap to a monetary policy shock of the same size[39]. Both the initial impact,

[39] The shock size is one standard deviation at the estimated parameter values for each case, which is

as well as the cumulative impact of the shock are larger under BLE. Furthermore the shock takes several quarters to reach its full impact under BLE, leading to hump-shaped responses for both inflation and output gap. This is consistent with previous studies in the literature, where a contractionary monetary policy shock typically leads to hump-shaped decreases in output and inflation with peaks after one to two years; see e.g. Leeper et al. (1996) and Christiano et al. (1999). This effect is absent at REE, and it shows that the persistence amplification at BLE is also indirectly reflected in the system's response to an exogenous monetary policy shock. As a consequence of this stronger transmission channel, inflation is more responsive to changes in interest rate at BLE, which leads to smaller fluctuations in ϕ_π^*. In particular for sufficiently large ω_r, ϕ_π^* stabilizes around 1.67 at BLE, which is the close to the standard value of inflation reaction associated with the Taylor rule. Using Woodford (1999) calibration of $(\omega_y, \omega_r) = (0.048, 0.236)$, the optimal rule at BLE is given by $(\phi_y^*, \phi_\pi^*) = (1.67, 1.56)$, while the optimal rule at REE falls into the indeterminacy region with $(\phi_y^*, \phi_\pi^*) = (0.94, 0.75)$.

The column $E[L^*]$ in Table 10 shows the value of the loss function at the optimal rule for a given pair of weights (ω_y, ω_r), and $\Delta E[L^*]$ shows the percentage improvement at the optimal rule relative to the loss function at the estimated parameters. It is readily seen that the values for $\Delta E[L^*]$ obtained at BLE are generally larger than REE. This is again a consequence of the stronger transmission channel at BLE, which allows monetary policy to have a larger influence in stabilizing the economy. In particular, the optimal variance of inflation obtained at BLE is typically less than half of the optimal variance of inflation at REE.

In our analysis above, the optimal parameters are inevitably influenced by the degree of interest rate smoothing ρ_r. Since this parameter is estimated at different values at BLE and REE with 0.85 and 0.8 respectively, we also examine optimal policies at BLE and REE when $\rho_r = 0$, which is illustrated at the bottom panel of Table 5. It is readily seen that our results continue to hold in this case: ϕ_π^* is less sensitive and ϕ_y^* is more sensitive at BLE to changes in policy weights, and the percentage changes $\Delta E[L^*]$ in the loss function are generally larger compared to REE. The only difference with the previous case is that the optimal parameters are generally smaller at both BLE and REE, which is expected since shutting off ρ_r allows for larger movements in interest rates, which in turn leads to smaller optimal parameters. This final exercise also reveals that, interest

the same under BLE and REE with 0.29.

rate smoothing is typically welfare improving at REE, while this is never the case at BLE. $E[L^*]$ is uniformly smaller and $\Delta E[L^*]$ uniformly larger at BLE when interest rate smoothing is shut off, while the opposite holds at REE. The result for REE is well-known from Woodford (1999), where a commitment to interest rate smoothing can have a stabilizing effect with forward-looking agents who anticipate and take into account future changes in interest rate. This is different than a BLE with backward-looking agents, where such a commitment does not have a stabilizing effect since agents are unaware of the commitment or do not take it into account when forming their expectations. Overall, these results show that optimal monetary policy at BLE can differ from that at REE in important ways.

5 Concluding Remarks

We have generalized the behavioral learning equilibrium concept to an n-dimensional linear stochastic framework and provided a general iterative method to approximate and estimate BLE. We have applied our framework to the 3-equation New Keynesian model. Boundedly rational agents use univariate AR(1) forecasting rules for all output gap and inflation. A BLE is parameter free, as along the BLE the two parameters of each rule are pinned down by two observable statistics: the unconditional mean and the first-order autocorrelation. Hence, to a first-order approximation the simple linear forecasting rule is optimal and consistent with observed market realizations. Agents gradually update the two coefficients –sample mean and first-order autocorrelation– of their linear rule through sample autocorrelation learning. In the long run, agents thus learn to coordinate on the best univariate linear forecasting rule for each endogenous state variable, without fully recognizing the more complex structure of the economy. In higher-dimensional systems, BLE exist under fairly general conditions and we provide simple stability conditions under learning. Coordination on a simple, parsimonious BLE is self-fulfilling and seems a plausible outcome of the coordination process of individual expectations in large complex socio-economic systems.

A striking feature of BLE is the strong *persistence amplification*: the persistence of output and inflation along a BLE is much higher, often near unit root, than the persistence in the exogenous shocks driving the economy. Due to these features, estimating the 3-equation model on historical data under BLE yields a substantially better model fit and a different propagation mechanism compared with the REE model. This leaves an important role for monetary policy with the goal of stabilizing inflation and output. Different from REE, we find finite optimal Taylor rule coefficients at the BLE in our benchmark calibration. Furthermore, we observe a stronger transmission channel of monetary policy at the estimated parameter values under BLE. A sufficiently aggressive Taylor rule may keep the economy in a stable region with relatively low volatility in inflation and output gap. Future work should study BLE and corresponding optimal policies in more general New Keynesian models.

Appendix

A Mean of the rational expectations equilibrium

Using (2.10-2.11) and (2.15-2.18) the mean satisfies

$$\begin{aligned}
\overline{\mathbf{x}^*} &= (\boldsymbol{I}-\boldsymbol{c}_1)^{-1}(\boldsymbol{c}_0+\boldsymbol{c}_2\overline{\boldsymbol{u}}) \\
&= (\boldsymbol{I}-\boldsymbol{c}_1)^{-1}(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{c}_1-\boldsymbol{b}_1)^{-1}(\boldsymbol{b}_0+\boldsymbol{b}_1\boldsymbol{c}_2\boldsymbol{a})+(\boldsymbol{I}-\boldsymbol{c}_1)^{-1}\boldsymbol{c}_2(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a} \\
&= (\boldsymbol{I}-\boldsymbol{c}_1)^{-1}(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{c}_1-\boldsymbol{b}_1)^{-1}[\boldsymbol{b}_0+(\boldsymbol{b}_1\boldsymbol{c}_2(\boldsymbol{I}-\boldsymbol{\rho})+(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{c}_1-\boldsymbol{b}_1)\boldsymbol{c}_2)(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}] \\
&= [(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{c}_1-\boldsymbol{b}_1)(\boldsymbol{I}-\boldsymbol{c}_1)]^{-1}[\boldsymbol{b}_0+\boldsymbol{b}_3(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}] \\
&= (\boldsymbol{I}-\boldsymbol{b}_1-\boldsymbol{b}_2)^{-1}[\boldsymbol{b}_0+\boldsymbol{b}_3(\boldsymbol{I}-\boldsymbol{\rho})^{-1}\boldsymbol{a}].
\end{aligned}$$

B Autocorrelation in the n-dimensional case

The purpose of this appendix is to show that the first-order autocorrelation coefficients of the stochastic stationary system (2.24) are continuous functions with respect to $(\beta_1,\beta_2,\cdots,\beta_n)$ and the other related parameters. Rewrite model (2.24) as

$$\begin{cases}\boldsymbol{x}_t-\overline{\boldsymbol{x}}=(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})+\boldsymbol{b}_3(\boldsymbol{u}_t-\overline{\boldsymbol{u}})+\boldsymbol{b}_4\boldsymbol{v}_t, \\ \boldsymbol{u}_t-\overline{\boldsymbol{u}}=\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})+\boldsymbol{\varepsilon}_t.\end{cases} \tag{B.1}$$

That is,

$$\begin{cases}\boldsymbol{x}_t-\overline{\boldsymbol{x}}=(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})+\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})+\boldsymbol{b}_3\boldsymbol{\varepsilon}_t+\boldsymbol{b}_4\boldsymbol{v}_t, \\ \boldsymbol{u}_t-\overline{\boldsymbol{u}}=\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})+\boldsymbol{\varepsilon}_t.\end{cases} \tag{B.2}$$

$$\begin{aligned}
\boldsymbol{\Gamma}(-1) &= E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'] \\
&= E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'+\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'+\boldsymbol{b}_3\boldsymbol{\varepsilon}_t(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})' \\
&\quad +\boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'\Big] \\
&= (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\boldsymbol{\Gamma}(0)+\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'] \\
&= (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\boldsymbol{\Gamma}(0)+\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})']. \qquad \text{(B.3)}
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\Gamma}(0) &= E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'] \\
&= E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'+\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'+\boldsymbol{b}_3\boldsymbol{\varepsilon}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'+\boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'\Big] \\
&= (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\boldsymbol{\Gamma}(1)+\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})']+\boldsymbol{b}_3E[\boldsymbol{\varepsilon}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})']+\boldsymbol{b}_4E[\boldsymbol{v}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'] \\
&= (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\boldsymbol{\Gamma}(1)+\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})']+\boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_3'+\boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4'. \qquad \text{(B.4)}
\end{aligned}
$$

Note that $E[\boldsymbol{\varepsilon}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'] = E\Big[\boldsymbol{\varepsilon}_t((\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}}))'+\boldsymbol{\varepsilon}_t(\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}}))'+\boldsymbol{\varepsilon}_t(\boldsymbol{b}_3\boldsymbol{\varepsilon}_t)'+\boldsymbol{\varepsilon}_t(\boldsymbol{b}_4\boldsymbol{v}_t)'\Big] = \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_3'$ and $E[\boldsymbol{v}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'] = E\Big[\boldsymbol{v}_t((\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}}))'+\boldsymbol{v}_t(\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}}))'+\boldsymbol{v}_t(\boldsymbol{b}_3\boldsymbol{\varepsilon}_t)'+\boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{b}_4\boldsymbol{v}_t)'\Big] = \boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4'$.

Based on (B.3), (B.4) and $\boldsymbol{\Gamma}(-1)=\boldsymbol{\Gamma}(1)'$,

$$
\begin{aligned}
\boldsymbol{\Gamma}(0) &= (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\boldsymbol{\Gamma}(0)(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)'+(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'](\boldsymbol{b}_3\boldsymbol{\rho})' \\
&\quad +\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})']+\boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_3'+\boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4'.
\end{aligned}
$$

In order to obtain the expression of $\boldsymbol{\Gamma}(0)$, we use column stacks of matrices. Suppose $vec(\boldsymbol{K})$ is the vectorization of a matrix $\boldsymbol{K}$ and $\otimes$ is the Kronecker product[40]. Under the assumption that all the eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2$ are inside the unit circle, based on the property of Kronecker product[41], it is easy to see all the eigenvalues of $(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)$ lie inside the unit circle and hence $[\boldsymbol{I}-(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)]^{-1}$ exist. Therefore,

$$
\begin{aligned}
vec(\boldsymbol{\Gamma}(0)) &= [\boldsymbol{I}-(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)]^{-1}[((\boldsymbol{b}_3\boldsymbol{\rho})\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2))vec(E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})']) \\
&\quad +(\boldsymbol{I}\otimes(\boldsymbol{b}_3\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'])+vec(\boldsymbol{b}_3\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_3'+\boldsymbol{b}_4\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{b}_4')]. \qquad \text{(B.5)}
\end{aligned}
$$

Thus in order to obtain $\boldsymbol{\Gamma}(1)$ and $\boldsymbol{\Gamma}(0)$, we need calculate $E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})']$ and $E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})']$.

$$
\begin{aligned}
& E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'] \\
=\; & E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'+\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'+\boldsymbol{b}_3\boldsymbol{\varepsilon}_t(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'+\boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'\Big] \\
=\; & E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'\boldsymbol{\rho}'+\boldsymbol{\varepsilon}_t']+\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'\boldsymbol{\rho}'+\boldsymbol{\varepsilon}_t'] \\
& +\boldsymbol{b}_3\boldsymbol{\varepsilon}_t[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'\boldsymbol{\rho}'+\boldsymbol{\varepsilon}_t']+\boldsymbol{v}_t[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'\boldsymbol{\rho}'+\boldsymbol{\varepsilon}_t']\Big] \\
=\; & (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)E[(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})']\boldsymbol{\rho}'+\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})']\boldsymbol{\rho}'+\boldsymbol{b}_3\Sigma_{\varepsilon}.
\end{aligned}
$$

[40]One property of column stacks is that the column stack of a product of three matrices is $vec(ABC) = (C'\otimes A)vec(B)$. For more details on this and related properties, see Magnus and Neudecker(1988, Chapter 2) and Evans and Honkapohja (2001, Section 5.7).

[41]The eigenvalues of $\widehat{A}\otimes\widehat{B}$ are the mn numbers $\lambda_r\mu_s, r=1,2,\cdots,m, s=1,2,\cdots,n$ where $\lambda_1,\cdots,\lambda_m$ are the eigenvalues of $m\times m$ matrix $\widehat{A}$ and $\mu_1,\cdots,\mu_n$ are the eigenvalues of $n\times n$ matrix $\widehat{B}$; see Lancaster and Tismenetsky (1985).

Correspondingly,

$$
\begin{aligned}
& vec(E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'])\\
= \ & [\boldsymbol{I}-\boldsymbol{\rho}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)]^{-1}[vec(\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})']\boldsymbol{\rho}')+vec(\boldsymbol{b}_3\boldsymbol{\Sigma}_\varepsilon)]\\
= \ & [\boldsymbol{I}-\boldsymbol{\rho}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)]^{-1}[(\boldsymbol{\rho}\otimes(\boldsymbol{b}_3\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'])+(\boldsymbol{I}\otimes\boldsymbol{b}_3)vec\boldsymbol{\Sigma}_\varepsilon)]\\
= \ & [\boldsymbol{I}-\boldsymbol{\rho}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)]^{-1}[(\boldsymbol{\rho}\otimes(\boldsymbol{b}_3\boldsymbol{\rho}))[\boldsymbol{I}-\boldsymbol{\rho}\otimes\boldsymbol{\rho}]^{-1}+(\boldsymbol{I}\otimes\boldsymbol{b}_3)]vec(\boldsymbol{\Sigma}_\varepsilon). \qquad \text{(B.6)}
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})']\\
= \ & E\Big[(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'+\boldsymbol{b}_3\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'+\boldsymbol{b}_3\boldsymbol{\varepsilon}_t(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'+\boldsymbol{b}_4\boldsymbol{v}_t(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'\Big]\\
= \ & (\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})']+\boldsymbol{b}_3\boldsymbol{\rho}E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'].
\end{aligned}
$$

Thus based on (B.6),

$$
\begin{aligned}
& vec(E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'])\\
= \ & (\boldsymbol{I}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2))vec(E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'])+(\boldsymbol{I}\otimes(\boldsymbol{b}_3\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'])\\
= \ & (\boldsymbol{I}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2))[\boldsymbol{I}-\boldsymbol{\rho}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2+\boldsymbol{b}_2)]^{-1}[(\boldsymbol{\rho}\otimes(\boldsymbol{b}_3\boldsymbol{\rho}))[\boldsymbol{I}-\boldsymbol{\rho}\otimes\boldsymbol{\rho}]^{-1}+(\boldsymbol{I}\otimes\boldsymbol{b}_3)]vec(\boldsymbol{\Sigma}_\varepsilon)\\
& +(\boldsymbol{I}\otimes(\boldsymbol{b}_3\boldsymbol{\rho}))[\boldsymbol{I}-\boldsymbol{\rho}\otimes\boldsymbol{\rho}]^{-1}vec(\boldsymbol{\Sigma}_\varepsilon). \qquad \text{(B.7)}
\end{aligned}
$$

Therefore based on (B.7), the expression of matrix $E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'$ can be obtained. Then by transposing the matrix $E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'$, we can obtain $vec(E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'])$. Furthermore, combining this with (B.6), we obtain the variance-covariance matrix $\boldsymbol{\Gamma}(0)$ from (B.5) and further $\boldsymbol{\Gamma}(1)$ from (B.3). Based on the properties of matrices operations, it is easy to see that the entries of matrices $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$ are smooth functions with respect to $(\beta_1,\beta_3,\cdots,\beta_n)$ and the other related parameters. Thus the first-order autocorrelation coefficients of the nontrivial stochastic stationary system (2.24) are continuous functions with respect to $(\beta_1,\beta_3,\cdots,\beta_n)$ and the other related parameters.

Moments for the zero-mean case

Taking (2.32) as the starting point and assuming $\bar{\gamma}=\mathbf{0}$, we have

$$S_t=\gamma_1 S_{t-1}+\gamma_2\boldsymbol{\beta}^2 S_{t-1}+\gamma_3\eta_t \qquad \text{(B.8)}$$

The first-order covariance matrix is given by:

$$\mathbb{E}[S_tS_{t-1}]=(\gamma_1+\gamma_2\boldsymbol{\beta}^2)\mathbb{E}[S_{t-1}S_{t-1}]+\gamma_3\mathbb{E}[\eta_tS_{t-1}] \qquad \text{(B.9)}$$

We have $\mathbb{E}[\eta_t S_{t-1}] = 0$, while $\mathbb{E}[\eta_{t-1} S_{t-1}] = \mathbb{E}[\eta_{t-1}((\gamma_1 + \gamma_2 \boldsymbol{\beta}^2) S_{t-1} + \gamma_3 \eta_t] = \gamma_3 \boldsymbol{\Sigma_\eta}$. Further denoting $(\gamma_1 + \gamma_2 \boldsymbol{\beta}^2) = M(\boldsymbol{\beta})$, the first-order covariance matrix $\mathbb{E}[S_t S_{t-1}] = \boldsymbol{\Gamma}(-1)$ and the variance-covariance matrix $\mathbb{E}[S_t S_t] = \boldsymbol{\Gamma}(0)$, the expression in (B.9) reduces to:

$$\boldsymbol{\Gamma}(-1) = M(\boldsymbol{\beta}) \boldsymbol{\Gamma}(0) \tag{B.10}$$

Taking the variance on both sides of (B.8) yields:

$$\boldsymbol{\Gamma}(0) = M(\boldsymbol{\beta}) \boldsymbol{\Gamma}(0) M(\boldsymbol{\beta})' + \gamma_3 \boldsymbol{\Sigma_\eta} \gamma_3' \tag{B.11}$$

Vectorizing both sides implies:

$$Vec(\boldsymbol{\Gamma}(0)) = Vec(M(\boldsymbol{\beta}) \boldsymbol{\Gamma}(0) M(\boldsymbol{\beta})' + \gamma_3 \boldsymbol{\Sigma_\eta} \gamma_3') \tag{B.12}$$

Using $Vec(ABC) = (C' \otimes A) Vec(B)$, and $Vec(A+B) = Vec(A) + Vec(B)$, the expression above reduces to:

$$Vec(\boldsymbol{\Gamma}(0)) = (M(\boldsymbol{\beta}) \otimes M(\boldsymbol{\beta})) Vec(\boldsymbol{\Gamma}(0)) + (\gamma_3 \otimes \gamma_3) Vec(\boldsymbol{\Sigma_\eta}) \tag{B.13}$$

Hence

$$Vec(\boldsymbol{\Gamma}(0)) = [I - M(\boldsymbol{\beta}) \otimes M(\boldsymbol{\beta})]^{-1} (\gamma_3 \otimes \gamma_3) Vec(\boldsymbol{\Sigma_\eta}) \tag{B.14}$$

which yields (2.33).

C Proof of Proposition 2 (stability under SAC-learning)

Set $\gamma_t = (1+t)^{-1}$. For the state dynamics equations in (2.28) and (2.8)[42], since all functions are smooth, the SAC-learning rule satisfies the conditions (A.1-A.3) of Section 6.2.1 in Evans and Honkapohja (2001, p.124).

In order to check the conditions (B.1-B.2) of Section 6.2.1 in Evans and Honkapohja (2001, p.125), we rewrite the system in matrix form by

$$\boldsymbol{X}_t = \widetilde{\boldsymbol{A}}(\boldsymbol{\theta}_{t-1}) \boldsymbol{X}_{t-1} + \widetilde{\boldsymbol{B}}(\boldsymbol{\theta}_{t-1}) \boldsymbol{W}_t,$$

where $\boldsymbol{\theta}_t' = (\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t, \boldsymbol{R}_t), \boldsymbol{X}_t' = (1, \boldsymbol{x}_t', \boldsymbol{x}_{t-1}', \boldsymbol{u}_t')$ and $\boldsymbol{W}_t' = (1, \boldsymbol{v}_t', \boldsymbol{\varepsilon}_t')$,

$$\widetilde{\boldsymbol{A}}(\boldsymbol{\theta}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \boldsymbol{b_0} + \boldsymbol{b_1}(\boldsymbol{I} - \boldsymbol{\beta}^2)\boldsymbol{\alpha} + \boldsymbol{b_2 a} & \boldsymbol{b_1}\boldsymbol{\beta}^2 & \mathbf{0} & \boldsymbol{b_2 \rho} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{a} & \mathbf{0} & \mathbf{0} & \boldsymbol{\rho} \end{pmatrix},$$

[42]For convenience of theoretical analysis, one can set $\mathbf{S_{t-1}} = \mathbf{R_t}$.

$$\widetilde{\boldsymbol{B}}(\boldsymbol{\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & \mathbf{I} & \mathbf{b_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

Based on the properties of eigenvalues, see e.g. Evans and Honkapohja (2001, p.117), all the eigenvalues of $\widetilde{\boldsymbol{A}}(\boldsymbol{\theta})$ include 0 (multiple $n+1$), the eigenvalues of $\boldsymbol{\rho}$ and $\boldsymbol{b_1}\boldsymbol{\beta}^2$. Thus based on the assumptions, all the eigenvalues of $\widetilde{\boldsymbol{A}}(\boldsymbol{\theta})$ lie inside the unit circle. Moreover, it is easy to see all the other conditions for Section 6.2.1 of Chapter 6 in Evans and Honkapohja (2001) are also satisfied.

Since $\boldsymbol{x}_t$ is stationary, then the limits

$$\sigma_i^2 := \lim_{t\to\infty} E(x_{i,t}-\alpha_i)^2, \qquad \sigma^2_{x_i x_{i,-1}} := \lim_{t\to\infty} E(x_{i,t}-\alpha_i)(x_{i,t-1}-\alpha_i)$$

exist and are finite. Hence according to Section 6.2.1 of Chapter 6 in Evans and Honkapohja (2001, p.126), the associated ODE is

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha},\boldsymbol{\beta}) - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{R}^{-1}[\boldsymbol{E} - \boldsymbol{\beta}\boldsymbol{\Omega}] = \boldsymbol{R}^{-1}\boldsymbol{\Omega}[\boldsymbol{E}\boldsymbol{\Omega}^{-1} - \boldsymbol{\beta}], \\ \dfrac{d\boldsymbol{R}}{d\tau} = \boldsymbol{\Omega} - \boldsymbol{R}, \end{cases} \tag{C.1}$$

where $\boldsymbol{R}$ is a diagonal matrix with the i-th diagonal entry R_i and $\boldsymbol{\Omega}$, $\boldsymbol{E}$ are also diagonal matrices as defined in Section 2. As shown in Evans and Honkapohja (2001), a BLE corresponds to a fixed point of the following ODE (C.2).

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha},\boldsymbol{\beta}) - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{G} - \boldsymbol{\beta}. \end{cases} \tag{C.2}$$

Note that $\boldsymbol{\beta}$ and $\boldsymbol{G}$ are both diagonal matrices. The Jacobian matrix of C.2 is, in fact, equivalent to

$$\begin{pmatrix} (\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^{*2})^{-1}(\boldsymbol{b}_1 - \boldsymbol{I}) & \boldsymbol{\varrho} \\ \mathbf{0} & \boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*) - \boldsymbol{I} \end{pmatrix},$$

where $\boldsymbol{DG}_{\boldsymbol{\beta}}$ is a Jacobian matrix with the (i,j)-th entry $\frac{\partial G_i}{\partial \beta_j}$ and the form of matrix $\boldsymbol{\varrho}$ is omitted since it is not needed in the proof. Therefore, if all the eigenvalues of $(\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^{*2})^{-1}(\boldsymbol{b}_1 - \boldsymbol{I})$ have negative real parts, and all the eigenvalues of $\boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*)$ have real parts less than 1, the SAC-learning $(\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t)$ converges to the BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ as time t tends to ∞.

D Local Stability Conditions

First we show that the condition $\rho(DG_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*)) < 1$ is not necessary for local stability of the Quasi-Newton iteration in (2.36). Note that the iteration is given as:

$$\boldsymbol{\beta^{(k+1)}} = \boldsymbol{\beta^{(k)}} - DF_{\boldsymbol{\beta}}(\boldsymbol{\beta^{(k)}}, \theta)^{-1} F(\boldsymbol{\beta^{(k)}}, \theta), \tag{D.1}$$

with $F(\boldsymbol{\beta^{(k)}}, \theta) = G(\boldsymbol{\beta^{(k)}}, \theta) - \boldsymbol{\beta^{(k)}}$. Defining[43] $H(\boldsymbol{\beta}) = \boldsymbol{\beta} - DF_{\boldsymbol{\beta}}(\boldsymbol{\beta})^{-1}F(\boldsymbol{\beta})$, we need to show that $H(\boldsymbol{\beta})$ is locally stable. Note that:

$$DH_{\boldsymbol{\beta}}(\boldsymbol{\beta}) = DF_{\boldsymbol{\beta}}(\boldsymbol{\beta})^{-2} D^2_{\boldsymbol{\beta}} F(\boldsymbol{\beta}) F(\boldsymbol{\beta}),$$

with $DF_{\boldsymbol{\beta}}(\boldsymbol{\beta}) = DG_{\boldsymbol{\beta}}(\boldsymbol{\beta}) - I$ and $D^2_{\boldsymbol{\beta}} F(\boldsymbol{\beta}) = D^2_{\boldsymbol{\beta}} G(\boldsymbol{\beta})$, which implies:

$$DH_{\boldsymbol{\beta}}(\boldsymbol{\beta}) = (D_{\boldsymbol{\beta}} G(\boldsymbol{\beta}) - I)^{-2} D^2_{\boldsymbol{\beta}} G(\boldsymbol{\beta})(G(\boldsymbol{\beta}) - \boldsymbol{\beta}).$$

Since $\boldsymbol{\beta}^* = G(\boldsymbol{\beta}^*)$ at a BLE by definition, it follows that $\rho(DH_{\boldsymbol{\beta}}(\boldsymbol{\beta^*})) < 1$. Hence (D.1) is locally stable at any BLE $\boldsymbol{\beta}^*$ and one can find a neighbourhood $\hat{D}$ around $\boldsymbol{\beta}^*$ such that:

$$lim_{k\to\infty} H^k(\boldsymbol{\beta^{(0)}}) = \boldsymbol{\beta^*}, \forall \boldsymbol{\beta^{(0)}} \in \hat{D}. \tag{D.2}$$

Importantly, this result holds for all E-stable and E-unstable BLE. Therefore the Quasi-Newton iteration may also converge E-unstable fixed-points.

Next we show that the local stability argument extends to this case after re-writing the maximization problem. Let $(\mathbf{0}, \boldsymbol{\beta^*})$ be an iteratively E-stable fixed-point at the estimated parameter values $\hat{\theta}^*$. Then the belief parameters $\boldsymbol{\beta^*}$ and the structural parameters $\hat{\theta}^*$ satisfy the conditions:

$$\begin{cases} \boldsymbol{\beta^*} = G(\boldsymbol{\beta^*}, \theta^*) \\ \theta^* = \underset{\theta}{\operatorname{argmax}} \; p(\theta | Y_{1:T}, \boldsymbol{\beta^*}). \end{cases} \tag{D.3}$$

Since the dataset $Y_{1:T}$ remains fixed at each step of the iteration, the second condition can be written as:

$$\theta^* = \underset{\theta}{\operatorname{argmax}} \; p(\theta | Y_{1:T}, \boldsymbol{\beta^*}) = p_m(\boldsymbol{\beta^*}), \tag{D.4}$$

for some $p_m(.)$. This implies that the estimated structural parameters θ^* are given as a function the equilibrium belief parameters $\boldsymbol{\beta^*}$ *for a given dataset and likelihood function.* Plugging this back into the first condition yields:

$$\boldsymbol{\beta^*} = G(\boldsymbol{\beta^*}, p_m(\boldsymbol{\beta^*})). \tag{D.5}$$

[43] For the remainder, we omit the dependence of $G(\beta, \theta)$ on the structural parameters θ for ease of notation.

(D.5) has the same functional form as the fixed-point iteration in (2.35). In this case the Jacobian matrix at the equilibrium $\boldsymbol{\beta^*}$ and θ^* is given by

$$DG_{\boldsymbol{\beta}}(\boldsymbol{\beta^*}) = \frac{\partial G}{\partial \boldsymbol{\beta}}_{|\boldsymbol{\beta^*},\theta^*} + \frac{\partial G}{\partial \theta^*}\frac{\partial \theta^*}{\partial \boldsymbol{\beta}}_{|\boldsymbol{\beta^*},\theta^*}. \tag{D.6}$$

Note that the first component in (D.6) corresponds to the Jacobian matrix of $G(\boldsymbol{\beta})$ in the case with fixed parameters, while the second component appears due to the fact that the structural parameters also depend on $\boldsymbol{\beta^*}$. Further note that, all three partial derivatives that appear in (D.6) can be numerically evaluated. If the eigenvalue condition $\rho(DG(\boldsymbol{\beta^*})) < 1$ is satisfied, it follows that $\boldsymbol{\beta^*} = G(\boldsymbol{\beta^*}, \theta^*)$ is locally stable under (2.40) and (2.41).

E Eigenvalues of matrix $\boldsymbol{B\beta}^2$

The characteristic polynomial of $\mathbf{B}\boldsymbol{\beta}^{\mathbf{2}}$ is given by $h(\nu) = \nu^2 + c_1\nu + c_2$, where

$$c_1 = -\frac{\beta_1^2 + [\gamma\varphi + \lambda(1+\varphi\phi_y)]\beta_2^2}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}, \quad c_2 = \frac{\lambda\beta_1^2\beta_2^2}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}.$$

Both of the eigenvalues of $\mathbf{B}\boldsymbol{\beta}^{\mathbf{2}}$ are inside the unit circle if and only if both of the following conditions hold (see Elaydi, 1999):

$$h(1) > 0, \quad h(-1) > 0, \quad |h(0)| < 1.$$

It is easy to see $h(-1) > 0, |h(0)| < 1$ for any $\beta_i \in [-1, 1]$. Note that

$$\begin{aligned} h(1) &= \frac{(1-\beta_1^2)(1-\lambda\beta_2^2) + \gamma\varphi\phi_\pi + \varphi\phi_y - (\gamma\varphi + \lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}, \\ &\geq \frac{\varphi[\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y]}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}. \end{aligned}$$

Thus if $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$, then $h(1) > 0$. Therefore, both eigenvalues of $\mathbf{B}\boldsymbol{\beta}^{\mathbf{2}}$ lie inside the unit circle.

F First-order autocorrelation coefficients of output gap and inflation

Now we calculate $\boldsymbol{G}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Define $\boldsymbol{z}_t = \boldsymbol{x}_t - \bar{\mathbf{x}}$. Then in order to obtain $\boldsymbol{G}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, we first calculate $\boldsymbol{E}(z_t z_{t-1}')$ and $\boldsymbol{E}(z_t z_t')$. Rewrite model (3.12) into its VARMA$(1, \infty)$

representation

$$\boldsymbol{z}_t = \boldsymbol{B\beta}^2 \mathbf{z}_{t-1} + \boldsymbol{C}\sum_{n=0}^{\infty}\rho^n \boldsymbol{\varepsilon}_{t-n}. \tag{F.1}$$

Since both eigenvalues of $\boldsymbol{B\beta}^2$ lie inside the unit circle under the assumption $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$ (see Appendix E), then

$$\boldsymbol{z}_t = \mathbf{C}[\rho\mathbf{I} - \mathbf{C}^{-1}\boldsymbol{B\beta}^2\mathbf{C}]^{-1}\sum_{n=0}^{\infty}[\rho^{n+1}\mathbf{I} - \mathbf{C}^{-1}(\boldsymbol{B\beta}^2)^{n+1}\mathbf{C}]\boldsymbol{\varepsilon}_{t-n}.$$

Note ρ is a scalar number and $\boldsymbol{I}$ is a 2×2 identity matrix. Based on i.i.d. assumption of $\boldsymbol{\varepsilon}_t$,

$$\begin{aligned}\boldsymbol{E}z_t\boldsymbol{z}_t' \;=\; & \mathbf{C}[\rho\mathbf{I} - \mathbf{C}^{-1}\boldsymbol{B\beta}^2\mathbf{C}]^{-1}\sum_{n=0}^{\infty}[\rho^{n+1}\mathbf{I} - \mathbf{C}^{-1}(\boldsymbol{B\beta}^2)^{n+1}\mathbf{C}]\boldsymbol{\Sigma}[\rho^{n+1}\mathbf{I} - (\mathbf{C}^{-1}(\boldsymbol{B\beta}^2)^{n+1}\mathbf{C})']\cdot \\ & [\rho\mathbf{I} - (\mathbf{C}^{-1}\boldsymbol{B\beta}^2\mathbf{C})']^{-1}\mathbf{C}', \end{aligned} \tag{F.2}$$

where $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$.

In the following we try to obtain the expression of the matrix $\boldsymbol{E}z_t\boldsymbol{z}_t'$ and hence we first calculate the matrix $\sum_{n=0}^{\infty}[\rho^{n+1}\mathbf{I} - \mathbf{C}^{-1}(\boldsymbol{B\beta}^2)^{n+1}\mathbf{C}]\boldsymbol{\Sigma}[\rho^{n+1}\mathbf{I} - (\mathbf{C}^{-1}(\boldsymbol{B\beta}^2)^{n+1}\mathbf{C})']$ and $\mathbf{C}^{-1}(\boldsymbol{B\beta}^2)^{n+1}\mathbf{C}$.

Note that

$$\boldsymbol{B\beta}^2 = \frac{1}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}\begin{bmatrix} \beta_1^2 & \varphi(1-\lambda\phi_\pi)\beta_2^2 \\ \gamma\beta_1^2 & (\gamma\varphi + \lambda(1+\varphi\phi_y))\beta_2^2 \end{bmatrix}.$$

$\boldsymbol{B\beta}^2$ has two eigenvalues[44]

$$\begin{aligned}\lambda_1 &= \frac{[\beta_1^2 + (\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2] + \sqrt{[\beta_1^2 + (\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2]^2 - 4\lambda\beta_1^2\beta_2^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}}{2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}, \\ \lambda_2 &= \frac{[\beta_1^2 + (\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2] - \sqrt{[\beta_1^2 + (\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2]^2 - 4\lambda\beta_1^2\beta_2^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}}{2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}.\end{aligned}$$

Their corresponding eigenvectors are

$$\begin{aligned}P_1 &= \Big[\frac{\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y},\ \lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big]', \\ P_2 &= \Big[\frac{\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y},\ \lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big]'.\end{aligned}$$

[44] In the special case $\lambda_1 = \lambda_2$, although $\boldsymbol{B\beta}^2$ is not diagonalizable, the expressions of first-order autocorrelations (3.13) and (3.14) still hold based on the Jordan normal form of matrix $\boldsymbol{B\beta}^2$. Without loss of generality, in the following we assume $\lambda_1 \neq \lambda_2$.

Let $\boldsymbol{P} = [P_1, P_2]$. Then

$$\boldsymbol{C^{-1}B\beta^2C} = \boldsymbol{C^{-1}P} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} (C^{-1}\boldsymbol{P})^{-1},$$

where

$$\begin{aligned} & \boldsymbol{C^{-1}P} \\ = & \begin{bmatrix} \frac{(1+\varphi\phi_y)\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \varphi\phi_\pi\left(\lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\right) & \frac{(1+\varphi\phi_y)\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \varphi\phi_\pi\left(\lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\right) \\ \frac{-\gamma\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \left(\lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\right) & \frac{-\gamma\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \left(\lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\right) \end{bmatrix} \\ =: & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}. \end{aligned}$$

Correspondingly

$$(C^{-1}\boldsymbol{P})^{-1} = \frac{1}{d_1d_4 - d_2d_3} \begin{bmatrix} d_4 & -d_2 \\ -d_3 & d_1 \end{bmatrix},$$

where

$$d_1d_4 - d_2d_3 = \det(\boldsymbol{C}^{-1}P) = \varphi(1-\lambda\phi_\pi)\beta_2^2(\lambda_2 - \lambda_1).$$

Hence

$$\begin{aligned} \boldsymbol{C^{-1}(B\beta^2)^{n+1}C} &= \boldsymbol{C^{-1}P} \begin{bmatrix} \lambda_1^{n+1} & 0 \\ 0 & \lambda_2^{n+1} \end{bmatrix} \boldsymbol{(C^{-1}P)^{-1}} \\ &= \frac{1}{d_1d_4 - d_2d_3} \begin{bmatrix} d_1d_4\lambda_1^{n+1} - d_2d_3\lambda_2^{n+1} & d_1d_2(\lambda_2^{n+1} - \lambda_1^{n+1}) \\ d_3d_4(\lambda_1^{n+1} - \lambda_2^{n+1}) & d_1d_4\lambda_2^{n+1} - d_2d_3\lambda_1^{n+1} \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} & \boldsymbol{\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C} = \\ & \frac{1}{d_1d_4 - d_2d_3} \begin{bmatrix} d_1d_4(\rho^{n+1} - \lambda_1^{n+1}) - d_2d_3(\rho^{n+1} - \lambda_2^{n+1}) & -d_1d_2(\lambda_2^{n+1} - \lambda_1^{n+1}) \\ -d_3d_4(\lambda_1^{n+1} - \lambda_2^{n+1}) & d_1d_4(\rho^{n+1} - \lambda_2^{n+1}) - d_2d_3(\rho^{n+1} - \lambda_1^{n+1}) \end{bmatrix}. \end{aligned}$$

Therefore

$$\boldsymbol{[\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^{n+1}I - (C^{-1}(B\beta^2)^{n+1}C)']} = \frac{1}{(d_1d_4 - d_2d_3)^2} \begin{bmatrix} s_1(n+1) & s_2(n+1) \\ s_2(n+1) & s_3(n+1) \end{bmatrix},$$

where

$$
\begin{aligned}
s_1(n+1) &= \sigma_1^2[d_1d_4(\rho^{n+1}-\lambda_1^{n+1})-d_2d_3(\rho^{n+1}-\lambda_2^{n+1})]^2+\sigma_2^2[d_1d_2(\lambda_2^{n+1}-\lambda_1^{n+1})]^2,\\
s_2(n+1) &= \sigma_1^2 d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})[d_1d_4(\rho^{n+1}-\lambda_1^{n+1})-d_2d_3(\rho^{n+1}-\lambda_2^{n+1})]+\\
&\quad \sigma_2^2 d_1d_2(\lambda_1^{n+1}-\lambda_2^{n+1})[d_1d_4(\rho^{n+1}-\lambda_2^{n+1})-d_2d_3(\rho^{n+1}-\lambda_1^{n+1})],\\
s_3(n+1) &= \sigma_1^2[d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})]^2+\sigma_2^2[d_1d_4(\rho^{n+1}-\lambda_2^{n+1})-d_2d_3(\rho^{n+1}-\lambda_1^{n+1})]^2.
\end{aligned}
$$

Correspondingly it is natural to have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}[\rho^{n+1}I-C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^{n+1}I-(C^{-1}(B\beta^2)^{n+1}C)']\\
&= \frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} \sum_{n=0}^{\infty}s_1(n+1) & \sum_{n=0}^{\infty}s_2(n+1)\\ \sum_{n=0}^{\infty}s_2(n+1) & \sum_{n=0}^{\infty}s_3(n+1)\end{bmatrix}\\
&= \frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} s_1^* & s_2^*\\ s_2^* & s_3^*\end{bmatrix}, \qquad \text{(F.3)}
\end{aligned}
$$

where

$$
\begin{aligned}
s_1^* &= \sigma_1^2\Big[(d_1d_4-d_2d_3)^2\frac{1}{1-\rho^2}-2d_1d_4(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_1}+(d_1d_4)^2\frac{1}{1-\lambda_1^2}\\
&\quad +2d_2d_3(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_2}-2d_1d_2d_3d_4\frac{1}{1-\lambda_1\lambda_2}+(d_2d_3)^2\frac{1}{1-\lambda_2^2}\Big]\\
&\quad +\sigma_2^2\Big[(d_1d_2)^2\big(\frac{1}{1-\lambda_2^2}-\frac{2}{1-\lambda_1\lambda_2}+\frac{1}{1-\lambda_1^2}\big)\Big], \qquad \text{(F.4)}\\
s_2^* &= \sigma_1^2\Big[d_3d_4\Big\{(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_2}-\frac{1}{1-\rho\lambda_1}\Big)+\frac{d_1d_4}{1-\lambda_1^2}-\frac{d_1d_4+d_2d_3}{1-\lambda_1\lambda_2}+\frac{d_2d_3}{1-\lambda_2^2}\Big\}\Big]+\sigma_2^2\cdot\\
&\quad \Big[d_1d_2\Big\{(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_1}-\frac{1}{1-\rho\lambda_2}\Big)+\frac{d_1d_4}{1-\lambda_2^2}-\frac{d_1d_4+d_2d_3}{1-\lambda_1\lambda_2}+\frac{d_2d_3}{1-\lambda_1^2}\Big\}\Big], \qquad \text{(F.5)}\\
s_3^* &= \sigma_1^2\Big[(d_3d_4)^2\big(\frac{1}{1-\lambda_2^2}-\frac{2}{1-\lambda_1\lambda_2}+\frac{1}{1-\lambda_1^2}\big)\Big]\\
&\quad +\sigma_2^2\Big[(d_1d_4-d_2d_3)^2\frac{1}{1-\rho^2}-2d_1d_4(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_2}+(d_1d_4)^2\frac{1}{1-\lambda_2^2}\\
&\quad +2d_2d_3(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_1}-2d_1d_2d_3d_4\frac{1}{1-\lambda_1\lambda_2}+(d_2d_3)^2\frac{1}{1-\lambda_1^2}\Big]. \qquad \text{(F.6)}
\end{aligned}
$$

Therefore based on (F.2) and (F.3), we can further obtain the expression of $\boldsymbol{E}z_t\boldsymbol{z}_t'$.

Note that

$$
[\rho I-C^{-1}(B\beta^2)C]^{-1}=\frac{1}{\widetilde{m}}\begin{bmatrix} d_1d_4(\rho-\lambda_2)-d_2d_3(\rho-\lambda_1) & d_1d_2(\lambda_2-\lambda_1)\\ d_3d_4(\lambda_1-\lambda_2) & d_1d_4(\rho-\lambda_1)-d_2d_3(\rho-\lambda_2)\end{bmatrix},
$$

where $\widetilde{m} = (d_1 d_4 - d_2 d_3)(\rho - \lambda_1)(\rho - \lambda_2)$, and

$$\boldsymbol{C}[\boldsymbol{\rho I} - \boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})\boldsymbol{C}]^{-1} = \frac{1}{\widetilde{m}(1 + \gamma\varphi\phi_\pi + \varphi\phi_y)} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix},$$

where

$$\begin{aligned} k_1 &= d_1 d_4(\rho - \lambda_2) - d_2 d_3(\rho - \lambda_1) - \varphi\phi_\pi d_3 d_4(\lambda_1 - \lambda_2), && \text{(F.7)} \\ k_2 &= d_1 d_2(\lambda_2 - \lambda_1) - \varphi\phi_\pi[d_1 d_4(\rho - \lambda_1) - d_2 d_3(\rho - \lambda_2)], && \text{(F.8)} \\ k_3 &= \gamma[d_1 d_4(\rho - \lambda_2) - d_2 d_3(\rho - \lambda_1)] + (1 + \varphi\phi_y) d_3 d_4(\lambda_1 - \lambda_2), && \text{(F.9)} \\ k_4 &= \gamma d_1 d_2(\lambda_2 - \lambda_1) + (1 + \varphi\phi_y)[d_1 d_4(\rho - \lambda_1) - d_2 d_3(\rho - \lambda_2)]. && \text{(F.10)} \end{aligned}$$

Thus we have

$$\boldsymbol{E} z_t \boldsymbol{z}_t' = \widetilde{k} \cdot \begin{bmatrix} k_1^2 s_1^* + 2k_1 k_2 s_2^* + k_2^2 s_3^* & k_1 k_3 s_1^* + (k_1 k_4 + k_2 k_3) s_2^* + k_2 k_4 s_3^* \\ k_1 k_3 s_1^* + (k_1 k_4 + k_2 k_3) s_2^* + k_2 k_4 s_3^* & k_3^2 s_1^* + 2k_3 k_4 s_2^* + k_4^2 s_3^* \end{bmatrix} \tag{F.11}$$

where $\widetilde{k} = \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2 (d_1 d_4 - d_2 d_3)^4 (\rho - \lambda_1)^2 (\rho - \lambda_2)^2}$, s_i^* is given in (F.4)-(F.6) and k_i is given in (F.7)-(F.10).

Through very technical and extremely complicated calculations[45], the variances of

[45]Because of limit of pages, we first drop the calculations here. In case the readers need the details, we can provide separately.

output gap and inflations can be further simplified as

$$
\begin{aligned}
& E(y_t^2) = \frac{1}{\widetilde{k}}(k_1^2 s_1^* + 2k_1 k_2 s_2^* + k_2^2 s_3^*) \\
= \quad & \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)} \\
& \Big\{\sigma_1^2\Big[[(1+\lambda^2\beta_2^4)-2\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)+(1+\lambda^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)] \\
& -\rho\lambda_1\lambda_2[(1+\lambda^2\beta_2^4)(\rho+\lambda_1+\lambda_2)-2\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+(1+\lambda^2\beta_2^4)\rho\lambda_1\lambda_2]\Big] \\
& +\sigma_2^2\Big[[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)-2\varphi\phi_\pi\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)+((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)] \\
& -\rho\lambda_1\lambda_2[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho+\lambda_1+\lambda_2)-2\varphi\phi_\pi\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) \\
& +((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)\rho\lambda_1\lambda_2]\Big]\Big\}, \qquad \text{(F.12)} \\
& E(\pi_t^2) = \frac{1}{\widetilde{k}}(k_3^2 s_1^* + 2k_3 k_4 s_2^* + k_4^2 s_3^*) \\
= \quad & \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)} \\
& \Big\{\sigma_1^2\Big[\gamma^2[1+\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2-\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)-(\rho\lambda_1\lambda_2)^2]\Big] \\
& +\sigma_2^2\Big[[((1+\varphi\phi_y)^2+\beta_1^4)-2(1+\varphi\phi_y)\beta_1^2(\rho+\lambda_1+\lambda_2)+((1+\varphi\phi_y)^2+\beta_1^4) \\
& (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]-\rho\lambda_1\lambda_2[((1+\varphi\phi_y)^2+\beta_1^4)(\rho+\lambda_1+\lambda_2)-2(1+\varphi\phi_y)\beta_1^2\cdot \\
& (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+((1+\varphi\phi_y)^2+\beta_1^4)\rho\lambda_1\lambda_2]\Big]\Big\}. \qquad \text{(F.13)}
\end{aligned}
$$

Note that here $E(y_t^2)$ and $E(\pi_t^2)$ in fact depend on the trace $\lambda_1+\lambda_2$ and determinant $\lambda_1\lambda_2$.

With the expression of covariance matrix $\boldsymbol{E}z_t\boldsymbol{z}_t'$, in order to obtain the expressions of first-order autocorrelation coefficient of output gap and inflation, we need to further calculate the first-order autocovariance $\boldsymbol{E}z_t\boldsymbol{z}_{t-1}'$.

Following the similar calculations to $\boldsymbol{E}z_t\boldsymbol{z}_t'$, we can obtain

$$
\begin{aligned}
\boldsymbol{E}z_t\boldsymbol{z}_{t-1}' &= \mathbf{C}[\rho\mathbf{I}-\mathbf{C}^{-1}\boldsymbol{B}\boldsymbol{\beta}^2\mathbf{C}]^{-1}\sum_{n=1}^{\infty}[\rho^{n+1}\mathbf{I}-\mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^{n+1}\mathbf{C}]\boldsymbol{\Sigma}[\rho^n\mathbf{I}-(\mathbf{C}^{-1}(\boldsymbol{B}\boldsymbol{\beta}^2)^n\mathbf{C})']\cdot \\
&\quad [\rho\mathbf{I}-(\mathbf{C}^{-1}\boldsymbol{B}\boldsymbol{\beta}^2\mathbf{C})']^{-1}\mathbf{C}' \\
&= \widetilde{k}\begin{bmatrix} k_1^2w_1^*+k_1k_2(w_2^*+w_3^*)+k_2^2w_4^* & k_1k_3w_1^*+k_1k_4w_2^*+k_2k_3w_3^*+k_2k_4w_4^* \\ k_1k_3w_1^*+k_2k_3w_2^*+k_1k_4w_3^*+k_2k_4w_4^* & k_3^2w_1^*+k_3k_4(w_2^*+w_3^*)+k_4^2w_4^* \end{bmatrix},
\end{aligned}
$$

where $\widetilde{k}$, k_i are given in (F.11) and (F.7)-(F.10), and

$$
\begin{aligned}
w_1^* &= \sigma_1^2\Big\{(d_1d_4-d_2d_3)^2\frac{\rho}{1-\rho^2} - d_1d_4(d_1d_4-d_2d_3)\frac{\rho+\lambda_1}{1-\rho\lambda_1} + (d_1d_4)^2\frac{\lambda_1}{1-\lambda_1^2} \\
&\quad + d_2d_3(d_1d_4-d_2d_3)\frac{\rho+\lambda_2}{1-\rho\lambda_2} - d_1d_2d_3d_4\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} + (d_2d_3)^2\frac{\lambda_2}{1-\lambda_2^2}\Big\} \\
&\quad + \sigma_2^2(d_1d_2)^2\Big[\frac{\lambda_2}{1-\lambda_2^2} - \frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} + \frac{\lambda_1}{1-\lambda_1^2}\Big], \\
w_2^* &= \sigma_1^2 d_3d_4\Big\{(d_1d_4-d_2d_3)\Big[\frac{\rho}{1-\rho\lambda_2} - \frac{\rho}{1-\rho\lambda_1}\Big] + \frac{d_1d_4\lambda_1}{1-\lambda_1^2} - \frac{d_1d_4\lambda_1+d_2d_3\lambda_2}{1-\lambda_1\lambda_2} + \frac{d_2d_3\lambda_2}{1-\lambda_2^2}\Big\} \\
&\quad + \sigma_2^2 d_1d_2\Big\{(d_1d_4-d_2d_3)\Big[\frac{\lambda_1}{1-\rho\lambda_1} - \frac{\lambda_2}{1-\rho\lambda_2}\Big] + \frac{d_2d_3\lambda_1}{1-\lambda_1^2} - \frac{d_1d_4\lambda_1+d_2d_3\lambda_2}{1-\lambda_1\lambda_2} + \frac{d_1d_4\lambda_2}{1-\lambda_2^2}\Big\}, \\
w_3^* &= \sigma_1^2 d_3d_4\Big\{(d_1d_4-d_2d_3)\Big[\frac{\lambda_2}{1-\rho\lambda_2} - \frac{\lambda_1}{1-\rho\lambda_1}\Big] + \frac{d_1d_4\lambda_1}{1-\lambda_1^2} - \frac{d_1d_4\lambda_2+d_2d_3\lambda_1}{1-\lambda_1\lambda_2} + \frac{d_2d_3\lambda_2}{1-\lambda_2^2}\Big\} \\
&\quad + \sigma_2^2 d_1d_2\Big\{(d_1d_4-d_2d_3)\Big[\frac{\rho}{1-\rho\lambda_1} - \frac{\rho}{1-\rho\lambda_2}\Big] + \frac{d_1d_4\lambda_2}{1-\lambda_2^2} - \frac{d_1d_4\lambda_2+d_2d_3\lambda_1}{1-\lambda_1\lambda_2} + \frac{d_2d_3\lambda_1}{1-\lambda_1^2}\Big\}, \\
w_4^* &= \sigma_1^2(d_3d_4)^2\Big[\frac{\lambda_2}{1-\lambda_2^2} - \frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} + \frac{\lambda_1}{1-\lambda_1^2}\Big] + \sigma_2^2\Big\{(d_1d_4-d_2d_3)^2\frac{\rho}{1-\rho^2} \\
&\quad - d_1d_4(d_1d_4-d_2d_3)\frac{\rho+\lambda_2}{1-\rho\lambda_2} + (d_1d_4)^2\frac{\lambda_2}{1-\lambda_2^2} + d_2d_3(d_1d_4-d_2d_3)\frac{\rho+\lambda_1}{1-\rho\lambda_1} \\
&\quad - d_1d_2d_3d_4\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} + (d_2d_3)^2\frac{\lambda_1}{1-\lambda_1^2}\Big\}.
\end{aligned}
$$

Again through very technical calculations, the first-order auto-covariances of output gap and inflations are further simplified as

$$
\begin{aligned}
&E(y_ty_{t-1}) = \frac{1}{\widetilde{k}}(k_1^2w_1^* + k_1k_2(w_2^*+w_3^*) + k_2^2w_4^*) \\
= \;& \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)} \\
&\Big\{\sigma_1^2\Big[(\rho+\lambda_1+\lambda_2-\lambda\beta_2^2)[1-\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)] + [\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) - \\
&\rho\lambda_1\lambda_2][(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) - \lambda\beta_2^2\rho\lambda_1\lambda_2]\Big] + \sigma_2^2\Big[(\varphi\phi_\pi(\rho+\lambda_1+\lambda_2) - \varphi\beta_2^2)) \\
&[\varphi\phi_\pi - \varphi\beta_2^2(\rho+\lambda_1+\lambda_2)] + [\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) - \varphi\phi_\pi\rho\lambda_1\lambda_2] \\
&[\varphi\phi_\pi(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) - \varphi\beta_2^2\rho\lambda_1\lambda_2]\Big]\Big\}, \qquad \text{(F.14)} \\
&E(\pi_t\pi_{t-1}) = \frac{1}{\widetilde{k}}(k_3^2w_1^* + k_3k_4(w_2^*+w_3^*) + k_4^2w_4^*) \\
= \;& \frac{1}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)} \\
&\Big\{\sigma_1^2\Big[\gamma^2[(\rho+\lambda_1+\lambda_2) - \rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\Big] + \sigma_2^2\Big[[(1+\varphi\phi_y)(\rho+\lambda_1+\lambda_2) - \beta_1^2]\cdot \\
&[(1+\varphi\phi_y) - \beta_1^2(\rho+\lambda_1+\lambda_2)] + [\beta_1^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) - (1+\varphi\phi_y)\rho\lambda_1\lambda_2]\cdot \\
&[(1+\varphi\phi_y)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) - \beta_1^2\rho\lambda_1\lambda_2]\Big]\Big\}. \qquad \text{(F.15)}
\end{aligned}
$$

Therefore, the first-order autocorrelation coefficients of output gap and inflation

$$G_1(\beta_1,\beta_2)=\frac{E(y_t y_{t-1})}{E(y_t^2)},\quad G_2(\beta_1,\beta_2)=\frac{E(\pi_t\pi_{t-1})}{E(\pi_t^2)},$$

i.e. the equations (3.13)-(3.14).

G Stability for the Taylor rule

Based on Proposition 2, we only need to show that both of the eigenvalues of $(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}-\boldsymbol{I})$ have negative real parts if $\gamma(\phi_\pi-1)+(1-\lambda)\phi_y>0$.

The characteristic polynomial of $(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}-\boldsymbol{I})$ is given by $h(\nu)=\nu^2-c_1\nu+c_2$, where c_1 is the trace and c_2 is the determinant of matrix $(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}-\boldsymbol{I})$. Direct calculation shows that

$$c_1 \;=\; \frac{-(1-\lambda)(1-\beta_1^2)-2\varphi(\gamma\phi_\pi+\phi_y)+\varphi(\gamma+\lambda\phi_y)(1+\beta_2^2)}{\triangle\;(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}, \tag{G.1}$$

$$c_2 \;=\; \frac{\varphi[\gamma(\phi_\pi-1)+(1-\lambda)\phi_y]}{\triangle\;(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}, \tag{G.2}$$

where $\triangle=\frac{(1-\beta_1^2)(1-\lambda\beta_2^2)+\gamma\varphi\phi_\pi+\varphi\phi_y-(\gamma\varphi+\lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$.

Both of the eigenvalues of $(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}-\boldsymbol{I})$ have negative real parts if and only if $c_1<0$ and $c_2>0$ (these conditions are obtained by applying the *Routh-Hurwitz criterion theorem*; see Brock and Malliaris, 1989). If $\gamma(\phi_\pi-1)+(1-\lambda)\phi_y>0$, from Appendix E it is easy to see $\triangle>0$. Furthermore,

$$c_1\le\frac{-2\varphi[(\gamma(\phi_\pi-1)+(1-\lambda)\phi_y]}{\triangle\;(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}<0,\quad c_2>0.$$

H Data Appendix

The observable variables used in our estimations follow the definitions in Smets & Wouters (2007). Accordingly:

$$\begin{cases} y_t^{obs}=100log(GDPC09_t/LNS_{index_t}) \\ \pi_t^{obs}=100log(\frac{GDPDEF09_t}{GDPDEF09_{t-1}}) \\ r_t^{obs}=100log(\frac{Funds_t}{4}) \end{cases} \tag{H.1}$$

where the time series are given as:

GDPC09: Real GDP, Billions of Chained 2009 Dollars, Seasonally Adjusted Annual

Rate. Source: Federal Reserve Economic Data (FRED).

GDPDEF09: GDP-Implicit Price Deflator, 2009=100, Seasonally Adjusted. Source: FRED.

LNU00000000: Unadjusted civilian noninstitutional population, Thousands, 16 years& over. Source: U.S. Bureau of Labor Statistics (BLS)

LNS10000000: Civilian noninstitutional populations, Thousands, 16 years & over, Seasonally Adjusted.

Source: BLS.

$LNS_{index} = \frac{LNS10000000}{LNS10000000(1992:03)}$

Source: FRED.

Funds:Federal Funds Rate, Daily Figure Averages in Percentages. Source: FRED.

The observable variable x_t^{obs} for the output gap in our main estimations is based on the HP-filtered series of y_t^{obs}, while the CBO-based output gap is defined as: $x_t = 100\frac{GDPC09-GDPPOT}{GDPPOT}$

with **GDPPOT**[46]: CBO's Estimate of the Potential Output, Billions of Chained 2009 Dollars, Not Seasonally Adjusted Quarterly Rate. Source: FRED.

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