Spatial Statistics

PM569 Lecture 12: Point Pattern 3

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Simple Inhibition Process

- This process is used to describe regular patterns
- Often related to interactions or contagions where the occurrence of an event raises or lowers the probability of subsequent events nearby
- Useful for modeling the spread of infectious disease (contagion) or an application where an event precludes the occurrence of other events in a nearby area such as animal territories (inhibition)
- **Contagion** typically refers to the increased likelihood of events occurring near other events
- **Inhibition** may be absolute, where there is a specified distance around which *no* other events may occur, or it may be probabilistic where there is small but positive probability of an event occurring near other events

Simple Inhibition Process

- Models for inhibition or contagion processes are Markov point processes or Gibbs processes
- The general idea is to take a CSR and "delete" points within a distance less than a threshold δ
- Under a Markov process, the existence of an event in a region depends on the locations of events in a neighbourhood (where neighbourhoods are within regions)
- There are two ways to do this: 1) to simulate CSR then delete all within a distance δ, and 2) to simulate CSR, record when event was simulated, then delete an event if it is within distance δ of an older event

Simple Inhibition Process

- We use the packing intensity to describe simple inhibition processes:

$$\tau = \lambda\pi(\delta/2)^2$$

Where λ is the intensity, giving τ to be the proportion of the region A covered by non-overlapping discs of diameter δ

Simple Inhibition Process

- For simple inhibition process 1) we take a a Poisson process with intensity ρ and thin it by the deletion of pairs of events that are less than δ apart
- In this case, the probability that an event "survives" is $\exp(-\pi\rho\delta^2)$ giving the intensity of a simple inhibition process as:

$$\lambda = \rho \exp(-\pi\rho\delta^2)$$

- The second order properties can be expressed as:

$$\lambda(h) = \rho^2 \exp(-\rho U_\delta(h)) \qquad\qquad h \geq \delta$$

- $\lambda(h) = 0$ when $0 < h < \delta$, and $U_\delta(h)$ is the area of the union of two discs with equal radius δ and centers distance h apart

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Simple Inhibition Process

- For simple inhibition process 2) we take a a Poisson process with intensity ρ and thin it by the deletion of pairs of "older" events that are less than δ apart
- The expressions are the same as for process 1) but with the addition of the sequential piece (this process is referred to as the simple sequential inhibition process)
- Let X_i be a sequence of n events in A, and $d(x, y)$ be the distance between two points x and y. Then:
  - X_1 is simulated from a uniform distribution in A
  - Given (past) $\{X_j = x_j, j = 1, ...(i-1)\}$, then X_i (present) is uniformly distributed on the intersection of A with $\{y : d(y, x_j) \geq \delta j = 1, ...(i-1)\}$
- So the simple sequential inhibition process has packing intensity:

$$\tau = \frac{n\pi(\delta/2^2)}{|A|}$$

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- In R (spatstat), the functions for thinning processes 1) and 2) described above are called rMaternI and rMaternII
- The simple sequential inhibition process, called rSSI is similar but slightly different:
  - Each new point is generated uniformly in the window and independently of preceding points
  - If a new point lies within distance δ from an existing point then it is rejected and another random point is generated
  - The SSI process ends when no points can be added

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Markov point processes

- The general idea of a Markov point process lies in conditioning, whereby the existence of an event in a finite region A depends on the locations of events in a neighbourhood
- Inhibition processes are a special form of Markov process: the conditional intensity of an event at a point x given the realization of the process in the remainder of the region A depends on the existence (or otherwise) of an event within distance δ of x
- General Markov processes were introduced by Ripley and Kelly (1977)
- Markov point processes are characterized by the likelihood ratio with respect to a Poisson process of unit intensity

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Markov point processes

- Let's call the likelihood ratio $f(\cdot)$
- If $\mathbf{X} = \{x_1, ..., x_n\}$ denotes a finite set of points in A then $f(\mathbf{X})$ indicates how much more likely is the configuration of events $\mathbf{X}$ than a homogeneous point process (with unit intensity)
- We can factorize the likelihood ratio to:

$$f(\mathbf{X}) = \alpha \prod_{i=1}^{n} g_i(x_i) \prod_{j>i} g_{ij}(x_i, x_j)...g_{12...n}(x_1, x_2, ..., x_n)$$

- Where α is a normalizing constant
- We also define two points x and y in A to be neighbours if $d(x, y) < \delta$ for some $\delta > 0$ where $d(x, y)$ is the distance between x and y
- We also define a clique (recall areal data) as a set of mutual neighbours, and the neighbourhood of x to be the set of points $\{y \in A : 0 < d(x, y), \delta\}$

Markov point processes

- The point process with these definitions is Markov with range δ if the conditional intensity at the point x given the configuration of the other events in A depends only on the configuration in the neighbourhood of x
- The g-functions from the above equation are unity *unless* the x form a clique

Examples of Markov point processes: the Strauss process

$$f(\mathbf{X}) = \alpha\beta^n\gamma^p$$

- Where α is the normalizing constant, β is the intensity of the process, γ is the interaction between neighbours, and p is the number of distinct pairs of neighbours in $\mathbf{X}$
- If $\gamma = 1$ then the Strauss process gives a Poisson process with intensity β
- if $\gamma = 0$ then the Strauss process gives a simple inhibition process because no two events may be neighbours
- In R spatstat, the Strauss process is simulated with `rStrauss`

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Examples of Markov point processes: the pairwise interaction process

$$f(\mathbf{X}) = \alpha\beta^n \prod_{i \neq j} h\{d(x_i, x_j)\}$$

- Where α is the normalizing constant, β is the intensity of the process, $h(d)$ is non-negative for all distances and the product is over all pairs of distinct points in **X**
- The additional restriction is that $h(d)$ is bounded and that $h(d) = 0$ for all distances less than some $\delta > 0$
- This restriction limits the number of events in A by imposing a minimum allowable distance δ between any two events
- The pairwise interaction process may be fit in R `spatstat` using the `rmh` function

Examples of Markov point processes: the pairwise interaction process

- The pairwise interaction process may be simulated using the following steps (MCMC):
    1. For the initial realization, consider n points $\{x_1, ...x_n\}$
    2. Delete one of the points in $\{x_1, ...x_n\}$
    3. Generate a point y from a uniform distribution in A, and accept y with probability $p(y)$
    4. Repeat 2-3 until the MCMC converges