Introduction to the Theory of Statistics Part 2

PM522b

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Slides 6, 2019

Topics covered

- Interval Estimation
    - Methods for finding intervals: Inverting a test statistic, pivotal quantities, pivoting the CDF
    - Methods for evaluating intervals: size and coverage, length

Interval Estimates

- We have examined point estimators of unknown distribution parameters $\theta_1, \theta_2, ..., \theta_n$ using MLE, numerical methods for MLE and MOM
- We have assessed our estimates by examining sufficiency, their bias, MSE, and MVUE.
- Even if we have minimized the squared error or have an minimum variance unbiased estimate, we have no idea if our parameter lies in an acceptable range (and where the parameter lies in that range).
- Interval estimates are calculated with our sample measurements and are two numbers that define endpoints.
- Ideally the interval has two properties: that it contains the target parameter θ and that it is relatively narrow. We want the interval to have high probability of containing θ.

Interval Estimates

Definition: Interval Estimator

Given a random sample $X_1, X_2, ..., X_n$, an interval estimate of an unknown parameter θ from probability distribution function $f(x|\theta)$ is any pair of functions $L(X_1, X_2, ..., X_n)$ and $U(X_1, X_2, ..., X_n)$ that satisfy $L(X) \leq U(X)$. $L(x)$ and $U(x)$ are the lower and upper limits of the interval, respectively.

When $X = x_1, x_2, ..., x_n$ is observed, the inference $L(x) \leq \theta \leq U(x)$ is made. $[L(x), U(x)]$ is the interval estimator.

Coverage Probability

Definition: Coverage probability

For an interval estimator $[L(x), U(x)]$, the probability that the interval contains the true parameter θ is defined by $P(\theta \in [L(x), U(x)]|\theta)$

Concept

Given a random sample $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$, we estimate μ, the expected value, with $\bar{X}$

However, there will be some estimation error between $\bar{X}$ and μ, as the probability of our estimate being exactly correct $P(\bar{X} = \mu)$ is 0. So it is more appropriate to define a range of values around $\bar{X}$ that has high probability of containing μ

We say that the probability of μ is covered by the interval $\bar{X} \pm c$ via:

$$P(\bar{X} - c \leq \mu \leq \bar{X} + c)$$

Coverage Probability

Example (CB 9.1.3)

Given a random sample $X_1, X_2, X_3, X_4 \sim N(\mu, 1)$, we estimate μ, the expected value, with $\bar{X}$

We have confidence interval $[\bar{X} - 1, \bar{X} + 1]$, and the probability that μ is covered by this interval is:

$$\begin{aligned} P(\mu \in [\bar{X} - 1, \bar{X} + 1]) &= P(\bar{X} - 1 \leq \mu \leq \bar{X} + 1) \\ &= P(-1 \leq \bar{X} - \mu \leq 1) \\ &= P(-2 \leq \frac{\bar{X} - \mu}{\sqrt{1/4}} \leq 2) \\ &= P(-2 \leq Z \leq 2) \\ &= 0.9544 \end{aligned}$$

Here, Z is the standard normal. And given these endpoints, we have over a 95% chance of covering the unknown μ with our interval estimator.

Coverage Probability and Confidence Coefficients

For an interval estimator of a parameter θ, $[L(x), U(x)]$, the confidence coefficient of $[L(x), U(x)]$ is the infimum of the coverage probabilities, $inf_\theta P_\theta(\theta \in [L(x), U(x)])$

Example (CB 9.1.6)

Example done in class

Confidence Intervals

Methods for finding interval estimators

- Interval estimators, in combination with a measure of confidence (via a confidence coefficient) are referred to as confidence intervals.
- $P(\theta \in [L(x), U(x)] | \theta) = 1 - \alpha$, where $1 - \alpha$ is the confidence coefficient
- The two primary methods for finding confidence intervals are by inverting a test statistic and the pivotal method.

Confidence Intervals: Inverting a Test Statistic

There is a strong relationship between hypothesis testing and confidence intervals in that a confidence interval can be obtained by inverting a hypothesis test (and vice versa). Specifically, the $1-\alpha$ confidence interval is obtained by inverting the acceptance region of the α-level test.
In general, under $H_0 : \theta = \theta_0$,

$$A(\theta_0) = \{\mathbf{x} : \text{the test accepts } H_0 : \theta = \theta_0\}$$

Where $A(\theta_0)$ is the acceptance region. Defining the 1-α confidence set $C(X_1, .., X_n)$:

$$C(X_1, ..., X_n) = \{\theta_0 : \mathbf{x} \in A(\theta_0)\}$$

The confidence interval is the set of all parameters for which the hypothesis would have accepted H_0. Namely, it is the set of θ given $X_1, .., X_n$ and for each $\theta_0 \in C(X)$ you would not reject $H_0 : \theta = \theta_0$

Confidence Intervals: Inverting a Test Statistic

Conversely we can say:

$$A(\theta_0) = \{\mathbf{x} : \theta_0 \in C(X_1, ..., X_n)\}$$

In this case $A(\theta_0)$ is the acceptance region of the α-level test of $H_0 : \theta = \theta_0$
Proof:
For the confidence set, since $A(\theta_0)$ is the acceptance region of the α-level test

$$P_{\theta_0}(\mathbf{X} \notin A(\theta_0)) \leq \alpha \leftrightarrow P_{\theta_0}(\mathbf{X} \in A(\theta_0)) \geq 1 - \alpha$$

Using θ more generally than θ_0, the coverage probability of the set $C((X)$ is

$$P_{\theta_0}(\theta \in C(\mathbf{X})) = P_{\theta_0}(\mathbf{X} \in A(\theta_0)) \geq 1 - \alpha$$

Showing that $C(\mathbf{X})$ is a $1 - \alpha$ confidence set. For the hypothesis test, the probability of Type I error for testing the null hypothesis $H_0 : \theta = \theta_0$ with acceptance region $A(\theta_0)$ is

$$P_{\theta_0}(\mathbf{X} \notin A(\theta_0)) = P_{\theta_0}(\theta_0 \notin C(\mathbf{X})) \leq \alpha$$

So this is the α-level test.

Confidence Intervals: Inverting a Test Statistic

In hypothesis testing, the acceptance region is the set of $X_1, .., X_n$ that are very likely for θ_0. We fix the parameter and find what sample values in the region are consistent with that value.
In interval estimation, the confidence interval is a set of θ's that make $X_1, .., X_n$ very likely.

Example: Inverting a Normal Test

Given $X_1, ..., X_n$ from a normal distribution where σ is known and we wish to test $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ for a fixed α level, we used the test statistic

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$$

This has critical region $\{x : |\bar{x} - \mu_0| > z_{\alpha/2}\sigma/\sqrt{n}\}$, or in other words, H_0 is accepted in the region defined by $|\bar{x} - \mu_0| \leq z_{\alpha/2}\sigma/\sqrt{n}$

Confidence Intervals: Inverting a Test Statistic

Example: Inverting a Normal Test, con't

This is written as

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Which has size α, so we can write $P(H_0 \text{ is rejected}|\mu = \mu_0) = \alpha$ or equivalently stated another way, $P(H_0 \text{ is accepted}|\mu = \mu_0) = 1 - \alpha$. Combining this,

$$P(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}|\mu = \mu_0) = 1 - \alpha$$

But the probability statement is true for every μ_0, so the above is written

$$P_\mu(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha$$

And the confidence interval $[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$ is obtained by inverting the acceptance region of the α-level test giving a $1 - \alpha$ confidence interval.

Confidence Intervals: Inverting a Test Statistic

For one sided tests, we have the following:

- L(X) is a lower confidence bound when $U(X)=\infty$
- U(X) is an upper confidence bound when $L(X)=-\infty$

Confidence Intervals: Inverting a LR Statistic

We can consider using $\lambda(X) = L(\theta_0)/L(\hat{\theta})$ as our statistic for inverting an α-level test of $H_0 : \theta = \theta_0$ versus $H_0 : \theta \neq \theta_0$.
We first define the LRT statistic by $\lambda(x)$ and then for a fixed θ_0 we define the acceptance region:

$$A(\theta_0) = \{x : \lambda(x) \geq k\}$$

Where k is a constant chosen to satisfy $P(X \in A(\theta_0)) = 1 - \alpha$. Since we are looking at the acceptance region, we are not splitting the region (as in UMP two sided which is non-existent). See CB Figure 9.2.2.
The associated 1-α confidence set is:

$$C(X) = \{\theta : \lambda(x) \geq k\}$$

The confidence interval can be expressed in the form:

$$\{\theta : L(\lambda(x)) \leq \theta \leq U(\lambda(x))\}$$

Where L and U are functions determined by the constraints that the set of the acceptance region has probability 1-α.

Confidence Intervals: Pivotal Method

Definition: Pivotal quantities

A random variable $Z(X,\theta) = Z(X_1, ..., X_n, \theta)$ is a pivotal quantity if the distribution of $Z(X,\theta)$ is independent of all the parameters θ
The function $Z(X,\theta)$ will usually contain both parameters and statistics, but for any set $\mathcal{A}$, $P_\theta(Z(X,\theta) \in \mathcal{A})$ cannot depend on θ.

- It is desirable to have the length U-L or mean length E(U-L) to be as short as possible.
- The pivotal method assures the minimization of the MSE of the constructed confidence interval (in most cases).

Confidence Intervals: Pivotal Method

- The pivotal method involves the following steps:
  1. Determine the point estimator for the unknown parameter θ. i.e. $\hat{\theta} = \hat{\theta}(X_1, X_2, ..., X_n)$
  2. Construct a function of $\hat{\theta}$ and θ, $Q = g(\hat{\theta}, \theta)$ with known distribution function $f_Q(q)$ which is independent of θ and any other unknown parameter. This is the pivotal quantity.
  3. Using the distribution function $f_Q(q)$, find two constants a and b with $a < b$ such that $P(a \leq Q \leq b) = 1 - \alpha$. Usually these constants are chosen so $P(Q < a) = P(Q > b) = \alpha/2$.
  4. Put everything together into a double inequality which constructs the confidence interval, $a \leq g(\hat{\theta}, \theta) \leq b$. In terms of the unknown parameter, the equivalent inequality is $L(X) \leq \theta \leq U(X)$.
  $$P(L(X) \leq \theta \leq U(X)) = P(a \leq Q \leq b) = 1 - \alpha$$
- Note that the interval does not depend on the unknown parameter.
- When choosing a and b such that $P(a \leq Q \leq b) = 1 - \alpha$ we want the interval length $b - a$ to be as small as possible because the shorter the interval, the more precise it is.
- Typically when the distribution of Q is symmetric, the interval is also symmetric.

Confidence Intervals: Pivotal Method

In location, scale and location scale families there are many possible pivotal quantities. For $X_1, X_2, ..., X_n$ we let $\bar{X}$ and S be the sample mean and standard deviation, respectively. The common pivotal quantities are shown below.

PDF	Type	Pivotal Quantity
$f(x-\mu)$	Location	$\bar{X}-\mu$
$\frac{1}{\sigma}f(\frac{x}{\sigma})$	Scale	$\frac{\bar{X}}{\sigma}$
$\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$	Location-Scale	$\frac{\bar{X}-\mu}{S}$

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 1

Given a random sample $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$ where σ^2 is known and we wish to construct a confidence interval for μ (Location type example).

1. The estimator for the unknown parameter μ is $\bar{X}$ and it has distribution $N(\mu, \sigma^2/n)$
2. The function $Q = (\bar{X}, \mu)$ defined by
$$Q = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$
has $f_Q(q) \sim N(0,1)$ which is independent of μ
3. Using $Q \sim N(0,1)$ we find the constants $P(Q < a) = \alpha/2$ so $a = -q_{1-\alpha/2}$ since the distribution of Q is symmetric. Similarly, $P(Q < b) = 1 - \alpha/2$ so $b = q_{1-\alpha/2}$. Note, q_α represents the upper α percentage of the standard normal distribution. In this case, $\pm q_{1-\alpha/2}$ are obtained from the standard normal distribution so we will use $\pm z_{1-\alpha/2}$.

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 1 con't

4. Putting this together into a double inequality,

$$-z_{1-\alpha/2} \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq z_{1-\alpha/2}$$

$$-\frac{\sigma}{\sqrt{n}} z_{1-\alpha/2} \leq \bar{X}-\mu \leq \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}$$

$$\bar{X}-\frac{\sigma}{\sqrt{n}} z_{1-\alpha/2} \leq \mu \leq \bar{X}+\frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}$$

So, the random interval $[\bar{X}-\frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}, \bar{X}+\frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}]$ is an exact confidence interval for μ with confidence coefficient $1-\alpha$.
Note: The length of the interval is constant, $l = 2\frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}$

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 2

Given a random sample $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$ where μ is known and σ is unknown, and we wish to construct a confidence interval for σ^2 (Scale type example).

1. The estimator for the unknown parameter σ^2 is $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X - \bar{X})^2$.
2. The function $Q = (S^2, \sigma^2)$ defined by
$$Q = \frac{(n-1)S^2}{\sigma^2}$$
has $f_Q(q) \sim \chi^2_{n-1}$ which is independent of σ^2
3. Using $Q \sim \chi^2_{n-1}$ we find the constants $P(Q < a) = \alpha/2$ and $P(Q < b) = 1 - \alpha/2$.

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 2 con't

4. Putting this together into a double inequality,

$$\chi^2_{n-1,\alpha/2} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{n-1,1-\alpha/2}$$

$$\frac{\chi^2_{n-1,\alpha/2}}{(n-1)S^2} \leq \frac{1}{\sigma^2} \leq \frac{\chi^2_{n-1,1-\alpha/2}}{(n-1)S^2}$$

$$\frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}}$$

So, the random interval $[(n-1)S^2/\chi^2_{n-1,1-\alpha/2}, (n-1)S^2/\chi^2_{n-1,\alpha/2}]$ is an exact confidence interval for σ^2 with confidence coefficient $1-\alpha$.
Note, this interval is not symmetric since the χ^2 distribution is not symmetric.

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 3

Given a random sample $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$ where μ and σ^2 are unknown, and we wish to construct a confidence interval for μ and σ^2 (Location-Scale type example).

1. The estimator for the unknown parameter μ is $\bar{X}$ and it has distribution $N(\mu, \sigma^2/n)$
2. As in the previous example, the function $Q = (\bar{X}, \mu)$ defined by
$$Q = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$
has $f_Q(q) \sim N(0, 1)$ which is independent of μ, but note that it also contains the unknown parameter σ. The confidence interval cannot contain an unknown parameter, so we must replace it with the unbiased estimator for σ^2, $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X - \bar{X})^2$.
The pivotal function becomes, $Q = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ which has a t_{n-1} distribution.

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 3 con't

3. Using $Q \sim t_{n-1}$ we find the constants $P(Q < a) = \alpha/2$ or $P(Q < a) = -(1-\alpha/2)$, so $a = -t_{1-\alpha/2,n-1}$ since the t-distribution is symmetric. Similarly, $P(Q < b) = 1-\alpha/2$ so $b = t_{1-\alpha/2,n-1}$.
4. Putting this together into a double inequality,

$$
\begin{aligned}
-t_{1-\alpha/2,n-1} \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq t_{1-\alpha/2,n-1} \\
-\frac{S}{\sqrt{n}} t_{1-\alpha/2,n-1} \leq \bar{X}-\mu \leq \frac{S}{\sqrt{n}} t_{1-\alpha/2,n-1} \\
\bar{X}-\frac{S}{\sqrt{n}} t_{1-\alpha/2,n-1} \leq \mu \leq \bar{X}+\frac{S}{\sqrt{n}} t_{1-\alpha/2,n-1}
\end{aligned}
$$

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 3 con't

So, the random interval $[\bar{X} - \frac{S}{\sqrt{n}} t_{1-\alpha/2,n-1}, \bar{X} + \frac{S}{\sqrt{n}} t_{1-\alpha/2,n-1}]$ is an exact confidence interval for μ with confidence coefficient $1 - \alpha$.

When the sample size is large ($n \geq 35$) the CLT states that the t-distribution is approximated by the standard normal distribution. Thus in this case, $t_{1-\alpha/2,n-1} = z_{1-\alpha/2}$ and the confidence interval for μ becomes $[\bar{X} - \frac{S}{\sqrt{n}} z_{1-\alpha/2}, \bar{X} + \frac{S}{\sqrt{n}} z_{1-\alpha/2}]$. This is an asymptotic result (Ch. 10).

Confidence Intervals: Pivotal Method

Practical Example: CI for μ given

Given a random sample $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$ of $n = 14$ gym-goers showed that the mean workout time was $\bar{X} = 45$ minutes with a sample standard deviation of $s = 14$ minutes. What is the population mean μ with confidence coefficient $1 - \alpha = 0.95$?

- The confidence coefficient $1 - \alpha = 0.95$ means $\alpha = 0.05$ and thus $\alpha/2 = 0.025$
- Use $Q \sim t_{n-1}$ as the pivot
- Using the table of standard normals, $t_{0.025,13} = 2.16$ and therefore

$$\bar{X} - \frac{S}{\sqrt{n}} t_{1-\alpha/2,n-1} = 45 - \frac{14}{\sqrt{14}} 2.16 = 36.92$$
$$\bar{X} + \frac{S}{\sqrt{n}} t_{1-\alpha/2,n-1} = 45 + \frac{14}{\sqrt{14}} 2.16 = 53.08$$

- Therefore the CI for the average workout time (μ) with $1 - \alpha = 0.95$ (also stated as $100(1 - \alpha) = 95\%$ confidence) is $[36.92, 53.08]$ minutes.

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 3 con't

1. The estimator for the unknown parameter σ^2 is S^2
2. The function $Q = (S^2, \sigma^2)$ defined by
$$Q = \frac{(n-1)S^2}{\sigma^2}$$
has $f_Q(q) \sim \chi^2_{n-1}$
3. The constants a and b are obtained with the χ^2_{n-1} distribution
$$P(\frac{(n-1)S^2}{\sigma^2} > a) = 1 - \alpha/2$$
$$a = \chi^2_{1-\alpha/2,n-1}$$
$$P(\frac{(n-1)S^2}{\sigma^2} > b) = \alpha/2$$
$$b = \chi^2_{\alpha/2,n-1}$$
4. Putting this together, we get the confidence interval for σ^2
$$\frac{(n-1)S^2}{\sqrt{\chi^2_{\alpha/2,n-1}}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\sqrt{\chi^2_{1-\alpha/2,n-1}}}$$
5. The confidence intervals for μ and σ^2 are constructed separately in this case.

Confidence Intervals: Pivotal Method

Pivotal method for Normal: Case 3 con't

Suppose we wish to find a simultaneous confidence interval for μ and σ^2.
In this situation one good option is to use the Bonferroni inequality (CB 1.2.9).
Recall $P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$. The probability that the interval covers μ is $P(A_1) = 1 - \alpha/2$, and similarly for σ^2, $P(A_2) = 1 - \alpha/2$. So, the inequality gives $2(1 - \alpha/2) - 1 = 1 - \alpha$.
We thus can use the same pivots shown above for μ and σ^2, however we require a and b to be defined by $\pm t_{n-1,\alpha/4}$ for μ and $a = \chi^2_{n-1,1-\alpha/4}, b = \chi^2_{n-1,\alpha/4}$ for σ^2.
The simultaneous $1 - \alpha$ CI for (μ, σ^2) is

$$\bar{X} - \frac{S}{\sqrt{n}} t_{1-\alpha/4,n-1} \leq \mu \leq \bar{X} + \frac{S}{\sqrt{n}} t_{1-\alpha/4,n-1}, \frac{(n-1)S^2}{\sqrt{\chi^2_{\alpha/4,n-1}}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\sqrt{\chi^2_{1-\alpha/4,n-1}}}$$

Confidence Intervals: Pivotal Method

The Gamma Pivot

In class

Confidence Intervals: Pivoting the CDF

The pivots, Q, defined previously were constructed with location and scale methods. We saw that they were straightforward to find for the normal distribution, and in CB (9.2.8) we see the use of the Gamma pivot. Another, more general pivot is using the CDF. We present this method for the continuous CDF:

Pivoting the continuous CDF

Assume T is a statistic with continuous CDF $F_T(t|\theta) = P(T \leq t)$. Given $\alpha_1 + \alpha_2 = \alpha, 0 < \alpha < 1$ (often $\alpha_1 = \alpha_2$), for each $t \in T$:

1) When $F_T(t|\theta)$ is a decreasing function of θ for each t, $\theta_L(t)$ and $\theta_U(t)$ can be defined by $F_T(t|\theta_U(t)) = \alpha_1$ and $F_T(t|\theta_L(t)) = 1 - \alpha_2$.

2) When $F_T(t|\theta)$ is an increasing function of θ for each t, $\theta_L(t)$ and $\theta_U(t)$ can be defined by $F_T(t|\theta_U(t)) = 1 - \alpha_2$ and $F_T(t|\theta_L(t)) = \alpha_1$.

Evaluating Interval Estimators

In evaluating the confidence intervals we have generated through the means described above, we look at two related quantities: length (size) and coverage probability. Interval length and coverage probability vie against each other. We discuss coverage probability previously, so here focus on defining the length of an interval.

Length: We choose the $1-\alpha$ interval such that is as short as possible through minimizing the expectation of $U(X) - L(X)$. It is important to note:

- when α is fixed, if n increases the confidence interval length decreases. So smaller n will result in larger intervals.
- when n is fixed, if $1-\alpha$ increases, a and b increase, and so will the confidence interval. Smaller $1-\alpha$ results in smaller a and b and thus a smaller interval.

Evaluating Interval Estimators

Theorem: Interval Length

Let f(x) be a unimodal pdf. If the interval [a,b] satisfies

1. $\int_a^b f(x)dx = 1 - \alpha$
2. $f(a) = f(b) > 0$
3. $a \leq x* \leq b$ where x* is the mode of f(x)

Then [a,b] is the shortest among all intervals that satisfy 1.

Evaluating Interval Estimators: Optimizing Length

Suppose we have $X_1, ..., X_n \sim N(\mu, \sigma^2)$ where both parameters are unknown. Find the shortest length $1-\alpha$ confidence interval for μ using a pivotal quantity.
Solution:
We saw before that our pivot for μ in this case is $Q = \frac{\bar{X}-\mu}{S/\sqrt{n}}$ which has a t_{n-1} distribution. We let

$$P(a \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq b) = 1-\alpha$$

$$P(\bar{X} - bS/\sqrt{n} \leq \mu \leq \bar{X} - aS\sqrt{n}) = 1-\alpha$$

So the length of the interval is $L = (b-a)/S/\sqrt{n}$. But what choice of a and b is best?

Evaluating Interval Estimators: Optimizing Length

We wish to find a and b such that L is minimized subject to

$$\int_a^b f(x)dx = 1 - \alpha$$

We can see that

$$\frac{dL}{da} = (\frac{db}{da} - 1)\frac{S}{\sqrt{n}}$$

$$f(b)\frac{db}{da} - f(a) = 0$$

$$\text{so } \frac{dL}{da} = (\frac{f(a)}{f(b)} - 1)\frac{S}{\sqrt{n}}$$

The minimum occurs at a=-b. We use $a = -b = -t_{n-1,\alpha/2}$ to get the shortest length $1 - \alpha$ confidence interval for the pivotal quantity $\sqrt{n}(\bar{X} - \mu)/S$