Spatial Statistics

PM569 Lecture 10: Point Pattern 1

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Point Pattern Data: A Historical Perspective

In 1946, R.D. Clarke wrote a report about heavily bombed region of South London.

AN APPLICATION OF THE POISSON DISTRIBUTION

By R. D. CLARKE, F.I.A.
of the Prudential Assurance Company, Ltd.

READERS of Lidstone's *Notes on the Poisson frequency distribution* (*J.I.A.* Vol. LXXI, p. 284) may be interested in an application of this distribution which I recently had occasion to make in the course of a practical investigation.

During the flying-bomb attack on London, frequent assertions were made that the points of impact of the bombs tended to be grouped in clusters. It was accordingly decided to apply a statistical test to discover whether any support could be found for this allegation.

An area was selected comprising 144 square kilometres of south London over which the basic probability function of the distribution was very nearly constant, i.e. the theoretical mean density was not subject to material variation anywhere within the area examined. The selected area was divided into 576 squares of $\frac{1}{4}$ square kilometre each, and a count was made of the numbers of squares containing 0, 1, 2, 3, ..., etc. flying bombs. Over the period considered the total number of bombs within the area involved was 537. The expected numbers of squares corresponding to the actual numbers yielded by the count were then calculated from the Poisson formula:

$$Ne^{-m}(1+m+m^2/2!+m^3/3!+\ldots),$$

where $\quad N=576 \quad$ and $\quad m=537/576.$

The result provided a very neat example of conformity to the Poisson law and might afford material to future writers of statistical text-books.

The actual results were as follows:

No. of flying bombs per square	Expected no. of squares (Poisson)	Actual no. of squares
0	226·74	229
1	211·39	211
2	98·54	93
3	30·62	35
4	7·14	7
5 and over	1·57	1
	576·00	576

The occurrence of clustering would have been reflected in the above table by an excess number of squares containing either a high number of flying bombs or none at all, with a deficiency in the intermediate classes. The closeness of fit which in fact appears lends no support to the clustering hypothesis.

Applying the χ^2 test to the comparison of actual with expected figures, we obtain $\chi^2=1\cdot17$. There are 4 degrees of freedom, and the probability of obtaining this or a higher value of χ^2 is ·88.

Point Pattern Data: A Historical Perspective

- During WWII, Germany launched 1,358 V-2 Rockets at London.
- The V-2 had speed and a trajectory that made it invulnerable to interception, but its guidance systems were primitive, so it was thought that it couldn't hit specific targets.
- After strikes began in 1944, bomb damage maps were interpreted as showing that impact sites were clustered.
- If the V-2 strikes were clustered, then the guidance systems were more sophisticated than thought.
- R.D. Clarke set out to analyze these data to determine if the data were clustered or not.

Point Pattern Data: A Historical Perspective

- Clarke took a 12 km x 12 km region and sliced it up in to a grid of 576 squares, (144 km^2, so each grid square is $1/4$ km^2).
- For each square, Clark recorded the total number of observed bomb hits. There were 537 total in the study area.
- He then recorded the number of squares with $k = 1, 2, 3, ...$ hits.
- The expected number of squares with k hits was derived from the Poisson distribution $\sum_{k=1}^{n} \frac{e^{-\lambda}\lambda^k}{k!}$ where $\lambda = \frac{537}{576}$ and $n = 576$.

No. of flying bombs per square	Expected no. of squares (Poisson)	Actual no. of squares
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Point Pattern Data: A Historical Perspective

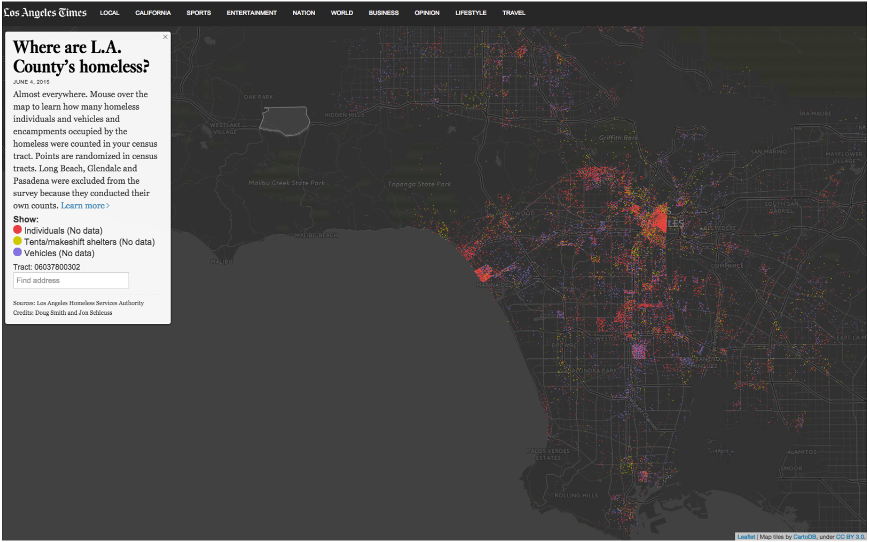
- Doing the cross tabulation of observed vs expected, he found $\chi^2 = 1.17$ which with 4 degrees of freedom (n-1 groups, the 0 group was excluded) has p-value=0.88.
- The occurrence of clustering would have been reflected in an excess of squares with a high number of bombs or none at all.
- The insignificant p-value and the closeness of fit of the data to the Poisson distribution indicates that the V-2 impact sites were random rather than clustered.

Point Pattern Data

- Goal in point pattern data analysis is to assess whether there is a spatial pattern in occurrences of an event
- Distinguish between a point and an event location
- In geostatistics our points were locations in a domain that we made a measurement. These points make up a set of spatial random variables for which we wanted to determine the spatial relationships (via the covariance function)
- Point patterns consist of event locations where we are concerned with the presence/absence of an event rather than the value of the measurement at a point.
- We ask the question: are the events that we observe in our domain from a completely random spatial process or are they exhibiting some type of pattern or clustering?
- For point pattern analysis in R we use the package `spatstat`.

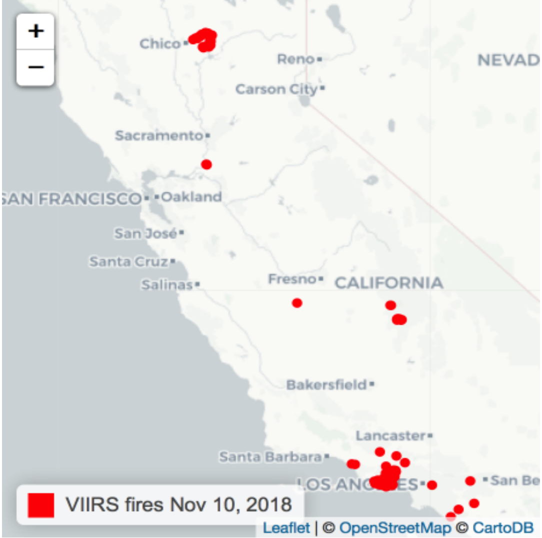
Point Pattern Data: Example

LA Homeless count: point locations of homeless



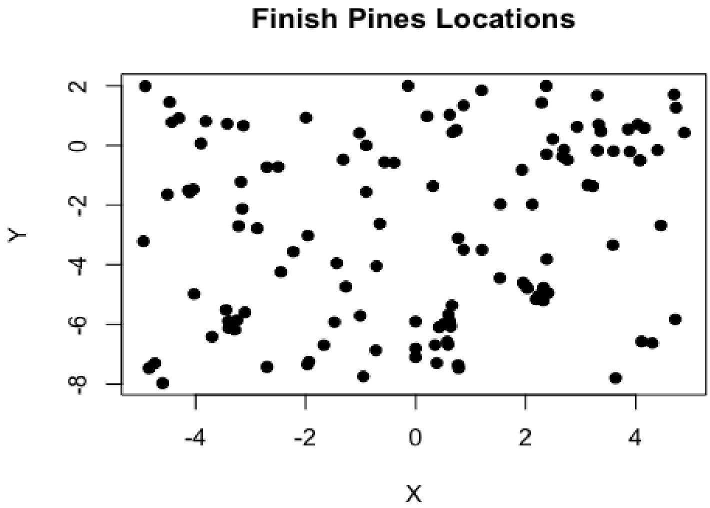
Point Pattern Data: Example

Locations of satellite detected fires in California



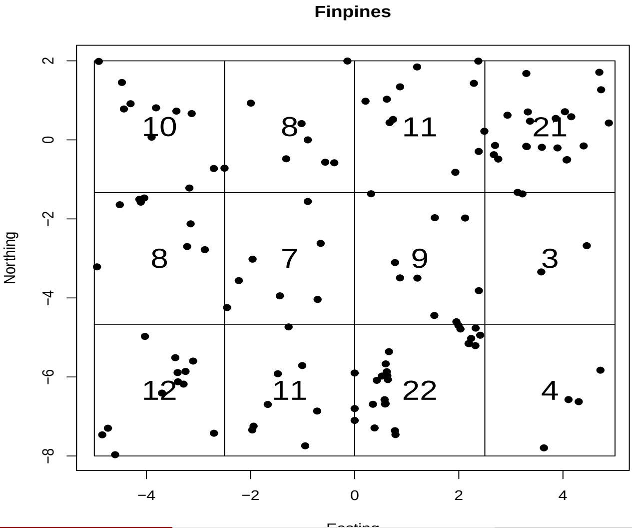
Point Pattern Data: Example

Locations of Finish pine trees in Finland forest



Point Pattern Data: Quadrant Count

A simple quadrant count of the Finnpines points.

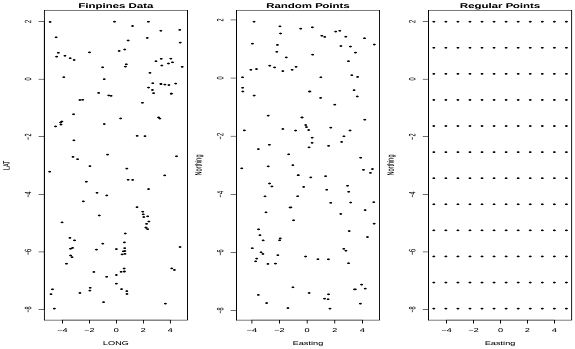


Point Pattern Data

Questions of interest are:

- Are points closer together than they would be by chance?
- Are the points more regularly spaced than they would be by chance?
- What model might reproduce our observed pattern?

Comparing the Finpines data to a spatially random process and a regular pattern:



Point Pattern Data

Objectives of statistical analysis of point pattern data

- Spatial location in (x,y) denoted as s.
- $Y(s)$ represents the presence or absence of Y where $Y(s) = 1$ if there is an observed case at location s, and$Y(s) = 0$ otherwise.
- Spatial domain of observed cases: D, $D = \{s; Y(s) = 1\}$.
- The null hypothesis: no spatial pattern (complete spatial randomness).
- Find a statistic to test whether the data is clustered, or regular.
- Develop a model to generate spatial pattern (Homogeneous Poisson Process (HPP), Inhomogeneous Poisson Process (IPP), Cluster Process,Cox Process, Simple Inhibition Process).

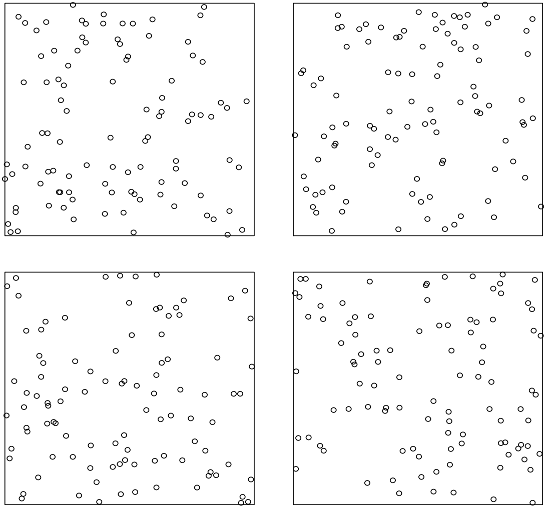
Point Pattern Data: Complete Spatial Randomness

Spatial Randomness

- Typical terms that are used are spatial randomness, random pattern, at random or by chance.
- Complete spatial randomness (CSR): events are uniformly distributed across a domain D and are independent of each other.
- CSR means an event is equally likely to occur at any location or region within D.
- Testing for CSR is the most basic test which can be performed on point pattern data.

Point Pattern Data: Complete Spatial Randomness

Here are 4 realizations of a random uniform process (CSR) in a 1x1 box



Point Pattern Data: Complete Spatial Randomness

- A point process which is CSR is defined as being a stationary **homogeneous spatial Poisson point process (HPP)**.
- The HPP is the building block of spatial point processes statistical analysis: It represents the simplest possible stochastic mechanism for the generation of spatial point patterns, and in applications is used as an idealized standard of CSR that, if strictly unattainable in practice, sometimes provides a useful approximate description of an observed pattern.

Homogeneous Poisson Process

Defining the homogeneous Poisson process (HPP)

- Homogeneity means that the intensity is constant across the study area. Homogeneity is similar to stationarity and isotropy for geostatistical data.
- In a HPP, the intensity, λ, is a constant equal to the expected number of events per unit area: $\lambda = \frac{n}{|D|}$.
- Let N be the number of events occurring in a region D. N is a random variable
- The pdf for N is the Poisson distribution:

$$P(N = k) = \frac{e^{-\lambda|D|}(\lambda|D|)^k}{k!}, k = 0, 1, 2, ...$$

- $E[N] = \lambda|D|$
- $\sigma^2 = \lambda|D|$

Homogeneous Poisson Process: Postulates

The HPP follows from the following mathematical postulates:

- **Independence**: The number of events in non-overlapping regions are statistically independent.
- **Stationarity**:

$$\lim_{|A| \to 0} \frac{\text{P[exactly one event in A]}}{|A|} = \lambda > 0$$

This implies that the probability of a single event depends only on the size of the area considered (is a constant independent of the location of the region A within a larger region D), and that as the area goes to 0 so does the probability.

- **Small area probabilities**:

$$\lim_{|A| \to 0} \frac{\text{P[two or more events in A]}}{|A|} = 0$$

This implies that the probability of a two or more events in the same location is 0.

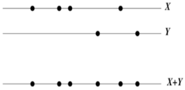
Homogeneous Poisson Process: Properties

Some properties of Homogeneous Poisson Processes:

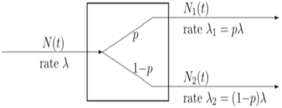
- The Poisson distribution allows the total number of observed events to vary from realization to realization while maintaining a fixed but unknown number of events per unit area.
- Under a HPP, the location of one point in space does not affect the probabilities of other points appearing nearby. The intensity of the point process in area A is a constant $\lambda > 0$.

Property: Independence of superimposed or pruned processes

- **Superposition of independent Poissons**: suppose $N_1(A)$ and $N_2(A)$ are independent Poisson random variables with rates λ_{N1} and λ_{N2}, respectively. Then $N_1(A) + N_2(A)$ is Poisson with rate $\lambda_{N1} + \lambda_{N2}$.



- **Pruning of Poissons**: Let $N(A)$ be a Poisson process with rate λ and assume that each arrival of $N(A)$ is assigned to a process $N_1(A)$ with probability p and to a process $N_2(A)$ with a probability $1 - p$, and all assignments are independent. Then $N_1(A)$ and $N_2(A)$ are independent Poisson processes with rates λp and $\lambda(1 - p)$.



Homogeneous Poisson Process: Properties

Property: Independence of superimposed or pruned processes

These facts are important for a few reasons:

- They allow us to define cluster processes
- They allow us to define **marked Poisson processes**: Poisson processes with different sorts of points. There are two equivalent ways to construct marked Poisson processes:
  1. Pruning: Drop all the points down according to the overall point process, then independently assign each point a marking.
  2. Alternatively, for each type of mark we can drop down an independent Poisson point process.
  3. These two models are equivalent by the superposition/pruning above.

Homogeneous Poisson Process: Properties

Conditional property: Given a fixed number of points, these are iid uniformly distributed

- In 2-dimensional space or one-dimensional, what happens to λ once the number of Poisson entities has been fixed?
- They vanish, and the unordered Poisson entities are iid uniformly over the area of interest.
- In other words, given that there are N points of the Poisson process in area D, these N points are conditionally independent and uniformly distributed in D.
- It's a little counter-intuitive that there exists no process parameter in this instance, but we can show why:

Homogeneous Poisson Process: Conditionality property

Proof: Uniform distribution given a fixed number of points
Let's consider a simple case: A single point in a line

- Given that a single arrival occurred in a line segment $[0, x]$, we can show that the pdf of the location is uniform over $[0, x]$.
- Consider an interval $[a, b]$ within segment $[0, x]$
- Let $l = b - a$
- Let X be the location of the point, and
- let E be the event that a single point occurs in $[0, x]$ (this is the given information).
- The probability for the location of X, given E, is then

$$P(X \in [a, b] | E) = \frac{P(X \in [a, b], E)}{P(E)}$$

Homogeneous Poisson Process: Conditionality property

Proof: Uniform distribution given a fixed number of points

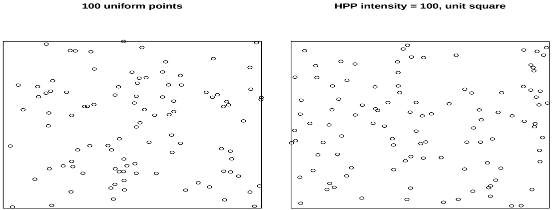
- $P(X \in [a, b], E)$ is the same as the probability that the Poisson process has exactly one point within the segment $[a, b]$ (of length l) $*$ the probability that the process has zero arrivals in $[0, a]$ and zero arrivals in $[b, x]$. These two intervals combine to a total length of xl (this is taking a **superimposed** process).
- So we have (plugging into the Poisson pdf):

$$
\begin{aligned}
P(X \in [a, b] | E) =& \frac{P(N = 1, A = l)P(N = 0, A = x - l)}{P(N = 1, A = l)} \\
=& \frac{(\lambda l e^{-\lambda l})(e^{-\lambda(x-l)})}{(\lambda x)(e^{-\lambda x})} \\
=& \frac{l}{x}
\end{aligned}
$$

Homogeneous Poisson Process: Conditionality property

Proof: Uniform distribution given a fixed number of points

- Recall that the uniform pdf is $f_u(X) = \frac{1}{x}$ for $X \in [0, x]$
- We can therefore see that the result $\frac{l}{x}$ is the same as the probability that a point ends up in interval l according to a uniform distribution
- This argument can be extended to a general case with any number of points, and 2 dimensional space.
- We will see this in the lab:



Homogeneous Poisson Process: Area proportionality

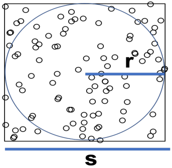
Property: Expected number of points is proportional to the area

- For any sub-region D of two-dimensional space, the expected number of points in D is proportional to the area of D: $E[N(X \wedge D)] = \lambda|D|$
- To drive in intuition on this property, we use this relationship to simulate the constant π

Homogeneous Poisson Process: Area proportionality

Application of uniformity intensity: Simulating π

- Generate a HPP in a square
- Inscribe a circle within that square

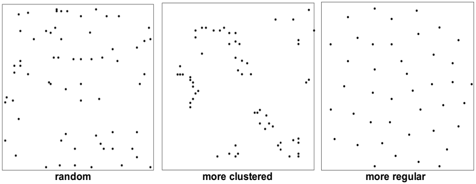


- Let $n =$ total number of points
- Let $m =$ number of points that land within circle
- Then we have, by the area proportionality property: $\frac{m}{n} = \frac{\pi r^2}{s^2}$
- So, our estimator for π is: $\hat{\pi} = \frac{ms^2}{nr^2}$
- Evaluate this in lab.

Point Pattern Data: Tests of CSR

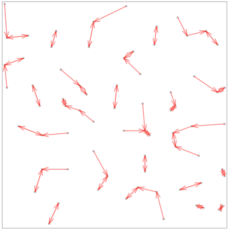
Testing for CSR Most analyses begin with a test of CSR, and there are several good reasons for this.

- A pattern for which CSR is not rejected scarcely merits any further formal statistical analysis.
- Rather than because rejection of CSR is of intrinsic interest, tests are used as a means of exploring a set of data, for example aiding in the formulation of hypotheses concerning pattern and its genesis.
- CSR acts as a dividing hypothesis to distinguish between patterns which are broadly classifiable as **clustered** or **regular**.

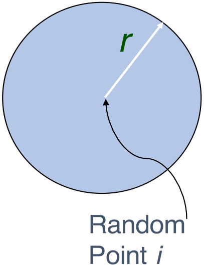


Exploring CSR: Nearest Neighbors

- For the i^{th} Poisson point, let D_i be the distance to its nearest neighbor.
- Construct a circle of radius r around i
- Let $G(r) = P(D_i \leq r)$, the CDF for the probability that the nearest neighbor distance is less than r.



Nearest Neighbor dists



Exploring CSR: Nearest Neighbors

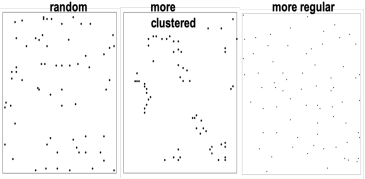
- We can derive this CDF $G(r) = P(D_i \leq r)$:

$$
\begin{aligned}
G(r) =& P(D_i \leq r) = 1 - P(D_i > r) \\
=& 1 - P(\text{no Poisson entities within circle of radius } r) \\
=& 1 - e^{-\lambda \pi r^2}
\end{aligned}
$$

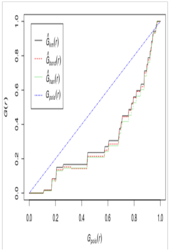
Exploring CSR: Nearest Neighbors

Evaluating CSR based on nearest neighbors:

- Let $\hat{G}(r)$ be the proportion of observed points with nearest neighbors less than r .
- The value of $\hat{G}(r)$ for any distance r tells what fraction of all nearest neighbor distances in the pattern are less than that distance.
- Compare $\hat{G}(r)$ with the exact $G(r) = 1 - e^{-\lambda \pi r^2}$
- **Interpretation**: If $\hat{G}(r)$ is much greater than $G(r)$, that means there is clustering, whereas if it is smaller that means there is regularity. Why?



NN Distance Distribution for "more regular" pattern



Point Pattern Data: Tests of CSR

Testing for homogeneous CSR using Monte Carlo

- Many tests of CSR use Monte Carlo methods.
- Compare the observed value of a test statistic to its distribution under the null hypothesis of CSR.
- Simulate a large number of CSR processes and compare the test statistic from N_{sim} to test statistic from observed.
- Also note, edge effects near the boundary of a region may need to be taken into account. (For example, if a point is less than distance r to the boundary, then this affects the probability that there is another point within radius r of it).

Point Pattern Data: Tests of CSR

Testing for homogeneous CSR: Ripley's K

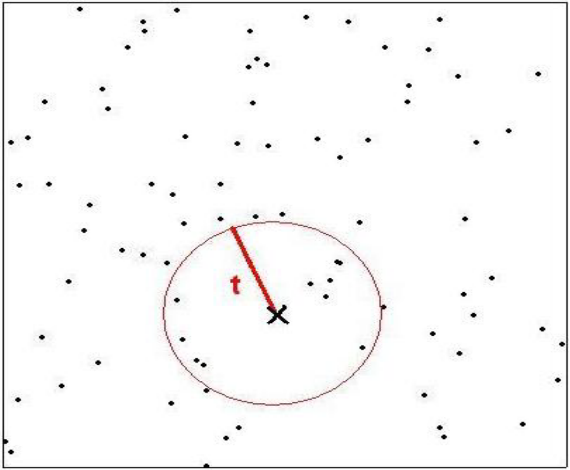
- Ripley's K, $K(r) = \lambda^{-1}E(N_0(r))$
- Where $N_0(r)$ is the number of events within a distance h of an arbitrary event
- $K(r)$ tests the expected number of events within distance h from an arbitrary event (excluding the chosen event itself) divided by the average number of events per unit area
- $K(r)$ is equivalent to showing the variance of the number of events occurring in subregion A (Ripley 1977) so is a second order property of the point process.

Testing for homogeneous CSR: Ripley's K

- Ripley's K, $K(r) = \lambda^{-1}E(N_0(r))$
- Under CSR $K(r) = \pi r^2$, the area of a circle of radius r

Point Pattern Data: Tests of CSR

Testing for homogeneous CSR: Ripley's K



Point Pattern Data: Tests of CSR

Testing for homogeneous CSR: Ripley's K

- For a process that is more regular than CSR we expect fewer events within distance r of a randomly chosen event
- For a process that is more clustered than CSR we expect more events within distance r of a randomly chosen event
- Estimating $K(h)$:

$$\hat{K}(h) = \hat{\lambda}^{-1} \frac{1}{N} \sum_i \sum_j \delta(d(i,j) < h)$$

- $i \neq j$ and $d(i,j)$ is the Euclidean distance between events and $\delta(d(i,j) < h) = 1$ if $d(i,j) < h$ and 0 otherwise

Point Pattern Data: Tests of CSR

Testing for homogeneous CSR: Ripley's K

- There is an alternate $\hat{K}(h)$ estimator that corrects for edges (boundaries of the region)
- Want to prevent including events that occur outside the boundary but within distance h
- Estimating $K(h)$ accounting for boundaries:

$$\hat{K}_{ec}(h) = \hat{\lambda}^{-1}\frac{1}{N}\sum_i\sum_j w_{ij}\delta(d(i,j) < h)$$

- where $w_{ij} = 1$ if the distance between i and j is less than the distance between event i and the boundary of the region

Point Pattern Data: Tests of CSR

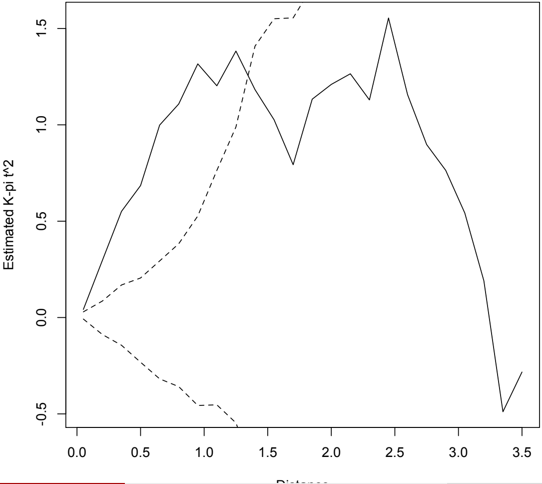
- Using the $K(h)$ function and determining p-values to test CSR.
- Plot $K(h)$; under CSR $K(h) = \pi h^2$ is a parabola.
- $(K(h)/\pi)^{1/2} = h$, so plot h vs $(\hat{K}(h)_{ec}/\pi)^{1/2} - h$.
- Under CSR, $(\hat{K}(h)_{ec}/\pi)^{1/2} - h = 0$.
- Departures from the horizontal line that defines CSR indicate clustering or regularity
- Deviations above the horizontal line indicate clustering because there are more events within distance h than expected.
- Deviations below the horizontal line indicate regularity because there are fewer events within distance h than expected.

Point Pattern Data: Tests of CSR

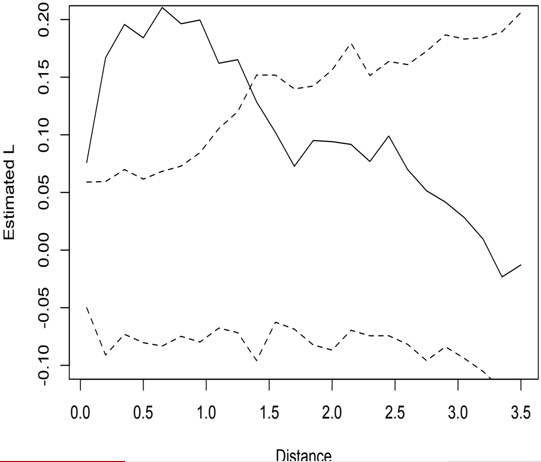
To test CSR based on Riply's K, use Monte Carlo method:

- Simulate k-1 samples (for example, k=100) of n points from a CSR process and compute $\hat{K}_2(h), ..., \hat{K}_n(h)$.
- For each distance h find the upper bound U(h)$=\max_i \hat{K}_i(h)$.
- For each distance h find the lower bound L(h)$=\min_i \hat{K}_i(h)$.
- For each distance find $\hat{K}_1(h)$ from data.
- Plot distance vs $\hat{K}_i(h) - \pi h^2$.
- Often plot distance (h) vs $\hat{L}(h) - h$ where $\hat{L}(h) = (\hat{K}(h)_{ec}/\pi)^{1/2}$ for better visualization.
- Under CSR, the expected value of $\hat{L}(h) - h = 0$, so we expect a CSR line to be around zero.

Testing for homogeneous CSR: Ripley's K



Testing for homogeneous CSR: Ripley's K (L version)



Point Pattern Data: Tests of CSR: Ripley's K

Test statistic

- Order the simulated values $\hat{K}_2(h), ..., \hat{K}_n(h)$, ie for i=1,...,n $\hat{K}_{(1)}(h) \geq ... \geq \hat{K}_{(n)}(h)$
- Let $\hat{K}_{(k)}(h)$ be the kth largest among $\hat{K}_i(h)$
- For each h, the probability that $\hat{K}_1(h)$ is lower than $L(h)$ or higher than $U(h)$ is n^{-1}, i.e.

$$P(\hat{K}_1(h) = \hat{K}_{(k)}(h)) = n^{-1}$$

- We reject the null hypothesis on the basis that $\hat{K}_1(h)$ ranks kth largest or higher. This gives us an exact one-sided test of size k/n

Point Pattern Data: Tests of CSR

- Can also test for CSR based on inter-event distances
- We let $H(h)$ be the theoretical distribution of inter-event distances, $H(h) = P(H \leq h)$
- We estimate $H(h)$ by the empirical distribution function:

$$\hat{H}(h) = \frac{\text{number of paired distances less than h}}{\text{total number of pairs}} = \frac{\#(h_{i,j} \leq h)}{0.5n(n-1)}$$

Point Pattern Data: Tests of CSR

- We know the theoretical $H(h)$ for inter-event distances $h_{i,j}$ if we have an area D which is square or circular.
- Example: Bartlett (1964) described $H(h)$ for a unit circle as

$$H(h) = 1 + \pi^{-1}[2(h^2 - 1)\cos^{-1}(h/2) - h(1 + h^2/2)\sqrt{(1 - h^2/4)}]$$

- for $0 \leq h \leq 2$

Point Pattern Data: Tests of CSR

- A visual test is to plot $\hat{H}(h)$ vs $H(h)$
- As with Ripley's K we need the distribution of our statistic $\hat{H}(h)$ under CSR
- Simulate k-1 samples of n points from CSR process and compute $\hat{H}_1(h), ..., \hat{H}_s(h)$
- For each inter-event distance h find the upper and lower bounds of $\hat{H}(h)$
- $\hat{H}_U(h) = max_i \hat{H}_i(h)$ and $\hat{H}_L(h) = min_i \hat{H}_i(h)$

Point Pattern Data: Tests of CSR

- Using nearest neighbour distances, let $G(h)$ be the theoretical distribution of NN distances
- Let h_i be the distance from the ith event to the nearest other event in D
- Our estimate of $G(h)$ is

$$\hat{G}(h) = \frac{\#(h_i \leq h)}{n}$$

- Now we need to test our statistic vs the statistic under CSR

Point Pattern Data: Tests of CSR

- Approximation of $G(h)$:
- For any event in our area D, under CSR we have $P(\text{event i is within distance h from j}) = \pi h^2 |D|^{-1}$ where area is represented by $|D|$
- The approximate distribution function of $G(h)$ is $G(h) \approx 1 - (1 - \pi h^2 |D|^{-1})^{n-1}$
- And when you have a large n, $G(h) \approx 1 - exp(-\lambda \pi h^2)$ where $\lambda = n/|D|$ is the intensity as we saw before
- Compare $\hat{G}(h)$ to $G(h)$, find envelope by simulation $\hat{G_U}(h)$ and $\hat{G_L}(h)$
- Plot $G(h)$ vs $\hat{G}(h)$

Point Pattern Data: Inhomogeneous Poisson Process

- A generalization of the HPP is the Inhomogeneous (heterogeneous) Poisson Process (IPP). The IPP occurs when the intensity λ is not constant over the region.
- Many cases homogeneity in intensity is not realistic, for example the locations of trees in a forest may be irregular due to geographic features such as soil, rock, slope or other terrain irregularities.
- In the case of IPP, the intensity is a function that varies spatially, $\lambda(s)$.
- The IPP does not define cluster process, but rather a

Point Pattern Data: Inhomogeneous Poisson Process

Inhomogeneous Poisson process

- We can estimate the intensity function in different ways: parametrically by defining a specific function or non-parametrically using kernel smoothing

$$\hat{\lambda}(s) = \frac{1}{h^2} \sum_i \kappa(\frac{||s - s_i||}{h})/q(||s||)$$

Where $\kappa(s)$ is a kernel function and $q||s||$ is a boundary correction. The distance h is our bandwidth for smoothing

Inhomogeneous Poisson process

- There are various kernel functions, but a quadratic function is often used:

$$\kappa(s) = \frac{3}{\pi}(1 - ||s||^2)^2$$

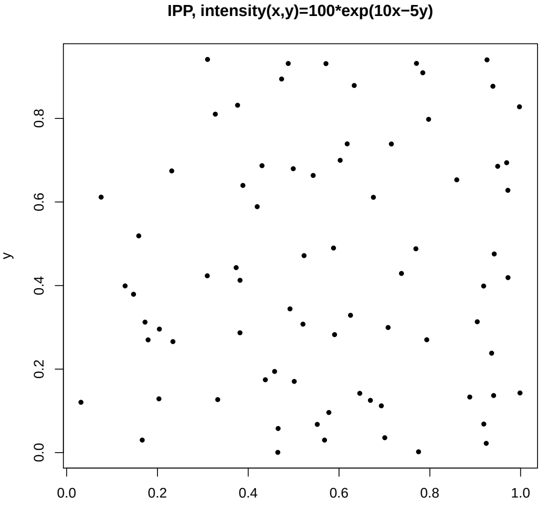
Point Pattern Data

Inhomogeneous Poisson Process

- Example of varying intensity function $\lambda(s)$ could be that intensity varies with location due to environmental heterogeneity
- Example if D i a square unit and N(D)=100
- $\lambda(x, y) = 100 * \exp(10x - 5y)$
- $\lambda(x, y) = 100 * \exp(-10x + 5y)$

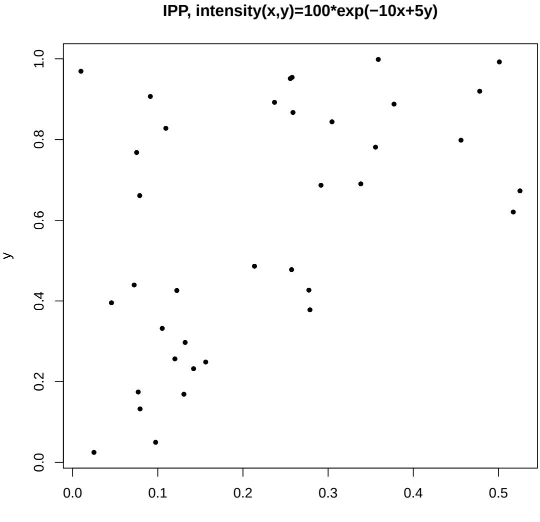
Point Pattern Data

Inhomogeneous Poisson Process



Point Pattern Data

Inhomogeneous Poisson Process



Point Pattern Data

Inhomogeneous Poisson Process

- Or we might see that cases of respiratory disease differ with respect to distance from a point source of environmental pollutions_0

$$\lambda(s) = \lambda_0(s) f(||s - s_0||, \theta)$$

- Where $\lambda_0(s)$ models the variation in population density
- f(u,θ) models how the impact of the source varies with distance u (s-s_0) and angle θ

Fitting point process models

- Given our set of observed point events $\{x_1, ...x_n\}$ in region A we wish to fit a model (which is stationary and isotropic)
- Model fitting is approached by estimating the parameters of the particular process
- Example: fitting a parametric form of the intensity of an inhomogeneous Poisson process requires estimating a trend
  - $\lambda(x, y) = \exp(\theta_0 + \theta_1 x + \theta_2 y)$
- We use familiar fitting methods: Least squares, Maximum Likelihood and non-parametric methods.

Fitting point process models

- In `spatstat` we require the intensity function $\lambda(x, y)$ to be loglinear in the θ parameters
  - $\log(\lambda(x, y)) = \theta S(x, y)$ where $S(x, y)$ is a function of the location referenced by x,y coordinates
- In practice $S(x, y)$ can be a function of the spatial coordinates, an observed covariate, or both.