Sequences and limits

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We start by defining what we mean when we say a sequence *converges* (if you have no idea what a "sequence" is, you might want to check the wikipedia entry `https://en.wikipedia.org/wiki/Sequence`; note that in this handout we are interested in "infinite sequences" only): A sequence $(a_n)_{n=1}^{\infty}$, where each $a_n \in \mathbb{R}$, converges to $a \in \mathbb{R}$ if for any $\varepsilon > 0$ we can find a N such that

$$|a_n - a| \leq \varepsilon \quad \text{for all} \quad n \geq N. \tag{1}$$

We then say that "a is the *limit* of the sequence $(a_n)_{n=1}^{\infty}$".

Example 1. *The sequence* $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}, \ldots$ *converges to 1: For a given* $\varepsilon > 0$, $\left|\frac{n+1}{n} - 1\right|$ *is less than* ε *for* n *high enough. More precisely,* $\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$ *for* $n > \frac{1}{\varepsilon}$*. Hence, by choosing* $N = \left[\frac{1}{\varepsilon}\right] + 1$ *(where* $[\,]$ *indicates rounding to the next integer) we know that* $|a_n - 1| \leq \varepsilon$ *for all* $n \geq N$ *and therefore the limit of the sequence is 1.*

However, the members of a sequence do not have to be real numbers. We can also have sequences of vectors: A sequence $(a_n)_{n=1}^{\infty}$, where each $a_n \in \mathbb{R}^m$, converges to a vector $a \in \mathbb{R}^m$ if for any $\varepsilon > 0$ we can find a N such that

$$||a_n - a|| \leq \varepsilon \quad \text{for all} \quad n \geq N. \tag{2}$$

In the last equation "$||\cdot||$" denotes a norm, e.g. for a vector $b = (b_1, \ldots, b_m)$ the Euclidean norm is

$$||b|| = \sqrt{b_1^2 + \cdots + b_m^2}. \tag{3}$$

Example 2. *Let* $a_n = \begin{pmatrix} \frac{n+1}{n} \\ \frac{1}{n} \end{pmatrix}$*. So, the sequence starts* $\begin{pmatrix} \frac{2}{1} \\ \frac{1}{1} \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \ldots$*. I claim that this sequence converges to* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$*. Note that* $||a_n - \begin{pmatrix} 1 \\ 0 \end{pmatrix}|| = || \begin{pmatrix} \frac{1}{n} \\ \frac{1}{n} \end{pmatrix} || = \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} =$

$\frac{\sqrt{2}}{n}$ *which is less or equal to* $\varepsilon > 0$ *if* $n \geq \frac{\sqrt{2}}{\varepsilon}$. *Hence, choosing* $N = \left[\frac{\sqrt{2}}{\varepsilon}\right] + 1$ *we have shown that* $||a_n - \begin{pmatrix} 1 \\ 0 \end{pmatrix}|| \leq \varepsilon$ *for all* $n \geq N$ *and therefore* $(a_n)_{n=1}^{\infty}$ *converges indeed to* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

You might have realized something in the previous example: The sequence in the first component of $\begin{pmatrix} \frac{n+1}{n} \\ \frac{1}{n} \end{pmatrix}_{n=1}^{\infty}$ is $a_n^1 = \left(\frac{n+1}{n}\right)_{n=1}^{\infty}$ which was the sequence we looked at in example 1. There we showed that this sequence converges to 1. The sequence in the second component of $a_n = \begin{pmatrix} \frac{n+1}{n} \\ \frac{1}{n} \end{pmatrix}$ is $a_n^2 = \left(\frac{1}{n}\right)_{n=1}^{\infty}$ and it is not so hard to see that this sequence converges to 0. If you take these two together it is hardly surprising that $\begin{pmatrix} \frac{n+1}{n} \\ \frac{1}{n} \end{pmatrix}_{n=1}^{\infty}$ is $a_n^1 = \left(\frac{n+1}{n}\right)_{n=1}^{\infty}$ converges to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The question is: Is this accidental or is it *always* the case that a vector sequence converges to the vector consisting of the limits of its component sequences? The following proposition says that this is always the case.

Proposition 1. *A sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^m$ *converges to a vector* $a \in \mathbb{R}^m$ *if and only if all component sequences converge.*

Proof. Let's write $a_n = (a_n^1, a_n^2, \ldots, a_n^m)$. We denote the sequence in the $i-th$ component as $(a_n^i)_{n=1}^{\infty}$. First we show that if each component sequence $(a_n^i)_{n=1}^{\infty}$ for $i \in \{1, 2, \ldots, m\}$ converges, then $(a_n)_{n=1}^{\infty}$ converges:

Denote the limit of the i-th component sequence as a^i and let $a = (a^1, a^2, \ldots, a^m)$. Take an $\varepsilon > 0$.

Since $(a_n^i)_{n=1}^{\infty}$ converges we can find N^i such that $|a_n^i - a^i| \leq \frac{\varepsilon}{\sqrt{m}}$ for all $n \geq N^i$. Now take $N = \max\{N^1, \ldots, N^m\}$. Then for all $n \geq N$, we have

$$\begin{aligned} ||a_n - a|| &= \sqrt{(a_n^1 - a^1)^2 + (a_n^2 - a^2)^2 + \cdots + (a_n^m - a^m)^2} \\ &\leq \sqrt{\left(\frac{\varepsilon}{\sqrt{m}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{m}}\right)^2 + \cdots + \left(\frac{\varepsilon}{\sqrt{m}}\right)^2} = \sqrt{m\frac{\varepsilon^2}{m}} = \varepsilon. \end{aligned}$$

Therefore, $(a_n)_{n=1}^{\infty}$ converges to a.

Second we have to show that when $(a_n)_{n=1}^{\infty}$ converges, then all component sequences

$(a^i_n)_{n=1}^\infty$ converge. This follows from

$$||a_n - a|| = \sqrt{(a^1_n - a^1)^2 + (a^2_n - a^2)^2 + \dots + (a^m_n - a^m)^2} \geq \sqrt{(a^i_n - a^i)^2} = |a^i_n - a^i|.$$

Since $(a_n)_{n=1}^\infty$ converges to a, $||a_n - a|| \leq \varepsilon$ for all $n \geq N$. But then the last expression implies that also $|a^i_n - a^i| \leq \varepsilon$. Consequently, $(a^i_n)_{n=1}^\infty$ converges to a^i. $\square$

We can define closed sets in $\mathbb{R}^m$ in terms of sequences:

Definition 1. *A set $S \in \mathbb{R}^m$ is* ***closed*** *if and only if every converging sequence in S has its limit in S.*

Put differently, let $(s_n)_{n=1}^\infty$ be an arbitrary sequence in S, i.e. $s_n \in S$ for $n = 1, 2, \dots,$ and let $(s_n)_{n=1}^\infty$ converge to s. Then $s \in S$ if S closed. If $s \in S$ for every converging sequence (that is entirely contained in S), then S is closed.

Example 3. *Let's look at the simple case where $m = 1$, i.e. we have normal real sequences. The interval $S = (1, 2]$ (including 2 but not including 1) is* not *closed: Our sequence from the first example $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, \frac{n+1}{n}, \dots$ is in S, i.e. each a_n is an element of S. But the limit of the sequence is 1 which is not in S.*

Now let's look at the set $S' = [1, 2]$. This set is closed and we can show this by contradiction: Suppose, S' was not closed. Then there would be a sequence $(a_n)_{n=1}^\infty$ such that (i) $a_n \in S'$ for all n, (ii) $(a_n)_{n=1}^\infty$ converges to a *and (iii) $a \notin S'$. Hence,* a *would be either strictly greater than 2 or strictly smaller than 1. Let's say* a *was strictly greater than 2 (the other case is similar). Then define $\varepsilon = (a - 2)/2$. As $(a_n)_{n=1}^\infty$ converges to* a, *there would have to be some a_n such that $|a_n - a| < \varepsilon$. But this would imply that $a_n > 2$ and therefore $a_n \notin S'$. But $a_n \notin S'$ contradicts (i). Hence, S' is closed.*

One important concept is a *subsequence*. Intuitively, a subsequence is what is left over after you delete some terms of the original sequence. For example, take the sequence $(1/1, 1/2, 1/3, 1/4, 1/5, \dots)$. One subsequence of this sequence would be $(1/1, 1/2, 1/4, 1/8, 1/16 \dots)$. Another subsequence would be $(1/2, 1/4, 1/10, 1/28, \dots)$ (in case you wondered: the rule here is $1/(3^{n-1} + 1)$). For a more formal definition of a subsequence, check your math books, e.g. Simon/Blume ch. 12 (p.256).

One theorem that we need a couple of times is the Bolzano-Weierstrass theorem.[1]

[1]Recall: A set S in $\mathbb{R}^m$ is *compact* if it is closed and bounded.

Theorem 1. *Let S be a compact set in $\mathbb{R}^m$ and let $(a_n)_{n=1}^{\infty}$ be a sequence that is entirely contained in S, i.e. $a_n \in S$ for all $n = 1, 2 \ldots$. Then $(a_n)_{n=1}^{\infty}$ has a convergent subsequence whose limit lies in S.*

Instead of a full-fledged proof we just look at the idea for the simple case where $m = 1$, i.e. the sequence is a sequence of real numbers. Then our compact set S is a closed interval and for simplicity we let $S = [0, 1]$. Now we construct a subsequence $(b_n)_{n=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ such that $(b_n)_{n=1}^{\infty}$ converges:

1. Let's split $[0, 1]$ in two subintervals $[0, 1/2]$ and $[1/2, 1]$. At least one of the two subintervals will contain an infinite number of a_n. If $[0, 1/2]$ contains an infinite number of elements of $(a_n)_{n=1}^{\infty}$, then we choose b_1 as an arbitrary a_{n_1} in the subinterval $[0, 1/2]$. If $[0, 1/2]$ does not contain an infinite number of a_ns, we let b_1 be some arbitrary a_{n_1} in $[1/2, 1]$.
2. Now we split $[0, 1]$ into four subintervals: $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$. We let b_2 be an arbitrary a_{n_2} that satisfies 2 conditions: (i) $n_2 > n_1$ and (ii) a_{n_2} is in the lowest of the four subintervals that contains an infinite number of a_ns.
3. Now we split $[0, 1]$ into 16 equally long closed subintervals. We let b_3 be some a_{n_3} such that: (i) $n_3 > n_2$ and (ii) a_{n_3} is in the lowest of the 16 subintervals that contains an infinite number of a_ns.
4. ...

The sequence $(b_n)_{n=1}^{\infty}$ constructed in this way converges. As $[0, 1]$ is closed, the limit of this sequence will be in $[0, 1]$.

For a more detailled exposition, you can check almost any "Maths for economists" book (there are a lot of them). One (of many) that is good to read is

Carl P. Simon and Lawrence Blume: *"Mathematics for Economists",* W.W. Norton & Company Inc., 1994

where all these results (and many more) can be found in chapter 12 (and some more advanced results in chapter 29).