Lecture Note 3a

Selections from the Nash equilibria I

1. Introduction

Let $\Gamma = (\Sigma_1, \ldots, \Sigma_n, \pi)$ be a finite game, i.e. the set of strategies Σ_i is finite for each player with $|\Sigma_i| = m_i$ for $i \in N$. We shall be interested in solution concepts which pick some, but not necessarily all, Nash equilibria in Γ. The need for selecting among the Nash equilibria can be seen from simple examples such as the following.

Example 1. Consider the two-player game

	L	R
T	$(1, 1)$	$(0, 0)$
B	$(0, 0)$	$(0, 0)$

There are two Nash equilibria in pure strategies, namely (T, L) and (B, R), but only (T, L) seems reasonable, while (B, R) appears as unstable – if a player expects the other player to choose her other pure strategy with some probability, however small, it would be better to choose the other pure strategy.

In this note, we take a look at the earlier attempts to specify what could be considered as a reasonable equilibrium. For the first attempts to exclude those Nash equilibria which are outright unreasonable, we need some more notation.

Since each set Σ_i of pure strategies is finite, we may write it as $\Sigma_i = \{1, \ldots, m_i\}$. A mixed strategy for player i is then a vector $s_i \in \mathbb{R}^{m_i}_+$ with $\sum_{k=1}^{m_i} s_i^k = 1$; the set of mixed strategies for player i is written S_i. An array of mixed strategies is a vector s in $S = S_1 \times \cdots \times S_n \subset \mathbb{R}^m$, where $m = m_1 + \cdots + m_n$; Given a strategy array $s \in S$, we write (k, s_{-i}) for the strategy array where the mixed strategy s_i of player i has been replaced by the pure strategy k.

A pure strategy k of player i is a best reply to the strategy array $s \in S$ if

$$\pi_i(k, s_{-i}) \geq \pi_i(l, s_{-i}), \text{ all } l \in \Sigma_i.$$

The set of best replies of player i to s is denoted $B_i(s)$, and $B(s) = B_1(s)\times\cdots\times B_n(s)$. Finally, for s_i a mixed strategy of player i, the carrier of s_i is the set

$$C(s_i) = \{k \in \Sigma_i \mid s_i^k > 0\}$$

of all pure strategies which have positive weight in s_i, and the carrier of the strategy array s is $C(s) = C_1(s_1) \times \cdots \times C_n(s_n)$.

Quasi-strong equilibria. Following Harsanyi (1973), we define a quasi-strong equilibrium as a strategy array which satisfies the condition

$$C(s) = B(s). \tag{1}$$

Thus, s is a quasi-strong equilibrium if it assigns positive weight to exactly those pure strategies which are best responses. It is easily checked that quasi-strong equilibria are Nash; indeed, if $C(s) \subseteq B(s)$, then each of the pure strategies in the carrier of s is a best reply, which is exactly the Nash equilibrium conditions. Thus, (1) is a stability condition which resembles that of Nash equilibria, but it goes further; in a Nash equilibrium, we have that $C(s) \subseteq B(s)$ but we do not necessarily have equality.

The equilibrium (B, R) in Example 1 is not a quasi-strong equilibrium, since the set of best replies of player 1 to R is $\{T, B\}$, and the carrier of (B, R) is just $\{B\}$ for player 1 and $\{R\}$ for player 2. Thus, the concept does quite well in this particular example, but not so in other, equally simple examples.

Example 2. Consider the game

	L	R
T	$(1, 1)$	$(1, 1)$
B	$(1, 1)$	$(0, 0)$

Here (T, L) is not quasi-strong, since $C(T, L) = (T, L)$, where as $B(T, L) = \{T, D\} \times \{L, R\}$. Moreover, if we take any mixed strategy (say of player 2) which puts nonzero weight on both strategies, then L is the only pure strategy which is a best reply. Arguing similarly for player 1 we see that there are no quasi-strong equilibria in this example, a somewhat unfortunate situation.

Since the quasi-strong equilibria are not in general well-behaved, they are not serious candidates for the 'reasonable Nash equilibrium solution. We have to add some new aspects to the discussion. Before we leave this topic, we comment briefly on the terminology: The equilibrium was called quasi-strong since Harsanyi had introduced another concept of a strong equilibrium, namely a mixed strategy s such that $C(s) = \{s\} = B(s)$. This condition is very restrictive since there should be pure strategy Nash equilibria, which is very often not the case.

This notion of a strong equilibrium should not be confused with another notion, dealing with cooperative games, and to which we return later. We shall not use Harsanyi's concept.

2. Perfect equilibria

It is natural to begin the discussion of selections in the context of normal form games with what comes closest to the notion of a subgame perfect equilibrium in an extensive form game. A suitable formalization of this may be obtained when we interpret the notion of subgame perfectness as a consequence of assuming that *make errors,* known in the literature as the *trembling hand hypothesis.* Even though players have chosen specific strategies, they may end up doing something different from what these strategies prescribe, not as a result of intentional deviation but just as a result of an error.

In our context of games with a finite number of pure strategies, we introduce a *tremble* as a vector $\eta \in \mathbb{R}^{m_1+\cdots+m_n}_{++}$, assigning a number $\eta_i^k > 0$ to each pure strategy $k \in \{1, \ldots, m_i\}$ of player i, for $i = 1, \ldots, n$. We interpret η_i^k as the probability that k is chosen by player i.

For $\eta \in \mathbb{R}^{m_1+\cdots+m_n}_{++}$ a tremble, we define the game $\Gamma(\eta) = (S_1(\eta), \ldots, S_n(\eta), \pi)$ with strategy sets

$$S_i(\eta) = \{s_i \in \triangle_{m_i} \mid s_i^k \geq \eta_i^k\}$$

(that is all the mixed strategies for player i in Γ which are such that the pure strategy k is chosen with probability at least η_i^k, $k = 1, \ldots, m_i$), and where π is the same as in Γ, so π is the function taking mixed strategies $s = (s_1, \ldots, s_n)$ in $S(\eta) = S_1(\eta) \times \cdots \times S_n(\eta)$ to

$$\sum_{(k_1,\ldots,k_n)} s_1^{k_1} \cdots s_n^{k_n} \pi(k_1, \ldots, k_n),$$

where the summation is over all pure strategy combinations in Γ. We notice in passing that if the numbers $\Gamma(\eta)$ satisfies the standard conditions for existence of Nash equilibria, at least for sufficiently small η so that all $S_i(\eta)$ are nonempty (convex and compact strategy sets, continuous payoff, and convex preferred sets of strategies for each player at each strategy combination). Thus, each $\Gamma(\eta)$ as at least one Nash equilibrium $s(\eta)$, which by the definition of $\Gamma(\eta)$ is *completely mixed,* in the sense that $s_i^k(\eta)$, the probability that i chooses her pure strategy $k \in \{1, \dots, m_i\}$ according to the mixed stratagy $s_i(\eta)$, is positive for all i and k.

We can now define perfectness as something which retains the equilibrium properties even if players make small errors.

Definition 1. *A Nash equilbriium s in Γ is a perfect equilibrium if there is a sequence $(\eta_t)_{t=1}^{\infty}$ of trembles converging to the zero vector, and Nash equilibria $s(\eta_t)$ in $\Gamma(\eta_t)$ for each t, such that the sequence $(s(\eta_t))_{t=1}^{\infty}$ converges to s.*

Since Nash equilibria exist for each tremble η, the question of whether there exists perfect equilibria (for a game of the type we consider here) can be answered in the affirmative.

Technically, choose a sequence of trembles converging to 0, and for each η in the sequence, choose a Nash equilibrium in $\Gamma(\eta)$. The sequence of Nash equilibria obtained in this way is not necessarily convergent, but since it consists of mixed strategies and thus belong to a compact set, it has a convergent subsequence. This subsequence may then be used instead of the original sequence, and its limit is indeed a perfect equilibrium.

In the above definition, we have used arbitrary trembles. It may be useful to restrict attention to trembles which are more easy to work with, namely such where all η_i^k have the same size ε. We define an *ε-perfect equilibrium* in Γ as a mixed strategy combination s which is completely mixed and satisfies

$$\pi_i(k, s_{-i}) < \pi_i(l, s_{-i}) \text{ implies } s_i^k \leq \varepsilon$$

for all pure strategies k and l of player i, $i = 1, \dots, n$. Thus, if the pure strategy k is inferior than pure strategy l, then i will put a very low weight on k (since using it reduces average payoff to i), at least below ε.

We have the following characterization of perfect equilibria:

Theorem 1. *Let s be a strategy array in Γ. Then the following are equivalent:*

(i) s is a perfect equilibrium,

(ii) s is a limit of ε-perfect equilibria, for ε going to 0,

(iii) s is the limit of a sequence $(s(\varepsilon))_{\varepsilon\to 0}$ of completely mixed strategies, such that s is best reply to $s(\varepsilon)$ (that is $s \in B(s(\varepsilon))$ for ϵ small enough.

Proof: (i)⇒(ii). If s is a perfect equilibrium, there is a sequence of Nash equilibria $s(\eta)$ in $\Gamma(\eta)$ converging to s for η going to 0. For each tremble η, let

$$\varepsilon(\eta) = \max_{k\in S_i, i\in N} \eta_i^k$$

be the largest of all the numbers η_i^k. Clearly $s(\eta)$ is completely mixed, and since it is Nash equilibrium in $\Gamma(\eta)$, we have that a pure strategy k for i which is inferior to another pure strategy l must have as small probability as possible in $S_i(\eta)$, meaning that $s_i^k = \eta_i^k \leq \varepsilon(\eta)$. We conclude that $s(\eta)$ is a $\varepsilon(\eta)$-equilibrium, and now (ii) follows.

(ii)⇒(iii). Let k be a pure strategy for i such that $s_i^k > 0$. Since s is a limit of the mixed strategies $s(\varepsilon)$, we have that $s(\varepsilon)_i^k > \varepsilon$ for ε small enough. But then k is a best reply to $s(\varepsilon)$, since otherwise $s(\varepsilon)_i^k \leq \varepsilon$. But this shows that s is a best reply to $s(\epsilon)$, since it has positive weight only on pure strategies which are best replies.

(iii)⇒(i). We show that $s(\varepsilon)$ are equilibria in $\Gamma(\eta)$ for suitable trembles η. For each i and $k \in \{1, \ldots, m_i\}$, if $s_i^k = 0$ choose $\eta_i^k = s(\varepsilon)_i^k$, and if $s_i^k > 0$, let η_i^k be an arbitrary very small positive number, e.g. $\eta_i^k = \min_l s(\varepsilon)_i^l$. If k is not a best reply of i to $s(\varepsilon)$, then $s_i^k = 0$, so by our definition of η, we have that $s(\varepsilon)_i^k = \eta_i^k$. But this means that $s(\varepsilon)$ is an equilibrium in $\Gamma(\eta)$, and we are done. □

It is seen that the for an equilibrium to be perfect, it should be the limit of a sequence of equilibria obtained for *some* tremble. Another, seemingly related concept is that of strict perfectness.

Definition 2. *An equilibrum s in Γ is strictly perfect if for all sequences of trembles η converging to 0 there is a sequence of equilibria $s(\eta)$ in $\Gamma(\eta)$ converging to s.*

The concept of strict perfectness is quite restrictive, actually so as to exclude existence in some otherwise rather standard games. Consider the game Γ with matrix

	L	M	R
T	$(1, 1)$	$(1, 0)$	$(0, 0)$
B	$(1, 1)$	$(0, 0)$	$(1, 0)$

Here both (T, L) and (B, L) are pure strategy equilibria. Also notice that each both M and R

are dominated, so all equilibria must have player 2 choosing L. Assume that s is a strictly perfect equilibrium in Γ. Now choose a sequence of trembles such that $\eta_2^M > \eta_2^R$; in each equilibrium $s(\eta)$ of $\Gamma(\eta)$ we then have that B is not a best reply to $s(\eta)$, meaning that $s(\eta)_1^B = 0$, and in the limit we get that $s_1^B = 0$. Choosing another sequence $(\hat{\eta})$ with $\hat{\eta}_2^M < \hat{\eta}_2^R$ we similarly obtain that $s_1^L = 0$. But we cannot have a mixed strategy for player 1 with zero weight on all the pure strategies, and we conclude that there is no strictly perfect equilibrium.

That the notion of perfectness does not solve all problems which initiated the search for a selection, can be seen from the following example.

Example 3. Consider the game

	L	C	R
T	$(1,1)$	$(0,0)$	$(-1,-2)$
M	$(0,0)$	$(0,0)$	$(0,-2)$
B	$(-2,-1)$	$(-2,0)$	$(-2,-2)$

This is the game of the Example 1, where we have added a dominated strategy for each player. Now the strategy array (M, C), which would not have been a perfect equilibrium without the new strategies, has become perfect. Indeed, for a sequence of trembles with higher value on R than on L, the strategy T is no longer a best reply and gets minimal weight, meaning that in the limit its weight is 0, and similarly for the column player, if the tremble assigns higher value to B than to T, then s will have zero weight on L. Since dominated strategies are never best replies, there must be full weight on M and C.

It is seen that the equilibrium (T, L), which is a perfect equilibrium in the game without the two added strategies, remains so in the new game.

3. Proper Equilibria

In the interpretation of the perfect equilibria, it is presumed that players make errors with some small probability. These errors may involve arbitrary pure strategies, including some which may be considered as quite harmful to the player. Thus, there is no distinction between strategies when it comes to selection by error, and this feature may be criticized, since we rather expect players to be more careful when an erroneous choice will inflict a big loss than when the difference in payoff is minor.

Such considerations lead to the notion of a *proper equilibrium* introduced by Myerson

(1978). Inspired by the previous discussion, we begin with the notion of ε-properness: The mixed strategy array $s(\varepsilon)$ is an ε-proper equilibrium, for $\varepsilon > 0$, if it is completely mixed and

$$\pi_i(k, s(\varepsilon)_{-i}) < \pi_i(l, s(\varepsilon)) \Rightarrow s(\varepsilon)_i^k > \varepsilon s(\varepsilon)_i^l.$$

A strategy array is a proper equilibrium if it is the limit of a sequence $(s(\epsilon))_{\varepsilon\downarrow 0}$ for ε going to 0.

The existence problem is taken care of rather easily.

Theorem 2. *Every normal form game with finite sets of pure strategies has a proper equilibrium.*

Proof: We show that there exist ε-proper equilibria for suitably small ε.

Let $0 < \varepsilon < 1$, and for each $i \in N$ let η_i^k, for $k \in \Sigma_i$, be given by

$$\eta_i^k = \frac{\varepsilon^{m_i}}{m_i},$$

where $m_i = |\Sigma_i|$ is the cardinality of the set of pure strategies of player i. We let $\eta = (\eta_1, \ldots, \eta_n)$, and let $S_i(\eta)$ denote the set of mixed strategies s of player i such that $s_i^k \geq \eta_i^k$ for all $k \in \Sigma_i$. Finally $S(\eta) = S_1(\eta_1) \times \cdots \times S_n(\eta_n)$.

We now define the correspondences $\Phi_i : S(\eta) \rightrightarrows S_i(\eta_i)$ by

$$\Phi_i(s) = \{\hat{s}_i \in S_i(\eta_i) \mid \pi_i(k, s_{-i}) < \pi_i(l, s_{-i}) \Rightarrow \hat{s}_i^k < \varepsilon s_i^l, \text{ all } k, l \in \Sigma_i\}.$$

We check that $\Phi_i(s) \neq \emptyset$ for all $s \in S(\eta)$. Indeed. let $\nu_i(s, k)$ be the number of pure strategies in Σ_i such that $\pi_i(k, s_{-i}) < \pi_i(l, s_{-i})$, and let

$$\hat{s}_i^k = \frac{\varepsilon^{\nu_i(s,k)}}{\sum_l \varepsilon^{\nu_i(s,l)}}.$$

Then $s_i^k \geq \eta_i^k$ for small enough ε, so that $\Phi_i(s) \neq \emptyset$. It is easily seen that $\Phi_i(s_i)$ is closed and convex, and that Φ_i is upper hemicontinuous.

Now, we may apply Kakutani's fixed point theorem to the correspondence $\Phi : S(\eta) \rightrightarrows S(\eta)$ defined by $S(\eta) = S_1(\eta_1) \times \cdots \times S_n(\eta_n)$ to get a strategy array s such that $s_i \in \Phi_i(s)$ for each i. It is easily checked that s is indeed an ε-proper equilibrium. □

One of the drawbacks of the perfect equilibria is that adding dominated strategies may enlarge the set of perfect equilibria. Unfortunately, this may happen also with proper equilibria, as it can be seen from the game

	L	R
T	$(1,1,1)$	$(0,0,1)$
B	$(0,0,1)$	$(0,0,1)$

I

	L	R
T	$(0,0,0)$	$(0,0,0)$
B	$(0,0,0)$	$(1,1,0)$

II

where player 1 chooses row, player 2 column, and player 3 matricx. Here the second strategy of player 3 is dominated, and it might therefore be considered as superfluous. In that case, only $(T, L.1)$ is a reasonable equilibrium, being the unique proper equilibrium in the game where player 3 has only the first matrix. But actually (B, R, I) is also a proper equilibrium.

4. Essential equilibria

While the refinements discussed above used the concept of trembles, thereby restricting the choices of the players to certain subsets of mixed strategies, there is another way of getting rid of undesired Nash equilibria, or rather another type of undesirablity to be considered. We might allow all strategies, mixed or pure, but instead apply the strategies to games which are almost the same but have small perturbations of the payoffs. If some Nash equilibrium disappears if the payoff structure is changed ever so little, we may have some reservations towards this equilibrium.

These considerations lead to the concept of an *essential equilibrium.* To define it, we need to introduce the notion of *distance* between two games (with the same player set and the same set of pure strategies). If $\Gamma = (\Sigma_1, \dots, \Sigma_n, \pi)$ and $\Gamma' = (\Sigma_1, \dots, \Sigma_n, \pi')$ are two games with the same sets of players and strategies, but possibly with different payoff functions, then we define the distance between Γ and Γ' as

$$d(\Gamma, \Gamma') = \max_{i \in N, \sigma \in \Sigma} \|\pi_i(\sigma) - \pi(\sigma')\|.$$

One can easily check that this notion of distance satisfy (i) $d(\Gamma, \Gamma') = d(\Gamma', \Gamma)$, (2) $d(\Gamma, \Gamma') \geq 0$ and $d(\Gamma, \Gamma') = 0$ if and only if $\Gamma = \Gamma'$, and also (iii) $d(\Gamma, \Gamma'') \leq d(\Gamma, \Gamma') + d(\Gamma', \Gamma'')$ for all $\Gamma, \Gamma', \Gamma''$ showing that $d(\cdot, \cdot)$ is a metric on the set of games with the fixed player and strategy sets. Notice also that the set of mixed strategy arrays can be considered as a subset of a suitable Euclidean space, so that there is also a well defined notion of distance between such strategy arrays.

Definition 3. *An equilibrium s in Γ is an essential equilibrium if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every game Γ' with $d(\Gamma, \Gamma') < \delta$ there is a Nash equilibrium s' in Γ' such that $\|s - s'\|$.*

The essential equilibria are points of (lower hemi-)continuity of the Nash equilibrium correspondence – each neighbourhood of the equilibrium will contain equilibria of the games which are sufficiently close to the game considered.

Although the concept of essential equilibria has some intuitive content – there is an equilibrium near by even if we change the payoff slightly – it is not easy to work with, and even existence is not trivial. It also turns out that essential equilibria are not necessarily better than non-essential ones, as is shown by the example below.

	L	C	R
T	$(1,1)$	$(0,0)$	$(0,0)$
M	$(0,0)$	$(2,2)$	$(2,2)$
B	$(0,0)$	$(2,2)$	$(2,2)$

Here (T, L) is an essential equilibrium (it will remain so even if all payoffs are changed as long as these changes are small). But there are many other Nash equilibria, including some which are better, such as (M, C). This equilibrium is not essential, however, since arbitrarily small positive changes in the payoff of player 1 at the strategy B will upset the equilibrium. The fact that there is a non-essential equilibrium which is better is not in itself a strong argument against the concept, but in our case there is a additional reason to discard (T, L) – in an actual play there would be some informal agreement among players of never using T, respectively L, since avoiding them would improve payoff to both.

Due to this and other shortcomings, the essential equilibria have only theoretical interest, and there has not been much discussion of them in the literature.

5. References

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