Quintic splines

In[1]:=
```
ClearAll["Global`*"]
```

This file explains the basic working of quintic splines, as we use them in the paper of Thomas Blanchet, Juliette Fournier and Thomas Piketty, “Generalized Pareto Curves: Theory and Applications”, 2017.

Basis functions

Determine the basis functions by solving the appropriate linear system:

In[2]:=
```
poly[x_] = Sum[a_i x^i, {i, 0, 5}]
```

Out[2]= $a_0 + x\,a_1 + x^2\,a_2 + x^3\,a_3 + x^4\,a_4 + x^5\,a_5$

In[3]:=
```
coef = CoefficientList[poly[x], x]
```

Out[3]= $\{a_0, a_1, a_2, a_3, a_4, a_5\}$

In[4]:=
```
h00[x_] = poly[x] /. First[Solve[{poly[0] == 1, poly[1] == 0,
      poly'[0] == 0, poly'[1] == 0, poly''[0] == 0, poly''[1] == 0}, coef]];
```

In[5]:=
```
h01[x_] = poly[x] /. First[Solve[{poly[0] == 0, poly[1] == 1,
      poly'[0] == 0, poly'[1] == 0, poly''[0] == 0, poly''[1] == 0}, coef]];
```

In[6]:=
```
h10[x_] = poly[x] /. First[Solve[{poly[0] == 0, poly[1] == 0,
      poly'[0] == 1, poly'[1] == 0, poly''[0] == 0, poly''[1] == 0}, coef]];
```

In[7]:=
```
h11[x_] = poly[x] /. First[Solve[{poly[0] == 0, poly[1] == 0,
      poly'[0] == 0, poly'[1] == 1, poly''[0] == 0, poly''[1] == 0}, coef]];
```

In[8]:=
```
h20[x_] = poly[x] /. First[Solve[{poly[0] == 0, poly[1] == 0,
      poly'[0] == 0, poly'[1] == 0, poly''[0] == 1, poly''[1] == 0}, coef]];
```

In[9]:=
```
h21[x_] = poly[x] /. First[Solve[{poly[0] == 0, poly[1] == 0,
      poly'[0] == 0, poly'[1] == 0, poly''[0] == 0, poly''[1] == 1}, coef]];
```

Expressions of the basis functions:

In[10]:=
```
TableForm[Map[# &, {h00[x], h01[x], h10[x], h11[x], h20[x], h21[x]}],
 TableHeadings → {{"h00", "h01", "h10", "h11", "h20", "h21"}}]
```

Out[10]//TableForm=

h00	$1 - 10\,x^3 + 15\,x^4 - 6\,x^5$
h01	$10\,x^3 - 15\,x^4 + 6\,x^5$
h10	$x - 6\,x^3 + 8\,x^4 - 3\,x^5$
h11	$-4\,x^3 + 7\,x^4 - 3\,x^5$
h20	$\frac{x^2}{2} - \frac{3\,x^3}{2} + \frac{3\,x^4}{2} - \frac{x^5}{2}$
h21	$\frac{x^3}{2} - x^4 + \frac{x^5}{2}$

Their first derivatives:

In[11]:=
```
TableForm[Map[D[#, x] &, {h00[x], h01[x], h10[x], h11[x], h20[x], h21[x]}],
  TableHeadings → {{"h00", "h01", "h10", "h11", "h20", "h21"}}]
```

Out[11]//TableForm=

h00	$-30x^2+60x^3-30x^4$
h01	$30x^2-60x^3+30x^4$
h10	$1-18x^2+32x^3-15x^4$
h11	$-12x^2+28x^3-15x^4$
h20	$x-\frac{9x^2}{2}+6x^3-\frac{5x^4}{2}$
h21	$\frac{3x^2}{2}-4x^3+\frac{5x^4}{2}$

Their second derivatives:

In[12]:=
```
TableForm[Map[D[#, {x, 2}] &, {h00[x], h01[x], h10[x], h11[x], h20[x], h21[x]}],
  TableHeadings → {{"h00", "h01", "h10", "h11", "h20", "h21"}}]
```

Out[12]//TableForm=

h00	$-60x+180x^2-120x^3$
h01	$60x-180x^2+120x^3$
h10	$-36x+96x^2-60x^3$
h11	$-24x+84x^2-60x^3$
h20	$1-9x+18x^2-10x^3$
h21	$3x-12x^2+10x^3$

Their third derivatives:

In[13]:=
```
TableForm[Map[D[#, {x, 3}] &, {h00[x], h01[x], h10[x], h11[x], h20[x], h21[x]}],
  TableHeadings → {{"h00", "h01", "h10", "h11", "h20", "h21"}}]
```

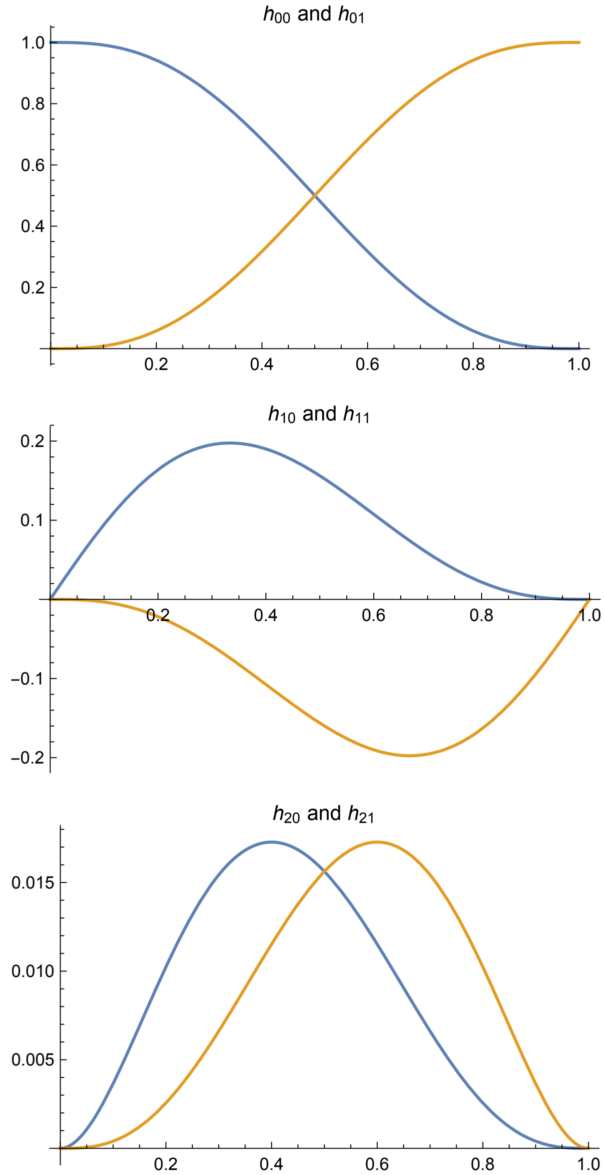
Out[13]//TableForm=

h00	$-60+360x-360x^2$
h01	$60-360x+360x^2$
h10	$-36+192x-180x^2$
h11	$-24+168x-180x^2$
h20	$-9+36x-30x^2$
h21	$3-24x+30x^2$

Plots of the basis functions:

In[14]:=
```
GraphicsColumn[{Plot[{h00[x], h01[x]}, {x, 0, 1}, PlotLabel → "h₀₀ and h₀₁"],
  Plot[{h10[x], h11[x]}, {x, 0, 1}, PlotLabel → "h₁₀ and h₁₁"],
  Plot[{h20[x], h21[x]}, {x, 0, 1}, PlotLabel → "h₂₀ and h₂₁"]}]
```

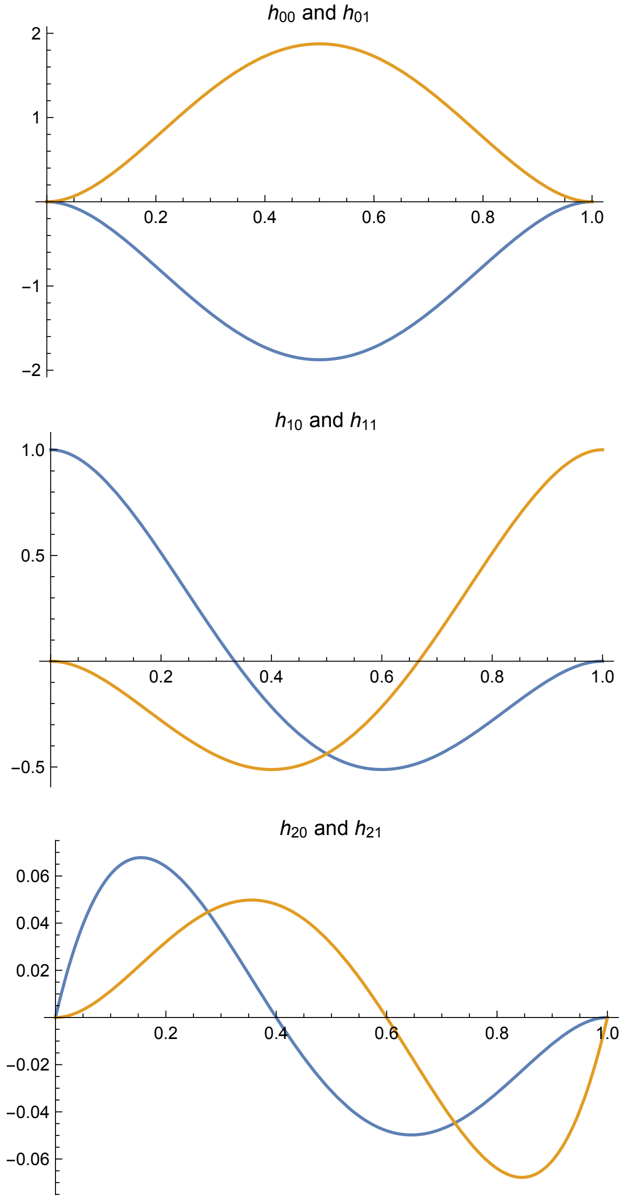
Out[14]=



Their first derivatives:

In[15]:=
```
GraphicsColumn[
 {Plot[Evaluate[{h00'[x], h01'[x]}], {x, 0, 1}, PlotLabel → "h00 and h01"],
  Plot[Evaluate[{h10'[x], h11'[x]}], {x, 0, 1}, PlotLabel → "h10 and h11"],
  Plot[Evaluate[{h20'[x], h21'[x]}], {x, 0, 1}, PlotLabel → "h20 and h21"]}]
```

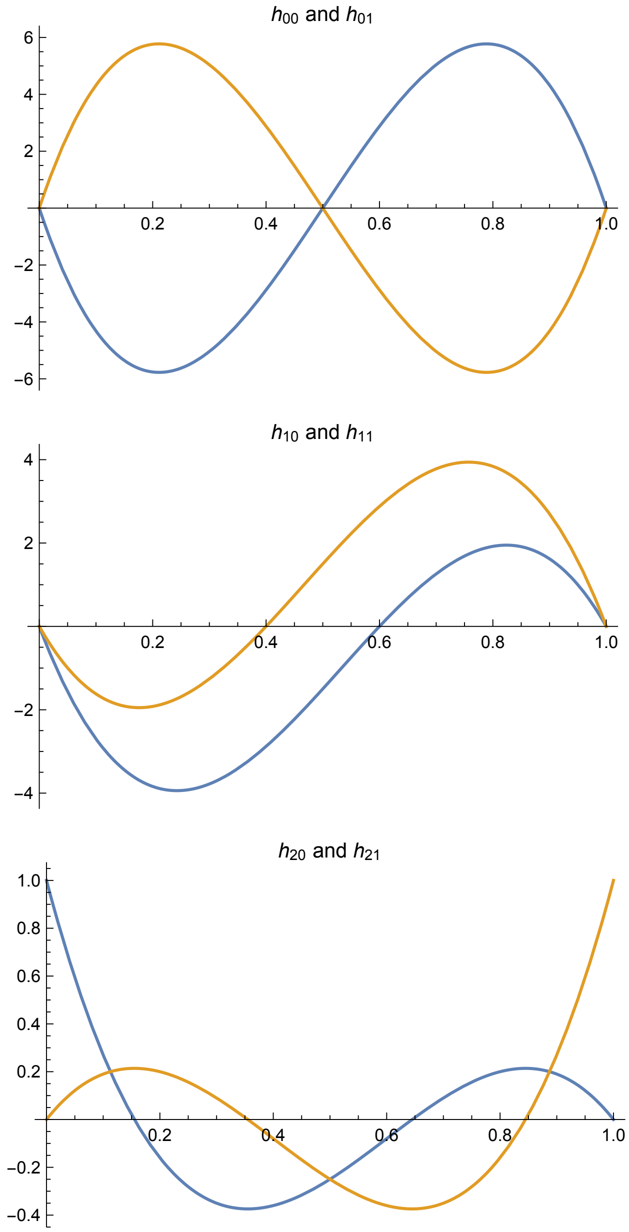
Out[15]=



Their second derivatives:

In[16]:=
```
GraphicsColumn[
 {Plot[Evaluate[{h00''[x], h01''[x]}], {x, 0, 1}, PlotLabel → "h00 and h01"],
  Plot[Evaluate[{h10''[x], h11''[x]}], {x, 0, 1}, PlotLabel → "h10 and h11"],
  Plot[Evaluate[{h20''[x], h21''[x]}], {x, 0, 1}, PlotLabel → "h20 and h21"]}]
```

Out[16]=



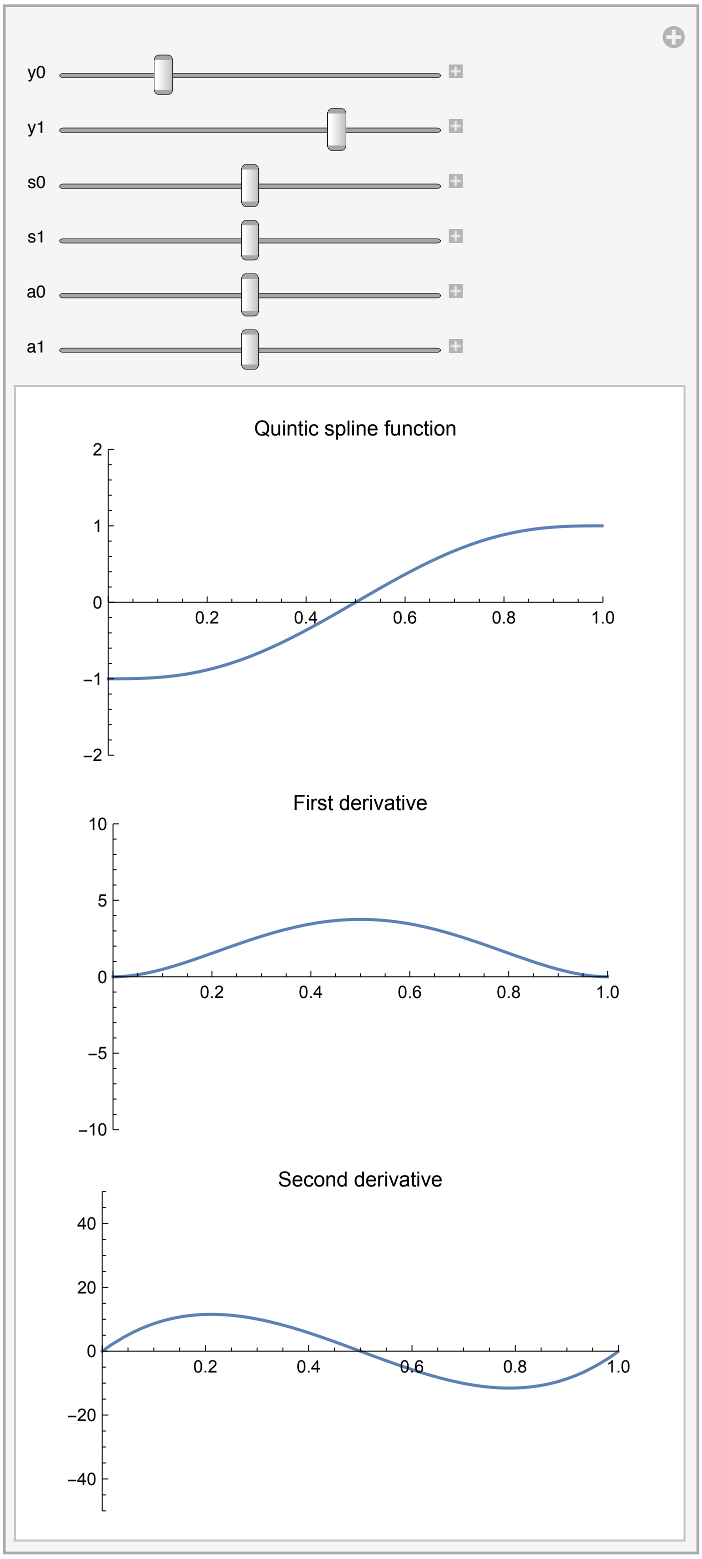
Interpolation function

The interpolating function over [0, 1] is a linear combination of those six basis functions:

In[17]:=
```
h[x_, y0_, y1_, s0_, s1_, a0_, a1_] =
  y0 * h00[x] + y1 * h01[x] + s0 * h10[x] + s1 * h11[x] + a0 * h20[x] + a1 * h21[x];
```

In[18]:=
```
Manipulate[GraphicsColumn[{Plot[h[x, y0, y1, s0, s1, a0, a1], {x, 0, 1},
    PlotRange → {{0, 1}, {-2, 2}}, PlotLabel → "Quintic spline function"],
   Plot[Evaluate[D[h[x, y0, y1, s0, s1, a0, a1], x]], {x, 0, 1},
    PlotRange → {{0, 1}, {-10, 10}}, PlotLabel → "First derivative"],
   Plot[Evaluate[D[h[x, y0, y1, s0, s1, a0, a1], {x, 2}]], {x, 0, 1},
    PlotRange → {{0, 1}, {-50, 50}}, PlotLabel → "Second derivative"]}],
 {{y0, -1}, -2, 2}, {{y1, 1}, -2, 2}, {{s0, 0}, -2, 2}, {{s1, 0}, -2, 2},
 {{a0, 0}, -20, 20}, {{a1, 0}, -20, 20}]
```

Out[18]=



We can extend this function to an arbitrary interval using an affine transformation:

In[19]:=
```
f[x_] = y0 * h00[(x - x0)/(x1 - x0)] + y1 * h01[(x - x0)/(x1 - x0)] +
  s0 * (x1 - x0) * h10[(x - x0)/(x1 - x0)] + s1 * (x1 - x0) * h11[(x - x0)/(x1 - x0)] +
  a0 * (x1 - x0) ^2 * h20[(x - x0)/(x1 - x0)] + a1 * (x1 - x0) ^2 * h21[(x - x0)/(x1 - x0)];
```

Check that the function and its derivatives have the right values:

In[20]:=
```
{f[x0], f[x1], f'[x0], f'[x1], f''[x0], f''[x1]}
```

Out[20]= {y0, y1, s0, s1, a0, a1}

Regularity conditions to determine the parameters of the spline

To determine the parameters of the spline, we impose the requirement that the third derivative is continuous at the jointures. That leads to the most regular curve possible in the sense that the third derivative has the lowest curvature possible.

We start by defining two splines over the knots (x_1, x_2, x_3). The spline over (x_1, x_2) if f_1, and the one over (x_2, x_3) is f_2.

In[21]:=
$$f1[x\_] = y1 * h00\left[\frac{x - x1}{x2 - x1}\right] + y2 * h01\left[\frac{x - x1}{x2 - x1}\right] + s1 * (x2 - x1) * h10\left[\frac{x - x1}{x2 - x1}\right] + s2 * (x2 - x1) * h11\left[\frac{x - x1}{x2 - x1}\right] + a1 * (x2 - x1)^2 * h20\left[\frac{x - x1}{x2 - x1}\right] + a2 * (x2 - x1)^2 * h21\left[\frac{x - x1}{x2 - x1}\right];$$

In[22]:=
$$f2[x\_] = y2 * h00\left[\frac{x - x2}{x3 - x2}\right] + y3 * h01\left[\frac{x - x2}{x3 - x2}\right] + s2 * (x3 - x2) * h10\left[\frac{x - x2}{x3 - x2}\right] + s3 * (x3 - x2) * h11\left[\frac{x - x2}{x3 - x2}\right] + a2 * (x3 - x2)^2 * h20\left[\frac{x - x2}{x3 - x2}\right] + a3 * (x3 - x2)^2 * h21\left[\frac{x - x2}{x3 - x2}\right];$$

Get the system in matrix form:

In[23]:=
```
system = CoefficientArrays[{f1'''[x2] == f2'''[x2]}, {a1, a2, a3}];
```

In[24]:=
```
system[[1]] // Normal
```

Out[24]=
$$\left\{-\frac{24\,s1}{(-x1 + x2)^2} - \frac{36\,s2}{(-x1 + x2)^2} + \frac{36\,s2}{(-x2 + x3)^2} + \frac{24\,s3}{(-x2 + x3)^2} - \frac{60\,y1}{(-x1 + x2)^3} + \frac{60\,y2}{(-x1 + x2)^3} + \frac{60\,y2}{(-x2 + x3)^3} - \frac{60\,y3}{(-x2 + x3)^3}\right\}$$

In[25]:=
```
system[[2]] // Normal
```

Out[25]=
$$\left\{\left\{-\frac{3}{-x1 + x2}, \frac{9}{-x1 + x2} + \frac{9}{-x2 + x3}, -\frac{3}{-x2 + x3}\right\}\right\}$$

We need two additional equations for the system to be uniquely determined. At the lower end (first knot), we impose the third derivative equal to zero.

In[26]:=
```
systemfirst = CoefficientArrays[{f1'''[x1] == 0}, {a1, a2, a3}];
```

In[27]:=
```
systemfirst[[1]] // Normal
```

Out[27]=
$$\left\{-\frac{36\,s1}{(-x1 + x2)^2} - \frac{24\,s2}{(-x1 + x2)^2} - \frac{60\,y1}{(-x1 + x2)^3} + \frac{60\,y2}{(-x1 + x2)^3}\right\}$$

In[28]:=
```
systemfirst[[2]] // Normal
```

Out[28]=
$$\left\{\left\{-\frac{9}{-x1 + x2}, \frac{3}{-x1 + x2}, 0\right\}\right\}$$

At the upper end (last knot), we estimate the second derivative directly using a two-points difference:

```
In[29]:= systemlast = CoefficientArrays[{f1''[x1] == (s3 - s2)/(x3 - x2)}, {a1, a2, a3}];

In[30]:= systemlast[[1]] // Normal

Out[30]= {-(-s2 + s3)/(-x2 + x3)}

In[31]:= systemlast[[2]] // Normal

Out[31]= {{1, 0, 0}}
```

Calculate explicit solution for testing purposes:

```
In[32]:= Simplify[Solve[{f1'''[x2] == f2'''[x2], f1'''[x1] == 0, f2'''[x3] == (s3 - s2)/(x3 - x2)},
            {a1, a2, a3}]] // InputForm

Out[32]//InputForm=
{{a1 -> (s3*(x1 - x2)^3*(x2^2 - 2*x2*(-18 + x3) + (-36 + x3)*x3) - s2*(x1 - x2)*
        (x2^3 - 2*x1*(x2^2 - x2*(-12 + x3) - 96*x3) + x1^2*(-84 + x2 - x3) + 24*x2
       12*(3*s1*(x1 - x2)*(8*x1 - x2 - 7*x3)*(x2 - x3)^2 - 10*(5*x2*x3^2*(y1 - y2)
          x1*(4*x3^2*(y1 - y2) + 8*x2*x3*(-y1 + y2) + x2^2*(4*y1 - y2 - 3*y3)) + x
          3*x1^2*x2*(-y2 + y3))))/(72*(x1 - x2)^2*(x1 - x3)*(x2 - x3)^2),
  a2 -> (s3*(x1 - x2)^3*(x2^2 - 2*x2*(-18 + x3) + (-36 + x3)*x3) - s2*(x1 - x2)*
        (x2^3 + x1^2*(-84 + x2 - x3) - 2*x1*x2*(-84 + x2 - x3) - 168*x2*x3 - x2^2*
       12*(3*s1*(x1 - x2)*(x2 - x3)^3 + 10*(x3^3*(y1 - y2) + 3*x2^2*(x3*(y1 - y2) -
          3*x2*(x3^2*(y1 - y2) + x1^2*(y2 - y3)) + x1^3*(y2 - y3) + x2^3*(-y1 + y3
  a3 -> (s3*(x1 - x2)^2*(x2 - x3)*(-x2^2 + 9*x1*(-28 + x2 - x3) + 8*x3*(36 + x3)
       s2*(x1 - x2)*(x2 - x3)*(x2^3 + 9*x1^2*(12 + x2 - x3) + 7*x2^2*x3 - 8*x2*(-3
        2*x1*(5*x2^2 - x2*(-12 + x3) - 4*(-24 + x3)*x3)) -
       12*(3*s1*(x1 - x2)*(x2 - x3)^3 - 10*(x3^3*(-y1 + y2) + x2^3*(y1 - y3) + 3*x
          x2^2*(x3*(3*y1 + y2 - 4*y3) + x1*(-y2 + y3)) + x2*(3*x3^2*(y1 - y2) + 8*
      (72*(x1 - x2)^2*(x1 - x3)*(x2 - x3)^2)}}
```

Constraint on the spline

The interpolation method requires certain conditions on the spline to get a nondecreasing quantile function. We search for a lower bound of the polynomial
$f''(x_0 + x(x_1 - x_0)) + f'(x_0 + x(x_1 - x_0))(1 - f'(x_0 + x(x_1 - x_0)))$ over [0, 1].

```
In[33]:= g = f''[x0 + x (x1 - x0)] + f'[x0 + x (x1 - x0)] (1 - f'[x0 + x (x1 - x0)]);

In[34]:= gcoefs = Simplify[CoefficientList[g, x]];
```

We rewrite the polynomial in its Bernstein form:

```
In[35]:= tobernstein[k_] :=
          Sum[Part[gcoefs, r + 1] * Binomial[k, r] / Binomial[Length[gcoefs] - 1, r], {r, 0, k}]

In[36]:= bernsteincoefs = FullSimplify[Map[tobernstein, Range[0, Length[gcoefs] - 1]]];
```

Hence, we require all the following expressions to be non-negative:

In[37]:=
```
TableForm[bernsteincoefs]
```

Out[37]//TableForm=
```
a0 + s0 - s0^2
a0 + s0 - s0^2 + ((x0-x1) (24 s1+2 s0 (18+a0 (x0-x1)^2)+(x0-x1) (3 a1+a0 (-9-x0+x1)))+60 (-y0+y1)) / (8 (x0-x1)^2)
(-480 (y0-y1)+(x0-x1) (16 s0^2 (x0-x1)-(a0 (34+5 x0-5 x1)+3 a1 (-6+x0-x1)+2 a0^2 (x0-x1)^2) (x0-x1)+24 s1 (7-x0+x1)+60 (y0-y1 / (56 (x0-x1)^2)
-300 (y0-y1)+(x0-x1) (96 s0^2 (x0-x1)+4 s1 (15-6 a0 x0^2+x1 (11-6 a0 x1)+x0 (-11+12 a0 x1))+2 s0 ((-18 a0+5 a1) x0^2+2 x0 (-5+22
- 100 a0 x0^2+100 a1 x0^2-20 a0 x0^3+20 a1 x0^3+9 a0^2 x0^4-26 a0 a1 x0^4+9 a1^2 x0^4+576 s0^2 (x0-x1)^2+576 s1^2 (x0-x1)^2-200 a0 x0 x1-200
(x0-x1) (96 s1^2 (x0-x1)+4 s0 (-15+6 a1 x0^2+x0 (-11+22 s1-12 a1 x1)+x1 (11-22 s1+6 a1 x1))-(x0-x1) (a0 (-1+x0 (-5+3 a1 x0)+5
(480 (y0-y1)-(x0-x1) ((34 a1+a1 (-5+2 a1 (x0-x1)) (x0-x1)-3 a0 (6+x0-x1)) (x0-x1)+16 s1^2 (-x0+x1)-24 s0 (-7+(-1+2 s1) x0+x1 / (56 (x0-x1)^2)
(-(x0-x1) (24 s0+8 s1^2 (x0-x1)-(3 a0+a1 (-1+x0-x1)) (x0-x1)+2 s1 (18+a1 x0^2-2 x0 (2+a1 x1)+x1 (4+a1 x1)))+60 (y0-y1)) / (8 (x0-x1)^2)
a1 + s1 - s1^2
```

In[38]:=
```
Map[InputForm[Simplify[D[#, a1]]] &, bernsteincoefs] // TableForm
```

Out[38]//TableForm=
```
0
3/8
(3*(6 + (-1 + 2*s0)*x0 + x1 - 2*s0*x1))/56
(1 - 3*a0*x0^2 + (5 - 10*s0)*x1 - 3*a0*x1^2 + x0*(-5 + 10*s0 + 6*a0*x1))/5
((13*a0 - 9*a1)*x0^2 + 2*(5 + 44*s0 + 36*s1)*x1 + (13*a0 - 9*a1)*x1^2 - 2*
(-35 - 3*a0*x0^2 + 6*a1*x0^2 + 24*s0*(x0 - x1) + 36*s1*(x0 - x1) + 6*a0*x0*
(-34 - 4*a1*x0^2 + 5*(-1 + 2*s1)*x1 - 4*a1*x1^2 + x0*(5 - 10*s1 + 8*a1*x1)
(-1 + x0 - 2*s1*x0 + (-1 + 2*s1)*x1)/8
1
```

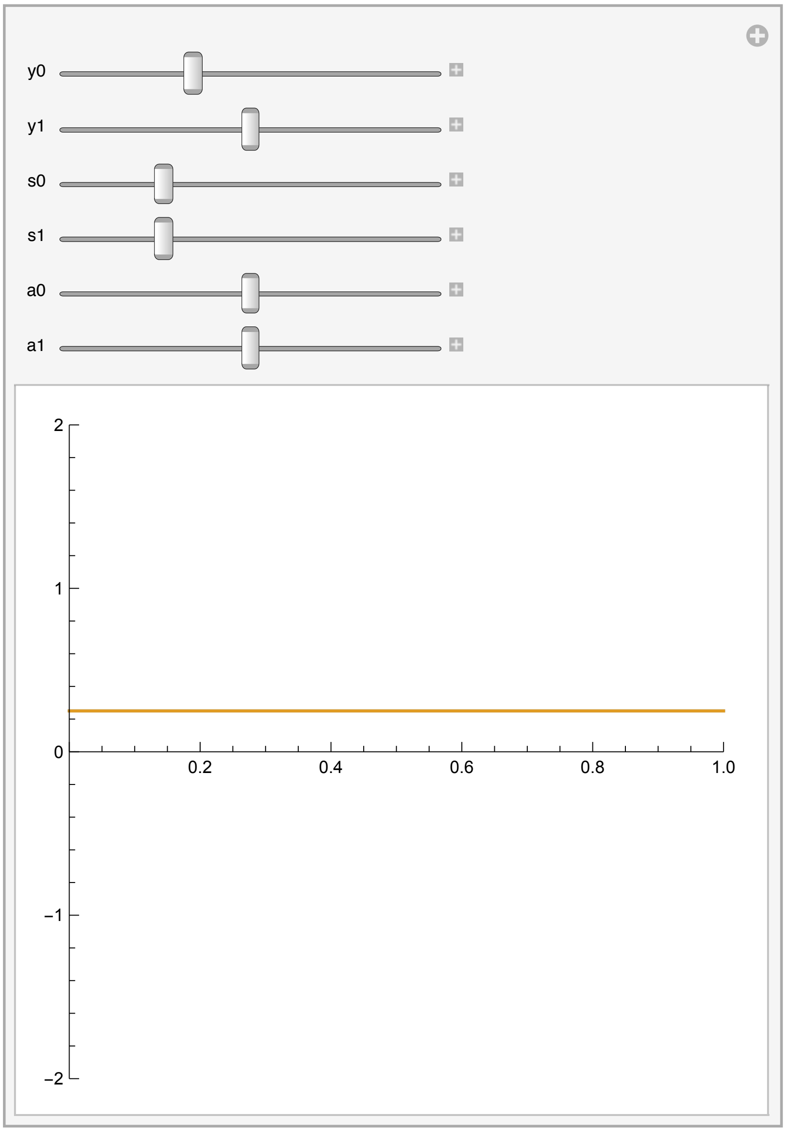
In[39]:=
```
ϕ[x_, x0_, x1_, y0_, y1_, s0_, s1_, a0_, a1_] = g;
```

In[40]:=
```
ψ[x0_, x1_, y0_, y1_, s0_, s1_, a0_, a1_] = Min[bernsteincoefs];
```

We can check the bound for different values of the parameters.

In[41]:=
```
Manipulate[Plot[{ϕ[x, 0, 1, y0, y1, s0, s1, a0, a1], ψ[0, 1, y0, y1, s0, s1, a0, a1]},
  {x, 0, 1}, PlotRange → {{0, 1}, {-2, 2}}, AspectRatio → 1],
 {{y0, 2}, 1, 4}, {{y1, 2.5}, 1, 4}, {{s0, 1/2}, 0, 2},
 {{s1, 1/2}, 0, 2}, {{a0, 0}, -10, 10}, {{a1, 0}, -10, 10}]
```

Out[41]=



In[42]:=
```
Map[InputForm[Simplify[#]] &, CoefficientList[g, x]] // TableForm
```

Out[42]//TableForm=
```
a0 + s0 - s0^2
a0 * (-9 + (-1 + 2 * s0) * x0 + x1 - 2 * s0 * x1) + (3 * (a1 * x0^2 + 12 * s0 * (x0 - x1) + 8 *
- (a0^2 * (x0 - x1)^2) - (9 * a0 * (-4 + (-1 + 2 * s0) * x0 + x1 - 2 * s0 * x1)) / 2 + (3 * (a
9 * a0^2 * (x0 - x1)^2 - a0 * (10 + 3 * a1 * x0^2 - 6 * (1 + 4 * s0 + 4 * s1) * x1 + 3 * a1 * x1
-15 * s0 - 294 * s0^2 - 15 * s1 - 402 * s0 * s1 - 144 * s1^2 + (5 * a0 * (x0 - x1)) / 2 + 221
1152 * s0^2 + 1776 * s0 * s1 + 672 * s1^2 + 240 * a1 * s0 * (x0 - x1) + 180 * a1 * s1 * (x0 - x
-1564 * s0^2 - 2692 * s0 * s1 - 1144 * s1^2 + 609 * a0 * s0 * (x0 - x1) + 531 * a0 * s1 * (x0
10 * (96 * s0^2 + 180 * s0 * s1 + 84 * s1^2 + 28 * a1 * s0 * (x0 - x1) + 26 * a1 * s1 * (x0 - x1
(-25 * (6 * s0 * x0 + 6 * s1 * x0 - a0 * x0^2 + a1 * x0^2 - 6 * s0 * x1 - 6 * s1 * x1 + 2 * a0 * x(
```

Optimization problem to satisfy the constraint

In[43]:=
```
A = Simplify[
  Integrate[((f[x] /. {y0 → y0new, y1 → y1new, s0 → s0new, s1 → s1new, a0 → a0new,
        a1 → a1new}) - f[x])^2, {x, x0, x1}]]
```

Out[43]=

$$
\begin{aligned}
&-\frac{1}{27\,720}\\
&(x0 - x1)\,\big(416\, s1^2\, x0^2 - 832\, s1\, s1new\, x0^2 + 416\, s1new^2\, x0^2 + 52\, a0\, s1\, x0^3 - 52\, a0new\, s1\, x0^3 +\\
&\quad 69\, a1\, s1\, x0^3 - 69\, a1new\, s1\, x0^3 - 52\, a0\, s1new\, x0^3 + 52\, a0new\, s1new\, x0^3 - 69\, a1\, s1new\, x0^3 +\\
&\quad 69\, a1new\, s1new\, x0^3 + 3\, a0^2\, x0^4 - 6\, a0\, a0new\, x0^4 + 3\, a0new^2\, x0^4 + 5\, a0\, a1\, x0^4 -\\
&\quad 5\, a0new\, a1\, x0^4 + 3\, a1^2\, x0^4 - 5\, a0\, a1new\, x0^4 + 5\, a0new\, a1new\, x0^4 - 6\, a1\, a1new\, x0^4 +\\
&\quad 3\, a1new^2\, x0^4 + 416\, s0^2\, (x0 - x1)^2 + 416\, s0new^2\, (x0 - x1)^2 - 832\, s1^2\, x0\, x1 +\\
&\quad 1664\, s1\, s1new\, x0\, x1 - 832\, s1new^2\, x0\, x1 - 156\, a0\, s1\, x0^2\, x1 + 156\, a0new\, s1\, x0^2\, x1 -\\
&\quad 207\, a1\, s1\, x0^2\, x1 + 207\, a1new\, s1\, x0^2\, x1 + 156\, a0\, s1new\, x0^2\, x1 - 156\, a0new\, s1new\, x0^2\, x1 +\\
&\quad 207\, a1\, s1new\, x0^2\, x1 - 207\, a1new\, s1new\, x0^2\, x1 - 12\, a0^2\, x0^3\, x1 + 24\, a0\, a0new\, x0^3\, x1 -\\
&\quad 12\, a0new^2\, x0^3\, x1 - 20\, a0\, a1\, x0^3\, x1 + 20\, a0new\, a1\, x0^3\, x1 - 12\, a1^2\, x0^3\, x1 +\\
&\quad 20\, a0\, a1new\, x0^3\, x1 - 20\, a0new\, a1new\, x0^3\, x1 + 24\, a1\, a1new\, x0^3\, x1 - 12\, a1new^2\, x0^3\, x1 +\\
&\quad 416\, s1^2\, x1^2 - 832\, s1\, s1new\, x1^2 + 416\, s1new^2\, x1^2 + 156\, a0\, s1\, x0\, x1^2 - 156\, a0new\, s1\, x0\, x1^2 +\\
&\quad 207\, a1\, s1\, x0\, x1^2 - 207\, a1new\, s1\, x0\, x1^2 - 156\, a0\, s1new\, x0\, x1^2 + 156\, a0new\, s1new\, x0\, x1^2 -\\
&\quad 207\, a1\, s1new\, x0\, x1^2 + 207\, a1new\, s1new\, x0\, x1^2 + 18\, a0^2\, x0^2\, x1^2 - 36\, a0\, a0new\, x0^2\, x1^2 +\\
&\quad 18\, a0new^2\, x0^2\, x1^2 + 30\, a0\, a1\, x0^2\, x1^2 - 30\, a0new\, a1\, x0^2\, x1^2 + 18\, a1^2\, x0^2\, x1^2 -\\
&\quad 30\, a0\, a1new\, x0^2\, x1^2 + 30\, a0new\, a1new\, x0^2\, x1^2 - 36\, a1\, a1new\, x0^2\, x1^2 + 18\, a1new^2\, x0^2\, x1^2 -\\
&\quad 52\, a0\, s1\, x1^3 + 52\, a0new\, s1\, x1^3 - 69\, a1\, s1\, x1^3 + 69\, a1new\, s1\, x1^3 + 52\, a0\, s1new\, x1^3 -\\
&\quad 52\, a0new\, s1new\, x1^3 + 69\, a1\, s1new\, x1^3 - 69\, a1new\, s1new\, x1^3 - 12\, a0^2\, x0\, x1^3 +\\
&\quad 24\, a0\, a0new\, x0\, x1^3 - 12\, a0new^2\, x0\, x1^3 - 20\, a0\, a1\, x0\, x1^3 + 20\, a0new\, a1\, x0\, x1^3 -\\
&\quad 12\, a1^2\, x0\, x1^3 + 20\, a0\, a1new\, x0\, x1^3 - 20\, a0new\, a1new\, x0\, x1^3 + 24\, a1\, a1new\, x0\, x1^3 -\\
&\quad 12\, a1new^2\, x0\, x1^3 + 3\, a0^2\, x1^4 - 6\, a0\, a0new\, x1^4 + 3\, a0new^2\, x1^4 + 5\, a0\, a1\, x1^4 -\\
&\quad 5\, a0new\, a1\, x1^4 + 3\, a1^2\, x1^4 - 5\, a0\, a1new\, x1^4 + 5\, a0new\, a1new\, x1^4 - 6\, a1\, a1new\, x1^4 +\\
&\quad 3\, a1new^2\, x1^4 + 1812\, s1\, x0\, y0 - 1812\, s1new\, x0\, y0 + 281\, a0\, x0^2\, y0 - 281\, a0new\, x0^2\, y0 +\\
&\quad 181\, a1\, x0^2\, y0 - 181\, a1new\, x0^2\, y0 - 1812\, s1\, x1\, y0 + 1812\, s1new\, x1\, y0 - 562\, a0\, x0\, x1\, y0 +\\
&\quad 562\, a0new\, x0\, x1\, y0 - 362\, a1\, x0\, x1\, y0 + 362\, a1new\, x0\, x1\, y0 + 281\, a0\, x1^2\, y0 -\\
&\quad 281\, a0new\, x1^2\, y0 + 181\, a1\, x1^2\, y0 - 181\, a1new\, x1^2\, y0 + 10\,860\, y0^2 - 1812\, s1\, x0\, y0new +\\
&\quad 1812\, s1new\, x0\, y0new - 281\, a0\, x0^2\, y0new + 281\, a0new\, x0^2\, y0new - 181\, a1\, x0^2\, y0new +\\
&\quad 181\, a1new\, x0^2\, y0new + 1812\, s1\, x1\, y0new - 1812\, s1new\, x1\, y0new + 562\, a0\, x0\, x1\, y0new -\\
&\quad 562\, a0new\, x0\, x1\, y0new + 362\, a1\, x0\, x1\, y0new - 362\, a1new\, x0\, x1\, y0new -\\
&\quad 281\, a0\, x1^2\, y0new + 281\, a0new\, x1^2\, y0new - 181\, a1\, x1^2\, y0new + 181\, a1new\, x1^2\, y0new -\\
&\quad 21\,720\, y0\, y0new + 10\,860\, y0new^2 + 3732\, s1\, x0\, y1 - 3732\, s1new\, x0\, y1 + 181\, a0\, x0^2\, y1 -\\
&\quad 181\, a0new\, x0^2\, y1 + 281\, a1\, x0^2\, y1 - 281\, a1new\, x0^2\, y1 - 3732\, s1\, x1\, y1 + 3732\, s1new\, x1\, y1 -\\
&\quad 362\, a0\, x0\, x1\, y1 + 362\, a0new\, x0\, x1\, y1 - 562\, a1\, x0\, x1\, y1 + 562\, a1new\, x0\, x1\, y1 +\\
&\quad 181\, a0\, x1^2\, y1 - 181\, a0new\, x1^2\, y1 + 281\, a1\, x1^2\, y1 - 281\, a1new\, x1^2\, y1 + 6000\, y0\, y1 -\\
&\quad 6000\, y0new\, y1 + 10\,860\, y1^2 + s0new\, (x0 - x1)\, \big(69\, a0\, x0^2 - 69\, a0new\, x0^2 + 52\, a1\, x0^2 -\\
&\qquad 52\, a1new\, x0^2 + 532\, s1\, (x0 - x1) - 532\, s1new\, (x0 - x1) - 138\, a0\, x0\, x1 +\\
&\qquad 138\, a0new\, x0\, x1 - 104\, a1\, x0\, x1 + 104\, a1new\, x0\, x1 + 69\, a0\, x1^2 - 69\, a0new\, x1^2 +\\
&\qquad 52\, a1\, x1^2 - 52\, a1new\, x1^2 + 3732\, y0 - 3732\, y0new + 1812\, y1 - 1812\, y1new\big) -\\
&\quad s0\, (x0 - x1)\, \big(-532\, s1new\, x0 + 69\, a0\, x0^2 - 69\, a0new\, x0^2 + 52\, a1\, x0^2 - 52\, a1new\, x0^2 +\\
&\qquad 832\, s0new\, (x0 - x1) + 532\, s1\, (x0 - x1) + 532\, s1new\, x1 - 138\, a0\, x0\, x1 +
\end{aligned}
$$

$$138\ a0new\ x0\ x1 - 104\ a1\ x0\ x1 + 104\ a1new\ x0\ x1 + 69\ a0\ x1^2 - 69\ a0new\ x1^2 +$$
$$52\ a1\ x1^2 - 52\ a1new\ x1^2 + 3732\ y0 - 3732\ y0new + 1812\ y1 - 1812\ y1new\Big) -$$
$$3732\ s1\ x0\ y1new + 3732\ s1new\ x0\ y1new - 181\ a0\ x0^2\ y1new + 181\ a0new\ x0^2\ y1new -$$
$$281\ a1\ x0^2\ y1new + 281\ a1new\ x0^2\ y1new + 3732\ s1\ x1\ y1new -$$
$$3732\ s1new\ x1\ y1new + 362\ a0\ x0\ x1\ y1new - 362\ a0new\ x0\ x1\ y1new +$$
$$562\ a1\ x0\ x1\ y1new - 562\ a1new\ x0\ x1\ y1new - 181\ a0\ x1^2\ y1new +$$
$$181\ a0new\ x1^2\ y1new - 281\ a1\ x1^2\ y1new + 281\ a1new\ x1^2\ y1new -$$
$$6000\ y0\ y1new + 6000\ y0new\ y1new - 21\,720\ y1\ y1new + 10\,860\ y1new^2\Big)$$