CS 530 Midterm 1 Cheatsheet

Complex Analysis

$z = a + ib = re^{i\theta} = r(cos(\theta) + isin(\theta))$
$\overline{z} = a - bi = re^{-i\theta}$
$|z| = \sqrt{a^2 + b^2}$
Inner product

$$\langle z, w \rangle = \sum_{n=0}^{N-1} z(n)\overline{w(n)}$$

$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$
Cauchy-Shwarz inequality $\overline{\langle u, v \rangle} \leq \|u\|\|v\|$

Fourier Transform

Fourier Basis for $\mathbb{C}^N$ is $\{F_0, F_1, \cdots, F_{N-1}\}$

$$F_m(n) = \frac{1}{N} e^{2\pi i \frac{mn}{N}}$$

Fourier basis is orthogonal, not orthonormal. Can be normalized the norm is $\|F_m\| = \frac{1}{\sqrt{N}}$

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi i \frac{mn}{N}}$$

$$z(n) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{z}(m) e^{2\pi i \frac{mn}{N}}$$

2D

$$\hat{z}(m_1, m_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} z(n_1, n_2) e^{-2\pi i \frac{m_1 n_1}{N_1}} e^{-2\pi i \frac{m_2 n_2}{N_2}}$$

$$z(n_1, n_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} \hat{z}(m_1, m_2) e^{2\pi i \frac{m_1 n_1}{N_1}} e^{2\pi i \frac{m_2 n_2}{N_2}}$$

Wavelets

Wavelet tools

$$(w * z)(m) = \sum_{n=0}^{N-1} w(n) z(m-n)$$

$(w * z)^\wedge(m) = \hat{w}(m)\hat{z}(m)$
$R_k z$ is called the translation of z by k.
$(R_k z)(n) = z(n-k)$
$(R_k z)^\wedge(m) = \hat{z}(m) e^{-2\pi i \frac{mk}{N}}$
$\langle z, R_k w \rangle = (z * \tilde{w})(k)$
$\langle z, R_k \tilde{w} \rangle = (z * w)(k)$
$\tilde{z}(n) = \overline{z(-n)}$
$(\tilde{z})^\wedge(m) = \overline{\hat{z}(m)}$
$z^*(n) = (-1)^n z(n)$
For the below, $N = 2M$
$(z^*)^\wedge(n) = \hat{z}(n+M)$

1st Stage Wavelet Basis

$A(n)$ is the system matrix of u and v

$$A(n) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{u}(n) & \hat{v}(n) \\ \hat{u}(n+M) & \hat{v}(n+M) \end{bmatrix}$$

$$B = \{R_{2k} v\}_{k=0}^{M-1} \cup \{R_{2k} u\}_{k=0}^{M-1}$$
$$= \{R_0 v, R_2 v, \ldots, R_{N-2} v, R_0 u, R_2 u, \ldots, R_{N-2} u\}$$

B (u and v) form a first stage wavelet basis if and only if the following properties are true:

1. $|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2$
2. $|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2$
3. $\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)}$

$\forall n = 0, 1, \cdots, M-1$

1st Stage Shannon Basis

$$\hat{u} = \begin{cases} \sqrt{2} & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \ldots, N-1 \\ 0 & \text{if } n = \frac{N}{4}, \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

$$\hat{v} = \begin{cases} 0 & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \ldots, N-1 \\ \sqrt{2} & \text{if } n = \frac{N}{4}, \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

Real Wavelet Basis

1st Stage Real Shannon Basis

$$\hat{u} = \begin{cases} \sqrt{2} & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4} + 1, \ldots, N-1 \\ i & \text{if } n = \frac{N}{4} \\ -i & \text{if } n = \frac{3N}{4} \\ 0 & \text{if } n = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

$$\hat{v} = \begin{cases} 0 & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4} + 1, \ldots, N-1 \\ 1 & \text{if } n = \frac{N}{4} \text{ or } n = \frac{3N}{4} \\ \sqrt{2} & \text{if } n = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

Misc

$$\sum_{k=a}^{b} r^k = \frac{r^a - r^{b+1}}{1-r}$$

Unitary matrix $\iff \overline{U^T} U = I$
orthogonal $\iff \langle u, v \rangle = 0$
Parseval's formula

$$\langle z, w \rangle = \frac{1}{N} \sum_{m=0}^{N-1} \hat{z}(m)\overline{\hat{w}(m)} = \frac{1}{N} \langle \hat{z}, \hat{w} \rangle$$

Plancherel's formula

$$\|z\|^2 = \frac{1}{N} \sum_{m=0}^{N-1} |\hat{z}(m)|^2 = \frac{1}{N} \|\hat{z}\|^2$$