CS 530 Midterm 2 Cheatsheet

Wavelets

Wavelet tools

$$(w * z)(m) = \sum_{n=0}^{N-1} w(m-n)z(n)$$

The convolution matrix is

$$C = \{R_0 w, R_1 w, \ldots, R_{N-1} w\}$$

For example, with $N = 4$

$$C = \{R_0 w, R_1 w, R_2 w, R_3 w\}$$

The convolution operation is simply

$$(w * z)(m) = C \cdot z$$

$(w * z)^\wedge(m) = \hat{w}(m)\hat{z}(m)$
$R_k z$ is called the translation of z by k.
$(R_k z)(n) = z(n-k)$
$(R_k z)^\wedge(m) = \hat{z}(m)e^{-2\pi i \frac{mk}{N}}$
$\langle z, R_k w\rangle = (z * \tilde{w})(k)$
$\langle z, R_k \tilde{w}\rangle = (z * w)(k)$
$\tilde{z}(n) = \overline{z(-n)}$
$(\tilde{z})^\wedge(m) = \overline{\tilde{z}(m)}$
$z^*(n) = (-1)^n z(n)$
For the below, $N = 2M$
$(z^*)^\wedge(n) = \hat{z}(n+M)$
Unitary matrix $\iff \overline{U^T}U = I$
Parseval's formula

$$\langle z, w\rangle = \frac{1}{N}\sum_{m=0}^{N-1} \hat{z}(m)\overline{\hat{w}(m)} = \frac{1}{N}\langle \hat{z}, \hat{w}\rangle$$

Plancherel's formula

$$\|z\|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\hat{z}(m)|^2 = \frac{1}{N}\|\hat{z}\|^2$$

Downsampling

$$z = \begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ z(3) \\ z(4) \\ z(5) \\ z(6) \\ z(7) \end{bmatrix} \xrightarrow{\downarrow 2} U_z = \begin{bmatrix} z(0) \\ z(2) \\ z(4) \\ z(6) \end{bmatrix}$$

Upsampling

$$z = \begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ z(3) \end{bmatrix} \xrightarrow{\uparrow 2} U_z = \begin{bmatrix} z(0) \\ 0 \\ z(1) \\ 0 \\ z(2) \\ 0 \\ z(3) \\ 0 \end{bmatrix}$$

1st Stage Wavelet Basis

$A(n)$ is the system matrix of u and v

$$A(n) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}(n) & \hat{v}(n) \\ \hat{u}(n+M) & \hat{v}(n+M) \end{bmatrix}$$

$$B = \{R_{2k} v\}_{k=0}^{M-1} \cup \{R_{2k} u\}_{k=0}^{M-1}$$
$$= \{R_0 v, R_2 v, \ldots, R_{N-2} v, R_0 u, R_2 u, \ldots, R_{N-2} u\}$$

B (u and v) form a first stage wavelet basis if and only if the following properties are true:

1. $|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2$
2. $|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2$
3. $\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)} = 0$

$\forall n = 0, 1, \cdots, M-1$

1st Stage Shannon Basis

$$\hat{u} = \begin{cases} \sqrt{2} & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \ldots, N-1 \\ 0 & \text{if } n = \frac{N}{4}, \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

$$\hat{v} = \begin{cases} 0 & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \ldots, N-1 \\ \sqrt{2} & \text{if } n = \frac{N}{4}, \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

Real Wavelet Basis

1st Stage Real Shannon Basis

$$\hat{u} = \begin{cases} \sqrt{2} & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4} + 1, \ldots, N-1 \\ i & \text{if } n = \frac{N}{4} \\ -i & \text{if } n = \frac{3N}{4} \\ 0 & \text{if } n = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

$$\hat{v} = \begin{cases} 0 & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4} + 1, \ldots, N-1 \\ 1 & \text{if } n = \frac{N}{4} \text{ or } n = \frac{3N}{4} \\ \sqrt{2} & \text{if } n = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

Wavelet Transform

Wavelet transform

$$z = \begin{cases} *\tilde{u} \to z * \tilde{u} \xrightarrow{\downarrow 2} \\ *\tilde{v} \to z * \tilde{v} \xrightarrow{\downarrow 2} \end{cases} = \begin{bmatrix} D(z * \tilde{u}) \\ D(z * \tilde{v}) \end{bmatrix}$$

Inverse wavelet transform

$$\begin{bmatrix} D(z * \tilde{u}) \\ D(z * \tilde{v}) \end{bmatrix} = \begin{matrix} \xrightarrow{\uparrow 2} \frac{(z*\tilde{u}) + (z*\tilde{u})^*}{2} * u \\ \xrightarrow{\uparrow 2} \frac{(z*\tilde{v}) + (z*\tilde{v})^*}{2} * v \end{matrix} \quad += z$$

Iterative Wavelet Construction

Misc