CS 530 Midterm 2 Cheatsheet

Wavelets

Wavelet tools

$$(w * z)(m) = \sum_{n=0}^{N-1} w(m-n)z(n)$$

The convolution matrix is

$$C = \{R_0 w, R_1 w, \ldots, R_{N-1} w\}$$

For example, with $N = 4$

$$C = \{R_0 w, R_1 w, R_2 w, R_3 w\}$$

The convolution operation is simply

$$(w * z)(m) = C \cdot z$$

$(w * z)^\wedge(m) = \hat{w}(m)\hat{z}(m)$
$R_k z$ is called the translation of z by k.
$(R_k z)(n) = z(n-k)$
$(R_k z)^\wedge(m) = \hat{z}(m)e^{-2\pi i \frac{mk}{N}}$
$\langle z, R_k w\rangle = (z * \tilde{w})(k)$
$\langle z, R_k \tilde{w}\rangle = (z * w)(k)$
$\tilde{z}(n) = \overline{z(-n)}$
$(\tilde{z})^\wedge(m) = \overline{\tilde{z}(m)}$
$(\tilde{u} * \tilde{v})(n) = \widetilde{(u * v)}(n)$
$z^*(n) = (-1)^n z(n)$
For the below, $N = 2M$
$(z^*)^\wedge(n) = \hat{z}(n+M)$
Unitary matrix $\iff \overline{U^T}U = I$
Parseval's formula

$$\langle z, w\rangle = \frac{1}{N}\sum_{m=0}^{N-1} \hat{z}(m)\overline{\hat{w}(m)} = \frac{1}{N}\langle \hat{z}, \hat{w}\rangle$$

Plancherel's formula

$$\|z\|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\hat{z}(m)|^2 = \frac{1}{N}\|\hat{z}\|^2$$

Up/Down Sampling

Downsampling

$$z = \begin{bmatrix} z(0)\\ z(1)\\ z(2)\\ z(3)\\ z(4)\\ z(5)\\ z(6)\\ z(7) \end{bmatrix} \xrightarrow{\downarrow 2} D_z = \begin{bmatrix} z(0)\\ z(2)\\ z(4)\\ z(6) \end{bmatrix}$$

Upsampling

$$z = \begin{bmatrix} z(0)\\ z(1)\\ z(2)\\ z(3) \end{bmatrix} \xrightarrow{\uparrow 2} U_z = \begin{bmatrix} z(0)\\ 0\\ z(1)\\ 0\\ z(2)\\ 0\\ z(3)\\ 0 \end{bmatrix}$$

Properties

1. $D(z) * w = D(z * U(w))$
2. $U(x) * U(y) = U(x * y)$

Suppose N is divisible by $2^l, x, y, w \in l^2(Z_{N/2^l})$ and $z \in l^2(Z_N)$ Then

1. $D^l(z) * w = D^l(z * U^l(w))$
2. $U^l(x) * U^l(y) = U^l(x * y)$

1st Stage Wavelet Basis

$A(n)$ is the system matrix of u and v

$$A(n) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}(n) & \hat{v}(n)\\ \hat{u}(n+M) & \hat{v}(n+M) \end{bmatrix}$$

$$B = \{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$$
$$= \{R_0 v, R_2 v, \ldots, R_{N-2}v, R_0 u, R_2 u, \ldots, R_{N-2}u\}$$

B (u and v) form a first stage wavelet basis if and only if the following properties are true:

1. $|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2$
2. $|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2$
3. $\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)} = 0$

$\forall n = 0, 1, \cdots, M-1$

1st Stage Shannon Basis

$$\hat{u} = \begin{cases} \sqrt{2} & \text{if } n = 0, 1, \ldots, \frac{N}{4}-1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4}+1, \ldots, N-1\\ 0 & \text{if } n = \frac{N}{4}, \frac{N}{4}+1, \ldots, \frac{3N}{4}-1 \end{cases}$$

$$\hat{v} = \begin{cases} 0 & \text{if } n = 0, 1, \ldots, \frac{N}{4}-1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4}+1, \ldots, N-1\\ \sqrt{2} & \text{if } n = \frac{N}{4}, \frac{N}{4}+1, \ldots, \frac{3N}{4}-1 \end{cases}$$

Real Wavelet Basis

1st Stage Real Shannon Basis

$$\hat{u} = \begin{cases} \sqrt{2} & \text{if } n = 0, 1, \ldots, \frac{N}{4}-1 \text{ or } n = \frac{3N}{4}+1, \ldots, N-1\\ i & \text{if } n = \frac{N}{4}\\ -i & \text{if } n = \frac{3N}{4}\\ 0 & \text{if } n = \frac{N}{4}+1, \ldots, \frac{3N}{4}-1 \end{cases}$$

$$\hat{v} = \begin{cases} 0 & \text{if } n = 0, 1, \ldots, \frac{N}{4}-1 \text{ or } n = \frac{3N}{4}+1, \ldots, N-1\\ 1 & \text{if } n = \frac{N}{4} \text{ or } n = \frac{3N}{4}\\ \sqrt{2} & \text{if } n = \frac{N}{4}+1, \ldots, \frac{3N}{4}-1 \end{cases}$$

Wavelet Transform

Wavelet transform

$$z = \begin{cases} *\tilde{u} \to z * \tilde{u} \xrightarrow{\downarrow 2}\\ *\tilde{v} \to z * \tilde{v} \xrightarrow{\downarrow 2} \end{cases} = \begin{bmatrix} D(z * \tilde{u})\\ D(z * \tilde{v}) \end{bmatrix}$$

Inverse wavelet transform

$$\begin{bmatrix} D(z * \tilde{u})\\ D(z * \tilde{v}) \end{bmatrix} = \begin{matrix} \xrightarrow{\uparrow 2} \frac{(z*\tilde{u})+(z*\tilde{u})^*}{2} * u\\ \xrightarrow{\uparrow 2} \frac{(z*\tilde{v})+(z*\tilde{v})^*}{2} * v \end{matrix} + = z$$

Iterative Wavelet Construction

$$z = \begin{cases} *\tilde{v_1} \to \downarrow 2 \xrightarrow{x_1}\\ *\tilde{u_1} \to \downarrow 2 \xrightarrow{y_1} \begin{cases} *\tilde{v_2} \to \downarrow 2 \xrightarrow{x_2}\\ *\tilde{u_2} \to \downarrow 2 \xrightarrow{y_2} \end{cases} \end{cases}$$

$x_1 = D(z * \tilde{v_1})$
$y_1 = D(z * \tilde{u_1})$
$x_2 = D(y_1 * \tilde{v_2}) = D(D(z * \tilde{u_1}) * \tilde{v_2}) = D^2(z * \widetilde{(u_1 * Uv_2)})$
$y_2 = D(y_1 * \tilde{u_2}) = D(D(z * \tilde{u_1}) * \tilde{u_2}) = D^2(z * \widetilde{(u_1 * Uu_2)})$
Let $f_1 = v_1, g_1 = u_1$
$f_2 = g_1 * Uv_1, g_2 = g_1 * Uu_2$
Then $x_1 = D(z * \tilde{f_1})$
$y_1 = D(z * \tilde{g_1})$
$x_2 = D^2(z * \tilde{f_2})$
$y_2 = D^2(z * \tilde{g_2})$

$$f_l = g_{l-1} * U^{l-1}(v_l)$$

$$f_\ell = u_1 * U(u_2) * U^2(u_3) * \cdots U^{\ell-2}(u_{\ell-1}) * U^{\ell-1}(v_\ell)$$

$$g_l = g_{l-1} * U^{l-1}(u_l)$$

$$g_\ell = u_1 * U(u_2) * U^2(u_3) * \cdots U^{\ell-2}(u_{\ell-1}) * U^{\ell-1}(u_\ell)$$

$$x_\ell = D^\ell(z * \tilde{f_\ell})$$

$$y_\ell = D^\ell(z * \tilde{g_\ell})$$

The ℓth system matrix is

$$A_\ell(n) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u_\ell}(n) & \hat{v_\ell}(n)\\ \hat{u_\ell}(n+\frac{N}{2^\ell}) & \hat{v_\ell}(n+\frac{N}{2^\ell}) \end{bmatrix}$$

$$\psi_{-j,k} = R_{2^j k}f_j \text{ and } \varphi_{-j,k} = R_{2^j k}g_j$$

for $j = 1, 2, \ldots, p$
Then

$$\{R_{2^1 k}f_1\}_{k=0}^{(N/2)-1} \cup \{R_{2^2 k}f_2\}_{k=0}^{(N/4)-1} \cup \ldots \cup \{R_{2^p k}f_p\}_{k=0}^{(N/2^p)-1} \cup \{R_{2^p k}g_p\}_{k}^{(}$$

$$\{\psi_{-1,k}\}_{k=0}^{(N/2)-1} \cup \{\psi_{-2,k}\}_{k=0}^{(N/4)-1} \cup \ldots \cup \{\psi_{-p,k}\}_{k=0}^{(N/2^p)-1} \cup \{\varphi_{-p,k}\}_{k=0}^{(N/2^p)}$$

More succinctly:

$$(\bigcup_{\ell=1}^{j} \{\psi_{-\ell,k}\}_{0 \le k \le (N/2^\ell)-1}) \cup \{\phi_{-j,k}\}_{0 \le k \le (N/2^\ell)-1}$$

is a p-th stage wavelet basis.

Haar construction

$$u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$u_\ell(0) = \frac{1}{\sqrt{2}}, u_\ell(1) = \frac{1}{\sqrt{2}}, \text{ and } u_\ell(n) = 0 \text{ for } 2 \le n \le (\frac{N}{2^{\ell-1}}) - 1$$

$$v_\ell(0) = \frac{1}{\sqrt{2}}, v_\ell(1) = \frac{1}{\sqrt{2}}, \text{ and } v_\ell(n) = 0 \text{ for } 2 \le n \le (\frac{N}{2^{\ell-1}}) - 1$$

$$f_\ell(n) = \begin{cases} 2^{-\ell/2}, & n = 0, 1, \ldots, 2^{\ell-1} - 1 \\ -2^{-\ell/2}, & n = 2^{\ell-1}, 2^{\ell-1} + 1, \ldots, 2^\ell - 1 \\ 0, & n = 2^\ell, 2^\ell + 1, \ldots, N - 1 \end{cases}$$

$$g_\ell(n) = \begin{cases} 2^{-\ell/2}, & n = 0, 1, \ldots, 2^{\ell-1} - 1 \\ 0, & n = 2^\ell, 2^\ell + 1, \ldots, N - 1 \end{cases}$$

$$\psi_{-\ell,k}(n) = \\ \begin{cases} 2^{-\ell/2}, & n = 2^\ell k, 2^\ell k + 1, \ldots, 2^\ell k + 2^{\ell-1} - 1 \\ -2^{-\ell/2}, & n = 2^\ell k + 2^{\ell-1}, 2^\ell k + 2^{\ell-1} + 1, \ldots, 2^\ell k + 2^\ell - 1 \\ 0, & n = 0, 1, \ldots, 2^\ell k - 1; 2^\ell k + 2^\ell, \ldots, N - 1 \end{cases}$$

$$\varphi_{-\ell,k}(n) = \begin{cases} 2^{-\ell/2}, & n = 2^\ell k, 2^\ell k + 1, \ldots, 2^\ell k + 2^{\ell-1} - 1 \\ 0, & n = 0, 1, \ldots, 2^\ell k - 1; 2^\ell k + 2^\ell, \ldots, N - 1 \end{cases}$$

Shannon construction

$$\hat{v}_\ell(n) = \begin{cases} \sqrt{2}, & \frac{N}{2^{\ell+1}} \le n \le \frac{3N}{2^{\ell+1}} - 1 \\ 0, & 0 \le n \le \frac{N}{2^{\ell+1}} - 1; \frac{3N}{2^{\ell+1}} \le n \le \frac{N}{2^{\ell-1}} - 1 \end{cases}$$

$$\hat{u}_\ell(n) = \begin{cases} \sqrt{2}, & 0 \le n \le \frac{N}{2^{\ell+1}} - 1; \frac{3N}{2^{\ell+1}} \le n \le \frac{N}{2^{\ell-1}} - 1 \\ 0, & \frac{N}{2^{\ell+1}} \le n \le \frac{3N}{2^{\ell+1}} - 1 \end{cases}$$

$$\hat{\psi}_{-\ell,0}(n) = \begin{cases} 2^{\ell/2}, & \frac{N}{2^{\ell+1}} \le n \le \frac{N}{2^\ell} - 1; \\ & N - \frac{N}{2^\ell} \le n \le N - \frac{N}{2^{\ell+1}} - 1 \\ 0, & 0 \le n \le \frac{N}{2^{\ell+1}} - 1; \frac{N}{2^\ell} \le n \le N - \frac{N}{2^{\ell-1}}; \\ & N - \frac{N}{2^{\ell+1}} \le n \le N - 1 \end{cases}$$

$$\hat{\varphi}_{-\ell,0}(n) = \begin{cases} 2^{\ell/2}, & 0 \le n \le \frac{N}{2^{\ell+1}} - 1; N - \frac{N}{2^{\ell+1}} \le n \le N - 1 \\ 0, & \frac{N}{2^{\ell+1}} \le n \le N - \frac{N}{2^{\ell+1}} - 1 \end{cases}$$

Optimization

Gradient

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Hessian

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Convex functions

A function $f : dom(f) \to \mathbb{R}$ is convex if

1. Its domain $dom(f) \in \mathbb{R}^n$ is convex
2. For all $x^{(1)}, x^{(2)} \in dom(f)$, and $0 \le \lambda \le 1$ we have

$$f(\lambda x^{(1)} + (1 - \lambda) x^{(2)}) \le \lambda f(x^{(1)}) + (1 - \lambda) f(x^{(2)})$$