

CS 530 Midterm 1 Cheatsheet

Linear Algebra

Linear combination

$$\sum_{i=1}^n \alpha_i v_i = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Span: the set of all possible linear combinations of the elements of a given matrix.

Linearly Independent

$$\alpha_1 v_1 + \dots + \alpha_n v_n \implies \alpha_1 = \dots = \alpha_n = 0$$

Support of a vector z is the number of non zero entries of z , denoted by $\|\text{supp } z\|$

For example,

$$w = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \text{supp } w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \|\text{supp } w\| = 2$$

Uncertainty principle

$$\|\text{supp } z\|^2 \cdot \|\text{supp } \hat{z}\|^2 \geq N$$

Complex Analysis

$$z = a + ib = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$$

$$\bar{z} = a - bi = re^{-i\theta}$$

$$|z| = \sqrt{a^2 + b^2}$$

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

$$(a_1 + ib_1) \times (a_2 + ib_2) = (a_1 \times a_2 - b_1 \times b_2) + i(a_1 \times b_2 + a_2 \times b_1)$$

$$z^{-1} = \frac{\bar{z}}{a^2 + b^2}$$

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Roots of unity

$$e^{i\theta} \text{ where } \theta = \frac{2k\pi}{n}, k = 0, 1, \dots, n-1$$

Inner product

$$\langle z, w \rangle = \sum_{n=0}^{N-1} z(n) \overline{w(n)}$$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$$

$$\text{Conjugate symmetry } \langle u, v \rangle = \overline{\langle v, u \rangle}$$

$$\text{Cauchy-Schwarz inequality } \overline{\langle u, v \rangle} \leq \|u\| \|v\|$$

$$\langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \iff u = \mathbf{0}$$

$$\text{orthogonal } \iff \langle u, v \rangle = 0$$

Fourier Transform

1D

Fourier Basis for \mathbb{C}^N is $\{F_0, F_1, \dots, F_{N-1}\}$

$$F_m(n) = \frac{1}{N} e^{2\pi i \frac{mn}{N}}$$

Fourier basis is orthogonal, not orthonormal. Can be normalized the norm is $\|F_m\| = \frac{1}{\sqrt{N}}$

Normalized Fourier Basis

$$\sqrt{N} \cdot F_m = \frac{F_m}{\|F_m\|} = \frac{F_m}{\frac{1}{\sqrt{N}}}$$

The Fourier Basis for \mathbb{C}^4 is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$
$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi i \frac{mn}{N}}$$
$$z(n) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{z}(m) e^{2\pi i \frac{mn}{N}}$$

2D

$$\hat{z}(m_1, m_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} z(n_1, n_2) e^{-2\pi i \frac{m_1 n_1}{N_1}} e^{-2\pi i \frac{m_2 n_2}{N_2}}$$

$$z(n_1, n_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} \hat{z}(m_1, m_2) e^{2\pi i \frac{m_1 n_1}{N_1}} e^{2\pi i \frac{m_2 n_2}{N_2}}$$

Wavelets

Wavelet tools

$$(w * z)(m) = \sum_{n=0}^{N-1} w(m-n) z(n)$$

The convolution matrix is

$$C = \{R_0 w, R_1 w, \dots, R_{N-1} w\}$$

For example, with $N = 4$

$$C = \{R_0 w, R_1 w, R_2 w, R_3 w\}$$

The convolution operation is simply

$$(w * z)(m) = C \cdot z$$

$$(w * z)^\wedge(m) = \hat{w}(m) \hat{z}(m)$$

$R_k z$ is called the translation of z by k .

$$(R_k z)(n) = z(n-k)$$

$$(R_k z)^\wedge(m) = \hat{z}(m) e^{-2\pi i \frac{mk}{N}}$$

$$\langle z, R_k w \rangle = \langle z * \tilde{w} \rangle(k)$$

$$\langle z, R_k \tilde{w} \rangle = \langle z * w \rangle(k)$$

$$\hat{z}(n) = z(-n)$$

$$(\hat{z})^\wedge(m) = \overline{\hat{z}(m)}$$

$$z^*(n) = (-1)^n z(n)$$

For the below, $N = 2M$

$$(z^*)^\wedge(n) = \hat{z}(n+M)$$

$$\text{Unitary matrix } \iff \overline{U^T} U = I$$

Parseval's formula

$$\langle z, w \rangle = \frac{1}{N} \sum_{m=0}^{N-1} \hat{z}(m) \overline{\hat{w}(m)} = \frac{1}{N} \langle \hat{z}, \hat{w} \rangle$$

Plancherel's formula

$$\|z\|^2 = \frac{1}{N} \sum_{m=0}^{N-1} |\hat{z}(m)|^2 = \frac{1}{N} \|\hat{z}\|^2$$

Downsampling

$$z = \begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ z(3) \\ z(4) \\ z(5) \\ z(6) \\ z(7) \end{bmatrix} \xrightarrow{\downarrow 2} D_z = \begin{bmatrix} z(0) \\ z(2) \\ z(4) \\ z(6) \end{bmatrix}$$

Upsampling

$$z = \begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ z(3) \end{bmatrix} \xrightarrow{\uparrow 2} U_z = \begin{bmatrix} z(0) \\ 0 \\ z(1) \\ 0 \\ z(2) \\ 0 \\ z(3) \\ 0 \end{bmatrix}$$

1st Stage Wavelet Basis

$A(n)$ is the system matrix of u and v

$$A(n) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{u}(n) & \hat{v}(n) \\ \hat{u}(n+M) & \hat{v}(n+M) \end{bmatrix}$$

$$B = \{R_{2k} v\}_{k=0}^{M-1} \cup \{R_{2k} u\}_{k=0}^{M-1} = \{R_0 v, R_2 v, \dots, R_{N-2} v, R_0 u, R_2 u, \dots, R_{N-2} u\}$$

B (u and v) form a first stage wavelet basis if and only if the following properties are true:

- $|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2$
- $|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2$
- $\hat{u}(n) \hat{v}(n) + \hat{u}(n+M) \hat{v}(n+M) = 0$

$$\forall n = 0, 1, \dots, M-1$$

1st Stage Shannon Basis

$$\hat{u} = \begin{cases} \sqrt{2} & \text{if } n = 0, 1, \dots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \dots, N-1 \\ 0 & \text{if } n = \frac{N}{4}, \frac{N}{4} + 1, \dots, \frac{3N}{4} - 1 \end{cases}$$

$$\hat{v} = \begin{cases} 0 & \text{if } n = 0, 1, \dots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \dots, N-1 \\ \sqrt{2} & \text{if } n = \frac{N}{4}, \frac{N}{4} + 1, \dots, \frac{3N}{4} - 1 \end{cases}$$

Real Wavelet Basis

1st Stage Real Shannon Basis

$$\hat{u} = \left\{ \begin{array}{ll} \sqrt{2} & \text{if } n = 0, 1, \dots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4} + 1, \dots, N - 1 \\ i & \text{if } n = \frac{N}{4} \\ -i & \text{if } n = \frac{3N}{4} \\ 0 & \text{if } n = \frac{N}{4} + 1, \dots, \frac{3N}{4} - 1 \end{array} \right.$$

$$\hat{v} = \left\{ \begin{array}{ll} 0 & \text{if } n = 0, 1, \dots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4} + 1, \dots, N - 1 \\ 1 & \text{if } n = \frac{N}{4} \text{ or } n = \frac{3N}{4} \\ \sqrt{2} & \text{if } n = \frac{N}{4} + 1, \dots, \frac{3N}{4} - 1 \end{array} \right.$$

Wavelet Transform

Wavelet transform

$$z = \left\{ \begin{array}{ll} *\tilde{u} \rightarrow z * \tilde{u} \xrightarrow{\downarrow 2} \\ *\tilde{v} \rightarrow z * \tilde{v} \xrightarrow{\downarrow 2} \end{array} \right. = \left[\begin{array}{l} D(z * \tilde{u}) \\ D(z * \tilde{v}) \end{array} \right]$$

Inverse wavelet transform

$$\left[\begin{array}{l} D(z * \tilde{u}) \\ D(z * \tilde{v}) \end{array} \right] = \begin{array}{l} \xrightarrow{\uparrow 2} \frac{(z * \tilde{u}) + (z * \tilde{u})^*}{2} * u \\ \xrightarrow{\uparrow 2} \frac{(z * \tilde{v}) + (z * \tilde{v})^*}{2} * v \end{array} + = z$$

Misc

$$\sum_{k=a}^b r^k = \frac{r^a - r^{b+1}}{1 - r}$$