CS 530 Midterm 1 Cheatsheet

Linear Algebra

Linear combination

$$\sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

Span: the set of all possible linear combinations of the elements of a given matrix.

Linearly Independent

$\alpha_1 v_1 + \cdots + \alpha_n v_n \implies \alpha_1 = \cdots = \alpha_n = 0$

Support of a vector z is the number of non zero entries of z, denoted by $\|\text{supp } z\|$

For example,

$$w = \begin{bmatrix}0.5\\0\\0\\2\end{bmatrix}, \text{supp } w = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \|\text{supp } w\| = 2$$

Uncertainty principle

$$\|\text{supp } z\|^2 \cdot \|\text{supp } \hat{z}\|^2 \geq N$$

Complex Analysis

$z = a + ib = re^{i\theta} = r(cos(\theta) + isin(\theta))$

$\overline{z} = a - bi = re^{-i\theta}$

$|z| = \sqrt{a^2 + b^2}$

$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

$(a_1 + ib_1) \times (a_2 + ib_2)$

$= (a_1 \times a_2 - b_1 \times b_2) + i(a_1 \times b_2 + a_2 \times b_1)$

$z^{-1} = \frac{\overline{z}}{a^2+b^2}$

$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Roots of unity

$$e^{i\theta} \text{ where } \theta = \frac{2k\pi}{n}, k = 0, 1, \ldots, n-1$$

Inner product

$$\langle z, w\rangle = \sum_{n=0}^{N-1} z(n)\overline{w(n)}$$

$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$

$\langle u + v, w\rangle = \langle u, w\rangle + \langle u, v\rangle$

$\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$

Conjugate symmetry $\langle u, v\rangle = \overline{\langle u, v\rangle}$

Cauchy-Shwarz inequality $\overline{\langle u, v\rangle} \leq \|u\|\|v\|$

$\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \iff u = \mathbf{0}$

orthogonal $\iff \langle u, v\rangle = 0$

Fourier Transform

1D

Fourier Basis for $\mathbb{C}^N$ is $\{F_0, F_1, \cdots, F_{N-1}\}$

$$F_m(n) = \frac{1}{N} e^{2\pi i \frac{mn}{N}}$$

Fourier basis is orthogonal, not orthonormal. Can be normalized the norm is $\|F_m\| = \frac{1}{\sqrt{N}}$

Normalized Fourier Basis

$$\sqrt{N} \cdot F_m = \frac{F_m}{\|F_m\|} = \frac{F_m}{\frac{1}{\sqrt{N}}}$$

The Fourier Basis for $\mathbb{C}^4$ is:

$$\begin{bmatrix}1 & 1 & 1 & 1\\1 & -i & -1 & i\\1 & -1 & 1 & -1\\1 & i & -1 & -i\end{bmatrix}$$

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi i \frac{mn}{N}}$$

$$z(n) = \frac{1}{N}\sum_{m=0}^{N-1} \hat{z}(m) e^{2\pi i \frac{mn}{N}}$$

2D

$$\hat{z}(m_1, m_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} z(n_1, n_2) e^{-2\pi i \frac{m_1 n_1}{N_1}} e^{-2\pi i \frac{m_2 n_2}{N_2}}$$

$$z(n_1, n_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1} \hat{z}(m_1, m_2) e^{2\pi i \frac{m_1 n_1}{N_1}} e^{2\pi i \frac{m_2 n_2}{N_2}}$$

Wavelets

Wavelet tools

$$(w * z)(m) = \sum_{n=0}^{N-1} w(m-n) z(n)$$

The convolution matrix is

$$C = \{R_0 w, R_1 w, \ldots, R_{N-1} w\}$$

For example, with $N = 4$

$$C = \{R_0 w, R_1 w, R_2 w, R_3 w\}$$

The convolution operation is simply

$$(w * z)(m) = C \cdot z$$

$(w * z)^\wedge(m) = \hat{w}(m)\hat{z}(m)$

$R_k z$ is called the translation of z by k.

$(R_k z)(n) = z(n-k)$

$(R_k z)^\wedge(m) = \hat{z}(m) e^{-2\pi i \frac{mk}{N}}$

$\langle z, R_k w\rangle = (z * \tilde{w})(k)$

$\langle z, R_k \tilde{w}\rangle = (z * w)(k)$

$\tilde{z}(n) = \overline{z(-n)}$

$(\tilde{z})^\wedge(m) = \overline{\hat{z}(m)}$

$z^*(n) = (-1)^n z(n)$

For the below, $N = 2M$

$(z^*)^\wedge(n) = \hat{z}(n + M)$

Unitary matrix $\iff \overline{U^T}U = I$

Parseval's formula

$$\langle z, w\rangle = \frac{1}{N}\sum_{m=0}^{N-1} \hat{z}(m)\overline{\hat{w}(m)} = \frac{1}{N}\langle \hat{z}, \hat{w}\rangle$$

Plancherel's formula

$$\|z\|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\hat{z}(m)|^2 = \frac{1}{N}\|\hat{z}\|^2$$

Downsampling

$$z = \begin{bmatrix}z(0)\\z(1)\\z(2)\\z(3)\\z(4)\\z(5)\\z(6)\\z(7)\end{bmatrix} \xrightarrow{\downarrow 2} U_z = \begin{bmatrix}z(0)\\z(2)\\z(4)\\z(6)\end{bmatrix}$$

Upsampling

$$z = \begin{bmatrix}z(0)\\z(1)\\z(2)\\z(3)\end{bmatrix} \xrightarrow{\uparrow 2} U_z = \begin{bmatrix}z(0)\\0\\z(1)\\0\\z(2)\\0\\z(3)\\0\end{bmatrix}$$

1st Stage Wavelet Basis

$A(n)$ is the system matrix of u and v

$$A(n) = \frac{1}{\sqrt{2}}\begin{bmatrix}\hat{u}(n) & \hat{v}(n)\\ \hat{u}(n+M) & \hat{v}(n+M)\end{bmatrix}$$

$$B = \{R_{2k} v\}_{k=0}^{M-1} \cup \{R_{2k} u\}_{k=0}^{M-1}$$
$$= \{R_0 v, R_2 v, \ldots, R_{N-2} v, R_0 u, R_2 u, \ldots, R_{N-2} u\}$$

B (u and v) form a first stage wavelet basis if and only if the following properties are true:

1. $|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2$
2. $|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2$
3. $\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)} = 0$

$\forall n = 0, 1, \cdots, M-1$

1st Stage Shannon Basis

$$\hat{u} = \begin{cases}\sqrt{2} & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \ldots, N-1\\ 0 & \text{if } n = \frac{N}{4}, \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1\end{cases}$$

$$\hat{v} = \begin{cases}0 & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \ldots, N-1\\ \sqrt{2} & \text{if } n = \frac{N}{4}, \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1\end{cases}$$

Real Wavelet Basis

1st Stage Real Shannon Basis

$$\hat{u} = \begin{cases} \sqrt{2} & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4} + 1, \ldots, N - 1 \\ i & \text{if } n = \frac{N}{4} \\ -i & \text{if } n = \frac{3N}{4} \\ 0 & \text{if } n = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

$$\hat{v} = \begin{cases} 0 & \text{if } n = 0, 1, \ldots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4} + 1, \ldots, N - 1 \\ 1 & \text{if } n = \frac{N}{4} \text{ or } n = \frac{3N}{4} \\ \sqrt{2} & \text{if } n = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1 \end{cases}$$

Wavelet Transform

Wavelet transform

$$z = \begin{cases} *\tilde{u} \to z * \tilde{u} \xrightarrow{\downarrow 2} \\ *\tilde{v} \to z * \tilde{v} \xrightarrow{\downarrow 2} \end{cases} = \begin{bmatrix} D(z * \tilde{u}) \\ D(z * \tilde{v}) \end{bmatrix}$$

Inverse wavelet transform

$$\begin{bmatrix} D(z * \tilde{u}) \\ D(z * \tilde{v}) \end{bmatrix} = \begin{matrix} \xrightarrow{\uparrow 2} \frac{(z*\tilde{u})+(z*\tilde{u})^*}{2} * u \\ \xrightarrow{\uparrow 2} \frac{(z*\tilde{v})+(z*\tilde{v})^*}{2} * v \end{matrix} + = z$$

Misc

$$\sum_{k=a}^{b} r^k = \frac{r^a - r^{b+1}}{1 - r}$$