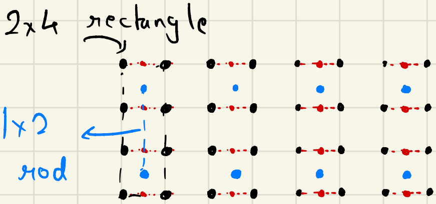
On the problem of Hard rectangles

– Soumyadeep Sarma

A hard rectangle can basically be extended from a $1\times k$ rod on a lattice system. Assume taking an $l\times k$ rectangle. However, it is not a complete analogy. A system of pure x-mers or y-mers can be analogus to k-mer on a lattice. Let's take $l = 2$ and see this construction:



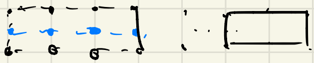
Black dots represent initial lattice
Blue dots represent final lattice

$(8\times 4) \to (4,2)$ lattice

Basically the idea is to clump lattice points together to form a smaller lattice. The other idea will be to find analogies in this system. A hole in this system can be a bit more complex to identify.

Let's consider a 1-D scenario firstly

Consider a $2\times L$ lattice system, we see that a



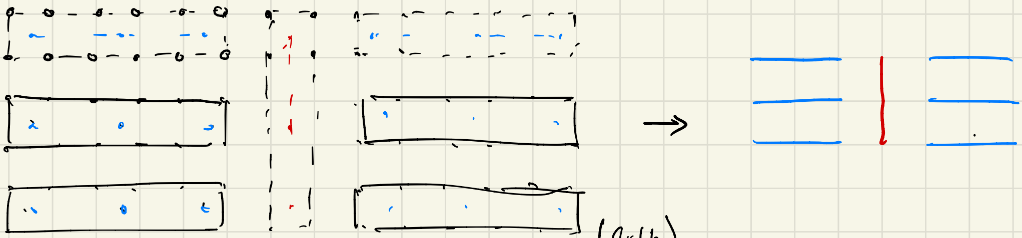
grand-partition function on this is essential the same given the analogous situation and follows the same recursion

$$\Omega_{1d}[L,z] = \Omega_{1d}[L-1,z] + z\,\Omega_{1d}[L-K,z]$$

The same treatment as the original paper follows here. But now, for the 2D case with $z_y \neq 0$, we have:

$$\frac{\Omega_{2d}(L,z,z^y)}{\Omega_{2d}(L,z,0)} = 1 + z_y L^2 \left[\frac{\Omega'_{1d}(L,z)}{\Omega_{1d}(L,z)}\right]^K + O(z_y^2)$$

Here $\Omega'_{1d}(L,z)$ refers to partition function where 2 adjacent sites on the analogous 1-D lattice are empty. This is different from the original paper and the ratio $\frac{\Omega'_{1d}}{\Omega_{1d}}$ cannot be marked as ε. However one other interpretation which takes analogous 1D lattice point to be the center is also possible which accounts for scaling in both directions



(6x14) (3x7)

If we take ratio to some t (let's say) we ha

$$\frac{\ln \Omega_{2d}(L, z, z_y)}{L^2} = z_y t^k + \underbrace{\frac{\ln\left(\lambda^{\frac{L^2}{2}} + \lambda^{\frac{L^2}{2} - L}\right)}{L^2}}_{\text{even}} \Big/ \underbrace{\frac{\ln(2(\lambda^{\frac{L^2-1}{2}}}{L^2}}_{\text{odd}}$$

Solutions for $k \times \infty$ strip :

Let $\Omega_{strip}(L, z, z_y)$ be partition function of a $K \times L$ strip .

$$GF \equiv \Omega(x, z, z_y) = \sum_{L=0}^{\infty} \Omega_{strip}(L, z, z_y)\, x^L$$

It can be split into blocks of nematic x-mers separated by y-mers . Let $R(x, z)$ be generating function for x-mers only :

$$\Omega(x, z, z_y) = R(x, z) + R(x, z)\, x^2 z_y\, R(x, z)$$
$$+ R(x, z)\, x^2 z_y\, R(x, Z)\, x^2 z_y\, R(x, z)$$
$$\cdots\cdots$$

(as width is 2 for y-mer $\equiv x^2$)

$$= \frac{R(x,z)}{1 - x^2 z_y R(x,z)}$$

Now, $R(x,z) = \sum_{L=0}^{\infty} \left[\Omega_{1d}(L,z)^{\frac{k}{2}} + \Omega_{1d}(L,z)^{\frac{k}{2}-1} \right] x^L$ (for k = even)

$$= \sum_{L=0}^{\infty} 2\left[\Omega_{1d}(L,z)\right]^{\frac{k-1}{2}} x^L \text{ (for } k = \text{odd)}$$

Reduced recursion relation is same as that for paper

$$Q_L(z) = \frac{Q_{L-1}(z)}{\lambda} + \frac{\lambda - 1}{\lambda} Q_{L-k}(z)$$

or $\Omega_{1d} = Q_L(z)\lambda^L$

$\therefore$ Equation becomes:

$\theta = x \lambda^{\frac{k}{2}}$

$$= \frac{\sum_{L=0}^{\infty} \left(Q_L(z)^{\frac{k}{2}} + \frac{Q_L(z)^{(\frac{k}{2}-1)}}{\lambda^L} \right) \theta^L}{1 - \theta^2 \frac{z_y}{z} (1 - \lambda^{-1}) \sum_{L=0}^{\infty} \left(Q_L(z)^{\frac{k}{2}} + \frac{Q_L(z)^{\frac{k}{2}-1}}{\lambda^L} \right) \theta^L}$$

$\forall$ k being even

$$= \frac{2\sum_{L=0}^{\infty} \frac{Q_L(z)^{\frac{k-1}{2}}}{\lambda^{L/2}} \theta^L}{1 - \frac{2\theta^2 z_y}{z} (1 - \lambda^{-1}) \sum_{L=0} \frac{Q_L(z)^{\frac{k-1}{2}}}{\lambda^{L/2}} \theta^L}$$

The denominator must be zero

$$1 - \frac{\theta^{*2} z_y}{z}(1-\lambda^{-1}) \sum_{L=0}^{\infty} \left(Q_L(z)^{\frac{K}{2}} + \frac{Q_L(z)^{\frac{K}{2}-1}}{\lambda^L} \right) \theta^{*L} = 0$$

$$1 - \frac{2\theta^{*2} z_y}{z}(1-\lambda^{-1}) \sum \frac{Q_L(z)^{\frac{K-1}{2}}}{\lambda^{L/2}} \theta^{*L} = 0$$

The solution to this gives x^* which then gives

$$\lim_{L\to\infty} \frac{\ln \Omega_{strip}(L, z, z_y)}{L} = -\ln x^*$$

5/1/2024

We know that generating function for dimer on 1D strip of length L, $\Omega(L,z) = \sum_N \binom{L-N}{N} z^N$. We also see that this $\Omega_L(z)$ satisfies:

$$\Omega(L,z) = \Omega(L-1,z) + z\,\Omega(L-2,z)$$

$$[\Omega(2,z) = 1+z,\ \Omega(1,z) = 1]$$

We can write it like this:

$$\vec{\Omega} = \begin{pmatrix} \Omega_L \\ \Omega_{L-1} \\ \vdots \\ \Omega_3 \\ \Omega_2 \\ \Omega_1 \end{pmatrix}_{(L-3)\times 1}, \quad \vec{A} = \begin{pmatrix} 1 & -1 & -z & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & -z & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & -z & \cdots & \vdots \\ \vdots & & & \cdots & 0 & 1 & -1 & -z \\ & & & & \cdots & 0 & 1 & 0 \\ & & & & & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \quad \vec{A} \cdot \vec{\Omega} = \vec{\ell}$$

We wish to understand how $\Omega_L(z) \sim \lambda^L$ is a valid solution to the recursion $\Omega_L(z) = \Omega_{L-1}(z) + z\,\Omega_{L-k}(z)$ or in other words, after substituting λ^L: $\lambda^k - \lambda^{k-1} = z$

Let us employ the Transfer Matrix method here: Label the sites of a rod with numbers $1, 2 \dots k$ and label empty sites as 0. Given a site, it might be empty (0). Then the immediate left neighbour is either empty (0) or k (right end of a rod). Similarly if our initial site is of type 1, then the immediate left site is either 0 or k again. For all other numbers $2, 3, 4 \cdots k$, the site immediately to the left can be $x-1$, if x is number of current site. We assign weight 1 to $0, 1 \dots k-1$ and z to k to denote presence of a rod. For the transfer matrix T thus, $T_{m,n}$ is the element in the the n^{th} row and m^{th} column

$$T_{i,0/1} = \begin{cases} 1, & \forall i = 0. \\ 0, & \forall 1 \le i \le k-1 \\ z, & \forall i = k \end{cases}$$

(the weight is assigned if type n is present in the immediate left of type m site) which gives us:

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \\ \vdots & \vdots & \vdots & & \ddots & \\ \vdots & \vdots & \vdots & & & 1 \\ z & z & 0 & \cdots & 0 & 0 \end{pmatrix}_{(k+1)\times(k+1)}$$

The characteristic equation for this matrix is $\det(T-\lambda I) = 0$

$$T-\lambda I = \begin{pmatrix} 1-\lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & & 1 \\ z & z & & & & -\lambda \end{pmatrix}$$

If we expand along the first row, we have:

$$(1-\lambda)\det\begin{pmatrix} -\lambda & 1 & 0 & 0 & \cdots \\ 0 & -\lambda & 1 & 0 & \\ \vdots & \vdots & & \ddots & 1 \\ z & 0 & & & -\lambda \end{pmatrix}_{k\times k} + (-1)\det\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ z & 0 & & & -\lambda \end{pmatrix}_{k\times k}$$

Let's look at the 2^{nd} det, by expanding along first row again:

$$(-1)\det\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ z & 0 & & & -\lambda \end{pmatrix}_{k\times k} = (-1)^2\det\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ z & 0 & \cdots & \cdots & -\lambda \end{pmatrix}_{(k-1)\times(k-1)}$$

$$\vdots$$

$$= (-1)^{k-1}\det\begin{pmatrix} 0 & 1 \\ z & -\lambda \end{pmatrix} = (-1)^k z$$

Similarly the first det by expanding along first row:

$$\det\begin{pmatrix} -\lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ & & & & \ddots & 1 \\ z & 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix}_{k\times k} = (-\lambda)\det\begin{pmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ & & & & 1 \\ 0 & 0 & & & -\lambda \end{pmatrix}_{(k-1)\times(k-1)}$$

$$+(-1)\det\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & -\lambda & 1 & \cdots & 0 \\ & \vdots & & & \vdots \\ z & 0 & \cdots & & -\lambda \end{pmatrix}_{(k-1)\times(k-1)}$$

The second part is something we have solved, and is equal to $(-1)^{k-1}z$. The first part is an upper triangular matrix and evaluates to $(-\lambda)^k$. Thus, our characteristic equation gives us:

$$\det(T-\lambda I) = (1-\lambda)\left[(-\lambda)^k - (-1)^k z\right] + (-1)^k z = 0$$

$$\Rightarrow \quad (1-\lambda)(-\lambda)^k + (-1)^k z\left(1-(1-\lambda)\right) = 0$$

$$\therefore \quad z = -\left[\frac{1-\lambda}{\lambda}(\lambda)^k\right] = \lambda^k - \lambda^{k-1}$$

This is the same recursion obtained by taking $\Omega_L(z) \sim \lambda^L$ where λ is the largest eigen value and:

$$\Omega_L(z) = \Omega_{L-1}(z) + z\,\Omega_{L-k}(z)$$

We can understand this solution in the following sense : The weight of one possible configuration is :

$$\langle k_1|T|k_2\rangle\langle k_2|T|k_3\rangle\langle k_3|T|k_4\rangle \cdots \langle k_{L-1}|T|k_L\rangle \overbrace{\underbrace{\langle k_L|T|k_1\rangle}_{=1\text{, always for valid config}}}^{\text{Term added for connecting}}$$

where $0 \leq k_i \leq k$. The sum over all configurations is the grand partition function :

$$\Omega_L(z) = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_L} \langle k_1|T|k_2\rangle\langle k_2|T|k_3\rangle \cdots \langle k_L|T|k_1\rangle$$

$$= \sum_{k_1}\sum_{k_L} \langle k_1|T^L|k_1\rangle = Tr(T^L)$$

$$= \sum_{i=0}^{k} \lambda_i^L$$

WLOG, let $\lambda_0 \equiv \lambda$ be the largest eigenvalue. At large L, clearly for $i \neq 0$, $\left(\frac{\lambda_i}{\lambda}\right)^L \to 0$, $\therefore$ $\Omega_L(z) \sim \lambda^L$.

---

* <u>Dimer problem on a ladder</u> : Given a strip of $2\times L$, find the generating function for placing dimers on this lattice .

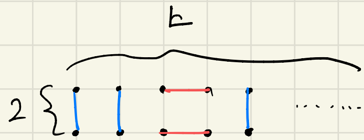
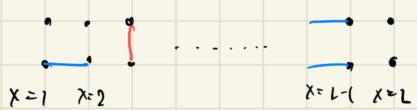


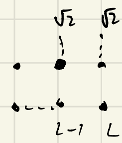
Fig: One configuration on a $2\times L$ strip

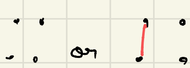
We use the techique developed earlier : We assign a site 0 (empty), 1 (head of a horizontal dimer, going from left to right), 2 (tail of a horizontal dimer), 3 (vertical dimer). Consider $\Omega_L(\sigma_1, \sigma_2)$ as the restricted grand partition function where σ_1 is the label for the top right position (i.e for $x = L$ column, upper site) and σ_2 is label for bottom right. We can have :

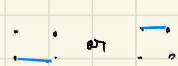
$$\Omega_{L+1}(\sigma_1, \sigma_2) = \sum_{\sigma_1', \sigma_2'} T(\sigma_1, \sigma_2; \sigma_1', \sigma_2') \, \Omega_L(\sigma_1', \sigma_2')$$

If $\Omega_L(\sigma_1, \sigma_2)$ is written as a column vector, T can become a 16×16 matrix. But we can reduce the dimensions by utilising symmetries in the problem. We associate a weight of $\sqrt{z}$ with each occupied site. The possible scenarios are :





1) No dimer goes across column $x = L-1$ and $x = L$ ()

2) Exactly one dimer goes across $x = L-1$ and $x = L$ ()

3) Two dimers go across $x = L-1$ and $x = L$ ( )

We label sum over each such configurations $Z_1(L)$, $Z_2(L)$, $Z_3(L)$

$$Z_1(L) = (1+z)Z_1(L-1) + Z_2(L-1) + Z_3(L-1)$$

$$Z_2(L) = 2z\,Z_1(L-1) + z\,Z_2(L-1)$$

$$Z_3(L) = z^2 Z_1(L-1)$$

which gives us the matrix equation:

$$\begin{bmatrix} Z_1(L) \\ Z_2(L) \\ Z_3(L) \end{bmatrix} = \begin{pmatrix} 1+z & 1 & 1 \\ 2z & z & 0 \\ z^2 & 0 & 0 \end{pmatrix} \begin{bmatrix} Z_1(L-1) \\ Z_2(L-1) \\ Z_3(L-1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Z_1(L+1) \\ Z_2(L+1) \\ Z_3(L+1) \end{bmatrix} = \underbrace{\begin{pmatrix} 1+z & 1 & 1 \\ 2z & z & 0 \\ z^2 & 0 & 0 \end{pmatrix}^{L-1}}_{T^{L-1}} \begin{bmatrix} Z_1(2) \\ Z_2(2) \\ Z_3(2) \end{bmatrix}$$

Clearly, $\Omega_L(z) = Z_1(L+1)$ and $Z_1(2) = 1+z$

$Z_2(2) = 2z$, $Z_3(2) = z^2$. Using spectral decomposition for matrix T, we have:

$$\det(T-\lambda I) = \det\begin{pmatrix} 1+z-\lambda & 1 & 1 \\ 2z & z-\lambda & 0 \\ z^2 & 0 & -\lambda \end{pmatrix}$$

$$= -\lambda(\lambda-(1+z))(\lambda-z) + 2\lambda z + (\lambda-z)z^2$$

$$= -\lambda\left(\lambda^2 - \lambda(1+2z) + z(1+z)\right) + \lambda(2z) + \lambda(z^2) - z^3$$

$$= -\lambda^3 + \lambda^2(1+2z) + \lambda(z) - z^3$$

Equating this to zero gives three eigenvalues which are functions of z as $\lambda_1(z)$, $\lambda_2(z)$, $\lambda_3(z)$, which are some highly complex equations in z (as checked on Mathematica). For $\Omega_L(z)$, we would then have

$$\Omega_L(z) \sim \sum_{i=1}^{3} c_i\, \lambda_i(z)^L \quad \text{or for large } L$$

$$\Omega_L(z) \sim \lambda(z)^L \quad (\lambda = \max(\lambda_1, \lambda_2, \lambda_3))$$

For finding zeroes of this form:

$$c_1\lambda_1^L + c_2\lambda_2^L + c_3\lambda_3^L = 0$$

To solve this eqn, we would need accurate information about the coefficients as well as, which is not available to us. We try to plot $\lambda_1^L + \lambda_2^L + \lambda_3^L$

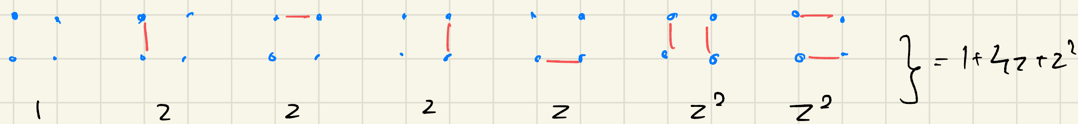
Let us find explicit expressions of $\Omega_L(z)$ for $L = 1, 2$ from the relation $\Omega_L(z) = Z_1(L+1)$

$$\Omega_1(z) = Z_1(2) = 1 + z$$

$$\Omega_2(z) = Z_1(3) = (1+z)\, Z_1(2) + Z_2(2) + Z_3(2)$$

$$= (1+z)^2 + 2z + z^2 = 1 + 4z + 2z^2$$

Matches



Now, $Z_2(3) = 2z\, Z_1(2) + z\, Z_2(2)$

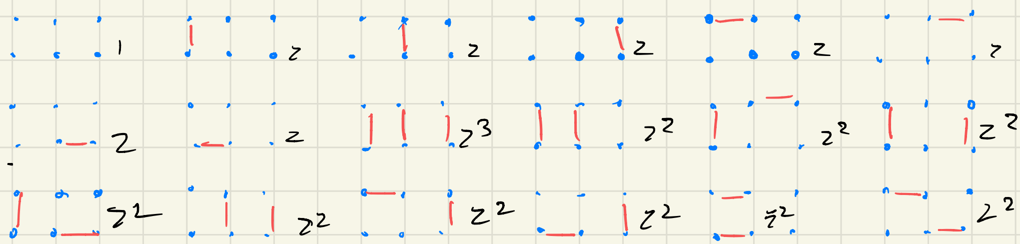
$$= 2z(1+z) + z\, 2z = 2z + 4z^2$$

$$Z_3(3) = z^2 Z_1(2) = z^2 + z^3$$

$$\Omega_3(z) = Z_1(4) = (1+z)\, Z_1(3) + Z_2(3) + Z_3(3)$$

$$= (1+z)(1+4z+2z^2) + 2z + 4z^2 + z^2 + z^3$$

$$= 1 + 7z + 11z^2 + 3z^3$$



$$= 1 + 7z + 11z^2 + 3z^3$$

$$Z_2(4) = 2z\, Z_1(3) + 2\, Z_2(3)$$

$$= 2z(1+4z+2z^2) + z(2z+4z^2)$$

$$= 2z + 10z^2 + 8z^3$$

$$Z_3(4) = z^2 Z_1(3) = z^2(1+4z+2z^2) = z^2 + 4z^3 + 2z^4$$

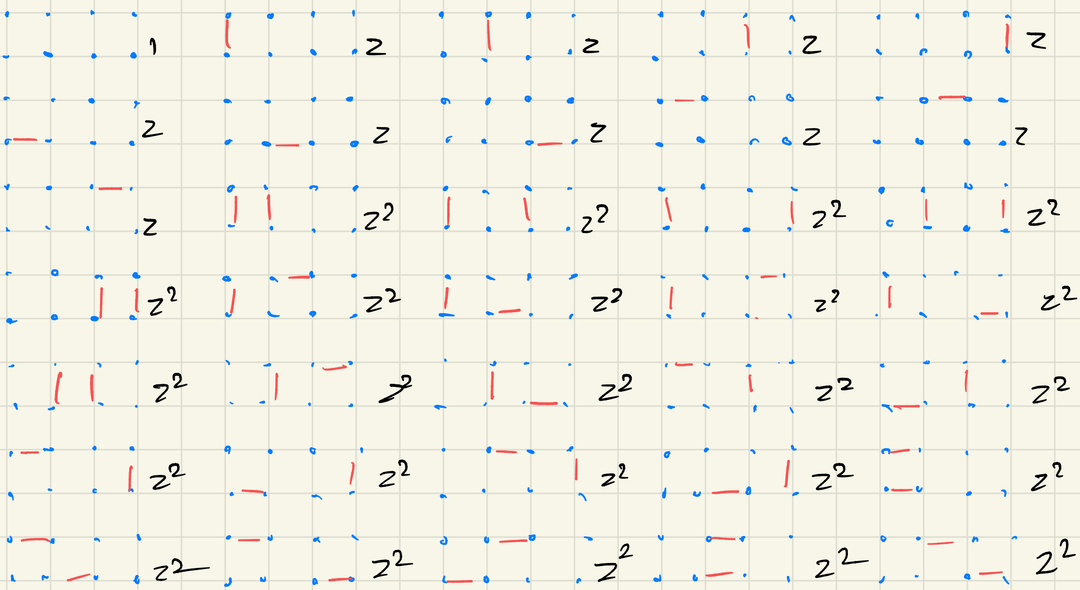
$$\Omega_4(z) = Z_1(5) = (1+z)Z_1(4) + Z_2(4) + Z_3(4)$$

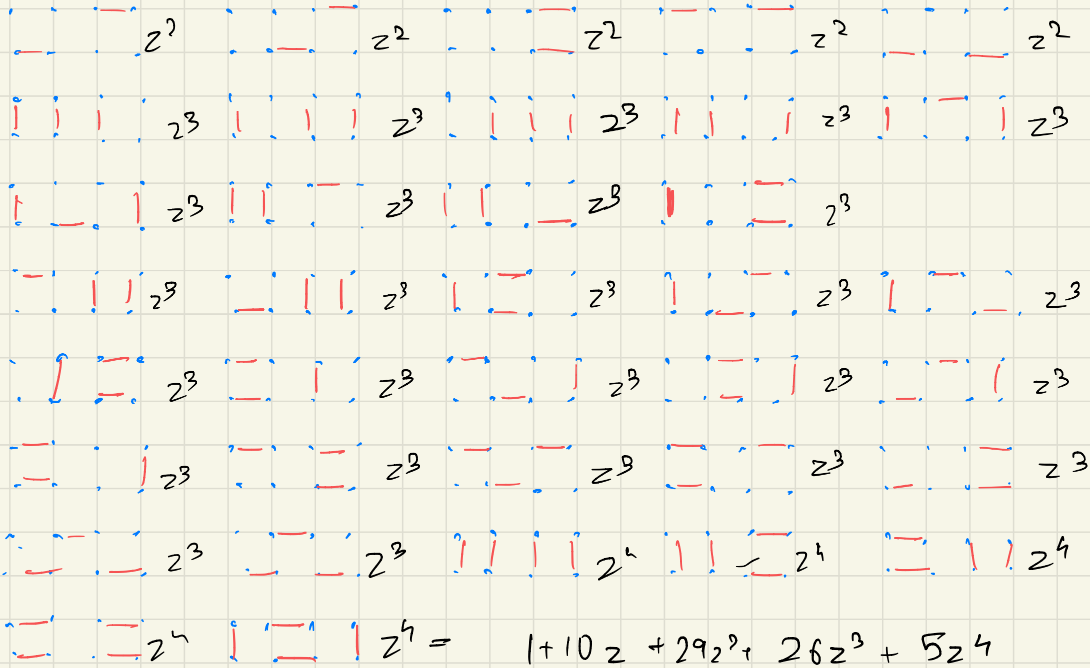
$$= (1+z)(1+7z+11z^2+3z^3) + 2z + 10z^2 + 8z^3$$

$$+ z^2 + 4z^3 + 2z^4$$

$$= 1 + 10z + 29z^2 + 26z^3 + 5z^4$$

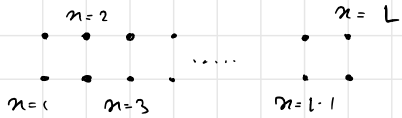
The configurations are:



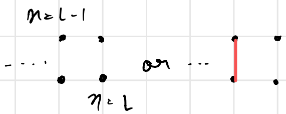


$$= 1 + 10z + 29z^2 + 26z^3 + 5z^4$$

We can see that the current recursion results match with the results we would obtain via counting.

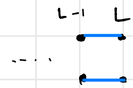


a)



b)

c)



d)