(1) $\vec{r}(t) = \hat{x}(t)\hat{i} + y(t)\hat{j}$

Let field be $\frac{\mu_0 i\,(y\hat{i} - x\hat{j})}{2\pi(x^2+y^2)}$ $\left(= \frac{\mu_0 i}{2\pi r}\hat{n}\right.$ where $\hat{n}$ is Tangential)

$$\frac{\mu_0 i\,(y\hat{i} - x\hat{j})}{2\pi(x^2+y^2)} = k\left(\frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j}\right)$$

constant

$$\Rightarrow \frac{\mu_0 i\, y\hat{i}}{2\pi(x^2+y^2)} = \frac{k\,dx}{dt} \quad \text{— i)}$$

$$\& \ -\frac{\mu_0 i\, x}{2\pi(x^2+y^2)} = \frac{k\,dy}{dt} \quad \text{— ii)}$$

i) / ii)

$$-\frac{y}{x} = \frac{dx}{dy}$$

$$\Rightarrow -y\,dy = dx \cdot x$$

$$\Rightarrow -\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$x^2 + y^2 = -2c \quad (c \text{ is } -ve)$$

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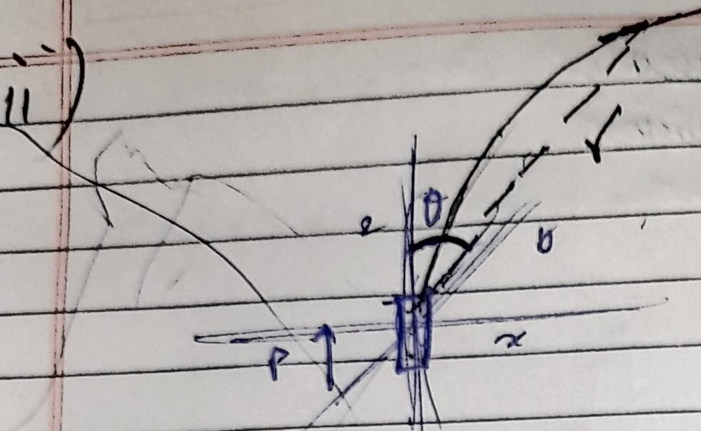
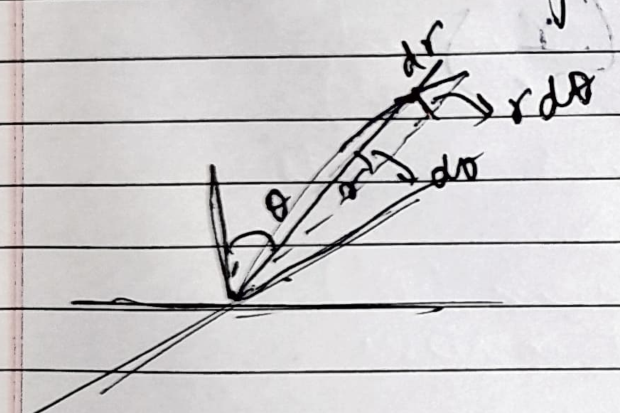
ii)



field due to dipole is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Now consider the following figure



The field line is tilted with $\vec{r}$ by angle α

Then clearly $\tan\alpha = \frac{r\,d\theta}{dr}$

Also this must be $\parallel$ to field
Angle made by field with with $\vec{r}$

$$\tan\alpha = \frac{B_\theta}{B_r} = \frac{\sin\theta}{2\cos\theta}$$

$$\frac{r\,d\theta}{dr} = \frac{\sin\theta}{2\cos\theta}$$

$$\frac{2\cos\theta\,d\theta}{\sin\theta} = \frac{dr}{r}$$

$$\Rightarrow \int_{\pi/2}^{\theta} \frac{2\cos\theta\,d\theta}{\sin\theta} = \int_{r_e}^{r} \frac{dr}{r}$$

$$\Rightarrow 2\ln(\sin\theta) = \ln\left(\frac{r}{r_e}\right)$$

$$\Rightarrow r = \sin^2\theta\, r_e$$

iii) Assume a field line $r = r_e \sin^2\theta$

Magnitude magnetic field

$$= \frac{\mu_0 p}{4\pi r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$$

$$= \frac{\mu_0 p}{4\pi r^3}\sqrt{1+3\cos^2\theta}$$

$$= \frac{\mu_0 p \sqrt{1+3\cos^2\theta}}{4\pi (r_e)^3 \sin^6\theta} = B(\theta) \longrightarrow \text{iii)}$$

Now if speed of particle $\perp$ to field line is v_2, we have

$$\frac{m v_2^2}{r} = q v_2 B$$

$$\Rightarrow \boxed{r = \frac{m v_2}{q B}} = \frac{m v_2 \, 4\pi (r_e)^3 \sin^6\theta}{q \mu_0 p \sqrt{1+3\cos^2\theta}}$$

Pitch = $v_1 \times$ Time period

$$= v_1 \times \frac{2\pi r}{v_2}$$

$$\boxed{= v_1 \frac{2\pi m}{qB}}$$

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iv Magnetic moment generated due to motion ⊥ to field line

$= \text{area} \times \text{effective current}$

$= \pi r^2 \times \frac{q}{T}$

$= \frac{\pi m^2 v_2^2}{q^2 B^2} \cdot \frac{q(q)B}{2\pi m} = \frac{m v_2^2}{2B}$

Now $\frac{m v_2^2}{2B}$ remains constant — iv)

But note that magnetic field does no work $\therefore v_1^2 + v_2^2 = \text{constant}$

Also note as proton moves towards poles $\theta \downarrow$ (iii)

$\therefore$ from (iii) $B \uparrow$

$\therefore$ from iv v_2^2 must increase.

But $v_1^2 + v_2^2 =$ constant,

$\therefore v_1$ should decrease and possible could become zero somewhere

Let latitude be α

$\therefore \theta = \frac{\pi}{2} - \alpha$.

When proton stops $v_1 = 0$

$v_2^2 = (v_{1eq})^2 + (v_{2eq})^2$ — v)

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$$\frac{m v_2^2}{2B(\theta)} = \frac{m (v_{2eq})^2}{2B\left(\frac{\pi}{2}\right)}$$

$$\Rightarrow (v_{1eq})^2 + (v_{2eq})^2 = (v_{2eq})^2 \left(\frac{B(\theta)}{B\left(\frac{\pi}{2}\right)}\right)$$

$$\Rightarrow 1 + \tan^2(\rho_\theta) = \tan^2(\rho_\theta) \frac{\frac{\mu_0 \rho \sqrt{1+3\cos^2\theta}}{4\pi (r_c)^3 \sin^6\theta}}{\frac{\mu_0 \rho}{4\pi (r_c)^3}}$$

$$\Rightarrow 1 + \tan^2(\rho_\theta) = \tan^2(\rho_\theta) \frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta} \longrightarrow \text{vi)}$$

$$\theta = \frac{\pi}{2} - \alpha$$

$$\Rightarrow \cancel{1+} \; 1 + \cot^2(\rho_\theta) = \frac{\sqrt{1+3\sin^2\alpha}}{\cos^6(\alpha)} \longrightarrow \text{vii)}$$

The RHS is clearly $\uparrow$ in $\left(0, \frac{\pi}{2}\right)$

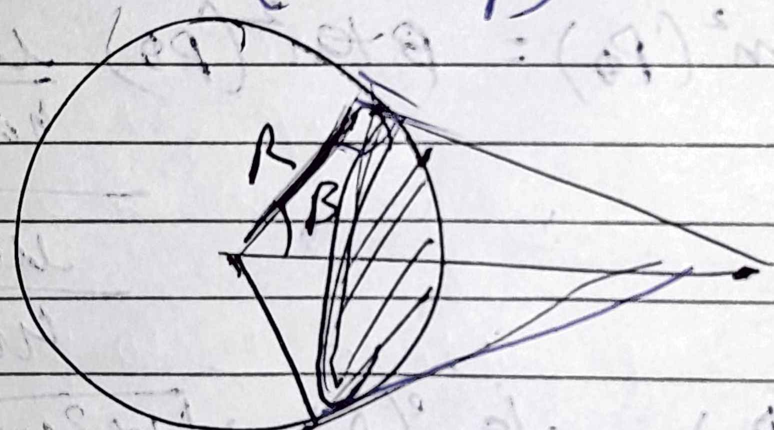
$\therefore$ We get only one solution for α in $\left(0, \frac{\pi}{2}\right)$

This can be solved numerically for any given value of ρ_θ

V. Let us say value of θ obtained from previous part is θ'

we have $r = r_e \sin^2(\theta')$

Now area of spherical cap is given by $2\pi(1-\cos\beta)$

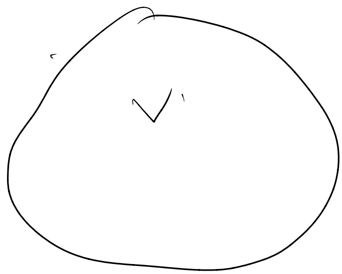


But $\cos\beta = \frac{R}{r} = \frac{R}{r_e \sin^2(\theta')}$

$\therefore$ Area $= 2\pi\left(1 - \frac{R}{r_e \sin^2(\theta')}\right)$

Q 2 Hints / Solutions

a) Note the typo in question, we meant parallel and perpendicular component of $\vec{B}$



Suppose there exists more than 1 solution $\vec{B}_1$ and $\vec{B}_2$

Define $\vec{B}_3 = \vec{B}_2 - \vec{B}_1$

$\vec{\nabla} \times \vec{A}_1 = \vec{B}_1$; $\vec{B}_3 = 0$ at surface

$\vec{\nabla} \times \vec{A}_2 = \vec{B}_2$

Here $\vec{A}_1$ and $\vec{A}_2$ are corresponding vector potentials

$\vec{A}_3 = \vec{A}_2 - \vec{A}_1$

$\vec{\nabla} \times \vec{A} = \vec{B}$

and $\vec{\nabla} \times \vec{B} = \vec{J}$ (working units $\mu_0 = 1$)

$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times (\vec{A}) = \vec{J}$

$= \vec{\nabla} (\cancel{\vec{\nabla} \cdot \vec{A}}) - \nabla^2 \vec{A} = \vec{J}$

0

$\vec{\nabla} \cdot \vec{A}$ (Choose)

$\therefore \quad -\nabla^2 \vec{A} = \vec{J}$

$$-\nabla^2 \vec{A_1} = \vec{J}$$
$$-\nabla^2 \vec{A_2} = \vec{J}$$

Now $\vec{\nabla} \times \vec{B_3} = \vec{\nabla} \times \vec{B_1} - \vec{\nabla} \times \vec{B_2} \longrightarrow$

$\vec{\nabla} \times \vec{A_3} = \vec{\nabla} \times A_2 - \vec{\nabla} \times \vec{A_1} = \vec{B_2} - \vec{B_1} = \vec{B_3}$

Now consider $\vec{\nabla} \cdot (\vec{B_3} \times \vec{A_3})$

$$= \vec{A_3} \cdot \underbrace{(\vec{\nabla} \times \vec{B_2})}_{0} - \vec{B_2} \cdot (\vec{\nabla} \times \vec{A_3})$$

$$= -(\vec{B_3} \cdot \vec{B_3})^2 \leq 0$$

But $\int \vec{\nabla} \cdot (\vec{B_3} \times \vec{A_3}) \, dv = \oint (\vec{B_3} \times \vec{A_3}) \cdot d\vec{a}$

$\vec{B_3} = 0$ at surface

$\int (B_3)^2 \, d\tau = 0$

$\therefore \quad B_3 = 0$

Q2 $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

But $\vec{E} = 0$

$\Rightarrow -\frac{\partial \vec{B}}{\partial t} = 0$

$\therefore \vec{B} = \text{constant}$

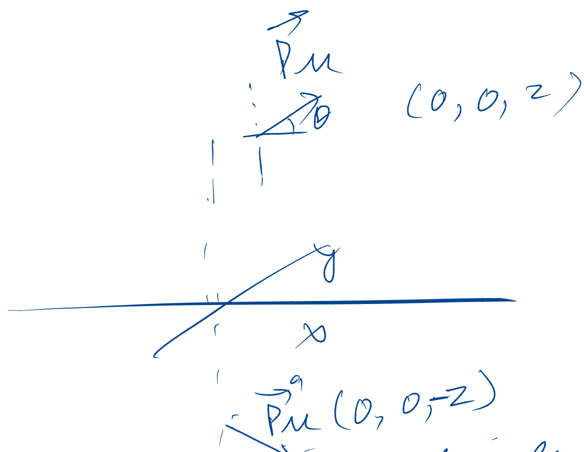
Q3
$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\vec{B} = 0$ and $\vec{E} = 0$

$\therefore \vec{J} = 0$ inside

Q4 we know $\vec{B} = 0$ inside a superconductor

$\therefore \vec{\nabla} \cdot \vec{B} = 0$, $B_\perp = 0$ near surface.



If we place image dipole $\vec{p_u}'$ at $(0,0,-z)$, it can be seen.

$B_\perp = 0$.

(use $\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{1}{r^3}\right)\left(3(\vec{p_u}\cdot\hat{r})\hat{r} - \vec{p_u}\right)$

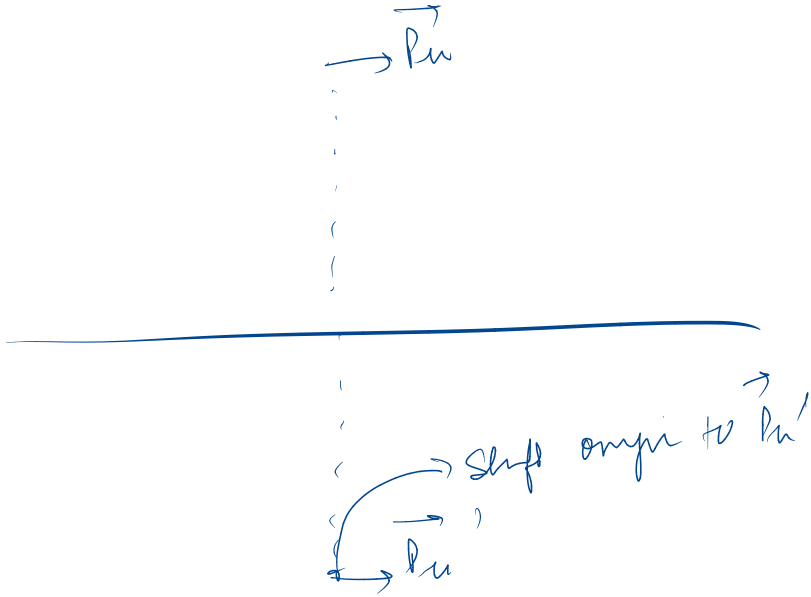
(Reference → Introduction to Electrodynamics by David J. Griffiths))

Force on dipole is given as

$$\vec{\nabla}'(\vec{p_u}\cdot\vec{B})$$

Position of one dipole is $(0,0,z)$ and $\vec{p_u}$ is constant use i) to find $\vec{B}$ and find the derivative

Q5



Force can be calculated as $\vec{\nabla}(\vec{P_m} \cdot \vec{B})$

use identity

$$\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A})$$
$$+ (\vec{A} \cdot \vec{\nabla}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{A}$$

Thus we get (considering only z component)

$$\vec{F} = (P\mu)^2 \left(\frac{\partial}{\partial x}\right)\left(\frac{\mu_0}{4\pi}\frac{3xz}{(x^2+y^2+z^2)^{3/2}(x^2+y^2+z^2)}\right)$$

$$= (P\mu)^2 \frac{3\mu_0}{4\pi}\frac{\partial}{\partial x}\left(\frac{xz}{(x^2+y^2+z^2)^{5/2}}\right)$$

$$= (P\mu)^2 \frac{3}{4\pi}\mu_0 \left(\frac{z}{(x^2+y^2+z^2)^{5/2}} + \cancel{}\right)$$

$$\boxed{= (P\mu)^2 \frac{3}{4\pi}\frac{\mu_0}{z^4}}$$

$z = 2h$

For eq.

$$(P\mu)^2 \frac{3}{4\pi}\frac{\mu_0}{(2h_0)^3} = mg$$

Q6 Force is repulsive and

$\downarrow$ with increase in z

$\therefore$ Equilibrium is stable.

$$F(h) = \frac{(P_n)^2 \, 3\mu_0}{16\pi h^4} - mg$$

$$F(h+\Delta h) - F(h)$$

$$= -\underbrace{\frac{(P_n)^2 \, 3}{16\pi} \frac{\mu_0}{(h_0)^4} \frac{4\Delta h}{h_0}}$$

$$\therefore \; m\omega^2 = \frac{(P_n)^2 \, 3\mu_0}{4\pi (h_0)^5}$$

$$\therefore \; T = \frac{2\pi}{\omega}.$$

Note - In parts 4, 5 and 6, partial marks would be awarded if only factors are wrong.

$u = AT^4$

$\frac{U}{V} = AT^4$

$P = \frac{u}{3}$

$= \frac{U}{3V} = \frac{AT^4}{3}$

1. $W = \int P\,dV$

$= \int_R^{2R} P \cdot 4\pi r^2\,dr$

$= \int_R^{2R} \frac{4\pi \cdot AT_0^4}{3} r^2\,dr$

$= \frac{4\pi A T_0^4}{3} \cdot \frac{7R^3}{9}$

$= \frac{28\pi A T_0^4}{9}$

b. For reversible adiabatic process

$dQ = 0$

$\Rightarrow dU + dW = 0$

$U = 3PV$

$dU = 3P\,dV + 3V\,dP$

$dW = P\,dV$

$\cancel{3PdV}$

$\Rightarrow 4P\,dV + 3V\,dP = 0$

$\Rightarrow 4\frac{dV}{V} + 3\frac{dP}{P} = 0$

$\Rightarrow 4\ln V + 3\ln P = a$

$\Rightarrow P^3 V^4 = c$

$\therefore PV^{4/3} = \text{constant}$

$P = \frac{u}{3} = \frac{AT^4}{3}$.

$\Rightarrow T^4 V^{4/3} = \text{const.}$

$\Rightarrow TV^{1/3} = \text{const.}$

Initial volume $= \frac{4}{3}\pi R^3$

Final volume $= \frac{4}{3}\pi (2R)^3$.

$T_0 \cdot \left(\frac{4}{3}\pi R^3\right)^{\frac{1}{3}} = T_f \left(\frac{4}{3}\pi (2R)^3\right)^{1/3}$.

$\Rightarrow T_f = \frac{T_0}{2}$

e) In adiabatic free expansion

$g = 0$.

$P = 0$. $\therefore W = 0$ (as free expansion).

$\therefore U_i = U_f$.

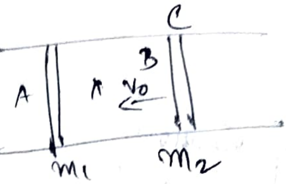
$U = AVT^4$.

$\therefore VT^4 = \text{constant}$

$\therefore T_0^4 R^3 = T_f^4 (4R)^3$.

$T_f = \frac{T_0}{4^{3/4}}$

d) X enclosed



Initial v_0, T_0 Final = V_f, T_f

Since there is no external force, linear momentum of system is conserved ~~and total energy~~ Total energy is conserved too.

Piston B decelerates while A accelerates, temp. of gas will be maximum when compression is maximum.

At this instant both pistons will be moving with same velocities, say v:

By COLM, $m_2 v_0 = (m_1+m_2)v$

$$\Rightarrow v = \frac{m_2 v_0}{m_1+m_2}$$

$$E_i = E_f$$

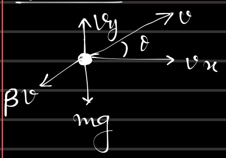
$$\frac{1}{2}m_2 v_0^2 + U_i = \frac{1}{2}(m_1+m_2)v^2 + U_f$$

$$\Rightarrow \frac{1}{2}m_2 v_0^2 + AV_0T_0^4 = \frac{1}{2}\frac{m_2^2 v_0^2}{(m_1+m_2)} + AV_f T_{max}^4$$

$$\Rightarrow T_{max} = \left(\frac{AV_0T_0^4 + \frac{1}{2}\frac{m_1 m_2 v_0^2}{(m_1+m_2)}}{AV_f}\right)^{1/4}$$

(Ans.)

Part C :—



$\therefore$ We have :— $\dot{v}_y = -g - \frac{\beta v \sin\theta}{m} = -g - \frac{\beta v_y}{m}$

$\dot{v}_x = -\frac{\beta v \cos\theta}{m} = -\frac{\beta v_x}{m}$.

$$\Rightarrow \frac{m\,dv_y}{mg+\beta v_y} + dt = 0$$

$$\Rightarrow \frac{m}{\beta}\ln\left(\frac{g+\beta v_y}{g+\beta u_y}\right) + t = 0 \Rightarrow \frac{g+\beta v_y}{g+\beta u_y} = e^{-\beta t/m} \Rightarrow v_y = \frac{(g+\beta u_y)e^{-\beta t/m} - g}{\beta}$$

$$\frac{dv_x}{v_x} + \frac{\beta\,dt}{m} = 0 \Rightarrow \ln\frac{v_x}{u_x} = -\frac{\beta t}{m} \Rightarrow v_x = u_x e^{-\beta t/m}$$

Time of flight :— $y = \int_0^t v_y\,dt = \frac{m(mg+\beta u_y)(1-e^{-\frac{\beta t}{m}})}{\beta^2} - \frac{gt}{\beta}$.

$$\Rightarrow t = \underbrace{\left(1+\frac{\beta}{g}\frac{u_y}{m}\right)}_{1/a}\left(1-e^{-\frac{\beta t}{m}}\right) \Rightarrow (1-at)(e^{\frac{\beta t}{m}}) = 1$$

Now, we calculate x :— $\int_0^t v_x\,dt = x = \frac{m u_x}{\beta}(1-e^{-\beta t})$

$$\frac{dt}{d\theta} = \left(1+\frac{\beta u\cos\theta}{mg}\right)\left(1-e^{-\frac{\beta t}{m}}\right) + \left(1+\frac{\beta u\sin\theta}{mg}\right)\frac{\beta}{m}e^{-\frac{\beta t}{m}}\frac{dt}{d\theta}$$

$$\Rightarrow \boxed{\frac{dt}{d\theta} = \frac{\left(1+\frac{\beta u\cos\theta}{mg}\right)\left(1-e^{-\frac{\beta t}{m}}\right)}{1-\left(1+\frac{\beta u\sin\theta}{mg}\right)\beta e^{-\frac{\beta t}{m}}}.}$$

$$\therefore \frac{dx}{d\theta} = \frac{-u\sin\theta}{\beta}(1-e^{-\beta t}) + \frac{u\cos\theta}{\beta}e^{-\beta t}\frac{dt}{d\theta}$$

for maximum range, $\frac{dx}{d\theta} = 0$.

$$\therefore \frac{\frac{m\cancel{u}\sin\theta}{\beta}\cancel{(1-e^{-\frac{\beta t}{m}})}}{\cancel{u}\cos\theta\,(e^{-\frac{\beta t}{m}})} = \frac{\left(1+\frac{\beta u\cos\theta}{mg}\right)\cancel{(1-e^{-\beta t})}}{1-\left(1+\frac{\beta u\sin\theta}{mg}\right)\frac{\beta e^{-\frac{\beta t}{m}}}{m}}$$

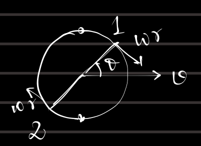
$\therefore$ We have a relation for maximal range in terms of t, θ, u.

Part - D :—

Magnus effect :— Due to rotation + translation of the ball, we see different velocities of the ball at peripheri. Due to Bernoulli's eq^{n}s, we know dynamics effect net pressure, hence there is a differential pressure force.

Approximation :—

Consider ω to be constant.



$P_1 + \frac{1}{2}\rho v_1^2 = P_0$

$P_2 + \frac{1}{2}\rho v_2^2 = P_0$

$\Rightarrow \Delta P = -2\rho v\omega r\cos\theta$

$v_1 = \sqrt{v^2+\omega^2 r^2 + 2v\omega r\cos\theta}$

$v_2 = \sqrt{v^2+\omega^2 r^2 - 2v\omega r\cos\theta}$

$\Rightarrow F = \frac{\pi\rho v\omega r}{2}$, perpendicular to propagation.

$\therefore \quad F_p = -\frac{\pi\rho\omega r\bar{v}_\perp}{2} \qquad \vec{F_v} = -\beta\bar{v}m. \qquad \bar{F}_{mg} = -mg\,\hat{y}$.

$$\bar{v}_\perp = (v_y+v_z)\,\hat{i} + (v_z+v_x)\,\hat{j} + (v_x+v_y)\,\hat{k}.$$

$$\Rightarrow \dot{v}_x = -\frac{\pi\rho\omega r(v_y+v_z)}{2m} - \frac{\beta v_x}{m} \;;\; \dot{v}_y = -\frac{\pi\rho\omega r(v_z+v_x)}{2m} - \frac{\beta v_y}{2m} - mg$$

Path integral

Kanishk

August 31, 2023

1 Introduction

The path integral formulation is a description in quantum mechanics that calculates the probability amplitude to go from a given initial state to a given final state by adding the amplitude of all the possible trajectories that the system can take between the two states. It replaces the classical notion of a single, unique classical trajectory for a system with a sum (or functional integral), over an infinity of quantum-mechanically possible trajectories to compute a quantum amplitude.

You might find it helpful to watch some videos on the path integral approach to quantum mechanics on YouTube if you are not already familiar with the basic idea behind such calculations. In this problem, we shall try to show how the maths of path integral actually works via an example.

Note:

- There are partial marks for the half solutions and procedures.

2 Finding the wave equation

Let us try to calculate the probability amplitude of a spin-zero free particle to go from one point in space to another point. To keep the calculations simple let us assume the space to have only 1 space direction (i.e. 1 dimensional) and a time dimension. Take the value of c to be 1 for this problem.

Suppose the particle can travel back and forth only with the speed of light. Let us consider the particle to travel only forward in time for this problem. Then in the x-t plane, all trajectories shuttle back and forth with slopes of $\pm 45°$, as in Fig. 1. The amplitude for such a path can be defined as follows: Suppose time is divided into small equal steps of length e. Suppose reversals of path direction can occur only at the boundaries of these steps, i.e., at $t = t_A + ne$, where n is an integer and A is the initial point.

Let there be R such reversals that the particle takes on a given path then the contribution of that path to the probability amplitude will be calculated by the following expression:

$$\phi = i\alpha^R \tag{1}$$

See the Fig. 2 for an example of calculating ψ for some points

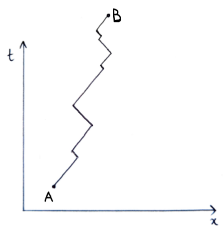


Figure 1: All lines at 45° angle

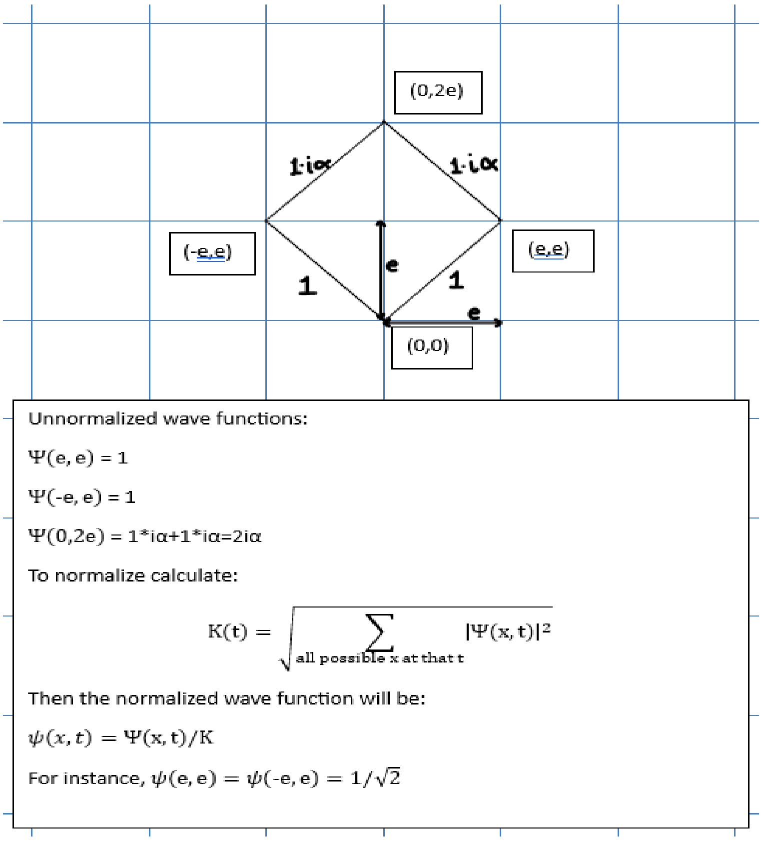


Figure 2: An example

Let's find the probability amplitude (ψ) of a particle to go from point A=(0,0) to point B=(x,t) on the xt plane , where x=Xe and t=Te

Note:

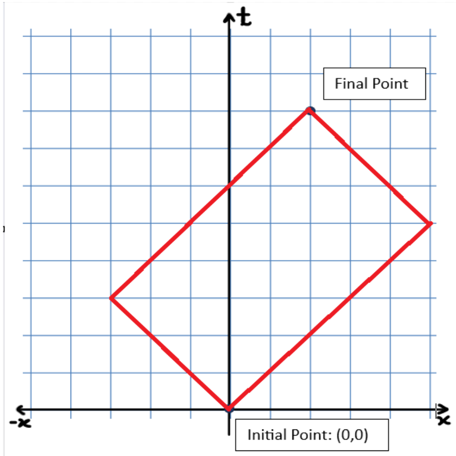
- X,$T \in \mathbb{Z}$
- x and t are a constant in this problem and in the limit e tends to zero X, T changes in a way so as to keep the value of x and t remains the same.

2.1

Find the conditions X and T should satisfy so that the probability of finding the particle on that point is non-zero.
Solution:
Since the particle can only travel at $\pm 45°$ with respect to the x-axis the condition that X and T must satisfy is that (X+T) must be divisible by 2.

2.2

There is a rectangular region within which all the paths lie. Draw the respective region for the initial and the final points in the discrete space-time lattice below:
Solution:
The rectangular region within which all the paths lie for the given initial and final points is shown below:



2.3

Calculate the number of paths connecting initial and final points having R reversals

Hint:

- It becomes a simple permutation and combination question if we count the number of paths having R reversals by breaking them into four cases:
  1. Paths that start with positive velocity and end with positive velocity ($N(R)_{++}$)
  2. Paths that start with positive velocity but end with negative velocity ($N(R)_{+-}$)
  3. Paths that start with negative velocity and end with negative velocity ($N(R)_{--}$)
  4. Paths that start with negative velocity but end with positive velocity ($N(R)_{-+}$)

  We get, $N(R) = [N(R)_{++}] + [N(R)_{+-}] + [N(R)_{--}] + [N(R)_{-+}]$

Solution:
The total number of 45° traversals made by the particle is equal to $\frac{T+X}{2}$ and the number of -45° traversals made by the particle is equal $\frac{T-X}{2}$. Let us call the +45° side of the rectangle, containing all the possible paths within, as side A and the -45° side as B.
In order to calculate $N(R)_{++}$ we have to find the number of ways we can choose $\frac{R}{2}$ points on side A (excluding the corners) and the number of ways we can choose $\frac{R}{2} - 1$ points on side B (excluding the corners) because there is a bijection between these set of set of these points and the set of paths that the particle can travel from the initial point to reach the final point with positive initial and final velocity.
The number of ways of choosing this set of points is:

$$N(R)_{++} = {}^{\frac{T+X}{2}-1}C_{\frac{R}{2}} \cdot {}^{\frac{T-X}{2}-1}C_{\frac{R}{2}-1} \tag{2}$$

Similarly, we get,

$$N(R)_{+-} = {}^{\frac{T+X}{2}-1}C_{\frac{R-1}{2}} \cdot {}^{\frac{T-X}{2}-1}C_{\frac{R-1}{2}} \tag{3}$$

$$N(R)_{--} = {}^{\frac{T+X}{2}-1}C_{\frac{R}{2}-1} \cdot {}^{\frac{T-X}{2}-1}C_{\frac{R}{2}} \tag{4}$$

$$N(R)_{-+} = {}^{\frac{T+X}{2}-1}C_{\frac{R-1}{2}} \cdot {}^{\frac{T-X}{2}-1}C_{\frac{R-1}{2}} \tag{5}$$

And we have, $N(R) = [N(R)_{++}] + [N(R)_{+-}] + [N(R)_{--}] + [N(R)_{-+}]$

2.4

Define,

$$a = \frac{T + |X|}{2} \tag{6}$$

$$b = \frac{T - |X|}{2} \tag{7}$$

Show that ψ can be written as the constant term in the binomial expression:

$$\left(1+\frac{1}{z^2}\right)^{a-1}\left(1+(i\alpha z)^2\right)^{b-1}\left(\frac{1}{z}+i\alpha z\right)^2 \tag{8}$$

Hint:

- It is best to calculate ψ by counting the number of paths $N(R)$ having R reversals connecting the initial and final point first and then writing ψ as shown in a summation shown below:

$$\psi = \sum_{all\ possible\ R} N(R)(i\alpha)^R \tag{9}$$

Solution:
Let us try to find out the constant term(c_0) in the given expression.

$$c_0 = \sum_{all\ possible\ r} \cdot^{a-1} C_r \cdot^{b-1} C_r \cdot (i\alpha)^{2r+1} + \tag{10}$$

$$\sum_{all\ possible\ r} \cdot^{a-1} C_r \cdot^{b-1} C_r \cdot (i\alpha)^{2r+1} + \tag{11}$$

$$\sum_{all\ possible\ r} {}^{a-1}C_{r+1} \cdot^{b-1} C_r \cdot (i\alpha)^{2r+2} +$$

$$\sum_{all\ possible\ r} {}^{a-1}C_r \cdot^{b-1} C_{r+1} \cdot (i\alpha)^{2r+2}$$

The first term in c_0 is equal to $\sum_{all\ possible\ R} N(R)_{+-}$.
The second term in c_0 is equal to $\sum_{all\ possible\ R} N(R)_{-+}$.
The third term plus second term term in c_0 is equal to $\sum_{all\ possible\ R}(N(R)_{++} + N(R)_{--})$.

2.5

Find the differential equation that ψ will satisfy at the point (x,t) in the limit e tends to zero.

Hint:

- We can write the following derivatives in a discrete way in the limit e tends to zero:
  - $\frac{\partial\psi(x,t)}{\partial x} = \frac{\psi(x+2e,t)-\psi(x,t)}{2e}$
  - $\frac{\partial\psi(x,t)}{\partial t} = \frac{\psi(x,t+2e)-\psi(x,t)}{2e}$
  - $\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\psi(x+2e,t)+\psi(x-2e,t)-2\psi(x,t)}{(2e)^2}$
  - $\frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{\psi(x,t+2e)+\psi(x,t-2e)-2\psi(x,t)}{(2e)^2}$
- Try to find an equation (L=0) in terms of the binomial expansion partial derivatives of ψ and ψ itself such that there is no constant term when we expand the binomial expressions. Then substitute back the partial derivatives of ψ and ψ in place of the binomial expressions.
- The value of $\frac{\alpha}{e}$ must be a finite constant as e tends to zero.
- The equation must have the same order in x and t since it is a realistically correct equation.

Solution:
Let us take $i\alpha = k/z^2$.
Let us take $\left(1+\frac{1}{z^2}\right)^{a-2}\left(1+(i\alpha z)^2\right)^{b-2}\left(\frac{1}{z}+i\alpha z\right)^2 = A_0$
Let $\gamma(x,t) = \left(1+\frac{1}{z^2}\right)^{a-1}\left(1+(i\alpha z)^2\right)^{b-1}\left(\frac{1}{z}+i\alpha z\right)^2$ Then,

$$\gamma(x,t) = A_0 \cdot (1+\frac{i\alpha}{k})(1+i\alpha k)\gamma(x+2e,t) = A_0 \cdot (1+\frac{i\alpha}{k})^2\gamma(x-2e,t) = A_0 \cdot (1+i\alpha k)^2\gamma(x,t+2e) = A_0 \cdot (1+\frac{i\alpha}{k})^2(1- \tag{12}$$

Let,

$$\frac{\partial\gamma(x,t)}{\partial t}_1 = \frac{\gamma(x,t)-\gamma(x,t-2e)}{2e} \tag{13}$$
$$\frac{\partial\gamma(x,t)}{\partial t}_2 = \frac{\gamma(x,t+2e)-\gamma(x,t-2e)}{4e} \tag{14}$$
$$\frac{\partial^2\gamma(x,t)}{\partial x^2} = \frac{\gamma(x+2e,t)+\gamma(x-2e,t)-2\gamma(x,t)}{(2e)^2} \tag{15}$$

Now,

$$\frac{\partial\gamma(x,t)}{\partial t}_1 = A_0[\frac{i\alpha k+(i\alpha)^2+\frac{i\alpha}{k}}{2e}] \tag{16}$$
$$\frac{\partial\gamma(x,t)}{\partial t}_2 = A_0[\frac{(i\alpha)^2k^2+(2i\alpha+2(i\alpha)^3)k+(4(i\alpha)^2+(i\alpha)^4)+\frac{(2i\alpha+2(i\alpha)^3)}{k}+\frac{(i\alpha)^2}{k^2}}{4e}] \tag{17}$$
$$\frac{\partial^2\gamma(x,t)}{\partial x^2} = A_0[\frac{(i\alpha)^2k^2-2(i\alpha)^2+\frac{(i\alpha)^2}{k^2}}{4e^2}] \tag{18}$$

Now let us try to make a linear equation of the form L=0 in terms of gamma and its derivatives. One such equation which is an identity for all values of x and t is given below:

$$4e\cdot\frac{\partial\gamma(x,t)}{\partial t}_2 - 2(1+(i\alpha)^2)[2e\frac{\partial\gamma(x,t)}{\partial t}_1] - 4e^2\frac{\partial^2\gamma(x,t)}{\partial x^2} + [4\alpha^2+\alpha^4](\gamma - 2e\frac{\partial\gamma(x,t)}{\partial t}_1) = 0 \tag{19}$$

Since the above equation is true irrespective of the value of z we have that the constant terms must satisfy the above equation also Simplifying the above equation and substituting α and e to be very small numbers we get the following final equation:

$$\frac{\partial^2\gamma(x,t)}{\partial x^2} = \left[\frac{\alpha}{e}\right]^2\gamma + \frac{\partial^2\gamma(x,t)}{\partial t^2} \tag{20}$$
$$\implies \frac{\partial^2\psi(x,t)}{\partial x^2} = \left[\frac{\alpha}{e}\right]^2\psi + \frac{\partial^2\psi(x,t)}{\partial t^2} \tag{21}$$

Now we have our desired differential equation that ψ given by Eq.(5) must satisfy.

3 Solving it

3.1

Let's try to apply method of Separation of Variables to solve the differential equation. This is one of the most common methods used to solve differential equations in multiple variables.

In this method, we will assume that $\psi(x,t)$ can be written as a product of two functions which are only functions of x or t but not both. After finding separable solutions we can write any physically allowed general solution as a sum of separable solutions (sometimes we need infinitely many of them). Let,

$$\psi(x,t) = f(x)\times g(t) \tag{22}$$

be a separable solution. Show that the separable solution of the differential equation is of the form:

$$f(x) = f_1 e^{\frac{-i}{\hbar}kx} + f_2 e^{\frac{i}{\hbar}kx} \tag{23}$$
$$g(t) = g_1 e^{\frac{-i}{\hbar}lt} + g_2 e^{\frac{i}{\hbar}lt} \tag{24}$$

Find the relation between k and l in terms of α and e.

Hint:

- You might find helpful to read https://andrealommen.github.io/PHY309/lectures/separation

Solution:
Applying the method of separation of variables we get,

$$\frac{\frac{\partial^2 f(x)}{\partial x^2}}{f(x)} - \frac{\frac{\partial^2 g(t)}{\partial t^2}}{g(t)} = \left[\frac{\alpha}{e}\right]^2 \tag{25}$$

Let $\frac{\frac{\partial^2 f(x)}{\partial x^2}}{f(x)} = c$ and $\frac{\frac{\partial^2 g(t)}{\partial t^2}}{g(t)} = d$ where l and k are constant numbers satisfying $c - d = \left[\frac{\alpha}{e}\right]^2$. Upon solving the above two differential equations we get that the solution of f(x) and g(x) must be of the form:

$$f(x) = f_1 e^{-\sqrt{c}x} + f_2 e^{\sqrt{c}x} \tag{26}$$

$$g(t) = g_1 e^{-\sqrt{d}t} + g_2 e^{\sqrt{d}t} \tag{27}$$

Thus the relation between k and l must be:

$$\left[i\frac{k}{\hbar}\right]^2 - \left[i\frac{l}{\hbar}\right]^2 = \left[\frac{\alpha}{e}\right]^2 \tag{28}$$

$$\implies l^2 - k^2 = \left[\frac{\alpha\hbar}{e}\right]^2 \tag{29}$$

3.2

Can you identify (/guess) what the various arbitrary constants in the separable solution of the above form symbolize in reality?
Solution:
k symbolizes momentum(p) and l symbolizes energy(E).

4 Particles with non-zero spin also satisfy this equation

4.1

It is only that in the case of particles with zero spin, it gives a complete description of the evolution of wave function. In other cases, this equation is not enough to describe the wave function but whatever the wave equation is it will also satisfy our differential equation. Thus we can say that all one-dimensional wave functions that exist in reality satisfy our differential equation.

Let's take the Dirac equation in 1 dimension for instance.

The wave function (ψ) for a spin half particle in 1 dimension has two components, $\psi = \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}$

Both components satisfy the following first-order differential equations:

$$i\hbar\frac{\partial\psi_1}{\partial t} = m\psi_1 - i\hbar\frac{\partial\psi_2}{\partial x} \tag{30}$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = -m\psi_2 - i\hbar\frac{\partial\psi_1}{\partial x} \tag{31}$$

Show that both ψ_1 and ψ_2 satisfy our differential equation.

Also, find the value of $\frac{\alpha}{e}$ by comparing our differential equation with the differential equation obtained in this question.
Solution:
Differentiating equation (30) with respect to time(t) and equation (31) with respect to space(x) and then subtracting one from the other gives us:

$$i\hbar\frac{\partial^2\psi_1}{\partial t^2} = m\frac{\partial\psi_1}{\partial t} + m\frac{\partial\psi_2}{\partial x} + i\hbar\frac{\partial^2\psi_1}{\partial x^2} \tag{32}$$

Using equation (30) we get,

$$i\hbar\frac{\partial^2\psi_1}{\partial t^2} = \frac{m^2}{i\hbar}\psi_1 + i\hbar\frac{\partial^2\psi_1}{\partial x^2} \tag{33}$$

$$\implies \frac{\partial^2\psi_1}{\partial x^2} = \left[\frac{m}{\hbar}\right]^2\psi_1 + \frac{\partial^2\psi_1}{\partial t^2} \tag{34}$$

Hence we get $\frac{\alpha}{e} = \pm\frac{m}{\hbar}$

5 Deriving Schrodinger's equation from our differential equation

5.1

In this section, we will show that Schrodinger's equation is the same as our differential equation in the non-relativistic limit.

Let $\psi(x,t) = Ae^{\frac{-i}{\hbar}(lt-kx)}$
Let $\phi(x,t) = \psi(x,t)\cdot e^{\frac{i}{\hbar}mt}$

Here ϕ only differs from ψ in phase. Since probability density only depends upon the magnitude square of the wave function the probability density given by both the wave functions at any given (x,t) will be the same.
Show that $\phi(x,t)$ satisfies the following differential equation in the limit k⋘m :

$$\frac{-\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} = i\hbar\frac{\partial\phi}{\partial t} \tag{35}$$

Note that the above differential equation is the same as Schrodinger's equation with no potential term (as expected for a non-relativistic free particle).

Hint:

- You might need to use the binomial approximation `https://en.wikipedia.org/wiki/Binomial_approximation`

Solution:
From equation (29) and problem 4.1, we get that $l^2 - k^2 = m^2$

$$\psi(x,t) = Ae^{\frac{-i}{\hbar}(\sqrt{k^2+m^2}t-kx)} \tag{36}$$

In non-relativistic limit:

$$\psi(x,t) = Ae^{\frac{-i}{\hbar}((m+\frac{k^2}{2m})t-kx)} \tag{37}$$

$$\implies \phi(x,t) = Ae^{\frac{-i}{\hbar}(\frac{k^2}{2m}t-kx)} \tag{38}$$

Now,

$$\frac{-\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} = \frac{-\hbar^2}{2m}\left[\frac{ik}{\hbar}\right]^2 = \frac{k^2}{2m} \tag{39}$$

$$i\hbar\frac{\partial\phi}{\partial t} = i\hbar\left[\frac{-ik^2}{2\hbar m}\right] = \frac{k^2}{2m} \tag{40}$$

Thus $\phi(x,t)$ satisfies the given differential equation in relativistic limits.

Minimal surface solution

Samanway

November 2023

1 Solution

We have a parametrization of the form:

$$\vec{X} : (x, y) \mapsto (x, y, z)$$

where:

$$z = a\left(ln\left(cos\left(\frac{y}{a}\right)\right) - ln\left(cos\left(\frac{x}{a}\right)\right)\right)$$

$$\vec{X}_x = \frac{\partial \vec{X}}{\partial x} = (1, 0, z_x)$$

$$\vec{X}_y = \frac{\partial \vec{X}}{\partial y} = (0, 1, z_y)$$

At a given point, these vectors span the tangent plane to the surface.
We calculate the components of the first fundamental form:

$$E = \langle \vec{X}_x, \vec{X}_x \rangle = 1 + z_x^2$$

$$F = \langle \vec{X}_x, \vec{X}_y \rangle = z_x z_y$$

$$G = \langle \vec{X}_y, \vec{X}_y \rangle = 1 + z_x^2$$

The unit normal is given by:

$$\vec{N} = \frac{\vec{X}_x \wedge \vec{X}_y}{\sqrt{EG - F^2}} = \frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}}$$

Note that:

$$\vec{X}_{xx} = (0, 0, z_{xx})$$

$$\vec{X}_{xy} = (0, 0, z_{xy})$$

$$\vec{X}_{yy} = (0, 0, z_{yy})$$

We calculate the components of the second fundamental form:

$$e = \langle \vec{X}_{xx}, \vec{N} \rangle = \frac{z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}}$$

$$f = \langle \vec{X}_{xy}, \vec{N} \rangle = \frac{z_{xy}}{\sqrt{1 + z_x^2 + z_y^2}}$$

$$g = \langle \vec{X}_{yy}, \vec{N} \rangle = \frac{z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}}$$

We use the formula for mean curvature:

$$H = \frac{Ge + Eg - 2fF}{2(EG - F^2)} = \frac{(1 + z_x^2)z_{yy} + (1 + z_y^2)z_{xx} - 2(z_x z_y)z_{xy}}{2(1 + z_x^2 + z_y^2)^{3/2}}$$

As defined, mean curvature of minimal surface is zero everywhere. We see that $H \equiv 0$ if and only if:

$$(1 + z_x^2)z_{yy} + (1 + z_y^2)z_{xx} - 2(z_x z_y)z_{xy} = 0$$

This equation is known as the **minimal surface equation**. Satisfying this equation is a necessary and sufficient condition to be a minimal surface.

For the first Scherk surface:

$$z_x = \frac{1}{\cos(x/a)} \sin(x/a) = \tan(x/a)$$

$$z_y = \frac{1}{\cos(y/a)} - \sin(y/a) = -\tan(y/a)$$

$$z_{xy} = 0$$

$$z_{xx} = \frac{\sec^2(x/a)}{a}$$

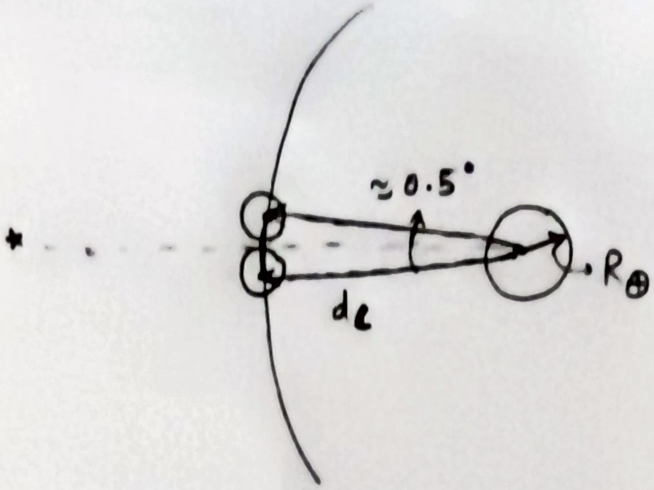
$$z_{yy} = \frac{-\sec^2(y/a)}{a}$$

$$(1+z_x^2)z_{yy}+(1+z_y^2)z_{xx}-2(z_x z_y)z_{xy} = \frac{-\sec^2(x/a)\sec^2(y/a)}{a} + \frac{sec^2(y/a)sec^2(x/a)}{a} + 0 = 0$$

Hence we are done.
(Please refer to “Differential geometry of curves and surfaces” by Manfredo P.do Carmo : Chapters 2 and 3 for First fundamental form, Second fundamental form and mean curvature.)

Solutions :

3.1



[Moon occultation]

$\theta = 32' = \left(\frac{32}{60}\right)^\circ$;

$d_☾ = 3.8 \times 10^8$ m

$R_\oplus = 6.34 \times 10^5$ m

$\omega_\oplus = 4.16 \times 10^{-3}$ deg/sec.

$\omega_☾ = 1.525 \times 10^{-4}$ deg/sec.

$t = \dfrac{\theta \cdot d_☾}{\omega_☾ \cdot d_☾ - \omega_\oplus R_\oplus}$; for t_{max}.

$t \simeq 1.78$ hrs.

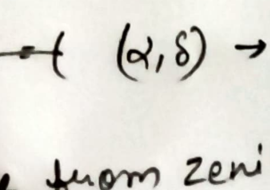
3.2

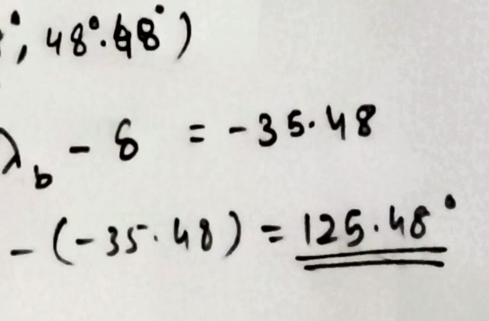
$\lambda_{bangalore} = 12.97°.N \approx 13°.N$

~~(α, δ)~~ $(\alpha, \delta) \rightarrow (\underline{\underline{239.28}}°, 48°.48°)$

angle from zenith $= \lambda_b - \delta = -35.48$

Altitude of star $= 90 - (-35.48) = \underline{\underline{125.48°}}$





$\left(\frac{360}{12} \rightarrow 30°\right)$

when sun is here it will be best time

→ So somewhere around May will be the best time

3.3

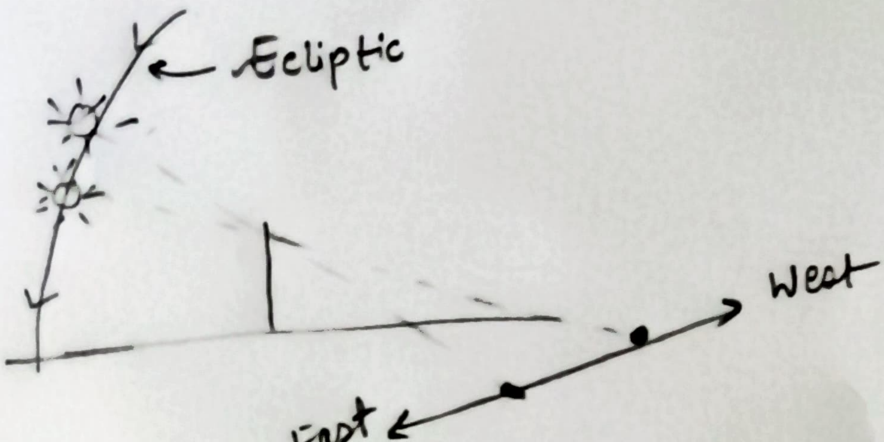
$$\sin(b) = \sin(\delta_{NGP})\sin(\delta) + \cos(\delta_{NGP})\cdot\cos(\delta)\cos(\alpha-\alpha_{NGP})$$

$$\cos(b)\cdot\sin(l_{NGP}-l) = \cos(\delta)\sin(\alpha-\alpha_{NGP})$$

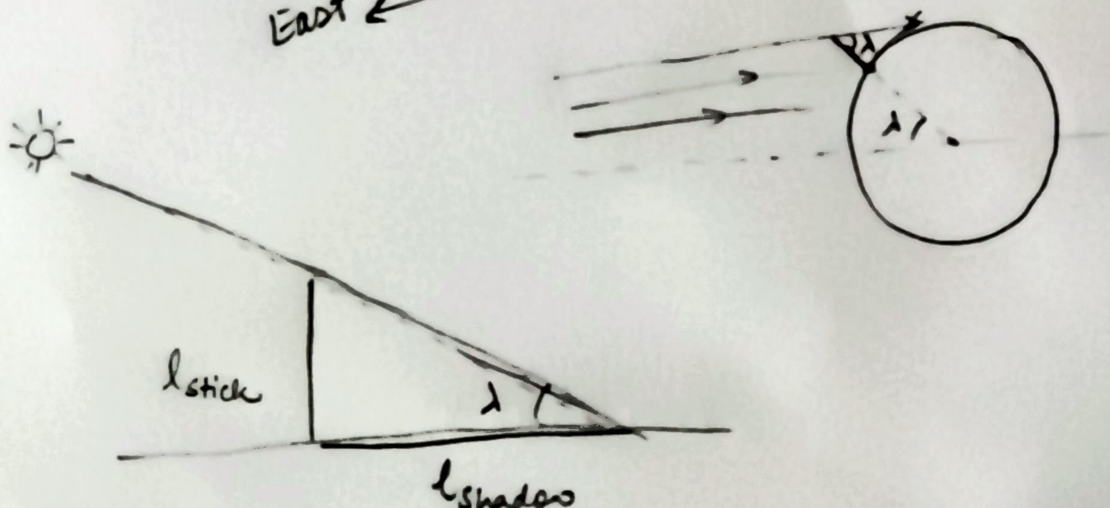
$$\cos(b)\cdot\cos(l_{NGP}-l) = \cos(\delta_{NGP})\sin(\delta) - \sin(\delta_{NGP})\cos(\delta)\cdot\cos(\alpha-\alpha_{NGP})$$

$\rightarrow \alpha_{NGP} = 12^h.51.4^m$; $\delta_{NGP} = 27.13^\circ$; $l_{NGP} = 122.933^\circ$

3.4.



3.5.



$$\lambda = \tan^{-1}\left(\frac{l_{stick}}{l_{shadow}}\right);$$

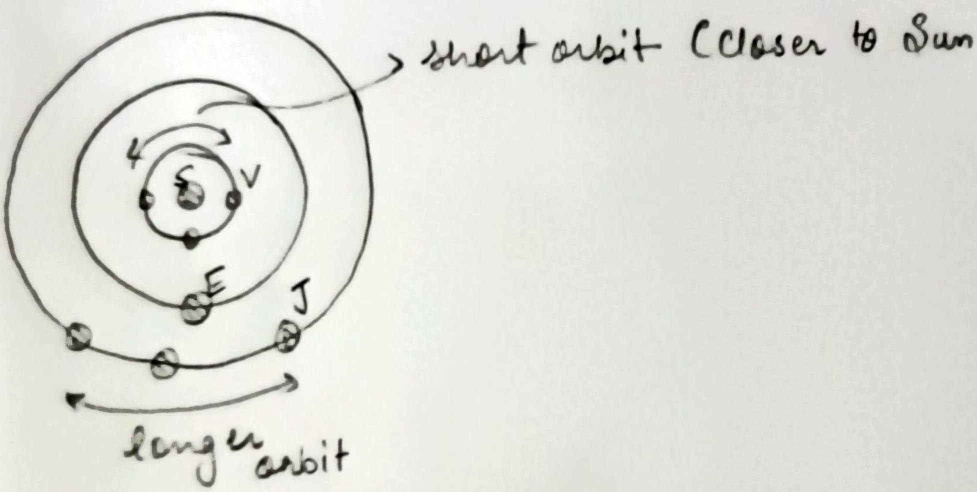
latitude.

3.6

(1) Akshank
(2) Akshank
(3) kenil
(4) Gaurav
(5) kenil
(6) Gaurav
(7) Hemansh
(8) kenil

~~this~~ Concept was to find t different using change in latitude.

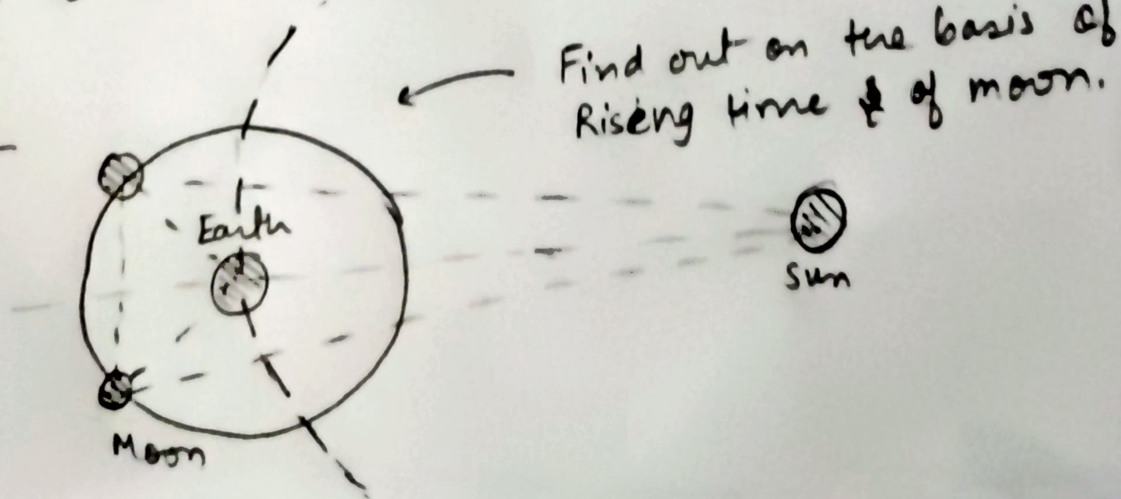
3.7



3.8

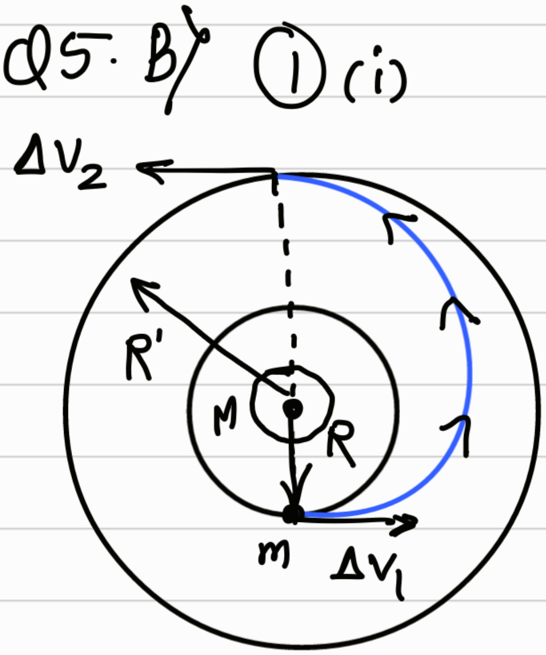
8 times

3.9



Orbit transfer.

Q5. B) ① (i)



← Hohmann's transfer orbit.

$$P_i = -\frac{GMm}{R} \quad ; \quad K_i = \frac{1}{2}mv_i^2$$

$$P_f = -\frac{GMm}{R'} \quad ; \quad K_f = \frac{1}{2}mv_2^2$$

* Angular momentum will be conserved about plant.

$\therefore mv_1 \cdot R = mv_2 \cdot R'$

$\therefore \boxed{v_1 R = v_2 R'}$

$\therefore PE_i + KE_i = PE_f + KE_f$

conservation of mechanical energy.

$$\therefore \frac{-GMm}{R} + \frac{1}{2}mv_1^2 = \frac{-GMm}{R'} + \frac{1}{2}mv_2^2$$

$$\therefore \frac{1}{2}m(v_1^2 - v_2^2) = GMm\left(\frac{R' - R}{RR'}\right)$$

$$v_2 = \frac{v_1 R}{R'}$$

$$\Rightarrow \frac{1}{2}m\left(\frac{(R')^2 - R^2}{(R')^2}\right)v_1^2 = \frac{GMm(R' - R)}{R \cdot R'}$$

$$\frac{1}{2}\frac{\cancel{(R' - R)}(R' + R)}{R'}v_1^2 = \frac{GM}{R}\cancel{(R' - R)}$$

$$\therefore v_1^2 = \frac{2GM}{R}\left(\frac{R'}{R' + R}\right)$$

$$\left\{ \therefore v_1 = \sqrt{\frac{2GMR'}{R(R' + R)}} \right\}$$

$$\Delta V_1 = V_1 - V$$

$V \rightarrow$ velocity of satellite while orbiting circle of radius R.

$$\Delta V_1 = \sqrt{\frac{GM(2R')}{R(R'+R)}} - \sqrt{\frac{GM}{R}}$$

$$\therefore \boxed{\Delta V_1 = \sqrt{\frac{GM}{R}}\left(\sqrt{\frac{2R'}{R+R'}} - 1\right)}$$

$$V_2 = \frac{V_1 \cdot R}{R'}$$

$$\therefore V_2 = \sqrt{\frac{2GM \cdot R'}{R(R'+R)}} \cdot \frac{R}{R'} = \sqrt{\frac{2GM \cdot R}{R'(R'+R)}}$$

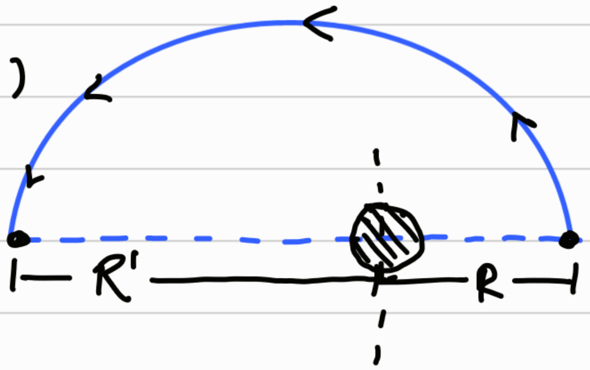
$$\Delta V_2 = V_0 - V_2$$

V_0 – velocity of satellite while orbiting circle of radius R'.

$$\Delta V_2 = \sqrt{\frac{GM}{R'}} - \sqrt{\frac{GM \cdot 2R'}{R'(R'+R)}}$$

$$\boxed{\Delta V_2 = \sqrt{\frac{GM}{R'}}\left(1 - \sqrt{\frac{2R'}{R'+R}}\right)}$$

(ii)



$$a(1-e) = R$$
$$a(1+e) = R'$$

$$\therefore\ 2a = R + R' \Rightarrow \boxed{a = \frac{R+R'}{2}}$$

$$\therefore\ 1 - e = \frac{2R}{R+R'}$$

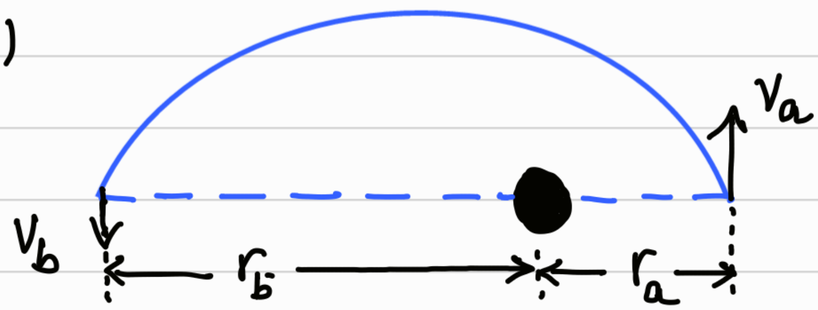
$$\rightarrow \boxed{e = \frac{R'-R}{R'+R}}$$

(iii) According to kepler's third law

$t_H = \pi\sqrt{\frac{R^3}{GM}}$ $\Rightarrow$ R in our case is $a = \frac{R'+R}{2}$

$$\therefore \boxed{t_H = \pi\sqrt{\frac{(R'+R)^3}{8GM}}}$$

(iv)



$$\frac{1}{2}mV_a^2 - \frac{GMm}{r_a} = \frac{1}{2}mV_b^2 - \frac{GMm}{r_b}$$

$\therefore$ so as we have calculated in (i) question

$$\frac{1}{2}V_a^2 = \frac{GM\,r_b}{r_a(r_a+r_b)}$$

$$\frac{1}{2}V_a^2 = \frac{GM(2a-r_a)}{r_a(2a)}$$

$$\therefore \frac{V_a^2}{2} = \frac{GM}{r_a} - \frac{GM}{2a}$$

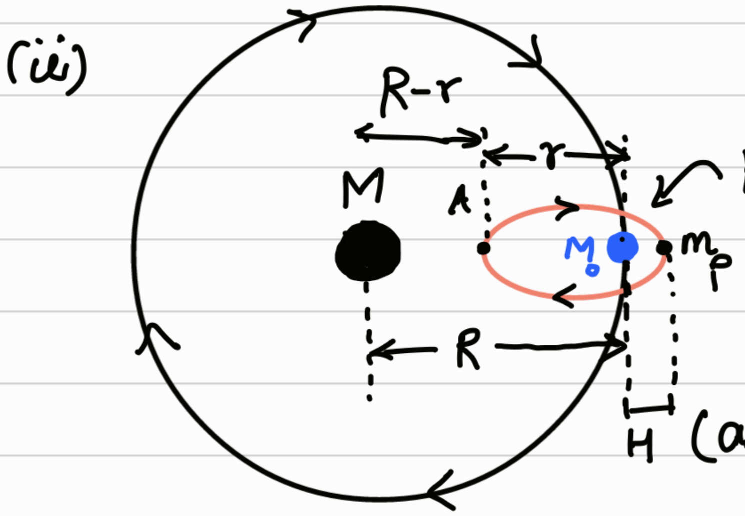
$$\frac{V_a^2}{2} = GM\left(\frac{1}{r_a} - \frac{1}{2a}\right)$$

∴ In general we can write

$$\boxed{v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

→ It's also called vis-viva equation.

(iii)



→ F.B.D of satellite at apogee.

$$F_p \longleftarrow\!\!\bullet\!\!\longrightarrow F_m$$

$$\longrightarrow a_c$$

$$F_m - F_p = m\,a_c$$

$$a_c = \underbrace{\omega_m^2}_{\text{angular velocity of moon}} \cdot r$$

$$\therefore \quad \frac{GMm}{(R-r)^2} - \frac{GM_0 m}{r^2} = m\omega_m^2 (R-r)$$

$$\omega_m = ??$$

$$\Rightarrow \frac{GMM_0}{R^2} = M_0 \omega^2 \cdot R$$

$$\therefore \omega = \sqrt{\frac{GM}{R^3}}$$

$$\therefore \frac{\cancel{G}M}{(R-r)^2} - \frac{\cancel{G}M_0}{r^2} = \frac{\cancel{G}M(R-r)}{R^3}$$

$$\frac{M}{R^2\left(1-\frac{r}{R}\right)^2} - \frac{M_0}{r^2} = \frac{M(R-r)}{R^3}$$

$$\frac{M}{R^2}\left(1-\frac{r}{R}\right)^{-2} - \frac{M_0}{r^2} = \frac{M(R-r)}{R^3}$$

$$R >> r$$

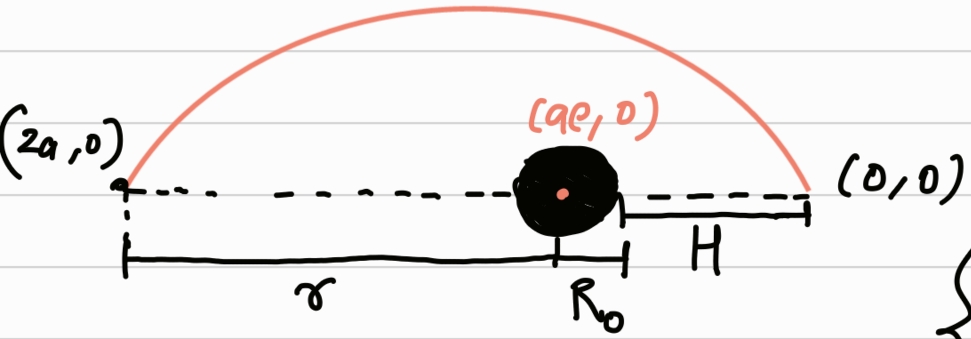
$$\therefore \frac{M}{R^2}\underbrace{\left(1+\frac{2r}{R}\right)}_{\text{binomial approx}} - \frac{M_0}{r^2} = \frac{M}{R^2} - \frac{Mr^2}{R^3}$$

$$\frac{2Mr}{R^3} + \frac{Mr}{R^3} = \frac{M_0}{r^2}$$

$$\frac{3Mr}{R^3} = \frac{M_0}{r^2}$$

$$\therefore \boxed{r = \sqrt[3]{\frac{M_0}{3M}} \cdot R}$$ (apogee distance)

so called radius of hill's sphere of the moon



$$\left\{ e = \frac{r - (R_0 + H)}{r + (R_0 + H)} \right\}$$

as earlier mentioned in solution.

$$\therefore \boxed{e = \frac{\sqrt[3]{\frac{M_0}{3M}}R - (R_0 + H)}{\sqrt[3]{\frac{M_0}{3M}}R + (R_0 + H)}}$$