Derivatives : forwards, futures, swaps and options $\longrightarrow$ derivative contracts

(written on stocks, currencies ...)

— risk management, pricing of contracts (prevent arbitrage)

— implementation of pricing concepts

Chapter 1. Introduction

A derivative is a financial instrument whose value depends on or derives from the value of an underlying asset.

$\Rightarrow$ limited time to maturity

Forward contract is an agreement to trade the underlying asset in the future (at T) for a fixed price specified today (forward price $F_t^{(T)}$ at time t)

pre determined

buy the asset (long position) $S_T > F_t^{(T)} \Rightarrow$ profit $S_T < F_t^{(T)} \Rightarrow$ loss

sell the asset (short position) $S_T < F_t^{(T)} \Rightarrow$ profit $S_T > F_t^{(T)} \Rightarrow$ loss

Transferring risks in the economy

OTC transactions > Exchange - Traded Markets

credit crunch : refuse to lend money

Hedgers

— reduces risk (foreign exchange rate derivatives)

looks like gambling but reduces uncertainty

payoff is negatively correlated with the corn price $\longrightarrow$ eliminate the price risk

Independent of the corn price you earn $\$x$.

Speculation (getting extra leverage)

E.g. Expect gold price ↑

→ take a long position in a forward or futures contract to lock in a price.

Arbitrage is a strategy that generates a positive payoff with 0 probability and a negative payoff with zero probability.

E.g. current stock price : \$571.5

You could enter a forward contract to buy 1000 shares of Apple for \$600000 1 year.

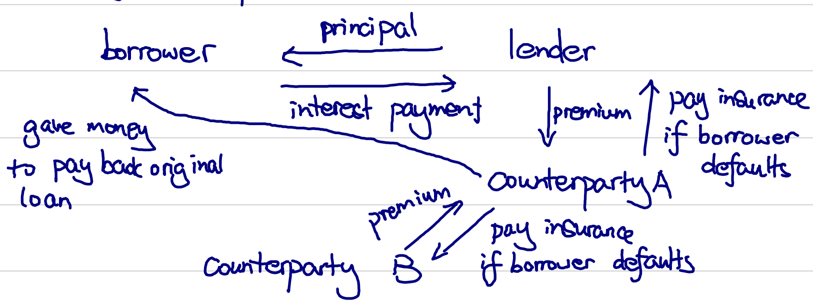
risk free rate : 1%

	$t=0$	$t=1$
buy 1000 shares :	$-\$571500$	$1000 \cdot S_1$
short position :	\$0	$600000 - 1000 \cdot S_1$
borrow :	$+\$571500$	$-\$571500 \cdot 1.01 = -\$577215$
	CF = \$0	\$22785

☆ Principle of Asset Pricing ⇒ No arbitrage

Traders can switch from hedgers to speculators or arbitrageurs to speculators

CDS : credit default swap



CDS (borrower 不给钱)

Chapter 2 Interest Rates

Treasury Rates : risk-free

Bills : 4 weeks to 1 year

Notes : 2–10 years } semi annual payments

Bonds : 20-30 years

TIPS : principal adjusted to CPI (risk free in real terms)

Federal Funds Rate (Interbank loan interest rate)

— overnight

— can directly borrow from Fed

Repurchase Agreement (Repo Rate)

— almost risk free

— better than collateralized 抵押 loan

LIBOR

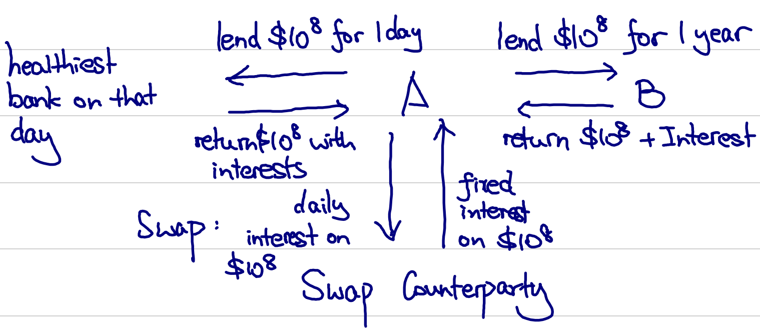
— funding cost of large banks

— risk free rate to price derivatives

— large banks might default (Lehman Brothers)

SOFR (based on repo)

Overnight Indexed Swap (OIS) Rates



$$EAR = \left(1+\frac{r}{m}\right)^{m(T-t)} - 1$$

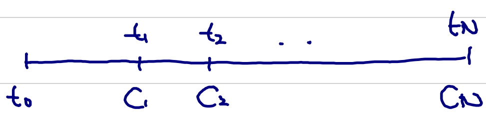
$$\lim_{m\to\infty} EAR = e^{r(T-t)} - 1 \qquad r: \text{ quoted rate}$$

$$A\left(1+\frac{r_m}{m}\right)^m = Ae^{r_c}$$

$$r_c = m\ln\left(1+\frac{r_m}{m}\right) \qquad r_m = m\left(e^{\frac{r_c}{m}} - 1\right)$$

Spot rates : $r_{t,T}$ $\quad t \xrightarrow{r_{t,T}} T$

Bond Price .



$$P_0 = C_1 e^{-r_{0,t_1}(t_1-0)} + C_2 e^{-r_{0,t_2}(t_2-0)} + \dots + C_n e^{-r_{0,t_N}(t_N-0)}$$

$$= \sum_{i=1}^{N} e^{-(r_{0,t_i})t_i} C_i \qquad \text{continuous}$$

$$P_0 = \sum_{i=1}^{N} \frac{C_i}{\left(1+\frac{r_{0,t_i}}{m}\right)^{mt_i}} \qquad \text{m-times compounding per annum}$$

Coupon = Face Value $\left(\frac{\text{Coupon rate}}{n}\right)$ $\quad$ n coupon payments per year

when $P_0 = F$ $\quad$ (traded at par)

* use specific spot rate for that period to discount

Bond yield : $P_0 = \sum_{i=1}^{N} e^{-y_c t_i} C_i$ $\quad$ weighted value of spot rates

$$P_0 = \sum_{i=1}^{N} \frac{C_i}{\left(1+\frac{y_m}{m}\right)^{mt_i}}$$

yield = coupon rate if trades at par

Duration : $D = \sum_{i=1}^{N} t_i \frac{C_i e^{-yt_i}}{P_0}$ $\quad$ (how long has to wait for cash flows)

D is large if the bond pays out much of its cash flows late

Bond Volatility $= \dfrac{\frac{\partial P_0(y)}{\partial y}}{P_0} = \dfrac{\frac{\partial}{\partial y}\left(\sum_{i=1}^{N} e^{-y t_i} C_i\right)}{P_0} = \dfrac{\sum_{i=1}^{N} (-t_i) e^{-y t_i} C_i}{P_0} = -D$ continuous

yield $\uparrow$, bond price $\downarrow$ $\quad$ $T\uparrow$, bond price $\downarrow\downarrow$

m-times c.p.a. $= \dfrac{\frac{\partial}{\partial y}\left(\sum_{i=1}^{N} \frac{C_i}{(1+\frac{y}{m})^{m t_i}}\right)}{P_0} = \dfrac{\sum_{i=1}^{N} \frac{(-t_i) C_i}{(1+\frac{y}{m})^{m t_i + 1}}}{P_0} = \dfrac{-D}{1+\frac{y}{m}}$ (modified duration)

$$\frac{\Delta P_0}{P_0} \approx \frac{\frac{\partial P_0(y)}{\partial y}}{P_0} \Delta y$$

Bond convexity $= \dfrac{\frac{\partial^2 P_0(y)}{\partial y^2}}{P_0}$ $\qquad$ $\dfrac{\Delta P_0}{P_0} \approx \dfrac{\frac{\partial P_0(y)}{\partial y}}{P_0} \Delta y + \dfrac{1}{2} \dfrac{\frac{\partial^2 P_0(y)}{\partial y^2}}{P_0} (\Delta y)^2$

c.c.

$$\underbrace{\underbrace{t \longrightarrow T_1}_{r_{t,T_1}} \underbrace{\longrightarrow T_2}_{f_t^{(T_1,T_2)}}}_{r_{t,T_2}}$$

$\underline{=\$0}$ $\qquad\qquad\qquad\qquad$ $\underline{>\$0}$

$-\$1 \xrightarrow{\quad =\$0 \quad} +\$e^{r_{t,T_2}(T_2-t)}$

$+\$1 \longrightarrow -\$e^{r_{t,T_1}(T_1-t)} \longrightarrow \$e^{r_{t,T_1}(T_1-t)} e^{f_t^{(T_1,T_2)}(T_2-T_1)}$

$+\$e^{r_{t,T_1}(T_1-t)}$

$$\Rightarrow r_{t,T_2}(T_2-t) = r_{t,T_1}(T_1-t) + f_t^{(T_1,T_2)}(T_2-T_1)$$

$$\Rightarrow f_t^{(T_1,T_2)} = \frac{r_{t,T_2}(T_2-t) - r_{t,T_1}(T_1-t)}{T_2-T_1}$$

$$f_t^{(T_1,T_2)} < \frac{r_{t,T_2}(T_2-t) - r_{t,T_1}(T_1-t)}{T_2-T_1}$$

borrow \$1 from t to T_1

sign FRA to borrow $\$e^{r_{t,T_1}(T_1-t)}$ from T_1 to T_2 of $f_t^{(T_1,T_2)}$

lend \$1 from t to T_2

Forward Rate Agreement (FRA)

Lender payoff at T_1: $L\left[\left(1+\frac{f_t^{(T_1,T_2)}}{m}\right)^{m(T_2-T_1)} e^{-r_{T_1,T_2}(T_2-T_1)} - 1\right]$

Lender value at $s \in [t, T_1]$:

$$L\left[\left(1+\frac{f_t^{(T_1,T_2)}}{m}\right)^{m(T_2-T_1)} e^{-r_{s,T_2}(T_2-s)} - e^{-r_{s,T_1}(T_1-s)}\right]$$

FRA only in OTC markets

Chp. 3 Forwards

Short selling a security you do not own (看跌)

300 (sell in the market) $\longrightarrow$ 250

$\updownarrow$ 50 buy back 250 $\longleftarrow$

If company issues dividend, you have to compensate lender

OTC traded ☆

Tailor made contracts

Network of dealers linked through telephones and computers

} flexibility

often cash settlement

collateral possible (给钱抵押)

Forward contract is worthless

long: $S_T > F_t^{(T)}$ (profit)

short: $S_T < F_t^{(T)}$ (profit)

Assumption:

no transaction costs

same tax rate on all trading profits

c.c.

exploit arbitrage

$100 now   expected $105   risk free rate = 10%

$t=0$	$t=1$	
$+\$10000000$	$-100000S_T$	shorting
$-\$10000000$	$+10000000e^{0.1}$	lending
0	$+100000S_T - \$10500000$	forward
$=\$0$	$>\$0$	

Fair price: borrow $F_t^{(T)} e^{-r_{t,T}(T-t)}$ $-F_t^{(T)}$

buy 1 unit of asset: $-S_t$ S_T

short forward : 0 $F_t^{(T)} \quad -S_T$

$F_t^{(T)} e^{-r_{t,T}(T-t)} - S_t = 0$ $\$0$

$$\Rightarrow F_t^{(T)} = S_t e^{r_{t,T}(T-t)}$$

$$\Leftrightarrow \underbrace{\frac{F_t^{(T)}}{S_t}}_{\text{perfect hedge}} = \underbrace{e^{r_{t,T}(T-t)}}_{\text{zero rate}}$$

$$\begin{array}{cccccc} t & s_1 & s_2 & \cdots & s_N & T \\ & I_{s_1} & I_{s_2} & \cdots & I_{s_N} & \end{array}$$

$$I_t = \sum_{s_i} I_{s_i} e^{-r_{t,s_i}(s_i-t)}$$

$$F_t^{(T)} = (S_t - I_t) e^{r_{t,T}(T-t)}$$

$$F_t^{(T)} = S_t e^{r_{t,T}(T-t)} - I_t e^{r_{t,T}(T-t)}$$

$$\frac{F_t^{(T)}}{S_t} = e^{r_{t,T}(T-t)} - \frac{I_t e^{r_{t,T}(T-t)}}{S_t}$$

$$\underbrace{\frac{F_t^{(T)}}{S_t}}_{\text{perfect hedge}} + \frac{\sum_{s_i} I_{s_i} \overbrace{e^{-r_{t,s_i}(s_i-t)} e^{r_{t,T}(T-t)}}^{e^{f_t(s_i,T)(T-s_i)}}}{S_t} = e^{r_{t,T}(T-t)}$$

$t=0$ ———— $s_1=\frac{2}{12}$ ———— $s_2=\frac{4}{12}$ ———— $T=\frac{6}{12}$

$I_{s_1}=2$ $\quad I_{s_2}=4$

	$t=0$	s_1	s_2	T
buy 10000 shares:	$-\$100000$	$+\$2000$	$+\$2000$	$1000S_T$
short forward :				$\$100000 - 1000S_T$
borrow $100000e^{-0.05\frac{6}{12}}$	97531	0	0	$-\$100000$
borrow $2000e^{-0.05\frac{2}{12}}$	1983	$-\$2000$	0	0
borrow $2000e^{-0.05\frac{4}{12}}$	1967	0	$-\$2000$	0
	$=\$1481$	0	0	0

t ———— $t+dt$ ———— $t+2dt$ ———— T $\qquad dt=\frac{1}{252}$

$S_t \qquad S_{t+dt} \qquad S_{t+2dt}$

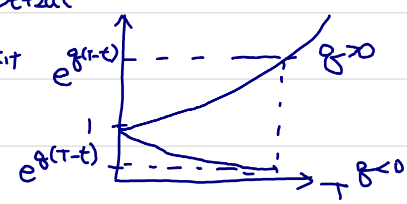
$D_{t+dt}=q\,dt\,S_{t+dt} \qquad D_{t+2dt}=q\,dt\,S_{t+2dt} \qquad \overbrace{D_{t+2dt}}^{1 \text{ unit}} + \underbrace{q\,dt}_{q\,dt \text{ unit}}\,D_{t+2dt}$

$$\boxed{\frac{D_{t+dt}}{S_{t+dt}}=q\,dt \text{ units}} \qquad \frac{(1+q\,dt)D_{t+2dt}}{S_{t+2dt}}=(1+q\,dt)q\,dt$$

1 unit $\longrightarrow 1+q\,dt \longrightarrow 1+q\,dt+(1+q\,dt)q\,dt \rightarrow\rightarrow\rightarrow e^{q(T-t)}$ units

$=(1+q\,dt)^2$

number of units we can buy.



$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} \qquad x=\frac{n}{r}$

$n=xr$

$\lim_{n\to\infty}\left(1+\frac{1}{x}\right)^{xrt}$

$=\left(\lim_{n\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)^{rt}$

$=e^{rt}$

$$F_t^{(T)}=S_t e^{(r_{t,T}-q_{t,T})(T-t)} \quad \text{or} \quad \ln\left(\frac{F_t^{(T)}}{S_t}\right)+q_{t,T}(T-t)=r_{t,T}(T-t)$$

t ———————— T

	t	T
borrow :	$+F_t^{(T)}e^{-r_{t,T}(T-t)}$	$-F_t^{(T)}$
buy :	$-e^{-q_{t,T}(T-t)}S_t$	$e^{-q_{t,T}(T-t)}e^{q_{t,T}(T-t)}S_t$
short forward	0	$F_t^{(T)}-S_T$
		$=\$0$

$$F_t^{(T)}e^{-r_{t,T}(T-t)}-e^{-q_{t,T}(T-t)}S_t=0$$

$$F_t^{(T)}=S_t e^{-q_{t,T}(T-t)}\cdot e^{r_{t,T}(T-t)}$$

$$\begin{tikzcd} t \arrow[r, dash] & T \\ \$1 \arrow[r] \arrow[d] & \$e^{r_{t,T}(T-t)} \\ \text{CHF}\,\frac{1}{E_t} \arrow[r] & \text{CHF}\,\frac{1}{E_t}\, e^{r^f_{t,T}(T-t)} \end{tikzcd}$$

$$= \$\, F_t^{(T)} \frac{1}{E_t}\, e^{r^f_{t,T}(T-t)}$$

$$\Rightarrow \quad F_t^{(T)} = E_t\, e^{(r_{t,T} - r^f_{t,T})(T-t)}$$

$$\Rightarrow \quad \frac{F_t^{(T)}}{E_t} = e^{(r_{t,T} - r^f_{t,T})(T-t)} \qquad \text{CIP}$$

$$\begin{tikzcd} t \arrow[r, dash] & T \\ \$1 \arrow[r] \arrow[d] & \$\left(1+\frac{r_{t,T}}{m}\right)^{m(T-t)} \\ \text{CHF}\,\frac{1}{E_t} \arrow[r] & \text{CHF}\,\frac{1}{E_t}\left(1+\frac{r^f_{t,T}}{m}\right)^{m(T-t)} \end{tikzcd}$$

$$= \$\, F_t^{(T)} \left(\frac{1}{E_t}\right)\left(1+\frac{r^f_{t,T}}{m}\right)^{m(T-t)}$$

$$\Rightarrow \quad F_t^{(T)} = E_t \, \frac{\left(1+\frac{r_{t,T}}{m}\right)^{m(T-t)}}{\left(1+\frac{r^f_{t,T}}{m}\right)^{m(T-t)}}$$

$$\frac{x}{E_t} e^{r^f_{t,T}(T-t)} E_t - x e^{r_{t,T}(T-t)}$$

$$= \frac{x}{E_t} e^{r^f_{t,T}(T-t)} \left(E_t - F_t^{(T)}\right)$$

borrow x in home currency

UIP. $$E\left[\frac{E_T}{E_t}\right] = e^{(r_{t,T} - r^f_{t,T})(T-t)}$$

Consumption asset : If $F_t^{(T)} < S_t e^{(r_{t,T} - q_{t,T})(T-t)}$ No arbitrage, since we can't short sell

$$\Rightarrow \quad F_t^{(T)} \leq S_t\, e^{(r_{t,T} - q_{t,T})(T-t)} \, / \qquad \text{Continuous}$$

$$F_t^{(T)} \leq (S_t - I_t)\, e^{r_{t,T}(T-t)} \qquad \text{discrete}$$

oc of holding, storage cost

$F_0 e^{yT} = S_0 e^{(r+u)T}$

$F_0 = S_0 e^{(r+u-y)T}$

Cost of carry: $C_{t,T} = r_{t,T} - \underbrace{q_{t,T}}_{\text{income}}$

Convenience yield: $y_{t,T}$

how large the opportunity costs are if one doesn't have the asset available during $[t, T]$ and has to postpone consumption until time T.

$F_t^{(T)} = S_t e^{(C_{t,T} - y_{t,T})(T-t)}$

Eg. $S_{4/6} = 5.63$ storage cost $= 2\%$

$F_t^{(T)}$ \$5.24 / b 10000b $r_{0,0.5} = 0.5\%$

$\hat{F}_t^{(T)} = 5.63 e^{(0.005+0.02)\times 1.5} = \$5.85$

t T

short sell: $e^{0.02\times 1.5} \times 5.63$ $-S_T$

receive 2% storage: help him store

long forward: $S_T - \$5.24$

lend: $-\$5.24 e^{-0.005\times 1.5}$ $\$5.24$

$= \$0.6$ $= \$0$

$F_t^{(T)} = S_t e^{(r_{t,T} + u - y_{t,T})(T-t)}$

hold the asset

t T

1 unit: $\longrightarrow$ $e^{(q_{t,T} + y_{t,T})(T-t)}$ units

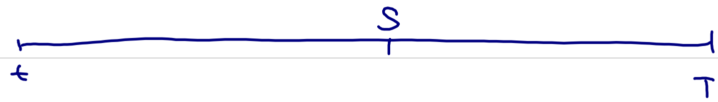
$S_t \longrightarrow e^{(q_{t,T} + y_{t,T})(T-t)} E_t[\$S_T]$ =

$\$1.S_t \longrightarrow e^{\overline{er}_{t,T}^{(S)}(T-t)} S_t$

$E_t[S_T] = \underbrace{S_t e^{(r_{t,T} - q_{t,T} - y_{t,T})(T-t)}}_{F_t^{(T)}} e^{(\overline{er}_{t,T}^{(S)} - r_{t,T})(T-t)}$ risk free rate cancels out

$\frac{E_t[S_T] - F_t^{(T)}}{F_t^{(T)}} = e^{(\overline{er}_{t,T}^{(S)} - r_{t,T})(T-t)} - 1$

Value ?



long forward: $S_T - F_t^{(T)}$

$$f_s^{(long,t,T)} = PV_s\left(S_T - F_t^{(T)}\right)$$

$$f_s^{(short,s,T)} = 0 \qquad F_s^{(T)} - S_T$$

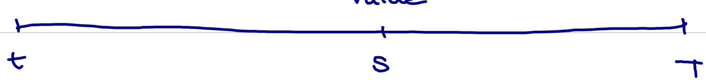
$$\overline{f_s^{(long,t,T)} + f_s^{(short,s,T)}} \qquad \overline{F_s^{(T)} - F_t^{(T)}}$$

$$f_s^{(long,t,T)} = PV_s\left(F_s^{(T)} - F_t^{(T)}\right) \longleftarrow$$

$$= \left(F_s^{(T)} - F_t^{(T)}\right) e^{-r_{s,T}(T-s)}$$

short forward:

Value



$F_t^{(T)} - S_T$

$$f_s^{(short,t,T)} = PV_s\left(F_t^{(T)} - S_T\right)$$

$$f_s^{(long,t,T)} = 0 \qquad S_T - F_s^{(T)}$$

$$\overline{f_s^{(short,t,T)} + f_s^{(long,t,T)}} \qquad \overline{F_t^{(T)} - F_s^{(T)}}$$

$$f_s^{(short,t,T)} = PV_s\left(F_t^{(T)} - F_s^{(T)}\right)$$

$$= \left(F_t^{(T)} - F_s^{(T)}\right) e^{-r_{s,T}(T-s)}$$

Futures Market

Margin requirements

No credit risk

Liquidity depends on contract type

At origination, forward and futures contracts are worthless

After origination, credit risk starts to grow for forward contracts

Settlement price : price used for calculating daily payoff

Open interest : total number of contracts outstanding

Hedging using futures

to lock in a price to buy or sell an asset in the future

Long hedge is a hedge that involves a long position in futures contracts

. hedger expects to purchase an asset at some time in the future

Imperfect hedge : mismatch between underlying of futures contract and asset to be hedged

maturity mismatch

$b_t = S_t - F_t^{(T)}$ $\qquad b_t = (S_t - S_t^*) + (S_t^* - F_t^{(T)})$

t_1 $\qquad t_2$ $\qquad T$ $\qquad$ futures expire at T

$-S_{t_2}$

$F_{t_1}^{(T)}$ $\qquad F_{t_2}^{(T)}$

long : $0 $\qquad \underline{F_{t_2}^{(T)} - F_{t_1}^{(T)}}$

$$CF = -S_{t_2} + F_{t_2}^{(T)} - F_{t_1}^{(T)}$$

$$= -\left[F_{t_1}^{(T)} + \underbrace{(S_{t_2} - F_{t_2}^{(T)})}_{b_{t_2}}\right]$$

sell an asset at t_2:

t_1 — t_2 — T

S_{t_2}

Short: $0 $\quad F_{t_1}^{(T)} - F_{t_2}^{(T)}$

$$CF = S_{t_2} + F_{t_1}^{(T)} - F_{t_2}^{(T)}$$
$$= -\left[F_{t_1}^{(T)} + \underbrace{(S_{t_2} - F_{t_2}^{(T)})}_{b_{t_2}}\right]$$

choose a delivery month that is as close as possible, but later than the end of the life of the hedge

Hedge for a long time, trade contracts with more open interests (lower bid ask spread)

size: 1000 bbl

$F_{15/6}^{(Dec)} = \$86$ — $S_{19/11} = \$88$, $F_{19/11}^{(Dec)} = 87.1$ — Dec.

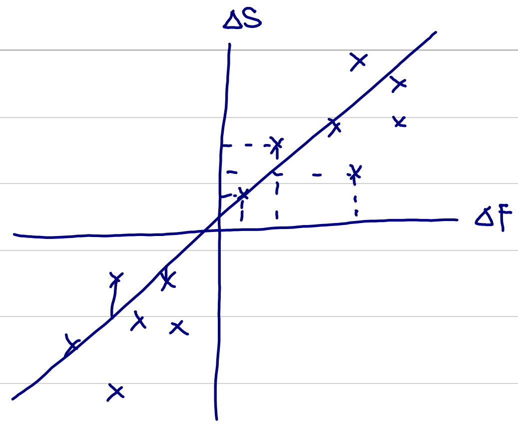
decide to purchase 20000 bbl

Gain on 20 long: $20000(87.1 - 86) = \$22000$

basis $= 20000(88 - 87.1) = \$18000$

Effective price paid $= 86 \times 20000 + 18000 = 1738000$

If maturity matches, Future price will converge to spot price



k - intercept

h : slope

$\Delta S = k + h\Delta F + \varepsilon \qquad E(\varepsilon) = 0$

$E(\Delta F \varepsilon) = 0 \qquad Cov(\Delta F, \varepsilon) = 0$

$$h = \frac{\sigma_{S,F}}{\sigma_F^2} = \frac{\rho_{S,F}\sigma_S\sigma_F}{\sigma_F^2}$$

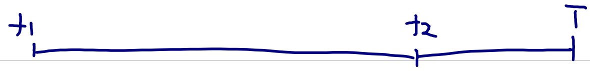
h short positions in futures

$\Delta F = +\$1 \longrightarrow \Delta S = k + \$h + \varepsilon$

futures $= -\$h$

$\Delta F = -\$1 \longrightarrow \Delta S = k - \$h + \varepsilon$

futures $= +\$h$



$\Delta S = S_{t_2} - S_{t_1}$

$\Delta F = F_{t_2}^{(T)} - F_{t_1}^{(T)}$

Q_S : units of S

N : short futures

Q_F : units per futures

$$Q_S \Delta S - N Q_F \Delta F = Q_S(k + h\Delta F + \varepsilon) - N Q_F \Delta F$$

$$\min_N \ var(Q_S(k + h\Delta F + \varepsilon) - N Q_F \Delta F)$$

$$= \min_N \ (Q_S h - N Q_F)^2 var(\Delta F) + Q_S^2 var(\varepsilon)$$

$N = \dfrac{Q_S h}{Q_F}$ ensures it's 0.

$r_S = \dfrac{\Delta S}{S}, \quad r_F = \dfrac{\Delta F}{F}$

$V_\tau^{(S)} = Q_S S_\tau$

$V_\tau^{(S)} r_{S,\tau} - N_\tau V_\tau^{(F)} r_{F,\tau}$

$$\min_N \ var(V_\tau^{(S)} r_{S,\tau} - N_\tau V_\tau^{(F)} r_{F,\tau})$$

$$= \min_N \ var(V_\tau^{(S)}(k + h r_{F,\tau} + \varepsilon) - N_\tau V_\tau^{(F)} r_{F,\tau})$$

$$= \min_N \ (V_\tau^{(S)} h - N_\tau V_\tau^{(F)})^2 Var(r_{F,\tau}) + V_\tau^{(S)2} var(\varepsilon)$$

$$\Rightarrow N_\tau = \frac{V_\tau^{(S)} h}{V_\tau^{(F)}}$$

$Q_S = 2000000$ gallons $\qquad Q_F = 42000$ gallons

$\sigma_F = 0.0313 \qquad \sigma_S = 0.0263 \qquad \rho = 0.928$

$S_0 = \$1.94 \qquad F_0^{(1/12)} = \$1.99$

$$V_0^{(S)} = S_0 Q_S = 3880000$$

$$V_0^{(F)} = F_0^{(1/12)} Q_F = 83580$$

$$h = \rho_{S,F} \frac{\sigma_S}{\sigma_F} = 0.7798$$

$$N = \frac{h Q_S}{Q_F} = 37.1 \qquad N_0 = \frac{h V_0^{(S)}}{V_0^{F}} = 36.2$$

Hedging a stock portfolio : $N_T = \beta_S \dfrac{V_T^{(S)}}{V_T^{(F)}}$

$I_0^{(S\&P)} = 1000 \qquad F_0^{(0.33)} = 990.05 \qquad V_0^{(F)} = 250\, F_0^{(0.33)} = 247512.46$

$V_0^{(S)} = \$5000000 \qquad r = 1\%$ (c.c.) $\qquad q = 4\% \qquad \beta_S = 1.5$

$$F_0^{(\frac{4}{12})} = 1000 e^{(0.01-0.04)\frac{4}{12}} = 990.05$$

$\Delta S - N_0 \Delta F = -82089 + 94668$

$= \$12579$

$0 \qquad\qquad 3/12 \qquad\qquad 4/12$

$\$5000000$

① Short $N_0 = 1.5 \dfrac{5000000}{247512.46}$

$= 30.3$ futures

$\$0$

$\$5000000 e^{-0.066(\frac{3}{12})} = 4917911$

$\Delta S = 4917911 - 5000000 = \underline{-82089}$

$I_{0.25}^{(S\&P)} = 980$ (what if)

$F_{0.25}^{(0.33)} = 980 e^{(0.01-0.04)(\frac{1}{12})} = 977.55$

$(990.05 - 977.55) \times 250 \times 30.3 = \underline{94668}$ (earning)

$$r_S = r_f + \cancel{\alpha_S} + \beta_S (r_m - r_f) + \cancel{\varepsilon_S} = 0.01 + 1.5(r_m - 0.01)$$

$I_0 = 1000 \rightarrow I_{\frac{3}{12}} = 990 \qquad \dfrac{e^{0.04(\frac{3}{12})} \cdot 990}{1000} = e^{r_m \cdot \frac{3}{12}} \Rightarrow r_m = -4.1\%$

$\Rightarrow r_S = -6.6\%$

$$5000000 \left(e^{0.01(\frac{3}{12})} - 1\right) = 12516$$

$N_T = (\beta_S - \beta^*) \frac{V_T^{(S)}}{V_T^{(F)}}$ To change the β from β_S to β^*

If $N > 0$ $(\beta_S > \beta^*)$ short position in futures

$N < 0$ $(\beta_S < \beta^*)$ long position in futures

$N = (1.5 - 0.5) \frac{5000000}{247512.46} = 20.2$ short positions $1.5 \to 0.5$

$N = (1.5 - 2) \frac{5000000}{247512.46} = -10.1$ short positions $1.5 \to 2$

Chp 5 Interest rate swaps & Futures

Eurodollar deposited in a bank outside the USA

maturity ranging from overnight to several years

Eurodollar Futures : written on the 3 mth Eurodollar deposit rate (3 mth LIBOR)

- A Eurodollar futures contract is settled in cash.
- One contract is on the rate earned on \$1 million.
- A change of one basis point (0.01%)

corresponds to a contract price change of $\$1000000 \times \frac{0.0001}{4} = 25$

The settlement price is $Q = 100 - R$ (actual 3-mth Eurodollar interest rate

in % points on the settlement date

with quarterly compounding and an

$\frac{\text{actual}}{360}$ day count convention)

Exchange defines contract price as $10000\,[100 - 0.25(100 - Q)] = 750000 + 2500Q$

Since $Q = 100 - R$ at the settlement: $= 1000000 - 2500R$

$\Rightarrow$ 1 basis point $\uparrow$ in R means long position loses (gains) 25 and the short position gains (loses) 25

Eg. 1/1 you know that you will have 1 million on Dec. 21 to invest for 3 mths.

1 long Eurodollar futures contract on Nov. 1 which expires on Dec. 21

As a long (short) position, R↓ (bond price ↑) / R↑ (bond price↓) gain

Date	R	Q	Payoff	Cumulative payoff
Nov 1	2.50%	97.5 (100−2.5)		
2	2.58%	97.42	−200	−200 (−25×8 basis)
⋮				
Dec. 21	2.25%	97.75	−50	625 (25×25 basis point)

Now we can invest 1 million with a rate of 2.25% for 3 mths.

$$\text{Interest} = 10^6\left(\frac{2.25\%}{4}\right)^{4\times\frac{3}{12}} = \$5625 \text{ on Mar. 21}$$

Also we earn \$625 from the long futures position,

which can be reinvested $= 625\left(1+\frac{2.25\%}{4}\right)^{4\times\frac{3}{12}} = 628.52$

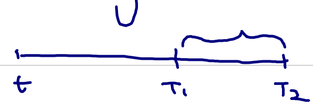
Earn interest of $\frac{5625+628.52}{1000000} = 0.625\%$

Annualized $= 0.625\% \times 4 = \underline{2.5\%}$

forward rate ≤ futures rate, strong correlation between futures prices and interest rates, while losses derived from a falling futures prices can more likely be invested at high interest rates, while losses derived from a falling futures price, can more likely be financed at a lower interest rate.

Long position has an advantage.

$$\text{Future Rate}_t = \text{Forward Rate}_t + \frac{1}{2}\sigma^2(T_1-t)(T_2-t)$$



Hedging against interest rate risk by matching the durations of assets and liabilities

Provides protection (only) against small, parallel shifts in the yield curve by making the duration of the entire position zero.

Consider Δy.

$$\Delta P = -PD_p \Delta y$$

$$\Delta F = -FD_F \Delta y$$

change of position if N contracts are sold short: $\Delta P - N\Delta F = -(PD_p - NFD_F)\Delta y$

$$N = \frac{P}{F} \cdot \frac{D_P}{D_F}$$

E.g. to hedge: \$10 million in gov. bonds.

3 mth T-bond futures contract has a current price of 93-02.

$100000 \times 0.930625 = 93062.5$

Duration in 3 mths is 6.8 years

Duration of cheapest to deliver bond is 9.2 years.

$\Rightarrow$ Short T-bond futures $N = \frac{10000000}{93062.5} \cdot \frac{6.8}{9.2} = 79.42$

Interest rate Swaps.

exchange CFs over a future period of time according to pre-specified conditions

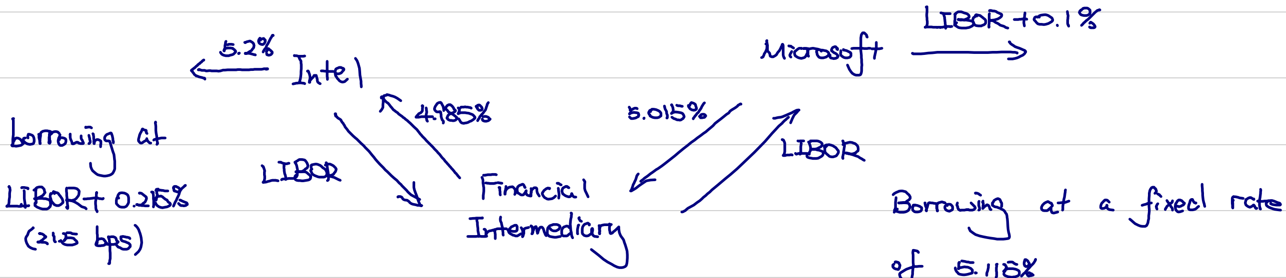
$$\boxed{\text{Intel}} \;\underset{\xrightarrow[\text{LIBOR}]{}}{\xleftarrow{5\%}}\; \boxed{\text{Microsoft}}$$

every 6 mths for 3 years on notional principal of 100 million

CF Microsoft

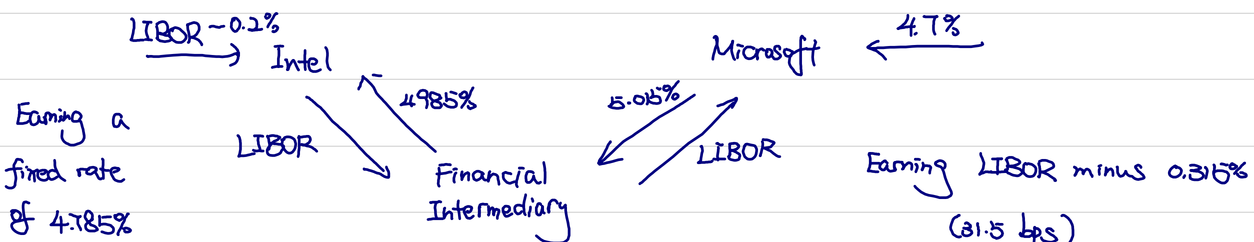
	LIBOR	Floating (10^6)	Fixed (10^6)	NCF
Jan 5 2012	4.5%			
July 5 2012	4.2%	2.25	-2.5	-0.25
Jan 5 2013	4.8%	2.1	-2.5	-0.4
July 5 2013	5.0%	2.4	-2.5	-0.1
Jan 5 2014	5.5%	2.5	-2.5	0
July 5 2014	6.1%	2.75	-2.5	0.25
Jan 5 2015	6.3%	3.05	-2.5	0.55

Converting a Liability using IRS



A profit of 3 bps
per year on the
notional principal
of $100 million

Converting an Asset using IRS



A profit of 0.03% (3 bps)
per year on the notional
principal of 100 million

Why do Swaps exist

Comparative advantage

	Fixed	Floating
Microsoft	5.15%	LIBOR+0.1%
Intel	5.2%	LIBOR+0.3%
	−0.05%	−0.2%

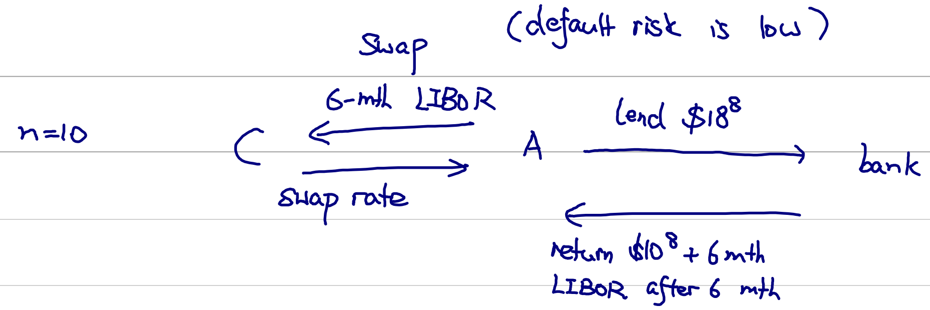
– Microsoft has an absolute advantage in borrowing money (fixed or floating)

disadvantage of intel is smaller for fixed borrowing

Without swap: Aggregate : LIBOR + 5.45%

With swap: Aggregate: LIBOR + 5.3%

(default risk is low)



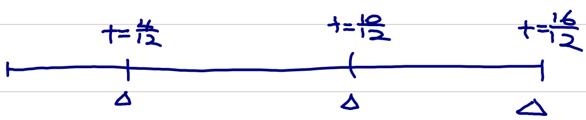
E.g. Valuation of IRS

→ A bank pays 6-mth LIBOR and receives 4.5% (2 times cpa) on a notional principal of 100 million

→ Swap has a remaining life of 1 year and 4 months

→ 4, 10, 16 mth spot rates are 4%, 5.5%, 6% (c.c.)

6-month LIBOR was 4.2% (2 times cpa) or 4.16% (cc) at the last payment date.



Time	Fixed (10^6)	Floating (10^6)	PV fixed	PV floating
$\frac{4}{12}$	2.25	$100\left(1+\frac{4.2\%}{2}\right)=102.1$	$2.25e^{-\frac{4}{12}(4\%)}=2.22$	$102.1e^{-\frac{4}{12}(4\%)}=100.75$
$\frac{10}{12}$	2.25		$2.25e^{-\frac{10}{12}(5.5\%)}=2.15$	
$\frac{16}{12}$	102.25		$102.25e^{-\frac{16}{12}(6\%)}=94.39$	
			98.76	100.75

Floating rate payer = -1990000

$$4\%\left(\frac{4}{12}\right)+f_0^{\left(\frac{4}{12},\frac{10}{12}\right)}\left(\frac{1}{2}\right)=5.5\%\left(\frac{10}{12}\right)$$
$$f_0^{\left(\frac{4}{12},\frac{10}{12}\right)}=6.5\%$$

$$5.5\%\left(\frac{10}{12}\right)+f_0^{\left(\frac{10}{12},\frac{16}{12}\right)}\left(\frac{1}{2}\right)=6\%\left(\frac{16}{12}\right)$$
$$f_0^{\left(\frac{10}{12},\frac{16}{12}\right)}=6.83\%$$

$r_0^{(fl)}=4.2\%$ $\quad r_{\frac{4}{12}}^{(fl)}=2\left(e^{\frac{0.065}{2}}-1\right)=6.61\%$ $\quad r_{\frac{10}{12}}^{(fl)}=2\left(e^{\frac{0.0683}{2}}-1\right)=6.95\%$

Time	Fixed	Floating	Net	PV(CF)
$\frac{4}{12}$	2.25	-2.1	0.15	0.15
$\frac{10}{12}$	2.25	-3.31	-1.06	-1.01
$\frac{16}{12}$	2.25	-3.48	-1.23	-1.13
				-1.99

Overnight Indexed Swaps

better proxy for the S-T risk free rate than LIBOR

Currency Swaps

E.g. Apple pays 3% (1 time c.p.a.) on GBP 100

and receives 4% (1 time cpa) on USD 150 for 4 years

	Time	CF (USD 10^6)	CF (GBP 10^6)
Apple:	0	−150	100
	1	6	−3
	2	6	−3
	3	6	−3
	4	156	−103

$E_0 = 1.56 \frac{USD}{GBP}$ $E_{2.5} = 1.3 \frac{USD}{GBP}$

$r_{US} = 2.5\%$ (c.c.) $r_{UK} = 1.6\%$ (c.c.)

	PV (USD)	PV (GBP)
1	5.85	2.95
2	5.71	2.91
3	5.57	2.86
4	141.15	96.59
	158.28	105.31

$V_0 = (158.28 - 1.56 \times 105.31 + 1.56 \times 100 - 150) \times 10^6 = 0$

$V_{2.5} = (146.72 e^{2.5 \times 0.025} - 1.3 \times 99.425 e^{2.8 \times 0.016}) \times 10^6$

$= 21.6 \times 10^6$

$e^{r^f_{t,T}(T-t)} F_t(T) = E_t e^{r_{t,T}(T-t)}$

Time	USD(10^6)	GBP(10^6)	FX	Net CF	PV
0	−150	100	1.56	6	6
1	6	−3	1.57	1.28	1.25
2	6	−3	1.59	1.24	1.18
3	6	−3	1.60	1.19	1.11
4	156	−103	1.62	−10.53	−9.53
					0

$V_0 = 0$

Time	USD(10^6)	GBP(10^6)	FX	Net CF (10^6)	PV (10^6)
2.5			1.3		
3	6	−3	1.31	2.08	2.06
4	156	−103	1.32	20.29	19.55
					21.6

$S_t = \frac{\text{Foreign}}{\text{Local}}$ $\qquad$ $P_{t,t+n} = (\ln F_{t,t+n} - \ln S_{t,t+n})\frac{1}{n} = y_{t,t+n} - y^{\$}_{t,t+n}$

$$F_{t,t+n}\, e^{n y^{\$}_{t,t+n}} = S_t\, e^{n y_{t,t+n}}$$

$$F_{t,t+n}\, e^{n y^{\$}_{t,t+n}} = S_t\, e^{n y_{t,t+n} + n x_{t,t+n}}$$

$$\ln F_{t,t+n} + n y^{\$}_{t,t+n} = \ln S_t + n y_{t,t+n} + n x_{t,t+n}$$

$$n x_{t,t+n} = \ln F_{t,t+n} - \ln S_t + n y^{\$}_{t,t+n} - n y_{t,t+n}$$

$$x_{t,t+n} = \frac{\ln F_{t,t+n} - \ln S_t}{n} + y^{\$}_{t,t+n} - y_{t,t+n}$$

$$= y^{\$}_{t,t+n} - y_{t,t+n} + P_{t,t+n}$$

Arbitrage with a negative basis:

USD lender

1USD $\downarrow$

Arbitrageur --- Fx Forward Dealer $\rightleftarrows$ Hedging

S_t JPY $\downarrow$

JPY borrower

USD lender

$\uparrow$ Pay back $1 + y^{\$}_{t,t+1}$ USD

$(1+y_{t,t+1})S_t$ Yen

$(1+y_{t,t+1})S_t / F_{t,t+1}$ USD $\uparrow$ receive $(1+y_{t,t+1})S_t$ JPY

JPY borrower

Assume $S_t = \frac{105.59 \text{ Yen}}{\text{USD}}$, $y^{\$}_{t,t+1} = 2\%$, $y_{t,t+1} = 1\%$

$t=0$		$t=1$	
$+\$1$ USD (borrow)	$\longrightarrow$	$e^{0.02} = 1.02020134$ USD $\approx (1+2\%) = 1.02$ USD	
$-1\$$USD $+105.59$ Yen	convert	$+105.59\, e^{0.01} = 106.6511971$	receive
-105.59 Yen	lend	$106.6511971 \times \frac{1}{103} = 1.0354$USD	Convert
0		$\pi = 0.015247176$ USD	

$F_T^{(T)} = 105.59\, e^{(1\% - 2\%)}$ $\qquad$ Suppose actual $F_t^{(i)} = \frac{103 \text{Yen}}{\text{USD}}$

$= 104.5393619 \frac{\text{JPY}}{\text{USD}}$

$x_{t,t+1} = 2\% - 1\% + \ln 103 - \ln 105.59 =$

Introduction: CIP does not hold (def[n] of CIP)

concept of basis

credit risk ↛ CIP deviation

By examining Repo and KfW bond.

ST ↗ (Repo) LT ↑ (KfW bond)

JPY, CHF, krone (-ve basis)

even after taking into transaction costs

Sharpe ratio: (no risk → profit) ∞

Explanation: Constraints on financial intermediaries

international imbalances in investment demand and funding supply

If unconstrained, supply of currency hedging should be perfectly elastic

4 main characteristic : 1. ↑ toward the quarter ends (face tighter balance sheet constraints renewed investor attention)

2. proxy for the shadow costs of banks' balance sheet (spread between IOER and federal funds rate)
accounts for $\frac{1}{3}$ of the CIP deviations

3. basis is +vely correlated with the nominal interest rates

4. cross-currency basis is correlated with other liquidity spreads

cross currency basis $= y^{\$}_{t,t+n}$ (local) $-(y_{t,t+n} - \rho_{t,t+n})$ synthetic dollar interest rate.

C.1. ST Libor Cross-Currency Basis

$$x^{Libor}_{t,t+n} \equiv y^{\$,Libor}_{t,t+n} - (y^{Libor}_{t,t+n} - \rho_{t,t+n})$$

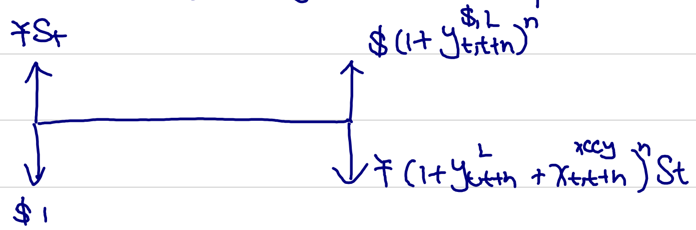
LT : spread on the cross-currency basis swap (exchange of cash flows linked to floating interest rates referenced to interbank rates in two currencies.

$x^{xccy}_{t,t+n}$: price at which swap counterparties are willing to exchange foreign currency floating cash flows against US dollar cash flows.

US against yen : negative basis

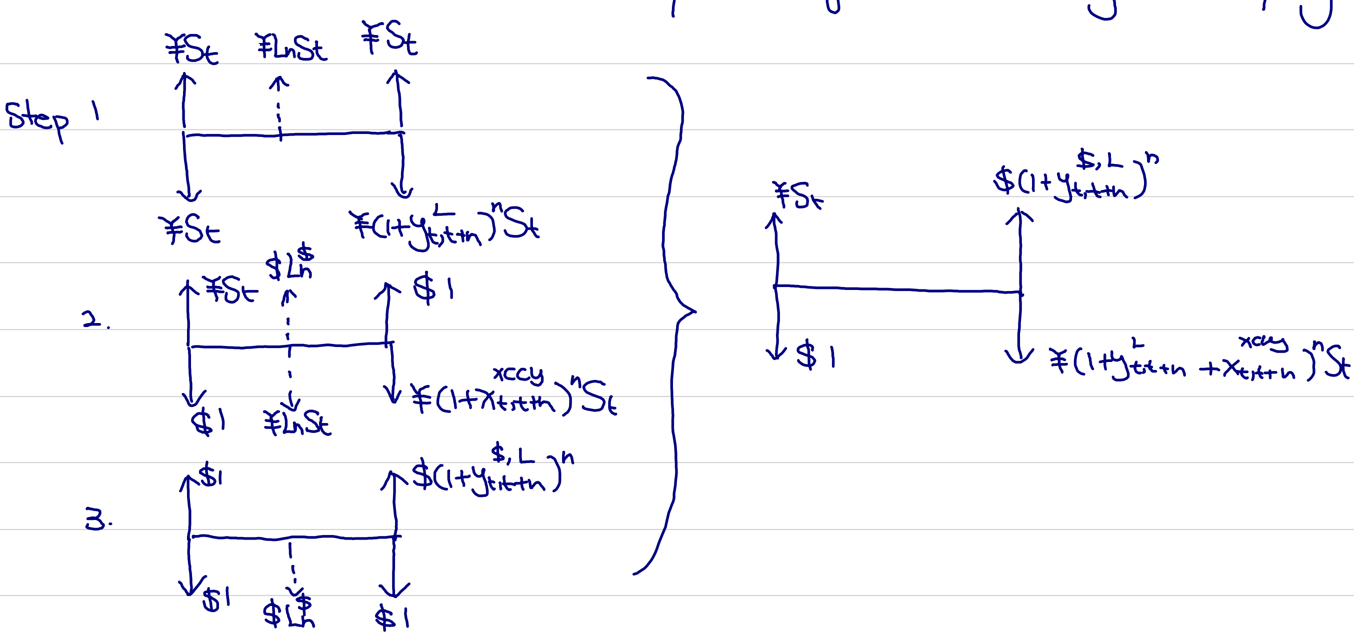
$$y_{t,t+j}^{Libor\,¥} + x^{xccy}_{t,t+n} < y_{t,t+j}^{Libor\,¥}$$

As long as the cross-currency basis swap is not zero, one counterparty benefits from the swap $\longrightarrow$ potential deviations.



$F_{t,t+n}\,\$(1+y^{\$,L}_{t,t+n})^n = (1+y^L_{t,t+n}+x^{xccy}_{t,t+n})^n S_t$ $\quad\quad \rho_{t,t+n} = \frac{1}{n}(f_{t,t+n} - s_t) = y^L_{t,t+n} + x^{xccy}_{t,t+n} - y^{\$L}_{t,t+n}$

Cross-currency basis swap , borrow and simultaneously lends the same value at current spot rates of a second currency to that party.



pays \$1 in exchange for S_t yen

receives $e^{n y_{t,t+n}^{\$ IRS}}$ USD and pays $\dfrac{e^{n y_{t,t+n}^{IRS} + n x_{t,t+n}^{xccy}} S_t}{F_{t,t+n}}$ USD

II CIP based Arbitrage Opportunities

existence of the repo and KfW basis implies CIP arbitrage opportunities free from currency and credit risk, even after taking into account transaction costs.

A. Credit Risk

① Can the arbitrageur really borrow and lend at the Libor rates?

Libor rates are only indicative, do not correspond to actual transactions (may be higher than the indicative Libor rate)

Transaction costs exist.

② Take on credit risk when lending at the yen Libor rate?

default risk premium ⟶ not risk free anymore (CIP deviations)

③ counterparty risk when entering an exchange rate forward contract?

(high degree of collateralization, second order importance)

Initial margins are also posted to cover the gap risk

collateral is seized by the other counterparty to cover the default cost

Default risk explanation of CIP deviations relies on cross-country differences in creditworthiness of different Libor panel banks.

$$x_t^{JPY/USD,\,Libor} = (y_t^{*USD} + sp_t^{USD}) - (y_t^{*JPY} + sp_t^{JPY} - \rho_t^{JPY/USD})$$
$$= [y_t^{*USD} - (y_t^{*JPY} - \rho_t^{JPY/USD})] + (sp_t^{USD} - sp_t^{JPY})$$

If CIP holds $x_t^{JPY/USD,\,Libor} = sp_t^{USD} - sp_t^{JPY}$

Yen basis can be negative if the yen Libor panel is riskier than the dollar panel

Repo Basis (to eliminate the credit risk associated with Libor-based CIP is to use secured borrowing and lending rates from the repo markets) SR

$$\pi^{Repo-}_{t,t+n} \equiv [y^{Repo}_{t,t+n,Bid} - \rho_{t,t+n,Ask}] - y^{\$,Repo}_{t,t+n,Ask}$$

$$\pi^{Repo+}_{t,t+n} \equiv y^{\$,Repo}_{t,t+n,Bid} - [y^{Repo}_{t,t+n} - \rho_{t,t+n,Bid}]$$

borrowing in the US GC repo market would require posting a US Treasury as collateral
$\Rightarrow$ scarcity of US Treasury bonds or difference in collateral value between US and foreign Treasury bonds could in theory be a source of CIP deviations.

borrow in US Libor and invest in foreign GC repo.
does not need to post US Treasury bonds as collateral
but receives the foreign T bonds.
Profits are similar $\Rightarrow$ can't be a main driver.

KfW Basis (LR) liabilities fully backed by the German government.

When KfW basis is negative, invest in the KfW bond in foreign currency
pay the cross-currency swap to convert into USD
short sell the KfW bond, paying the shorting fee

Even taken into account of bid ask spreads, exhibit significant arbitrage opportunities (except Australian dollar)

Neither credit risk nor the cov. between credit risk and currency risk seem to explain CIP deviations for KfW yields.

III Potential Explanations

(1) Balance sheet Constraints of Financial Intermediaries

regulatory reforms ↑ balance sheet costs of arbitrage

(1) Non-Risk-Weighted Capital Requirements

leverage ratio requires bank to maintain a minimum amount of capital against all on-balance-sheet assets and off-balance-sheet exposure

ST arbitrage trades involve lending and borrowing, make the leverage ratio requirement more binding

Assume leverage ratio equal to 3% that binds,

if overall objective return is around 10%,

banks need at least a $3\% \times 10\% = 30$ bps cross currency basis.

(2) Risk-Weighted Capital Requirements (increased the costs of LT CIP arbitrage)

capital ratio Tier 1 : $4\% \rightarrow 9.5\% \sim 13\%$

Total capital ratio $8\% \rightarrow 11.5\% \sim 15\%$

99% VaR = 0 capital charge theoretically increases 10 times

(3) Banking Regulations

① Volcker rule forbids banks from actively engaging in proprietary trading activities

(can only facilitate the arbitrage activities of their clients)

② OTC market sets higher capital and minimum margin requirements for cross-currency swaps

③ holding high-quality liquidity assets (HQLAs) liquidity coverage ratio: hold an amount of high quality liquid assets that's enough to fund cash outflows for 30 days.

(4) Limits to arbitrage facing other potential arbitrageurs

turn to hedge funds, MMFs, reserve managers, and corporate issuers.

persistence of CIP deviations suggests that potential arbitrageurs take only limited positions

reforms on banks have some spillover effects on the cost of leverage because hedge funds need to obtain funding from their prime brokers. Hedge funds would need to lever up the arbitrage strategy 10-20 times (borrowing costs ↑ significantly as their positions show up in their prime brokers' balance sheets.

dollar funding from US prime MMFs become scarcer, cross currency basis also widened notably.

(5) Testable Hypothesis

(i) CIP deviations are wider when banks' balance sheet costs are higher, particularly toward quarter end financial reporting dates

(ii) CIP deviations are of similar magnitude to the balance sheet costs associated with wholesale dollar funding

(iii) CIP deviations are correlated with other near-risk-free fixed income spreads

B. International Imbalances

Search-for-yield motives create large customer demand for investments in high-interest-rate currencies and a large supply of savings in low interest rate currencies.

financial intermediaries hedge the currency exposure of their forward and swap positions by going long in low interest rate currencies and short in high interest rate currencies.

The profit per unit of notional is equal to the absolute value of the cross-currency basis, compensating the intermediary for the cost of capital associated with the trade.

Predict 2: Cross currency basis is increasing in the nominal interest rate differential between the foreign currency and the USD.

Intuition. lower the foreign currency interest rate compared to the US rate
higher the demand for USD denominated investment opportunities
greater currency hedging demand to sell USD and buy foreign currencies in the forward or the swap market.
Providing the contracts is costly
⇒ basis has to become more negative to justify the balance sheets cost

IV characteristics of the basis

A. Quarter - End Dynamics

CIP deviations tend to increase at the quarter ends

tightened balance sheet constraints translate into wider CIP deviations.

B. CIP arbitrage based on excess reserves at central banks

global institutions have held large amounts of excess reserves

Excess reserve are 补偿 at IOER

Arbitrage: banks borrow in the federal funds market and deposit the proceeds in the form of excess reserves, earning the IOER fed funds spread.

IOER Libor spread to account for banks' balance sheet costs and investing at the foreign central bank significantly reduces the size of CIP deviations

Fed balances are more desirable for fulfilling banks' dollar liquidity needs compared to swapped foreign central bank balances.

D. Cross currency basis and nominal interest rates

low interest rate currencies tend to have most negative basis

high . . . less negative

relationship is strong at long maturities

6.(a) 6-mth HKD : 1.5% cc $\qquad E = \frac{5\,HKD}{NZD}$

6-mth NZD : 3.5% cc

$F\, e^{0.03(\frac{1}{2})} = 5e^{0.01(\frac{1}{2})}$ $\qquad F^{(L)} = 5.03 \frac{HKD}{NZD}$

$F = 4.950249169 \frac{HKD}{NZD}$ $\qquad F^{(S)} = 4.97 \frac{HKD}{NZD}$

(b) Yes

T_0	T_1
borrow 1 HKD	repay $e^{0.01(\frac{1}{2})} = 1.005012521$ HKD
convert 0.2 NZD	Receive $0.2e^{0.03(\frac{1}{2})} = 0.203022612$
Invest	Convert $= 1.015113065$
Sign forward	$T_1 =$
0

(d) $x_{t,t+h} = 3.5\% - (1.5\% - \frac{\ln(5.03)-\ln(5.01)}{0.5}$ (negative basis)

borrow 1 HKD $\qquad$ repay $e^{0.015(\frac{1}{2})} = 1.007528195$ HKD

Convert: $\frac{1}{5.01} = 0.19960798$ NZD $\qquad$ receive 0.2031245555

0 $\qquad$ Convert 1.00952904 HKD

Chp 6 Options

Right vs Obligation

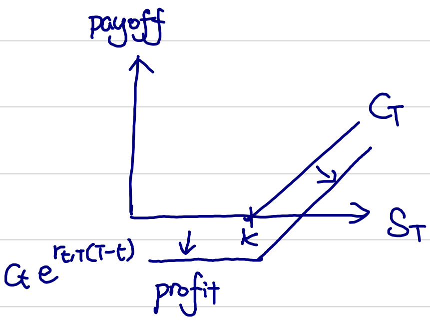
Call option : long position has the right to buy the underlying in future for a fixed price specified today

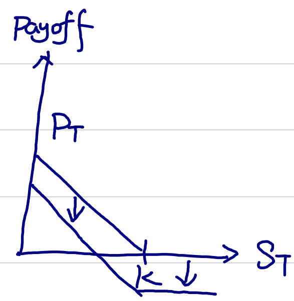
$$Payoff_T^{(long\ call)} = max(S_T - K, 0) \quad \text{execute or not}$$

$$Payoff_T^{(short\ call)} = -max(S_T - K, 0) = min(K - S_T, 0) \quad \text{always negative}$$

$S_T > K \Rightarrow$ long will exercise

Value of the option pay to the short





Put option long position has the right to sell the underlying in future for a fixed price specified today

$$Payoff_T^{(long\ put)} = max(K - S_T, 0)$$

$$Payoff_T^{(short\ put)} = -max(K - S_T, 0) = min(S_T - K, 0) \quad \text{always negative}$$

European (focus)

exercised only at maturity

American

exercised at any time during its life

most exchange traded

$$P_{USA} \geq P_{Euro}$$

OTC and exchange traded

underlyings can be nearly everything

Market makers to facilitate options trading

quote both bid and ask prices

hedge exposure using other derivatives

earn bid ask spreads and compete with one another

Exchange · Margins are required (short position)

brokers can adjust the margin requirement

closed by taking offsetting positions

randomly selects an equivalent short

? Warrants are options that are issued by an institution (OTC)

long time to maturity

dilutive

Convertible bonds · can be exchanged for equity

interest is lower (due to exchange)

dilutive

Employee Stock options: long time to maturity

vested period of up to 5 years

american call options as compensation to executives

dilutive (issue new stocks)

Moneyness (call) in the money if $S_t > K$ ✓

at the money if $S_t = K$

out-of-the money if $S_t < K$

deep-in-the-money if $S_t >> K$ ✓

deep-out-of-the-money if $S_t << K$

(put) in the money $S_t < K$ ✓

at the money $S_t = K$

out of the money $S_t > K$

deep in the money $S_t << K$ ✓

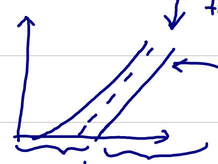
deep out of the money $S_t >> K$

Only true if no dividend (Call priced)

lower bound ($S_t - PV(K)$)

time value

intrinsic value



Intrinsic value Call: $\max\{0, S_t - K\}$

Put: $\max\{0, K - S_t\}$

buy the asset

borrow, PV of K

O_t = intrinsic value + time value

Time value is the largest near the strike price

(Cherry picking) we can wait to exercise the options.

if $S_t \to 0$, will not exercise $S_t \to \infty$,

exercise for sure

t —— T

C_t	$S_T > K$	$S_T < K$
	$S_T - K$	0
$S_t - Ke^{-r_{t,T}(T-t)}$	$S_T - K$	$S_T - K$ (−ve)

(payoff)

$C_t \geq S_t - Ke^{-r_{t,T}(T-t)} > S_t - K$

if $r_{t,T} > 0$

注意到 Call option payoff

永远大于等于 strategy

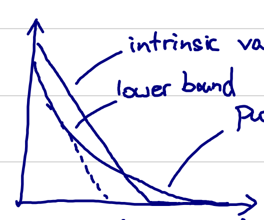
European Put option (no dividends) short sell and invest PV(K)

t —— T

intrinsic value

lower bound

Put price



exercise do not exercise

American put option early

$O_{Euro} < O_{USA}$

P_t	$S_T > K$	$S_T < K$
$Ke^{-r_{t,T}(T-t)} - S_t$	0	$K - S_T$
(pay today the amount we lend sell the asset)	$K - S_T < 0$	$K - S_T$

(payoff)

$P_t \geq \underbrace{Ke^{-r_{t,T}(T-t)} - S_t}_{\text{lower bound}} < \underbrace{K - S_t}_{\text{intrinsic value}}$ if $r_{t,T} > 0$

Put - Call Parity : Replicate Long Risk free asset

long risk free asset : $Ke^{-r(T-t)} = P_t - C_t + S_t - D_{t,T}$ Discrete

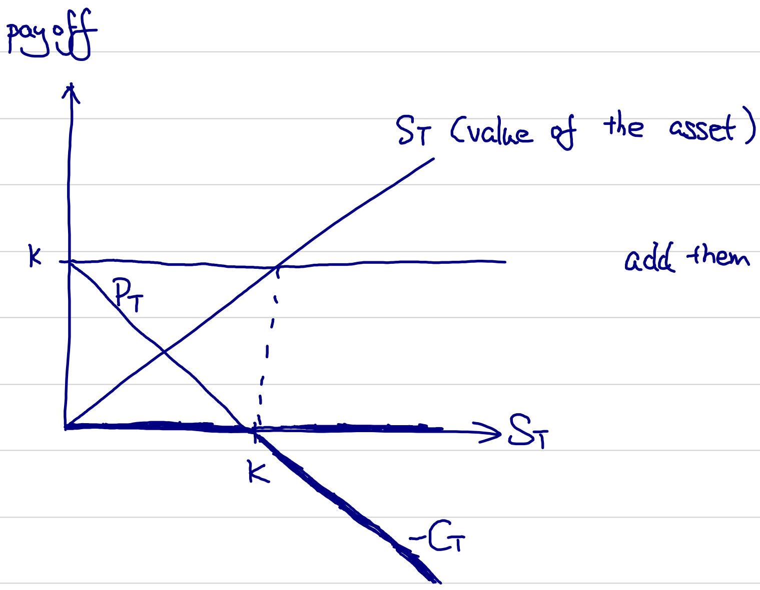
Time $t<T$: Long put , short call , long underlying , borrow $D_{t,T}$

Time T :

		$S_T \le K$	$S_T > K$
	long RF Asset:	$\underline{\underline{K}}$	$\underline{\underline{K}}$
P_t	long put	$K - S_T$	0
$-C_t$	short call	0	$-(S_T - K)$
S_t	long underlying	$S_T + e^{r(T-t)} D_{t,T}$	$S_T + e^{r(T-t)} D_{t,T}$
$-D_{t,T}$	borrow	$-e^{r(T-t)} D_{t,T}$	$-e^{r(T-t)} D_{t,T}$
	Total	$\underline{\underline{K}}$	$\underline{\underline{K}}$

(zero coupon bond)

two strategies have the same payoff no matter what .



add them together

Only holds for European option , same underlying asset , strike price, expiration date

pays a stream of dividends $\delta_{t,T}$, Put Call parity :

$$P_t - C_t + S_t e^{-\delta_{t,T}(T-t)} = e^{-r(T-t)} K$$

Continuous

t —————————————— T

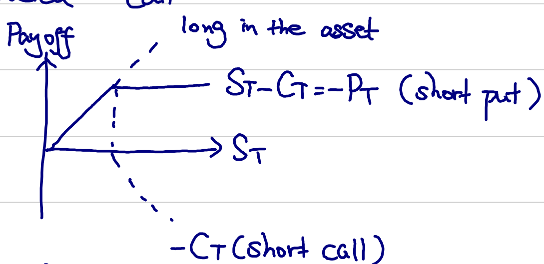
		$S_T < K$	$S_T \geq K$
long RF asset : $Ke^{-R_{t,T}(T-t)}$		$\underline{\underline{K}}$	
long put :	P_t have to pay	$K - S_T$	0
short call :	$-C_t$ short sell	0	$K - S_T$
buy $e^{-q_{t,T}(T-t)}$	$e^{-q_{t,T}(T-t)}S_t$	S_T	S_T
	$= P_t - C_t + e^{-q_{t,T}(T-t)}S_t$ $= Ke^{-R_{t,T}(T-t)}$	$\underline{\underline{K}}$	$\underline{\underline{K}}$

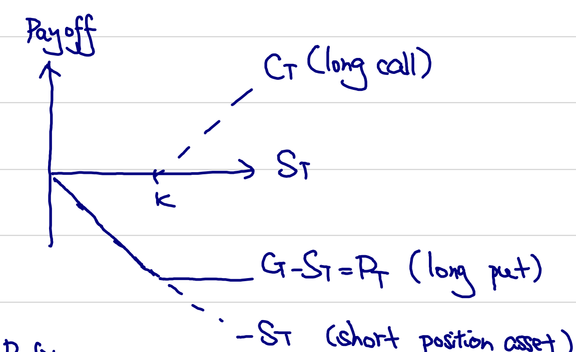
We can rearrange the eqt to generate other synthetic securities. P21

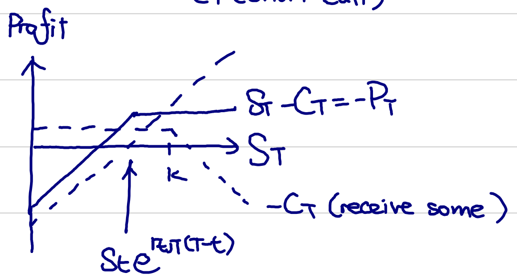
Option Trading Strategies

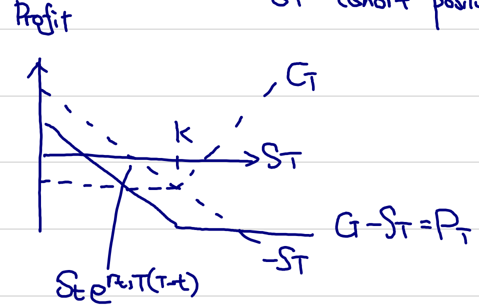
Payoff = Final Payoff − Future Value of Initial Cost

Covered Call

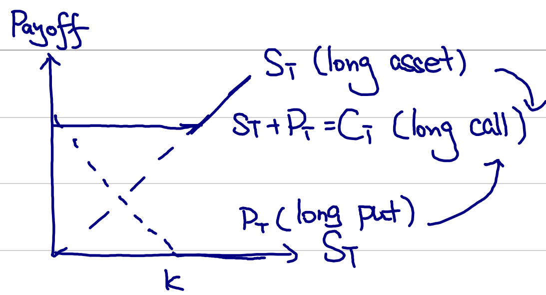


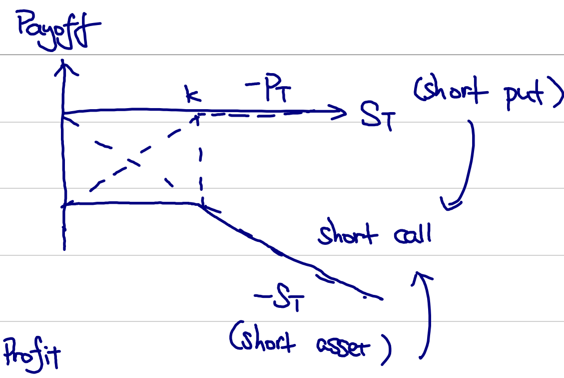


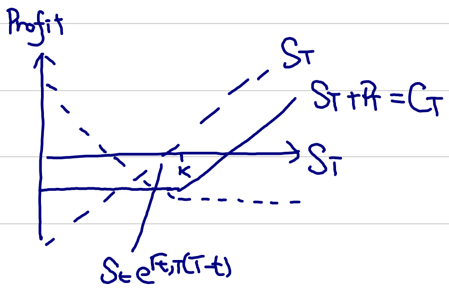


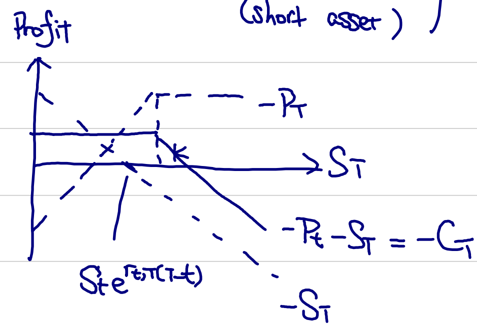


Protective Put

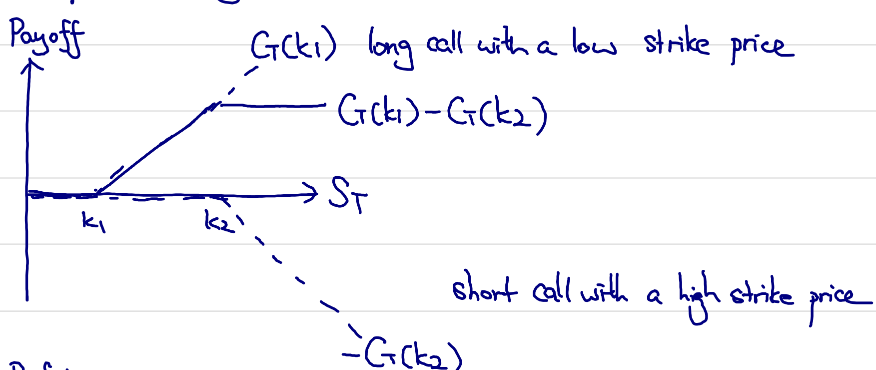


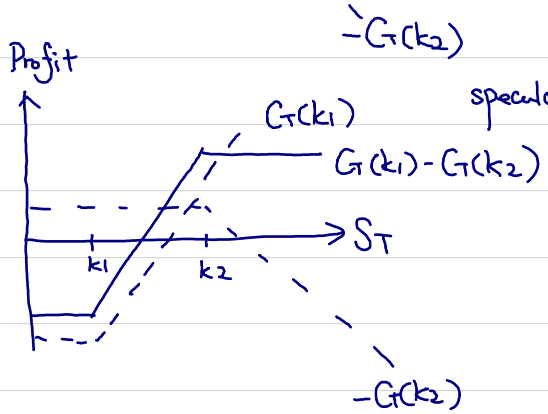






Bullspread using Calls





speculation is less risky (limit the upside profit and downside risk)

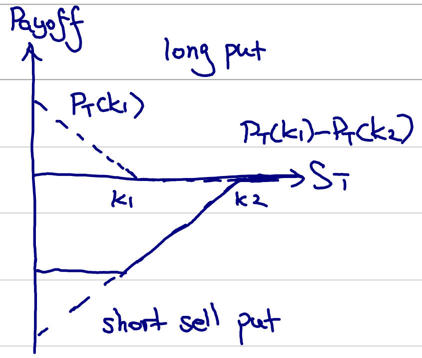
higher the strike price, discount more

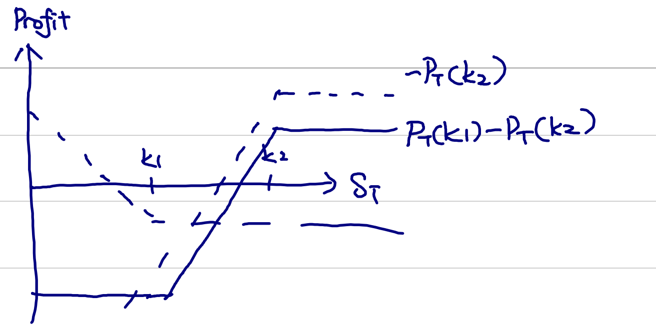
initial investment

$$-\underline{(k_2 - k_1)} e^{-r_{t,T}(T-t)} < C_t(k_2) - C_t(k_1) < 0$$

initial cost can't exceed of PV(payoff)

State	$C_T(k_1)$	$-C_T(k_2)$	$Payoff_T$
$S_T < k_1$	0	0	0
$k_1 < S_T < k_2$	$S_T - k_1$	0	$S_T - k_1 \geq 0$
$k_2 < S_T$	$S_T - k_1$	$-(S_T - k_2)$	$\underline{k_2 - k_1} > 0$



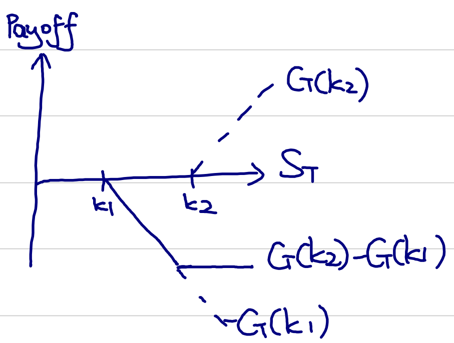


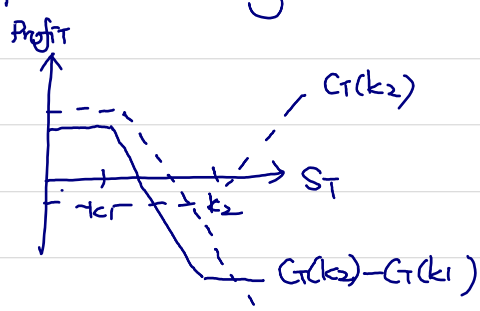
State	$P_T(k_1)$	$-P_T(k_2)$	Payoff I	
$S_T<k_1$	k_1-S_T	$-(k_2-S_T)$	k_1-k_2	<0
$k_1<S_T<k_2$	0	$-(k_2-S_T)$	$-(k_2-S_T)$	≤ 0
$k_2<S_T$	0	0	0	

$$0<P_t(k_2)-P_t(k_1)<(k_2-k_1)e^{r_{t,T}(T-t)}$$

Bearspread using Calls (speculate price of underlying decreases)

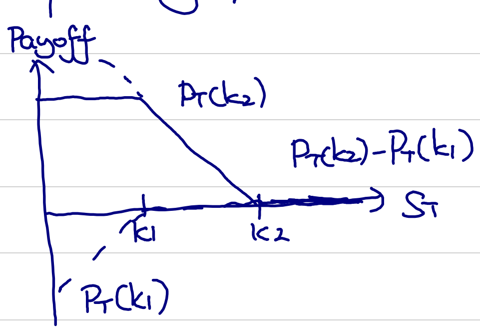
Sell a call with a low strike price (k_1) and buy a call

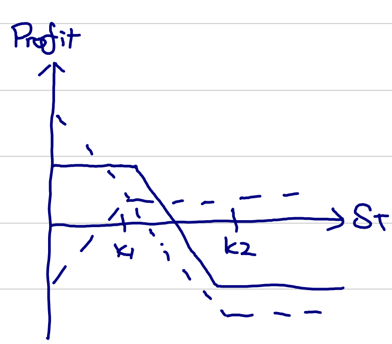




State	$-C_T(k_1)$	$C_T(k_2)$	Payoff I
$S_T<k_1$	0	0	0
$k_1<S_T<k_2$	$-(S_T-k_1)$	0	$-(S_T-k_1)$
$k_2<S_T$	$-(S_T-k_1)$	S_T-k_2	k_1-k_2

Bearspread using puts





Chp. 7 Option pricing in discrete time

$t_i \qquad t_{i+1}$

$u_{t_i} > e^{r_{t_i}(t_{i+1}-t_i)} > d_{t_i}$

$t_i : \Delta_{t_i},\ b_{t_i}$

S_{t_i}

B_{t_i}

$\Delta_{t_i} S_{t_i} + b_{t_i} B_{t_i} = O_{t_i}$

buying < selling

short sell >

$S^{(u)}_{t_{i+1}} = S_{t_i} u_{t_i}$

$B_{t_{i+1}} = B_{t_i} e^{r_{t_i}(t_{i+1}-t_i)}$

$$O^{(u)}_{t_{i+1}} = \begin{cases} \max\{S^{(u)}_{t_{i+1}} - K, 0\} & \text{if call} \\ \max\{K - S^{(u)}_{t_{i+1}}, 0\} & \text{if put} \end{cases}$$

$= \Delta_{t_i} S^{(u)}_{t_{i+1}} + b_{t_i} B_{t_{i+1}}$

$S^{(d)}_{t_{i+1}} = S_{t_i} d_{t_i}$

$B_{t_{i+1}} = B_{t_i} e^{r_{t_i}(t_{i+1}-t_i)}$

$$O^{(d)}_{t_{i+1}} = \begin{cases} \max\{S^{(d)}_{t_{i+1}} - K, 0\} & \text{if call} \\ \max\{K - S^{(d)}_{t_{i+1}}, 0\} & \text{if put} \end{cases}$$

$= \Delta_{t_i} S^{(d)}_{t_{i+1}} + b_{t_i} B_{t_{i+1}}$

If American, at up state,

exercise early

$\Rightarrow \max\{110-105, 5.4953\} = 5.4953$

down state

$\max\{95-105, 0\} = 0$

(check the payoff at every state)

$$O_T = \begin{cases} \max\{S_T - K, 0\} & \text{if call} \\ \max\{K - S_T, 0\} & \text{if put} \end{cases}$$

$O^{(u)}_{t_{i+1}} = \Delta_{t_i} S^{(u)}_{t_{i+1}} + b_{t_i} B_{t_{i+1}}$

$O^{(d)}_{t_{i+1}} = \Delta_{t_i} S^{(d)}_{t_{i+1}} + b_{t_i} B_{t_{i+1}}$

$$\Delta_{t_i} = \frac{O^{(u)}_{t_{i+1}} - O^{(d)}_{t_{i+1}}}{S^{(u)}_{t_{i+1}} - S^{(d)}_{t_{i+1}}} = \frac{O^{(u)}_{t_{i+1}} - O^{(d)}_{t_{i+1}}}{S_{t_i}(u_{t_i} - d_{t_i})}$$

$$b_{t_i} = \frac{1}{B_{t_{i+1}}}\left[O^{(u)}_{t_{i+1}} - \Delta_{t_i} S^{(u)}_{t_{i+1}}\right] = \frac{e^{-r_{t_i}(t_{i+1}-t_i)}}{B_{t_i}}\left[\frac{-d_{t_i}}{u_{t_i}-d_{t_i}} O^{(u)}_{t_{i+1}} + \frac{u_{t_i}}{u_{t_i}-d_{t_i}} O^{(d)}_{t_{i+1}}\right]$$

$K = 105$

$S_0 = 100$

$B_0 = 1$

$O_0 = \Delta 100 + b(1)$

$= 1.891$

$S^{(u)}_{0.08} = 110$

$B_{0.08} = 1.0017$

$O^{(u)}_{0.08} = \Delta 110 + b(1.0017)$

$= 5.4953$

$S^{(d)}_{0.08} = 95$

$B_{0.08} = 1.0017$

$O^{(d)}_{0.08} = \Delta 95 + b(1.0017)$

$= 0$

$\Delta = 0$

$b = 0$

$S^{(uu)}_{0.17} = 121$

$B_{0.17} = 1.0033$

$O^{(uu)}_{0.17} = 16 = \Delta 121 + b(1.0033)$

$S^{(ud)}_{0.17} = S^{(du)}_{0.17} = 104.5$

$B_{0.17} = 1.0033$

$O^{(ud)}_{0.17} = O^{(du)}_{0.17} = 0 = \Delta 104.5 + b(1.0033)$

$S^{(dd)}_{0.17} = 90.25$

$B_{0.17} = 1.0033$

$O^{(dd)}_{0.17} = 0 = \Delta 90.25 + b(1.0033)$

European Call with Dividends

$S_0=100$ $\quad u=1.15$ $\quad d=0.9$ $\quad$ every 3 mth, a dividend of 10 $\quad r=5\%$ (c.c.) $\quad K=100$

$S_0=100$
$B=1$
$O_0=4.103$

$S^{(u)}_{0.25-}=115$
$S^{(u)}_{0.25}=105$
$B_{0.25}=1.0126$
$O^{(u)}_{0.25}=9.2254$

$S^{(d)}_{0.25-}=90$
$S^{(d)}_{0.25}=80$
$B_0=1.0126$
$O^{(d)}_{0.25}=0$

$S^{(uu)}_{0.5}=120.75$
$B_{0.5}=1.0253$
$O^{(uu)}_{0.5}=20.75$

$S^{(ud)}_{0.5}=94.5$
$B_{0.5}=1.0253$
$O^{(ud)}_{0.5}=0$

$S^{(du)}_{0.5}=92$
$B_{0.5}=1.0253$
$O^{(du)}_{0.5}=0$

$S^{(dd)}_{0.5}=72$
$B_{0.5}=1.0253$
$O^{(dd)}_{0.5}=0$

$$\Delta^{(u)}_{0.25}=\frac{20.75-0}{120.75-94.5}=0.79048$$

$$b^{(u)}_{0.25}=\frac{20.75-0.79048\times120.75}{1.0253}=-72.857$$

$$O^{(u)}_{0.25}=0.79048\times\underline{105}-1.0126\times72.857=9.2254$$

(after dividend)

$$\Delta^{(d)}_{0.25}=\frac{0-0}{92-72}=0$$

$$b^{(d)}_{0.25}=\frac{0+0\times92}{1.0253}=0$$

$$\Delta_0=\frac{9.2254-0}{115-90}=0.36902 \quad \text{(use before)}$$

$$b_0=\frac{9.2254-0.36902\times115}{1.0126}=-32.799$$

$$O_0=0.36902\times100-32.799=4.103$$

pay for the portfolio

receive dividends immediately

$$\underbrace{\Delta115+b(1.0126)}_{\text{before dividend}}-\Delta10 \quad \text{(cost of building p)}$$

$=\Delta(115-10)+b(1.0126)$

$=\Delta(105)+b(1.0126)$

$\Delta90+b(1.0126)-\Delta10$

$=\Delta(80)+b(1.0126)$

before dividend $\qquad$ liquidate after the dividend

$$\Delta115+b(1.0126)=\Delta10+\underbrace{\Delta105+b(1.0126)}_{\text{payoff of selling p}}$$

$=\Delta115+b(1.0126)$

$\Delta90+b(1.0126)=\Delta10+\Delta80+b(1.0126)$

$=\Delta90+b(1.0126)$

American: $O^{(d)}_{0.25}=\max\{90-100,\,80-100,\,0\}=0$

$O^{(u)}_{0.25}=\max\{115-100,\,105-100,\,9.2254\}=15$

$$\Delta_0=\frac{15-0}{115-90}=0.6$$

$$b_0=\frac{15-0.6\times115}{1.0126}=-53.328$$

$$O_0=0.6\times100-53.328=6.672$$

Arrow Debreu Securities

pays off \$1 only if state i is realized and nothing in all other states in the world

If an Arrow Debreu security exists for each possible state of the world, markets are complete

t_0	rain	no rain
$A_{1,t_0} = 0.7$	$A_{1,t_1}^{(rain)} = 1$	$A_{1,t_1}^{(no)} = 0$
$A_{2,t_0} = 0.25$	$A_{2,t_1}^{(rain)} = 0$	$A_{2,t_1}^{(no)} = 1$
$A_{1,t_0} + A_{2,t_0} = 0.95$	$A_{1,t_1}^{(r)} + A_{2,t_1}^{(r)} = 1$	$A_{1,t_1}^{(n)} + A_{2,t_2}^{(n)} = 1$

$$A_{1,t_0} + A_{2,t_0} = e^{-r_{t_0}(t_1 - t_0)} = 0.95$$

Suppose $P(rain) = 0.6$

$$E\left(\frac{A_{1,t_1} - A_{1,t_0}}{A_{1,t_0}}\right) = \frac{0.6 + 0.4 \times 0 - 0.7}{0.7} = -14.286\%$$ (pay premium for the insurance)

$$E\left(\frac{A_{2,t_1} - A_{2,t_0}}{A_{2,t_0}}\right) = \frac{0.6 \times 0 + 0.4 \times 1 - 0.25}{0.25} = 60\%$$

$$E\left(\frac{A_{1,t_1} + A_{2,t_1} - (A_{1,t_0} + A_{2,t_0})}{A_{1,t_0} + A_{2,t_0}}\right) = \frac{0.6 + 0.4 - 0.95}{0.95} = 5.2632\%$$

K possible states $\Rightarrow K$ Arrow Debreu securities

$$A_{t_i}^{(k)} = [A_{1,t_i}^{(k)}, \dots, A_{k,t_i}^{(k)}]$$

Let $\hat{S}_{t_i} = [S_{t_i}^{(1)}, S_{t_i}^{(2)}, \dots, S_{t_i}^{(k)}]$ and $1_{(1 \times K)} = [1, 1, \dots, 1]$

S_{t_0}

B_{t_0}

A_{t_0}

$S_{t_1}^{(1)}$

$B_{t_1} = 1$

$A_{t_i}^{(1)} = [1, 0, 0, \dots, 0]$

$S_{t_2}^{(2)}$

$B_{t_1} = 1$

$A_{t_2}^{(2)} = [0, 1, \dots, 0]$

$\vdots$

$S_{t_1}^{(k)}$

$B_{t_1} = 1$

$A_{t_1}^{(k)} = [0, 0, \dots, 1]$

$$1_{(1\times k)}(A_{t_0})^T = \sum_{h=1}^{k} A_{h,t_0} = B_{t_0} = e^{-r_{t_0}(t_1-t_0)}$$

$$\hat{S}_{t_1}(A_{t_0})^T = \sum_{h=1}^{k} S_{t_1}^{(h)} A_{h,t_0} = S_0$$

$r = 6.839\%$

E.g.

$t=0$	$t=0.75$			$A_{1,0.75}^{(1)}$			
	$k=1$	$S_{0.75}^{(1)}=140$	$B_1=1$	1	0	0	0
$S_0=100$	$k=2$	$S_{0.75}^{(2)}=130$	$B_1=1$	0	1	0	0
$B_0=e^{-0.06839\times0.75}=0.95$	$k=3$	$S_{0.75}^{(3)}=105$	$B_1=1$	0	0	1	0
$A_{1,0}=0.15$	$k=4$	$S_{0.75}^{(4)}=80$	$B_1=1$	0	0	0	1

$A_0=(0.15, 0.15, 0.3, 0.35)$

Buy each Arrow Debreau Security

In every state we get \$1.

$q_1 = q_2 = \frac{0.15}{0.95} = 0.15789$

$q_3 = \frac{0.3}{0.95} = 0.31579$

$q_4 = \frac{0.35}{0.95} = 0.36842$

$A_{1,0}+A_{2,0}+A_{3,0}+A_{4,0}$

$=0.15+0.15+0.3+0.35$

$=0.95=B_0$

$140A_{1,0}+130A_{2,0}+105A_{3,0}+80A_{4,0}$

$=140\times0.15+130\times0.15+105\times0.3+80\times0.35$

$=100$

$S_0 = e^{-r_0(t_1-t_0)}E_0^Q[S_{0.75}]$

$= e^{-0.06839\times0.75}\sum_{k=1}^{4} q_k S_{0.75}^{(k)}$

$= e^{-0.06839\times0.75}\left(\frac{0.15}{0.95}\times140+\frac{0.15}{0.95}\times130 + \frac{0.3}{0.95}\times105+\frac{0.35}{0.95}\times80\right)$

$=100$

Risk Neutral Probabilities

$q_k = \frac{A_{k,t_0}}{\underbrace{1_{(1\times k)}(A_{t_0})^T}_{B_{t_0}}} = A_{k,t_0}e^{r_{t_0}(t_1-t_0)}$ $\quad q_k\in(0,1)$ $\quad \sum_k q_k = 1$ (looks like a probability ONLY!)

$B_{t_0} = \sum_k q_k e^{-r_{t_0}(t_1-t_0)} = e^{-r_0(t_1-t_0)}E_{t_0}^Q[1] = e^{-r_{t_0}(t_1-t_0)}$

$S_{t_0} = \underbrace{\sum_k S_{t_1}^{(k)} q_k}_{\text{expectation}} e^{-r_{t_0}(t_1-t_0)} = e^{-r_{t_0}(t_1-t_0)}E_{t_0}^Q[S_{t_1}]$ $\qquad \frac{E_0^Q[S_{0.75}]}{S_0} = 1.0526 = e^{0.06839\times0.75}$

Fundamental Theorem of Asset Pricing

If there is no arbitrage, then the price of an asset equals its expected future value under the 'risk neutral' probability measure Q discounted by the risk free interest rate.

$$E_{t_0}^{Q}\left[\frac{S_{t_1}}{S_0}\right] = e^{r_{t_0}(t_1-t_0)}$$

expected value of the future payoff = risk free rate

Radon-Nikodym Derivative and stochastic discount factor

$$B_{t_0} = \sum_k g_k e^{-r_{t_0}(t_1-t_0)} = \sum_k p_k \frac{\xi_{k,t_1}}{\xi_{t_0}} e^{-r_{t_0}(t_1-t_0)} = E_{t_0}^{P}\left[\frac{\xi_{t_1}}{\xi_{t_0}} e^{-r_{t_0}(t_1-t_0)}\right] = E_{t_0}^{P}\left[\frac{\pi_{t_1}}{\pi_{t_0}}\right]$$

$\frac{\xi_{k,t_1}}{\xi_{t_0}} = \frac{g_k}{p_k} > 0$ (Radon-Nikodym derivative) $\frac{\pi_{t_1}}{\pi_{t_0}} = \frac{\xi_{t_1}}{\xi_{t_0}} e^{-r_{t_0}(t_1-t_0)}$ (stochastic discount factor)

$$S_{t_0} = \sum_k g_k S_{t_1}^{(k)} e^{-r_{t_0}(t_1-t_0)} = \sum_k p_k \frac{\xi_{k,t_1}}{\xi_{t_0}} S_{t_1}^{(k)} e^{-r_{t_0}(t_1-t_0)} = E_{t_0}^{P}\left[\frac{\xi_{t_1}}{\xi_{t_0}} e^{-r_{t_0}(t_1-t_0)} S_{t_1}\right] = E_{t_0}^{P}\left[\frac{\pi_{t_1}}{\pi_{t_0}} S_{t_1}\right]$$

True : $p_k = 0.25$

$$\frac{\xi_{1,0.75}}{\xi_0} = \frac{\xi_{2,0.75}}{\xi_0} = \frac{0.15789}{0.25} = 0.63158 \qquad \frac{\xi_{3,0.75}}{\xi_0} = \frac{0.31579}{0.25} = 1.2632$$

$$\frac{\xi_{4,0.75}}{\xi_0} = \frac{0.36842}{0.25} = 1.4737$$

$$\frac{\pi_{1,0.75}}{\pi_0} = \frac{\pi_{2,0.75}}{\pi_0} = 0.6, \quad \frac{\pi_{3,0.75}}{\pi_0} = 1.2, \quad \frac{\pi_{4,0.75}}{\pi_0} = 1.4$$

$\frac{\xi_{k,t_1}}{\xi_{t_0}} > 0 \ \forall k$ since $g_k = p_k \xi_k$ has to be positive

$\frac{\xi_{k,t_1}}{\xi_{t_0}} > 0$ is true if we construct g_k from Arrow Debreu security prices and there's no arbitrage

$E_{t_0}^{P}[\xi_{t_1}] = \xi_{t_0}$ since $E_{t_0}^{P}\left[\frac{\xi_{t_1}}{\xi_{t_0}}\right] = \sum_{k=1}^{K} p_k \frac{\xi_{k,t_1}}{\xi_{t_0}} = \sum_{k=1}^{K} p_k \frac{g_k}{p_k} = \sum_{k=1}^{K} g_k = 1$

Since $\frac{\xi_{1,0.75}}{\xi_0} = \frac{\xi_{2,0.75}}{\xi_0} < 1 < \frac{\xi_{3,0.75}}{\xi_0} < \frac{\xi_{4,0.75}}{\xi_0}$

$$\frac{E_0^{P}(S_{0.75} - S_0)}{S_0} = \frac{0.25\times 140 + 0.25\times 130 + 0.25\times 105 + 0.25\times 80 - 100}{100} = 13.75\% \Rightarrow 17.18\%\ (c.c.) > r = 6.839\%\ (c.c.)$$

$A_{boom,t_0} < A_{recession,t_0}$ (willing to insure against recession)

$q_{boom} < 0.5 < q_{recession} \Rightarrow \frac{\xi_{boom,t_1}}{\xi_{t_0}} < 1 < \frac{\xi_{recession,t_1}}{\xi_{t_0}}$

$$\underline{\underline{S_{t_0}}} = e^{-r_{t_0}(t_1-t_0)} E^{Q}_{t_0}[S_{t_1}]$$

$$= e^{-r_{t_0}(t_1-t_0)} \left[q_{boom} S_{t_1}^{(boom)} + q_{rec.} S_{t_1}^{(rec)} \right]$$

$$\underline{= e^{-r_{t_0}(t_1-t_0)} E^{P}_{t_0}\left[\frac{\xi_{t_1}}{\xi_{t_0}} S_{t_1}\right]}$$

$$= e^{-r_{t_0}(t_1-t_0)} \left[P_{boom} \frac{\xi_{b,t_1}}{\xi_{t_0}} S_{t_1}^{(b)} + P_{rec.} \frac{\xi_{rec,t_1}}{\xi_{t_0}} S_{t_1}^{(rec.)} \right]$$

$$\frac{E^{P}_{t_0}[S_{t_1}]}{S_{t_0}} - 1 = \underline{\underline{\frac{E^{P}_{t_0}[S_{t_1}]}{e^{-r_{t_0}(t_1-t_0)} E^{P}_{t_0}\left[\frac{\xi_{t_1}}{\xi_{t_0}} S_{t_1}\right]}}} - 1$$

more weight on recession

$\uparrow$ good, $\downarrow$ bad

$\Rightarrow$ numerator > denominator

pro-cyclical asset: pays a lot during boom, little during recession ($S_{t_1}^{(boom)} > S_{t_1}^{(rec.)}$)

$S_{t_0} = E^{P}_{t_0}\left[\frac{\xi_{t_1}}{\xi_{t_0}} S_{t_1}\right] < E^{P}_{t_0}[S_{t_1}]$ payoff$^{(b)}$ are weighted less

counter-cyclical asset ($S_{t_1}^{(b)} < S_{t_1}^{(re)}$) has a relatively high price

$S_{t_0} = E^{P}_{t_0}\left[\frac{\xi_{t_1}}{\xi_{t_0}} S_{t_1}\right] > E^{P}_{t_0}[S_{t_1}]$

$\frac{\xi_{k,t_1}}{\xi_{t_0}} > 1 \Rightarrow$ bad state assign higher probability to state k under measure Q.

Assigning heavier weights to bad states $\Rightarrow$ low payoff in bad states has a low price and a high expected return.

E.g.

$P_1 = P_2 = P_3 = \frac{1}{3}$

$X_0 = 100$

$Y_0 = 100$

$B_0 = e^{-0.1 \times 0.25}$

$A_{1,0} =$

$A_{2,0}$

$A_{3,0} =$

$X_{0.25}^{(1)} = 140 = 140 A_{1,0.25}^{(1)} + 110 A_{2,0.25}^{(1)} + 80 A_{3,0.25}^{(1)}$

$Y_{0.25}^{(1)} = 150$

$B_{0.25}^{(1)} = 1$

$X_{0.25}^{(2)} = 110 = 140 A_{1,0.25}^{(2)} + 110 A_{2,0.25}^{(2)} + 80 A_{3,0.25}^{(2)}$

$Y_{0.25}^{(2)} = 170$

$B_{0.25}^{(1)} = 1$

$X_{0.25}^{(3)} = 80 = 140 A_{1,0.25}^{(3)} + 110 A_{2,0.25}^{(3)} + 80 A_{3,0.25}^{(3)}$

$Y_{0.25}^{(3)} = 40$

$B_{0.25}^{(1)} = 1$

$$E(R_X) = \frac{\ln\left(\frac{\frac{1}{3}\times 140 + \frac{1}{3}\times 110 + \frac{1}{3}\times 80}{100}\right)}{0.25} = 38.124\%$$

$$E(R_Y) = \frac{\ln\left(\frac{\frac{1}{3}\times 150 + \frac{1}{3}\times 170 + \frac{1}{3}\times 40}{100}\right)}{0.25} = 72.929\%$$

$$\begin{pmatrix} 140 & 110 & 80 \\ 150 & 170 & 40 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_{1,0} \\ A_{2,0} \\ A_{3,0} \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ e^{-0.1\times 0.25} \end{pmatrix}$$

$\Rightarrow A_{1,0} = 0.22826 \quad A_{2,0} = 0.27600 \quad A_{3,0} = 0.47106$ (bad state)

$q_k = \frac{A_{k,0}}{A_{1,0} + A_{2,0} + A_{3,0}}$ $\quad \underline{q_1 = 0.23404 \quad q_2 = 0.28298 \quad q_3 = 0.48298}$ $\quad \sum_{i=1}^{3} q_i = 1$

FTAP :

$$\begin{pmatrix} 140\, e^{-0.1\times 0.25} & 110\, e^{-0.1\times 0.25} & 80\, e^{-0.1\times 0.25} \\ 150\, e^{-0.1\times 0.25} & 170\, e^{-0.1\times 0.25} & 40\, e^{-0.1\times 0.25} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ 1 \end{pmatrix}$$

$\frac{\xi_{k,0.25}}{\xi_0} = \frac{q_k}{p_k}$ $\quad \frac{\xi_{1,0.25}}{\xi_0} = 0.70212 \quad \frac{\xi_{2,0.25}}{\xi_0} = 0.84894, \quad \frac{\xi_{3,0.25}}{\xi_0} = 1.4489$

$\frac{\pi_{k,0.25}}{\pi_0} = \frac{\xi_{k,0.25}}{\xi_0} e^{-0.1\times 0.25}$ $\quad \frac{\pi_{1,0.25}}{\pi_0} = 0.68478 \quad \frac{\pi_{2,0.25}}{\pi_0} = 0.82798 \quad \frac{\pi_{3,0.25}}{\pi_0} = 1.4131$ (emphasize)

FTAP

S_{t_i}

B_{t_i}

O_{t_i}

A_{1,t_i}

A_{2,t_i}

$S^{(u)}_{t_{i+1}} = S_{t_i} u_{t_i}$

$B_{t_{i+1}} = 1$

$O^{(u)}_{t_{i+1}} =$

$A^{(u)}_{1,t_{i+1}} = 1$

$A^{(d)}_{2,t_{i+1}} = 0$

$S^{(d)}_{t_{i+1}} = S_{t_i} d_{t_i}$

$B_{t_{i+1}} = 1$

$O^{(d)}_{t_{i+1}}$

$A^{(d)}_{1,t_{i+1}} = 0$

$A^{(d)}_{2,t_{i+1}} = 1$

$q_{d,t_i} = 1 - q_{u,t_i}$

$$S_{t_i} = e^{-r_{t_i}(t_{i+1}-t_i)} E^Q[S_{t_{i+1}}]$$
$$= e^{-r_{t_i}(t_{i+1}-t_i)}\left[S^{(u)}_{t_{i+1}} q_{u,t_i} + S^{(d)}_{t_{i+1}} q_{d,t_i}\right]$$
$$= e^{-r_{t_i}(t_{i+1}-t_i)}\left[S^{(u)}_{t_{i+1}} q_{u,t_i} + S^{(d)}_{t_{i+1}} (1 - q_{u,t_i})\right]$$
$$q_{u,t_i} = \frac{e^{r_{t_i}(t_{i+1}-t_i)} S_{t_i} - S^{(d)}_{t_{i+1}}}{S^{(u)}_{t_{i+1}} - S^{(d)}_{t_{i+1}}} = \frac{e^{r_{t_i}(t_{i+1}-t_i)} - d_{t_i}}{u_{t_i} - d_{t_i}}$$

$$S_{t_i} = S^{(u)}_{t_{i+1}} A_{1,t_i} + S^{(d)}_{t_{i+1}} A_{2,t_i}$$
$$B_{t_i} = A_{1,t_i} + A_{2,t_i}$$
$$O_{t_i} = O^{(u)}_{t_{i+1}} A_{1,t_i} + O^{(d)}_{t_{i+1}} A_{2,t_i}$$
$$q_{u,t_i} = \frac{A_{1,t_i}}{A_{1,t_i} + A_{2,t_i}} \qquad q_{d,t_i} = \frac{A_{2,t_i}}{A_{1,t_i} + A_{2,t_i}}$$

$$O_{t_i} = e^{-r_{t_i}(t_{i+1}-t_i)} E^Q[O_{t_{i+1}}] = e^{-r_{t_i}(t_{i+1}-t_i)}\left[O^{(u)}_{t_{i+1}} q_{u,t_i} + O^{(d)}_{t_{i+1}} q_{d,t_i}\right]$$

$$O_{t_i} = \Delta_{t_i} S_{t_i} + b_{t_i} B_{t_i}$$
$$= \frac{B_{t_i}}{B_{t_{i+1}}}\left[\frac{\frac{B_{t_{i+1}}}{B_{t_i}} - S^{(d)}_{t_{i+1}}}{\frac{S^{(u)}_{t_{i+1}}}{S_{t_i}} - \frac{S^{(d)}_{t_{i+1}}}{S_{t_i}}} O^{(u)}_{t_{i+1}} + \frac{\frac{S^{(u)}_{t_{i+1}}}{S_{t_i}} - \frac{B_{t_{i+1}}}{B_{t_i}}}{\frac{S^{(u)}_{t_{i+1}}}{S_{t_i}} - \frac{S^{(d)}_{t_{i+1}}}{S_{t_i}}} O^{(d)}_{t_{i+1}}\right]$$
$$= e^{-r_{t_i}(t_{i+1}-t_i)}\left[\frac{e^{r_{t_i}(t_{i+1}-t_i)} - d_{t_i}}{u_{t_i} - d_{t_i}} O^{(u)}_{t_{i+1}} + \frac{u_{t_i} - e^{r_{t_i}(t_{i+1}-t_i)}}{u_{t_i} - d_{t_i}} O^{(d)}_{t_{i+1}}\right]$$
$$= \frac{B_{t_i}}{B_{t_{i+1}}}\left[q_{u,t_i} O^{(u)}_{t_{i+1}} + q_{d,t_i} O^{(d)}_{t_{i+1}}\right]$$

Eg. (previous)

$k = 105$

$S_0 = 100$

$B_0 = 1$

$O_0 = \Delta 100 + b(1)$

$= 1.891$

$r = 2\%$

$S_{0.08}^{(u)} = 110$

$B_{0.08} = 1.0017$

$O_{0.08}^{(u)} =$

$=$

$S_{0.08}^{(d)} = 95$

$B_{0.08} = 1.0017$

$O_{0.06}^{(d)} =$

$\Delta = 0$

$b = 0$

$S_{0.17}^{(uu)} = 121$

$B_{0.17} = 1.0033$

$O_{0.17}^{(uu)} = 16$

$S_{0.17}^{(ud)} = S_{0.17}^{(du)} = 104.5$

$B_{0.17} = 1.0033$

$O_{0.17}^{(ud)} = O_{0.17}^{(du)} = 0$

$S_{0.17}^{(dd)} = 90.25$

$B_{0.17} = 1.0033$

$O_{0.17}^{(dd)} = 0$

$$100 = e^{-0.02 \times \frac{1}{12}} \left[110 q_u + 95(1 - q_u) \right]$$

$$\Rightarrow q_u = \frac{e^{0.02 \times \frac{1}{12}} - 0.95}{1.1 - 0.95} = 0.34445$$

$$O_{0.08}^{(u)} = e^{-0.02 \times \frac{1}{12}} \left[16 \times 0.34445 + 0 \times (1 - 0.34445) \right] = 5.5021$$

$$O_{0.06}^{(d)} = e^{-0.02 \times \frac{1}{12}} \left[0 \times 0.34445 + 0 \times (1 - 0.34445) \right] = 0$$

$$O_0 = e^{-0.02 \times \frac{1}{12}} \left[5.5021 \times 0.34445 + 0 \times (1 - 0.34445) \right] = 1.8921$$

Put refer to my notes

S_{t_i-} (cum-dividend prices) $\quad S_{t_i-} = S_{t_i} + D_{t_i}$

S_{t_i} (ex-dividend price)

$t_i \qquad t_{i+1}$

b4 dividend $\quad$ after $\quad$ b4 $\quad$ after

$S_{t_i}^- = D_{t_i} + S_{t_i} \qquad S_{t_{i+1}}^- = D_{t_{i+1}} + S_{t_{i+1}}$

receive immediately

$$S_{t_i} - = E^Q_{t_i-}\left[D_{t_i} + e^{-r_{t_i}(t_{i+1}-t_i)} S_{t_{i+1}}-\right]$$

$$= E^Q_{t_i-}\left[D_{t_i} + e^{-r_{t_i}(t_{i+1}-t_i)}\left(S_{t_{i+1}} + D_{t_{i+1}}\right)\right]$$

$$S_{t_i} = E^Q_{t_i}\left[e^{-r_{t_i}(t_{i+1}-t_i)} S_{t_{i+1}}-\right] = E^Q_{t_i}\left[e^{-r_{t_i}(t_{i+1}-t_i)}\left(S_{t_{i+1}} + D_{t_{i+1}}\right)\right]$$

European Call with Dividends

$S_0 = 100 \qquad u = 1.15 \qquad d = 0.9 \qquad$ every 3 mth, a dividend of 10 $\qquad r = 5\%$ (c.c.) $\qquad K = 100$

$S_0 = 100$
$B = 1$
$O_0 = 4.1038$

$S^{(u)}_{0.25-} = 115$
$S^{(u)}_{0.25} = 105$
$B_{0.25} = 1.0126$
$O^{(u)}_{0.25} = 9.2254$

$S^{(d)}_{0.25-} = 90$
$S^{(d)}_{0.25} = 80$
$B_0 = 1.0126$
$O^{(d)}_{0.25} = 0$

$S^{(uu)}_{0.5} = 120.75$
$B_{0.5} = 1.0253$
$O^{(uu)}_{0.5} = 20.75$

$S^{(ud)}_{0.5} = 94.5$
$B_{0.5} = 1.0253$
$O^{(ud)}_{0.5} = 0$

$S^{(du)}_{0.5} = 92$
$B_{0.5} = 1.0253$
$O^{(du)}_{0.5} = 0$

$S^{(dd)}_{0.5} = 72$
$B_{0.5} = 1.0253$
$O^{(dd)}_{0.5} = 0$

$$100 = e^{-0.05\times 0.25}\left[115 q_u + 90(1-q_u)\right] \quad \text{(b4 dividend)}$$

$$q_u = \frac{e^{0.05\times 0.25} - 0.9}{1.15 - 0.9} = 0.45031$$

$$O^{(u)}_{0.25} = e^{-0.05\times 0.25}\left[20.75 \times 0.45031 + 0(1-0.45031)\right] = 9.2279$$

$$O^{(d)}_{0.25} = e^{-0.05\times 0.25}\left[0 \times 0.45031 + 0(1-0.45031)\right] = 0$$

$$O_0 = e^{-0.05\times 0.25}\left(9.2279 \times 0.45031 + 0(1-0.45031)\right) = 4.1038$$

n time steps

$$S_{t_n}^{(uu\cdots u)} = u^n S_{t_0}$$

$$S_{t_n}^{(du\cdots u)} = u^k d^{n-k} S_{t_0}$$

$$S_{t_n}^{(dd\cdots d)} = d^n S_{t_0}$$

$$O_{t_n} = f(S_{t_n}) \qquad q_u = \frac{e^{r(t_{i+1}-t_i)} - d}{u-d} \qquad q_d = 1 - q_u = \frac{u - e^{r(t_{i+1}-t_i)}}{u-d}$$

$$O_{t_i}^{(\{h\})} = e^{-r(t_{i+1}-t_i)}\left(q_u \underline{O_{t_{i+1}}^{(\{h\}u)}} + q_d \underline{\underline{O_{t_{i+1}}^{(\{h\}d)}}}\right)$$

$\{h\}$ defines the history of 'up's and 'down's since t_0

$$O_{t_i}^{(\{h\})} = e^{-r(t_{i+1}-t_i)} q_u \left(\underline{e^{-r(t_{i+1}-t_i)}\left(q_u O_{t_{i+2}}^{(\{h\}uu)} + q_d O_{t_{i+2}}^{(\{h\}ud)}\right)}\right)$$

$$+ e^{-r(t_{i+1}-t_i)} q_d \left(\underline{\underline{e^{-r(t_{i+1}-t_i)}\left(q_u O_{t_{i+2}}^{(\{h\}du)} + q_d O_{t_{i+2}}^{(\{h\}dd)}\right)}}\right)$$

$$= e^{-r(t_{i+2}-t_i)}\left(q_u^2 O_{t_{i+2}}^{(\{h\}uu)} + 2 q_u q_d O_{t_{i+2}}^{(\{h\}ud)} + q_d^2 O_{t_{i+2}}^{(\{h\}dd)}\right)$$

$$O_{t_0} = e^{-r(t_n-t_0)} \sum_{k=0}^{n} \binom{n}{k} q_u^k q_d^{n-k} \overbrace{f(\underbrace{u^k d^{n-k} S_{t_0}}_{S_T})}^{O_T}$$

$$f(S_T) = \begin{cases} \max\{S_T - K, 0\} & \text{if call} \\ \max\{K - S_T, 0\} & \text{if put} \end{cases}$$

Chp. 9 Greek Letters

Hedging using options: $\Delta S + bB = 0$

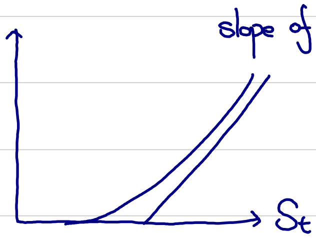
$$S - \frac{1}{\Delta} O = \frac{bB}{\Delta} \quad \text{(risk-free)}$$

Call: $\boxed{\Delta_t = \dfrac{\partial O(S,\sigma,r,T-t)}{\partial S}}$ $\quad C = S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2)$

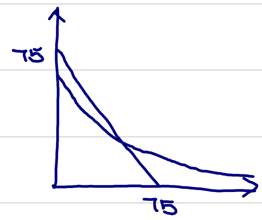
$$\Delta_t = \Phi(d_1)$$

Put: $\Delta_t = -\Phi(-d_1) = \Phi(d_1) - 1$

slope of call price $\quad \Delta_t \in (0,1)$



$\Delta_t \in (-1, 0)$



Delta hedge is only perfect under BS world.

$\lim_{S_t \to \infty} \Delta_t = 1 \qquad \lim_{S_t \to 0} \Delta_t = 0 \qquad \lim_{T-t \to 0} \Delta_t = \lim_{\sigma \to 0} \Delta_t = \begin{cases} 1 & S_t > e^{-r(T-t)}K \\ \frac{1}{2} & S_t = e^{-r(T-t)}K \\ 0 & S_t < e^{-r(T-t)}K \end{cases}$ 看图

$\Delta_t^{(j)} = \dfrac{\partial V_{j,t}}{\partial S} \qquad P_t = \sum_j w_{j,t} V_{j,t}$ (many assets) $V_{j,t}$: value of asset j at time t.

$$\Delta_t^{(P)} = \frac{\partial P_t}{\partial S} = \sum w_{j,t} \frac{\partial V_{j,t}}{\partial S} = \sum w_{j,t} \Delta_t^{(j)}$$

$$\Theta_t = \frac{\partial O(S,\sigma,r,T-t)}{\partial t} = -\frac{\partial O(S,\sigma,r,T-t)}{\partial (T-t)}$$

$\Theta_t = -\dfrac{S_t \sigma}{2\sqrt{T-t}} \phi(d_1) - rKe^{-r(T-t)} \Phi(d_2)$: call $\quad < 0$

$\Theta_t = -\dfrac{S_t \sigma}{2\sqrt{T-t}} \phi(d_1) + rKe^{-r(T-t)} \Phi(-d_2)$: put $\quad$ can be positive

Same as above

$$\Theta_t^{(P)} = \frac{\partial P_t}{\partial t} = \sum w_{j,t} \frac{\partial V_{j,t}}{\partial t} = \sum w_{j,t}^{(j)} \Theta_t^{(j)}$$

$$\Gamma_t = \frac{\partial \Delta_t}{\partial S} = \frac{\partial^2 O(S,\sigma,r,T-t)}{\partial S^2}$$

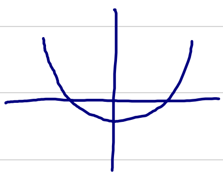
According to the put-Call parity, Gamma of put and call options are the same.

essence of optionality ; greatest when close to at the money

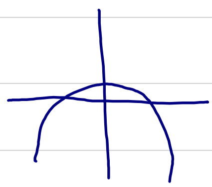
$$C_t - (\Delta_t S_t + b_t B_t)$$

$$\Gamma_t = \frac{1}{S_t \sigma \sqrt{T-t}} \phi(d_1) > 0$$

$C_t - (\Delta_t S_t + b_t B_t)$ buy a call, short sell synthetic option



long Gamma



short Gamma

$$\lim_{S_t \to \infty} \Gamma_t = \lim_{S_t \to 0} \Gamma_t = 0$$

$$\lim_{T-t \to 0} \Gamma_t = \lim_{\sigma \to 0} \Gamma_t = \begin{cases} 0 & S_t > e^{-r(T-t)}K \\ \infty & S_t = e^{-r(T-t)}K \\ 0 & S_t < e^{-r(T-t)}K \end{cases}$$

$$\Gamma_t^{(P)} = \frac{\partial^2 P_t}{\partial S^2} = \sum w_{j,t} \frac{\partial^2 V_{j,t}}{\partial S^2} = \sum w_{j,t} \Gamma_t^{(j)}$$

Vega:

$$\nu_t = \frac{\partial O(S,\sigma,r,T-t)}{\partial \sigma}$$ call - put the same

forward option has no gamma but vega

$$\nu_t = S_t \sqrt{T-t}\, \phi(d_1) > 0$$

close to at the money, vega is larger (cherry picking right)

$$\nu_t^{(P)} = \frac{\partial P_t}{\partial \sigma} = \sum w_{j,t} \frac{\partial V_{j,t}}{\partial \sigma} = \sum w_{j,t} \nu_t^{(j)}$$

Rho:

$$\rho_t = \frac{\partial O(S,\sigma,r,T-t)}{\partial r} = (T-t)Ke^{-r(T-t)}\Phi(d_2) > 0 \ : \text{call}$$

$$-(T-t)Ke^{-r(T-t)}\Phi(-d_2) < 0 \ : \text{put}$$

$$\rho_t^{(P)} = \frac{\partial P_t}{\partial r} = \sum w_{j,t} \frac{\partial V_{j,t}}{\partial r} = \sum w_{j,t}^{(j)} \rho_t^{(j)}$$

$$P_{t_{i+1}} - P_{t_i} = \frac{\partial P}{\partial t}(t_{i+1} - t_i) + \frac{\partial P}{\partial S}(S_{t_{i+1}} - S_{t_i}) + \frac{\partial P}{\partial \sigma}(\sigma_{t_{i+1}} - \sigma_{t_i})$$
$$+ \frac{\partial P}{\partial r}(r_{t_{i+1}} - r_{t_i}) + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}(S_{t_{i+1}} - S_{t_i})^2$$
$$= \sum w_{j,t}\Big[\Theta_t^{(j)}(t_{i+1} - t_i) + \Delta_t^{(j)}(S_{t_{i+1}} - S_{t_i}) + v_t^{(j)}(\sigma_{t_{i+1}} - \sigma_{t_i})$$
$$+ \rho_t^{(j)}(r_{t_{i+1}} - r_{t_i}) + \frac{1}{2}\Gamma_t^{(j)}(S_{t_{i+1}} - S_{t_i})^2$$

E.g.

P:	$\Delta = 0$	$\Gamma = -5000$	$v = -8000$
O_1:	$\Delta = 0.6$	$\Gamma = 0.5$	$v = 2$
O_2.	$\Delta = 0.5$	$\Gamma = 0.8$	$v = 1.2$

S. $\underline{\Delta = 1}$ $\Gamma = v = 0$ $\frac{\partial S}{\partial S} = 1$ $\frac{\partial^2 S}{\partial S^2} = 0$

$O(S, \sigma, r, T-t) = S$

What position in option 1 and S will make P Δ, Γ neutral

Long 10000 options, short $\underline{6000 \text{ of stock } S}$.

. . . make Δ, v neutral

Long 4000 options, short 2400 of stock S.

. . . option 1, 2 make Δ, Γ, v neutral

$$\begin{cases} -5000 + 0.5w_1 + 0.8w_2 + 0w_3 = 0 & (\Gamma) \\ -8000 + 2.0w_1 + 1.2w_2 + 0w_3 = 0 & (v) \\ 0 + 0.6w_1 + 0.5w_2 + w_3 = 0 & (\Delta) \end{cases}$$

$\Rightarrow w_1 = 400$ $w_2 = 6000$, $w_3 = -3240$

400 and 6000 long positions in option 1 and option 2,

3240 short positions in stock S.