1.1

11. F为事件域，$A_n \in F$，$n=1,2,\cdots$，Prove

(1) $\phi \in F$ Since $\Omega \in F$，$\bar{\Omega} \in F$，$\phi \in F$

(2) $\bigcup_{i=1}^{n} A_i \in F$，$n \geq 1$ $A_{n+1}=A_{n+2}=\cdots=\phi$ $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} A_i \in F$

(3) $\bigcap_{i=1}^{n} A_i \in F$，$n \geq 1$ $A_{n+1}=A_{n+2}=\cdots=\phi$ $\bigcup_{i=1}^{n} \bar{A_i} \in F$，$\overline{\bigcup_{i=1}^{n} \bar{A_i}} \in F$，$\bigcap_{i=1}^{n} A_i \in F$

(4) $\bigcap_{i=1}^{\infty} A_i \in F$，$\bar{A_1},\cdots,\bar{A_n},\cdots \in F$，$\bigcup_{i=1}^{\infty} \bar{A_i} \in F$，$\overline{\bigcup_{i=1}^{\infty} \bar{A_i}} \in F \Rightarrow \bigcap_{i=1}^{\infty} A_i \in F$

(5) $A_1 - A_2 \in F$ $\bar{A_2} \in F$，$A_1 \cap \bar{A_2} \in F$，$A_1 - A_2 \in F$

1.2

1. (3) ${}^nC_0 + {}^nC_1 + \cdots + {}^nC_n$

$= (1+1)^n$

$= 2^n$

(4) ${}^nC_1 + 2\,{}^nC_2 + \cdots + n\,{}^nC_n = n\, 2^{n-1}$

$r\,{}^nC_r = \frac{r\,n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r)!} = n\frac{(n-1)!}{(r-1)!(n-r)!} = n\,{}^{n-1}C_{r-1}$

$= n\left({}^{n-1}C_0 + \cdots + {}^{n-1}C_{n-1}\right)$

$= n \cdot 2^{n-1}$

(5) ${}^aC_0\,{}^bC_n + {}^aC_1\,{}^bC_{n-1} + \cdots + {}^aC_n\,{}^bC_0 = {}^{a+b}C_n$，$n = \min\{a,b\}$

$(1+x)^a$ $(1+x)^b$

$= {}^aC_0 + {}^aC_1 x + \cdots + {}^aC_a x^a$ $= {}^bC_0 + {}^bC_1 x + \cdots + {}^bC_b x^b$

Coefficient of $x^n = {}^aC_0\,{}^bC_n + {}^aC_1\,{}^bC_{n-1} + \cdots + {}^aC_n\,{}^bC_0$

$(1+x)^a(1+x)^b = (1+x)^{a+b} = {}^{a+b}C_0 + {}^{a+b}C_1 x + \cdots + {}^{a+b}C_{a+b} x^{a+b}$

${}^aC_0\,{}^bC_n + \cdots + {}^aC_n\,{}^bC_0 = {}^{a+b}C_n$

10. $1,2,\cdots,n$ 取2个，其中1个小于k，1个大于k概率？ $\frac{{}^{k-1}C_1\,{}^{n-k}C_1}{{}^nC_2}$

14. n个人围一桌，甲乙相邻？ $\frac{{}^2C_1\,(n-2)!}{(n-1)!} = \frac{2}{n-1}$ 本人认为：$\frac{{}^nC_1\,{}^2C_1\,(n-2)!}{n!}$ 更易理解

15. 5枚骰子，两枚点数一样？ $\frac{{}^5C_2\,{}^6C_1\,{}^5P_3}{6^5} = 0.4630$

2对？ $\frac{{}^6C_2\,{}^5C_2\,{}^3C_2\,{}^4C_1}{6^5} = 0.2315$ 直接取2点，5枚取2枚，剩下3枚取2枚

四枚一样 $\frac{{}^6C_1\,{}^5C_4 \times 5}{6^5} = 0.0193$

16. 六根草，头两两相接，尾两两相接，一个环的概率？ $\frac{4\times2}{5\times3}=\frac{8}{15}$

17. n个0，n个1，没有2个1连一起？ $\frac{n!\,{}^{n+1}P_n}{(2n)!}$

19. n男，m女，任意2女不相邻？ $(m\le n+1)$ $\frac{\frac{(n+1)!}{m!(n+1-m)!}}{\frac{(n+m)!}{n!\ m!}}=\frac{n!\,(n+1)!}{(n+m)!\,(n+1-m)!}$

22. 将n个完全相同的球，随机放入N个盒子

(1) 指定盒子恰好有k个球 $M={}^{N+n-1}C_n$ $A_1={}^{N-1+(n-k)-1}C_{n-k}$

(2) 恰好有m个空盒 $P(A_1)=\frac{(N+n-k-2)!\,n!\,(N-1)!}{(N+n-1)!\,(n-k)!\,(N-2)!}$

(3) 指定的m个盒子恰好有j个球

$A_2={}^{N}C_m\,{}^{n-1}C_{n-(N-m)}$ $A_3={}^{m+j-1}C_j\ {}^{(N-m)+(n-j)-1}C_{n-j}$

28. $a>0$, $0<x<a$, $0<y<a$, $P(xy<\frac{a^2}{4})$ $xy=\frac{a^2}{4}$

$y=\frac{a^2}{4}\frac{1}{x}$

$$P(A)=\frac{a^2-\int_{\frac{a}{4}}^{a}(a-\frac{a^2}{4x})dx}{a^2}=\frac{1}{4}+\frac{1}{4}\ln 4$$

1.3

2. (1) $P(AB)=0$ A和B可相容也可能不相容 $A=B=\Omega$

10. $P(A)=1$ $\bar{A}B\subset\bar{A}$ $P(\bar{A}B)\le P(\bar{A})=1-P(A)=0$ $P(\bar{A}B)=P(B-A)=P(B)-P(AB)=0$

11. 2n+1次硬币，正面数大于反面数？ $P(A)=0.5$

16. $P(AB)=P(\bar{A}\bar{B})$ $P(A)=p$ $P(\bar{A}\bar{B})=P(\overline{A\cup B})=1-P(A\cup B)=1-P(A)-P(B)+P(AB)$ $P(B)=1-p$

19. (1) $P(AB)+P(AC)-P(BC)\le P(A)$

$P(AB\cup AC)=P(AB)+P(AC)-P(ABC)$ $(AB\cup BC)\subset A$, $ABC\subset BC$

$P(AB\cup AC)\le P(A)$, $P(ABC)\le P(BC)$

$P(AB)+P(AC)-P(BC)=P(AB\cup AC)+P(ABC)-P(BC)\le P(AB\cup AC)\le P(A)$

(2) $P(AB)+P(AC)+P(BC)\ge P(A)+P(B)+P(C)-1$

Since $P(A)+P(B)+P(C)-1$

$=P(A)+P(B)+P(C)-(P(A)+P(B)+P(C)-P(AB)-P(BC)-P(AC)+P(ABC))$

$=P(AB)+P(BC)+P(AC)-P(ABC)$

$\Rightarrow P(AB)+P(AC)+P(BC)\ge P(A)+P(B)+P(C)-1$

20 $P(A)=a$ $P(B)=2a$ $P(C)=3a$ $P(AB)=P(AC)=P(BC)=b$ Prove $a\le\frac{1}{4}$, $b\le\frac{1}{4}$

$P(B\cup C)=P(B)+P(C)-P(BC)=5a-b$ $a=P(A)\geq P(AB)=b$

Since $a\geq b$ $\Rightarrow 5a-b\geq 4a$ $4a\le 1$

$a\le\frac{1}{4}$ $b\le a\le\frac{1}{4}$

21. $P(A)=P(B)=P(C)=\frac{1}{2}$ $P(ABC)=P(\bar{A}\cap\bar{B}\cap\bar{C})$ Prove $2P(ABC)=P(AB)+P(AC)+P(BC)-\frac{1}{2}$

$=P(\overline{A\cup B\cup C})=1-P(A\cup B\cup C)$ ☆

$=1-P(A)-P(B)-P(C)+P(AB)+P(BC)+P(AC)-P(ABC)$

$2P(ABC)=1-\frac{3}{2}+P(AB)+P(BC)+P(AC)$

$=P(AB)+P(AC)+P(BC)-\frac{1}{2}$

22.(1) $P(AB)\geq P(A)+P(B)-1$ $P(A\cup B)=P(A)+P(B)-P(AB)$

$P(AB)=P(A)+P(B)-P(A\cup B)\geq P(A)+P(B)-1$

23. $|P(AB)-P(A)P(B)|\le\frac{1}{4}$

$P(AB)-P(A)\underline{P(B)}=P(AB)-P(A)(\underline{P(AB)+P(\bar{A}B)})$ 构造可以记住

$=P(AB)(1-P(A))-P(A)P(\bar{A}B)$

$0\le P(AB)(1-P(A))\le P(A)(1-P(A))$

$0\le P(A)P(\bar{A}B)\le P(A)P(\bar{A})=P(A)(1-P(A))$

$|P(AB)-P(A)P(B)|=|P(AB)(1-P(A))-P(A)P(\bar{A}B)|$

$\le\max\{P(AB)[1-P(A)],P(A)P(\bar{A}B)\}\le P(A)-(P(A))^2=\frac{1}{4}-[P(A)-\frac{1}{2}]^2\le\frac{1}{4}$

1.4

11. 1白1黑，从中任取1个，取白停止，取黑放回再加1个黑

(1) 第n次，没结束？ $\frac{1}{2}\times\frac{2}{3}\times\cdots\times\frac{n}{n+1}=\frac{1}{n+1}$

(2) 第n次，结束？ $\frac{1}{2}\times\cdots\times\frac{1}{n+1}=\frac{1}{n(n+1)}$

18. 4项单选，现答对，做对概率

(1) given 猜和做对都是$\frac{1}{2}$. $P(做对|对)=\dfrac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}\times\frac{1}{4}}=0.8$

21. n根绳子的2n个头任意相接，结成n个圈 $\frac{1}{(2n-1)!!}$

22. m个人传球，从甲开始，传给剩余m-1个人，第n次由甲传出？

$$P(A_k) = (1-P(A_{k-1}))\left(\frac{1}{m-1}\right) = \frac{1}{m-1} - \frac{1}{m-1}P(A_{k-1})$$

$$P(A_n) = \frac{1}{m-1} - \frac{1}{m-1}P(A_{n-1}) = \frac{1}{m-1} - \frac{1}{m-1}\left(\frac{1}{m-1} - \frac{1}{m-1}P(A_{n-1})\right)$$

$$= \frac{1}{m-1} - \left(\frac{1}{m-1}\right)^2 + \left(\frac{1}{m-1}\right)^2 P(A_{n-1})$$

$$= \frac{1}{m-1} - \left(\frac{1}{m-1}\right)^2 + \cdots + (-1)^{n-2}\left(\frac{1}{m-1}\right)^{n-1} + (-1)^{n-1}\left(\frac{1}{m-1}\right)^{n-1} P(A_1)$$

$$= \frac{1}{m-1} - \left(\frac{1}{m-1}\right)^2 + \cdots + (-1)^{n-3}\left(\frac{1}{m-1}\right)^{n-2}$$

$$= \frac{\frac{1}{m-1}\left(1-\left(-\frac{1}{m-1}\right)^{n-2}\right)}{1-\left(-\frac{1}{m-1}\right)} = \frac{1}{m}\left(1-\left(\frac{-1}{m-1}\right)^{n-2}\right)$$

23. 甲乙轮流掷骰子，甲先，掷出1点给对方，第n次由甲掷？

$P(A_1)=1 \qquad P(A_k) = P(A_{k-1})\frac{5}{6} + (1-P(A_{k-1}))\frac{1}{6} = \frac{1}{6} + \frac{2}{3}P(A_{k-1})$

$$P(A_n) = \frac{1}{6} + \frac{2}{3}P(A_{n-1}) = \frac{1}{6} + \frac{2}{3}\left(\frac{1}{6} + \frac{2}{3}P(A_{n-2})\right) = \frac{1}{6} + \frac{2}{3}\cdot\frac{1}{6} + \left(\frac{2}{3}\right)^2 P(A_{n-2})$$

$$= \frac{1}{6} + \frac{2}{3}\cdot\frac{1}{6} + \cdots + \left(\frac{2}{3}\right)^{n-2}\cdot\frac{1}{6} + \left(\frac{2}{3}\right)^{n-1}P(A_1)$$

$$= \frac{1}{2} + \frac{1}{2}\cdot\left(\frac{2}{3}\right)^{n-1}$$

24. 甲口袋有1黑，2白，乙3白，两口袋交换一球，n次后，黑在甲袋

$P(A_1)=1 \qquad P(A_k) = \frac{2}{3}P(A_{k-1}) + (1-P(A_{k-1}))\cdot\frac{1}{3} = \frac{1}{3} + \frac{1}{3}P(A_{k-1})$

$$P(A_n) = \frac{1}{2} + \frac{1}{2}\cdot\left(\frac{1}{3}\right)^n$$

26. b个黑，r个红，取一个放回，再加入c个同色球，第k次取黑概率为 $\frac{b}{b+r}$，$k=1,2,\cdots$

$P(A_1) = \frac{b}{b+r} \qquad P(A_k) = \frac{b}{b+r}\cdot\frac{b+c}{b+r+c} + \frac{r}{b+r}\cdot\frac{r+c}{b+r+c} = \frac{b}{b+r}$

$P(A_{k-1}) = \frac{b}{b+r}$

27. a白，b黑，n红，一个个取不返回，P(白比黑早) $= \frac{a}{a+b}$

1st: P(白) $= \frac{a}{a+b+n} \qquad P(B_n) = \frac{a}{a+b+n} + \frac{n}{a+b+n}\cdot\frac{a}{a+b} = \frac{a}{a+b}$

28. $P(A)>0$ Prove $P(B|A) \geq 1-\frac{P(\bar{B})}{P(A)}$

$$P(B|A)=\frac{P(AB)}{P(A)}=1-\frac{P(A\bar{B})}{P(A)} \geq 1-\frac{P(\bar{B})}{P(A)}$$

29. A,B互不相容，$P(\bar{B})\neq 0$ Prove $P(A|\bar{B})=\frac{P(A)}{1-P(B)}$

$$P(A|\bar{B})=\frac{P(A\bar{B})}{P(\bar{B})}=\frac{P(A)}{1-P(B)} \qquad A\subset\bar{B}$$

30. $A\subset B$, $P(B)>0$, $P(A)\leq P(A|B)$ Prove $P(A|B)=\frac{P(AB)}{P(B)}=\frac{P(A)}{P(B)}\geq P(A)$

Since $A\subset B \Rightarrow \frac{P(AB)}{P(B)}=\frac{P(A)}{P(B)}$

Since $P(A)\leq\frac{P(AB)}{P(B)}=\frac{P(A)}{P(B)}$

31 $P(A|B)>P(A|\bar{B})$ Prove $P(B|A)>P(B|\bar{A})$

$$\Rightarrow \frac{P(AB)}{P(B)}>\frac{P(A\bar{B})}{1-P(B)}$$

$$\Rightarrow (1-P(B))P(AB)>P(B)(P(A)-P(AB))$$

$$P(AB)-P(B)P(AB)>P(A)P(B)-P(B)P(AB)$$

$$\Rightarrow P(AB)>P(A)P(B) \qquad P(AB)(1-P(A))>P(A)(P(B)-P(AB))$$

$$P(B|A)=\frac{P(AB)}{P(A)}>\frac{P(B)-P(AB)}{1-P(A)}=\frac{P(\bar{A}B)}{P(\bar{A})}=P(B|\bar{A})$$

33. $P(A|B)=1 \Rightarrow P(\bar{B}|\bar{A})=1$

Since $\frac{P(AB)}{P(B)}=1$ $\quad P(AB)=P(B)$ $\quad P(A\cup B)=P(A)$ $\quad P(\bar{A})=1-P(A\cup B)=P(\bar{A}\bar{B})$

$$\Rightarrow P(\bar{B}|\bar{A})=1$$

32. $P(A)=p$ $\quad P(B)=1-\varepsilon$ Prove $\frac{p-\varepsilon}{1-\varepsilon}\leq P(A|B)\leq\frac{p}{1-\varepsilon}$

$P(AB)\leq P(A)=p$ $\quad P(AB)=P(A)+P(B)-P(A\cup B)\geq P(A)+P(B)-1=p-\varepsilon$

$$p-\varepsilon\leq P(AB)\leq\varepsilon$$

$$\frac{p-\varepsilon}{1-\varepsilon}\leq P(A|B)\leq\frac{p}{1-\varepsilon}$$

1.5

8. $P(A)=0.4$ $P(A\cup B)=0.9$ Find $P(B)$

(2) A, B独立 $P(A\cup B)=P(A)+P(B)-P(A)P(B)$ $P(B)=\frac{5}{6}$

9. A, B, C 两两独立, $ABC=\phi$

(1) $P(A)=P(B)=P(C)=x$, x的最大值

$ABC=\phi$, $P(AB\cup AC)=P(AB)+P(AC)-P(ABC)=2x^2$

$2x^2=P(AB\cup AC)\leq P(A)=x$ $x\leq 0.5$

If $P(A)=P(B)=0.5$ $P(AB)=0.25$ $C=A\bar{B}\cup\bar{A}B$

$P(C)=P(A\bar{B}\cup\bar{A}B)=P(A\bar{B})+P(\bar{A}B)=P(A)-P(AB)+P(B)-P(AB)=0.5$

$P(AC)=P(A\bar{B})=P(A)-P(AB)=0.25=P(A)P(C)$ A, C独立

$P(BC)=P(\bar{A}B)=P(B)-P(AB)=0.25$ B, C 独立

$\Rightarrow \max x=0.5$

(2) $P(A)=P(B)=P(C)<\frac{1}{2}$ $P(A\cup B\cup C)=\frac{9}{16}$ Find $P(A)$

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(AB)-P(BC)-P(AC)+P(ABC)$

$=3x-3x^2=\frac{9}{16}$

$\Rightarrow (x-\frac{1}{4})(x-\frac{3}{4})=0 \Rightarrow x=\frac{1}{4}$

10. A, B独立, 仅A发生或仅B发生概率为$\frac{1}{4}$, $P(A)$, $P(B)$?

$P(\bar{A}B)=P(A\bar{B})=\frac{1}{4}$ $P(\bar{A})P(B)=(1-P(A))P(B)$

$=P(B)-P(A)P(B)$

$(1-P(B))P(A)=P(A)-P(A)P(B)$

$\Rightarrow P(A)=P(B)$ $P(A)=P(B)=\frac{1}{2}$

16. 100米处开第一枪, 命中概率0.5, 没打中, 打第2枪, 相距150米, 第3枪, 相距200米, 第3枪不中逃逸, 命中概率与距离成反比, 求击中概率?

$P=\frac{k}{x}$ $0.5=\frac{k}{100}$ $k=50$ $P(2)=\frac{50}{150}=\frac{1}{3}$

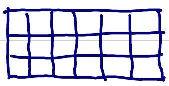
$P(3)=\frac{50}{200}=\frac{1}{4}$ $P(击中)=1-(0.5)(\frac{2}{3})(\frac{3}{4})=\frac{3}{4}$

Tutorial 1

1. 8 tubes, 3 balls, fall into the same tube 50, three adjacent tube 10, two 5. EX?

$P(\text{same}) = {}^8C_1\left(\frac{1}{8}\right)^3 = \frac{1}{64}$ $\quad P(3 \text{ adj.}) = \frac{{}^6C_1 3!}{8^3} = \frac{9}{128}$ $\quad P(2 \text{ adj.}) = \frac{{}^7C_1 2! \, {}^3C_2}{8^3} = \frac{21}{256}$

$EX = 50\left(\frac{1}{64}\right) + 10\left(\frac{9}{128}\right) + 5\left(\frac{21}{256}\right) = \frac{485}{256}$

4. 

${}^4C_2 \times {}^7C_2 = 126$

7. Maximum number of intersections of 20 circles.

${}^{20}C_2 \times 2 = 380$

14. 10 distinct points, 4 collinear. No other 3 points collinear, number of triangles?

${}^{10}C_3 - {}^4C_3 = 116$

20. 甲，乙，丙三人比赛，每局2人，胜者与第3人比，依次循环，直至一人连胜2场，实力55开，甲乙先比，求各人得冠概率？

$P(甲)=(0.5^2+0.5^5+0.5^8+\cdots)+(0.5^4+0.5^7+\cdots) \Rightarrow P(乙)=\frac{5}{14}$

$=\frac{0.5^2}{1-0.5^3}+\frac{0.5^4}{1-0.5^3}=\frac{5}{14}$

$P(丙)=(0.5^3+0.5^6+\cdots)+(0.5^3+0.5^6+\cdots)=\frac{2}{7}$

21. 甲，乙 55开，谁先赢一定局数就赢，中途打断，如何分配赌本

(1) 各需赢k局 $P(甲)=P(乙)=0.5$ 各一半

(2) 甲再赢2局，乙3局？ $P(甲)=(\frac{1}{2})^2+{}^2C_1(0.5)(0.5)^2+{}^3C_2(0.5)^4=0.6875$

$P(乙)=1-0.6875=0.3125$

(3) 甲n局，乙m局？

$P(甲)=0.5^n+{}^nC_1 0.5^{n+1}+{}^nC_2 0.5^{n+2}+\cdots+{}^{n+m-2}C_{m-1}0.5^{n+m-1}$ ☆

$P(乙)=1-P(甲)$

24. $0<P(A)<1$，$0<P(B)<1$，$P(A|B)+P(\bar{A}|\bar{B})=1$ Prove A, B independent

$$\frac{P(AB)}{P(B)}+\frac{P(\overline{A\cup B})}{1-P(B)}=1$$

$$\frac{P(AB)(1-P(B))+P(B)(1-P(A\cup B))}{P(B)(1-P(B))}=1$$

$$P(AB)-P(B)P(AB)+P(B)-P(B)P(A\cup B)=P(B)-(P(B))^2$$

$$P(AB)-\cancel{P(B)P(AB)}+\cancel{P(B)}-P(B)P(A)-\cancel{(P(B))^2}+\cancel{P(B)P(AB)}=\cancel{P(B)}-\cancel{(P(B))^2}$$

$$\Rightarrow P(AB)=P(A)P(B)$$

2.1

2. 一颗骰子抛两次，求分布列

(1) X表示两次所得的最小点数 $P\{X=1\}=\frac{11}{36}$ $(1,1)=(1,1)$

10. $P\{X\geq x_1\}=1-\alpha$ $P\{X\leq x_2\}=1-\beta$ $x_1<x_2$ $P\{x_1\leq X\leq x_2\}$?

$P\{x_1\leq X\leq x_2\}=F(x_2)-P\{X<x_1\}=1-\beta-1+1-\alpha=1-\alpha-\beta$

14. $p(x)=\begin{cases} A\cos x & -\frac{\pi}{2}\leq x\leq\frac{\pi}{2} \\ 0 & \text{else}\end{cases}$ (1) Find A (2) $P\{0<x<\frac{\pi}{4}\}$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A\cos x=1$ $2\int_0^{\frac{\pi}{2}} A\cos x=1$ $\Rightarrow A=\frac{1}{2}$ $\frac{1}{2}\int_0^{\frac{\pi}{4}}\cos x\,dx=\frac{\sqrt{2}}{4}$

15. $F(x)=\begin{cases} 0 & x<0 \\ Ax^2 & 0\leq x<1 \\ 1 & x\geq 1\end{cases}$ (1) Find A (2) $P\{0.3<x<0.7\}$ (3) $f(x)$

(1) $\lim\limits_{x\to 1^-} F(x)=1$ $A=1$ (2) $\int_{0.3}^{0.7} x^2\,dx=0.4$ (3) $f(x)=\begin{cases} 2x & 0\leq x<1 \\ 0 & \text{else}\end{cases}$

18. X, Y 同分布 $p(x)=\begin{cases}\frac{3}{8}x^2 & 0<x<2 \\ 0 & \text{else}\end{cases}$ $A=\{X>a\}$, $B=\{Y>a\}$ 独立 $P(A\cup B)=\frac{3}{4}$ Find a

$P(A\cup B)=P(A)+P(B)-P(A)P(B)=\frac{3}{4}\Rightarrow\int_a^2\frac{3}{8}x^2dx+\int_a^2\frac{3}{8}x^2dx-\left(\int_a^2\frac{3}{8}x^2dx\right)^2=\frac{3}{4}$

$a=\sqrt[3]{4}$

19. 连续型X的p.d.f.是偶函数，Prove

(1) $F(-a)=1-F(a)=0.5-\int_0^a p(x)\,dx$

$F(a)=\int_{-\infty}^0 p(x)dx+\int_0^a p(x)\,dx)$

$=0.5+\int_0^a p(x)dx$

$F(-a)=\int_a^{\infty} p(x)dx=1-F(a)$

(2) $P\{|X|<a\}=P\{-a<x<a\}$

$=F(a)-F(-a)$

$=F(a)-1+F(a)$

$=2F(a)-1$

2.2

5. 3组砝码 甲：1, 2, 2, 5, 10 乙：1, 2, 3, 4, 10 ，丙 1, 1, 2, 5, 10 ，只能用一组

当物品质量为 1, 2, ⋯, 10 的概率相同，哪组用的平均砝码数最少？

甲：1, 1, 2, 2, 1, 2, 2, 3, 3, 1 $P\{X=1\}=0.4$ $P\{X=2\}=0.4$ $P\{X=3\}=0.2$

乙：1, 1, 1, 1, 2, 2, 2, 3, 3, 1 $P\{Y=1\}=0.5$ $P\{Y=2\}=0.3$ $P\{Y=3\}=0.2$

丙：1, 1, 2, 3, 1, 2, 2, 3, 4, 10 $P\{Z=1\}=0.3$ $P\{Z=2\}=0.3$ $P\{Z=3\}=0.2$ $P\{Z=4\}=0.1$

$EX=1.8$ $EY=1.7$ $EZ=2$

17. $p(x)=\begin{cases}\frac{1}{2}\cos\frac{x}{2} & 0\le x\le\pi\\ 0 & \text{else}\end{cases}$ X独立重复4次，Y表示观察值大于 $\frac{\pi}{3}$ 的次数，求 EY^2

$P\{X>\frac{\pi}{3}\}=\int_{\frac{\pi}{3}}^{\pi}\frac{1}{2}\cos\frac{x}{2}\,dx=\frac{1}{2}$ $P\{Y=0\}=\frac{1}{16}$ $P\{Y=1\}={}^{4}C_1(\frac{1}{2})(\frac{1}{2})^3=\frac{4}{16}$

$P\{Y=2\}=\frac{6}{16}$ $P\{Y=3\}=\frac{4}{16}$ $P\{Y=4\}=\frac{1}{16}$

$EY^2=0^2\times\frac{1}{16}+1^2\times\frac{4}{16}+\cdots+4^2\times\frac{1}{16}=5$

19. Prove $\sum_{k=1}^{\infty}P\{X\ge k\}=\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}P\{X=n\}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}P\{X=n\}=\sum_{n=1}^{\infty}nP\{X=n\}=EX$

20. Prove $EX=\int_0^{\infty}(1-F(x))dx-\int_{-\infty}^{0}F(x)dx$

$EX=\int_{-\infty}^{0}xp(x)dx+\int_0^{\infty}xp(x)dx$

Since $\int_0^{\infty}xp(x)dx=\int_0^{\infty}(\int_0^x dy)\,p(x)dx=\int_0^{\infty}dx\int_0^x p(x)dy=\int_0^{\infty}dy\int_y^{\infty}p(x)dx=\int_0^{\infty}dy\ F(x)\Big|_y^{\infty}$

$=\int_0^{\infty}(1-F(y))\,dy=\int_0^{\infty}(1-F(x))dx$

$\int_{-\infty}^{0}xp(x)\,dx=\int_{-\infty}^{0}(-\int_x^0 dy)\,p(x)dx=-\int_{-\infty}^{0}dx\int_x^0 p(x)\,dy=-\int_{-\infty}^{0}dy\int_{-\infty}^{y}p(x)dx=-\int_{-\infty}^{0}dy\cdot F(x)\Big|_{-\infty}^{y}$

$=-\int_{-\infty}^{0}F(y)dy=-\int_{-\infty}^{0}F(x)\,dx$

$EX=\int_0^{\infty}(1-F(x))\,dx-\int_{-\infty}^{0}F(x)dx$

2.3

4. $p\{x=0\}=1-p\{x=1\}$ $\quad EX=3Var(X)$, $\quad p\{x=0\}$?

$p\{x=0\}=1-p \quad p\{x=1\}=p \qquad p=3p(1-p) \Rightarrow p=\frac{2}{3}$ or $p=0$

$\Rightarrow p\{x=0\}=\frac{1}{3}$ or 1

8. $F(x)=1-e^{-x^2}$ $\quad x>0$ $\qquad EX, VarX$?

$f(x)=2xe^{-x^2} \qquad EX=2\int_0^{\infty}x^2e^{-x^2}dx = \int_0^{\infty}x\,d(-e^{-x^2}) = -xe^{-x^2}\Big|_0^{\infty}+\int_0^{\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

$EX^2=2\int_0^{\infty}x^3e^{-x^2}dx=\int_0^{\infty}x^2d(-e^{-x^2})=1 \qquad VarX=1-\frac{\pi}{4}=\frac{3\pi}{4}$

9. Prove $\forall c\neq EX$, $\quad VarX=E(X-EX)^2<E(X-C)^2$

$E(X-C)^2=EX^2-2CEX+C^2=EX^2-(EX)^2+(EX)^2-2CEX+C^2$

$=E(X-EX)^2+(EX-C)^2>E(X-EX)^2=Var\,X$

10. X在[a,b]上取值 Prove $a\le EX\le b$ $\quad VarX\le\left(\frac{b-a}{2}\right)^2$

$x\ge a \quad x-a\ge 0 \quad E(x-a)=EX-a\ge 0 \quad EX\ge a$

$\Rightarrow a\le EX\le b$

$x\le b \quad x-b\le 0 \quad E(x-b)=EX-b\le 0 \quad EX\le b$

Since $a\le x\le b$, $-\frac{b-a}{2}\le x-\frac{a+b}{2}\le\frac{b-a}{2}$

$(x-\frac{a+b}{2})^2\le(\frac{b-a}{2})^2$ 妙！

$E((x-\frac{a+b}{2})^2-(\frac{b-a}{2})^2)=E(x-\frac{a+b}{2})^2-(\frac{b-a}{2})^2\le 0 \quad E(x-\frac{a+b}{2})^2\le(\frac{b-a}{2})^2$

$Var\,X=EX^2-(EX)^2\le E(X-\frac{a+b}{2})^2\le(\frac{b-a}{2})^2$

推广 11. $x_1\le\cdots\le x_n$ 的概率 $p_1,\dots,p_n$, $\sum_{i=1}^{n}p_i=1$ $\quad$ Prove $Var\,X\le\left(\frac{x_n-x_1}{2}\right)^2$

$x_1\le X\le x_n \qquad -\frac{x_n-x_1}{2}\le X-\frac{x_1+x_n}{2}\le\frac{x_n-x_1}{2}$

$(X-\frac{x_1+x_n}{2})^2\le(\frac{x_n-x_1}{2})^2$

$Var\,X=EX^2-(EX)^2\le E(X-\frac{x_1+x_n}{2})^2\le E(\frac{x_n-x_1}{2})^2=(\frac{x_n-x_1}{2})^2$

12. $g(x)$ 为随机变量 x 取值集合上的非负不减函数，且 $E(g(x))$ 存在. Prove $\forall \varepsilon>0$ $P\{x>\varepsilon\} \leqslant \frac{E(g(x))}{g(\varepsilon)}$

$$x>\varepsilon \ , \ g(x) \geqslant g(\varepsilon) > 0 \ , \ \frac{g(x)}{g(\varepsilon)} \geqslant 1$$

$$P\{x>\varepsilon\} = \int_{\varepsilon}^{\infty} p(x)dx \leqslant \int_{\varepsilon}^{\infty} \frac{g(x)}{g(\varepsilon)} p(x)\,dx = E\left(\frac{g(x)}{g(\varepsilon)}\right) = \frac{E(g(x))}{g(\varepsilon)}$$

13. X 为非负随机变量，$a>0$，若 $E(e^{aX})$ 存在，Prove $\forall x>0$，$P\{X\geqslant x\} \leqslant \frac{E(e^{aX})}{e^{ax}}$

$$P\{X\geqslant x\} = \int_{x}^{\infty} p(u)du \leqslant \int_{x}^{\infty} \frac{e^{au}}{e^{ax}} p(u)du \leqslant \int_{-\infty}^{\infty} \frac{e^{au}}{e^{ax}} p(u)\,du = E\left(\frac{e^{aX}}{e^{ax}}\right) = \frac{E(e^{aX})}{e^{ax}}$$

14. $\mu=7.3\times10^{9}$ $\sigma=0.7\times10^{9}$，用切比雪夫不等式估计 X 在 5.2×10^{9} 至 9.4×10^{9} 之间的概率下界。

$$P\{|X-EX|\geqslant\varepsilon\} \leqslant \frac{Var(X)}{\varepsilon^2} \qquad \varepsilon=2.1\times10^{9}$$

$$P\{|X-EX|<\varepsilon\} \geqslant 1-\frac{Var(X)}{\varepsilon^2} = \frac{8}{9}$$

2.4

9. 来的数量 $X\sim P(\lambda)$，每个来了购物的概率为 p，证来购物的数量 $Y\sim P(\lambda p)$

$$P\{Y=r\} = \sum_{k=r}^{\infty} P\{X=k\}P\{Y=r|X=k\}$$

$$= \sum_{k=r}^{\infty} \frac{\lambda^k}{k!}e^{-\lambda}\, {}^{k}C_r\, p^r(1-p)^{k-r}$$

$$= \sum_{k=r}^{\infty} \frac{\lambda^k}{k!}e^{-\lambda} \frac{k!\,p^r(1-p)^{k-r}}{r!\,(k-r)!}$$

$$= \frac{p^r e^{-\lambda}}{r!} \sum_{k=r}^{\infty} \frac{\lambda^k(1-p)^{k-r}}{(k-r)!} \qquad (n=k-r)$$

$$= \frac{p^r e^{-\lambda}}{r!} \sum_{n=0}^{\infty} \frac{\lambda^{n+r}(1-p)^n}{n!}$$

$$= \frac{\lambda^r p^r e^{-\lambda}}{r!} \sum_{n=0}^{\infty} \frac{(\lambda(1-p))^n}{n!}$$

$$= \frac{(\lambda p)^r e^{-\lambda}}{r!} e^{-\lambda(1-p)} = \frac{(\lambda p)^r}{r!} e^{-\lambda p} \qquad r=0,1,2,\cdots$$

$$\Rightarrow Y\sim P(\lambda p)$$

10. 一个人一年内感冒次数 $\sim P(5)$，药对75%有效（减为 $\lambda=3$），25%人无效，某人服用后，

得了2次感冒，药有效的可能性？

$$P(\text{有效}) = \frac{0.75(0.423-0.199)}{75\%(0.423-0.199)+25\%(0.125-0.040)} = 0.8877$$

12. m白，n黑，返回地摸球，摸到白停止，取到黑球数的期望？

X表示取到的黑，$X+1 \sim Geo\left(\frac{m}{m+n}\right)$ $\quad E(X+1)=\frac{1}{p}=\frac{m+n}{m}$

$$EX = \frac{m+n}{m} - 1 = \frac{n}{m}$$

13. 缺陷数X服从分布列：$P\{X=k\} = \frac{1}{2^{k+1}}$，$k=0,1,\cdots$，求平均缺陷数？

$$X+1 \sim Geo\left(\tfrac{1}{2}\right) \qquad E(X+1)=2 \qquad EX=1$$

15. 不合格率为0.1，每次抽10件，不合格多于1件就调整，每天抽4次，调整的平均数？

$X \sim B(10,0.1)$ $\quad P\{X\geq 2\} = 1-0.9^{10} - {}^{10}C_1(0.1)(0.9)^9 = 0.2639$

$Y \sim B(4,0.2639)$ $\quad EY = \sum_{k=0}^{4} k\, {}^4C_k (0.2639)^k (1-0.2639)^{n-k} = 4\times 0.2639 = 1.0556$

16. 系统由多个元件组成，各元件是否正常相互独立，且各个正常工作的概率为P，若有至少一半元件在工作，称系统有效. P取何值时，5个元件比3个元件更有效？

$X \sim B(5,p)$ $\quad P\{X\geq 3\} > P\{Y\geq 2\}$

$Y \sim B(3,p)$ $\quad {}^5C_3 p^3(1-p)^2 + {}^5C_4 p^4(1-p) + {}^5C_5 p^5 > {}^3C_2 p^2(1-p) + {}^3C_3 p^3$

$$\Rightarrow 3p^2(1-p)^2(1-2p) < 0 \quad \Rightarrow p > 0.5$$

17. $X \sim P(\lambda)$ prove $EX^n = \lambda E(X+1)^{n-1}$ and find EX^3 $\qquad m=k-1$

☆ $$EX^n = \sum_{k=0}^{\infty} k^n \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} k^{n-1} \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \sum_{m=0}^{\infty} (m+1)^{n-1} \frac{\lambda^{m+1}}{m!} e^{-\lambda} = \lambda \sum_{m=0}^{\infty} (m+1)^{n-1} \frac{\lambda^m}{m!} e^{-\lambda} = \lambda E(X+1)^{n-1}$$

$$EX^3 = \lambda E(X+1)^2 = \lambda EX^2 + 2\lambda EX + \lambda = \lambda^2 E(X+1) + 2\lambda EX + \lambda = \lambda^2(\lambda+1) + 2\lambda^2 + \lambda = \lambda^3 + 3\lambda^2 + \lambda$$

18. $X(n,p) \sim B(n,p)$，Prove $P\{X(n,p) \leq i\} = 1 - P\{X(n,1-p) \leq n-i-1\}$

$$P\{X(n,p)\leq i\} = 1 - P\{X(n,p) \geq i+1\} = 1 - \sum_{k=i+1}^{n} P\{X(n,p)=k\} = 1 - \sum_{k=i+1}^{n} {}^nC_k p^k (1-p)^{n-k}$$

现在由 $i+1 \to n$

目的是0开始

$m=i+1=0$

$$= 1 - \sum_{m=0}^{n-i-1} {}^nC_{n-m} p^{n-m} (1-p)^m = 1 - \sum_{m=0}^{n-i-1} {}^nC_m (1-p)^m p^{n-m} = 1 - P\{X(n,1-p) \leq n-i-1\}$$

19. $X \sim Geo(p)$ $\quad P\{x=k\} = (1-p)^{k-1}p^k \quad k=1,2,\cdots$

$$E\left(\frac{1}{x}\right) = \sum_{k=0}^{\infty} \frac{1}{k}(1-p)^{k-1}p^k = \frac{p}{1-p}\sum_{k=0}^{\infty}\frac{(1-p)^k}{k}$$

$$f(x) = \sum_{k=1}^{\infty}\frac{x^k}{k} \qquad f'(x) = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x} \qquad f(x) = \int_0^x \frac{1}{1-u}du = -\ln(1-u)\Big|_0^x = -\ln(1-x)$$

$$\Rightarrow E\left(\frac{1}{x}\right) = \frac{p}{1-p} f(1-p) = \frac{-p\ln p}{1-p}$$

20. $X \sim B(n,p)$ $\quad$ prove $\quad E\left(\frac{1}{x+1}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}$

$$p\{x=k\} = \sum_{k=0}^{n} {}^{n}C_k\, p^k(1-p)^{n-k}$$

$$E\left(\frac{1}{x+1}\right) = \sum_{k=0}^{n}\frac{1}{k+1}\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$$

$$= \sum_{k=0}^{n}\frac{1}{n+1}\frac{(n+1)!}{(k+1)!(n-k)!}p^k(1-p)^{n-k}$$

$$= \frac{1}{(n+1)p}\sum_{k=0}^{n} {}^{n+1}C_{k+1}\, p^{k+1}(1-p)^{n-k}$$

$$= \frac{1}{(n+1)p}\sum_{m=1}^{n} {}^{n+1}C_m\, p^m(1-p)^{n+1-m}$$

$$= \frac{1}{(n+1)p}\sum_{m=0}^{n}\left({}^{n+1}C_m\, p^m(1-p)^{n+1-m} - {}^{n+1}C_0\, p^0(1-p)^{n+1}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

2.5 $\quad$ 4. $K \sim N(\mu, \sigma^2)$ $\quad x^2+4x+K=0$ 无实根概率为0.5，求μ

$$\Delta = 16-4K < 0$$

$$\Rightarrow \quad P\{K>4\} = 0.5 \qquad P\{K>\mu\} = 0.5 \quad \Rightarrow \mu = 4$$

7. 需求$\sim U(10,30)$，进货数为$(10,30)$的某一整数，每卖1单位赚500，供大于求则削价处理，每处理1单位亏100，供不应求，则调剂，每单位赚300，期望利润不少于9280，最少进货量？

$$f(x) = \begin{cases} \frac{1}{20} & x\in(10,30) \\ 0 & \text{else} \end{cases}$$

$x \le a$, $Y = 500x - 100(a-x) = 600x - 100a$

$x > a$, $Y = 500a + 300(x-a) = 300x + 200a$

$$Y = g(x) = \begin{cases} 600x - 100a & x\le a \\ 300x + 200a & x > a \end{cases}$$

$$EY = \int_{10}^{a}(600x-100a)\frac{1}{20}dx + \int_{a}^{30}(300x+200a)\frac{1}{20}dx$$

$$= -\frac{15}{2}a^2 + 350a + 5250 \geq 9280$$

$$\Rightarrow \frac{62}{3} \le a \le 26$$

$$\min\{a\} = 21$$

8. 矿山发生10人或10人以上死亡的两次事故间的时间T服从均值为241的指数分布，求$P\{50\le T\le 100\}$

$ET=\frac{1}{\lambda}=241 \Rightarrow \lambda=\frac{1}{241}$

$f(x)=\begin{cases}\frac{1}{241}e^{-\frac{x}{241}} & x\ge 0\\ 0 & else\end{cases}$

$P\{50\le T\le 100\}$

$=\int_{50}^{100}\frac{1}{241}e^{-\frac{x}{241}}dx$

$=0.1523$

28. $X\sim N(\mu,\sigma^2)$ σ增大，$P\{|X-\mu|<\sigma\}$?

$P\{|X-\mu|<\sigma\}=\Phi_0\left(\frac{\mu+\sigma-\mu}{\sigma}\right)-\Phi_0\left(\frac{\mu-\sigma-\mu}{\sigma}\right)=\Phi_0(1)-1+\Phi_0(1)=0.6826$

30. $X\sim N(\mu,\sigma^2)$ $E|X-\mu|$?

$p(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty<x<\infty$ Let $y=x-\mu$

$E|X-\mu|=\int_{-\infty}^{+\infty}|y|\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}}dy$

$=2\int_0^{\infty}y\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}} = \frac{2}{\sqrt{2\pi}\sigma}\int_0^{\infty}e^{-\frac{y^2}{2\sigma^2}}d\left(\frac{y^2}{2}\right)=\frac{2}{\sqrt{2\pi}\sigma}\sigma^2=\sigma\sqrt{\frac{2}{\pi}}$

31. $X\sim N(0,\sigma^2)$ Prove $E|X|=\sigma\sqrt{\frac{2}{\pi}}$

$p(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}$

$E|X|=2\int_0^{\infty}x\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}=\frac{2}{\sqrt{2\pi}\sigma}\int_0^{\infty}xe^{-\frac{x^2}{2\sigma^2}}dx=\frac{2}{\sqrt{2\pi}\sigma}\int_0^{\infty}e^{-\frac{x^2}{2\sigma^2}}d\left(\frac{x^2}{2}\right)=\sigma\sqrt{\frac{2}{\pi}}$

2.6

4. $X\sim U[0,1]$ 求$1-X$的分布列

$f(x)=\begin{cases}1 & 0<x<1\\ 0 & else\end{cases}$

$F(x)=\begin{cases}\int_0^x dt=x & 0<x<1\\ 0 & else\end{cases}$

$F_Y(x)=P\{Y\le x\}=P\{1-X\le x\}=P\{X\ge 1-x\}$

$=1-P\{X<1-x\}=1-F(1-x)=x$

$f_Y(y)=\begin{cases}1 & 0<y<1\\ 0 & else\end{cases}$

5. $X \sim U[-\frac{\pi}{2}, \frac{\pi}{2}]$ Find p.d.f. of $Y = \cos X$ $0<y<1$

$$f(x) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2}<x<\frac{\pi}{2} \\ 0 & \text{else} \end{cases}$$

$$F_X(x) = \begin{cases} \frac{1}{\pi}x & -\frac{\pi}{2}<x<\frac{\pi}{2} \\ 0 & \text{else} \end{cases}$$

when $y<0$, $F_Y(y)=0$

when $0\le y<1$, $F_Y(y) = P\{\cos X \le y\} = P\{-\frac{\pi}{2} \le X \le -\cos^{-1}y\} + P\{\cos^{-1}y \le X \le \frac{\pi}{2}\}$

$= F_X(-\cos^{-1}y) - F_X(-\frac{\pi}{2}) + F_X(\frac{\pi}{2}) - F_X(\cos^{-1}y)$

$= -\frac{1}{\pi}\cos^{-1}y + \frac{1}{2} + \frac{1}{2} - \frac{1}{\pi}\cos^{-1}y$

$= 1 - \frac{2}{\pi}\cos^{-1}y$

When $y \ge 1$, $F_Y(y) = 1$

when $0\le y<1$, $f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}$

$$P_Y(y) = \begin{cases} \frac{2}{\pi\sqrt{1-y^2}} & 0<y<1 \\ 0 & \text{else} \end{cases}$$

6. $d \sim U(0,1)$, p.d.f. of circle's area $Y = \frac{1}{4}\pi d^2$ $0<y<\frac{\pi}{4}$

$$f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{else} \end{cases}$$

$F_Y(y) = P\{\frac{1}{4}\pi d^2 \le y\} = P\{d^2 \le \frac{4y}{\pi}\} = P\{-2\sqrt{\frac{y}{\pi}} \le d \le 2\sqrt{\frac{y}{\pi}}\}$

$= F_X(2\sqrt{\frac{y}{\pi}}) - F_X(-2\sqrt{\frac{y}{\pi}})$

$= 2\sqrt{\frac{y}{\pi}}$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{\pi y}} & 0<y<\frac{\pi}{4} \\ 0 & \text{else} \end{cases}$$

7. $X \sim U[1,2]$ p.d.f of $Y = e^{2X}$

$$f_X(x) = \begin{cases} 1 & 1<x<2 \\ 0 & \text{else} \end{cases}$$

$e^2<y<e^4$ $F_Y(y) = P\{e^{2X} \le y\} = P\{X \le \frac{\ln y}{2}\} = F_X(\frac{\ln y}{2})$

$$f_Y(y) = \begin{cases} \frac{1}{2y} & e^2<y<e^4 \\ 0 & \text{else} \end{cases}$$

9 $X \sim U(-1,1)$ (1) $P\{|X| > \frac{1}{2}\}$ (2) p.d.f of $Y = |X|$

$$f_X(x) = \begin{cases} \frac{1}{2} & -1<x<1 \\ 0 & \text{else} \end{cases}$$

$0\le y\le 1$ $F_Y(y) = P\{|X| \le y\} = P\{-y \le X \le y\} = F_X(y) - F_X(-y)$

$= \frac{y+1}{2} - \frac{1-y}{2}$

$= \frac{y+1-1+y}{2}$

$= y$

$P\{|X| > \frac{1}{2}\}$

$= \int_{-1}^{-\frac{1}{2}} \frac{1}{2}dx + \int_{\frac{1}{2}}^{1} \frac{1}{2}dx$

$= \left[\frac{x}{2}\right]_{-1}^{-\frac{1}{2}} + \left[\frac{x}{2}\right]_{\frac{1}{2}}^{1}$

$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$

$= \frac{1}{2}$

$$f_Y(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{else} \end{cases}$$

10. $X \sim U(0,1)$ p.d.f of $Y = |\ln X|$

$$f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{else} \end{cases} \qquad 0 \le y \le \infty \qquad F_Y(y) = P\{Y \le y\} = P\{|\ln x| \le y\}$$

$$= P\{-y \le \ln x \le y\} = P\{e^{-y} \le X \le e^{y}\} = F_X(e^{y}) - F_X(e^{-y})$$

$$F_X(x) = \begin{cases} x & 0<x<1 \\ 0 & \text{else} \end{cases} \qquad = -e^{-y}$$

$$f_Y(y) = \begin{cases} e^{-y} & y>0 \\ 0 & y \le 0 \end{cases}$$

13. $X \sim N(\mu, \sigma^2)$ $\quad Y = e^{x}$, Find EY and $VarY$.

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$$EY = \int_{-\infty}^{\infty} e^{x} \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2 + 2x\mu - \mu^2 + 2x\sigma^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2(\mu+\sigma^2)x + (\mu+\sigma^2)^2 - 2\mu\sigma^2 - \sigma^4}{2\sigma^2}} dx$$

$$= e^{\frac{2\mu\sigma^2 + \sigma^4}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}} dx$$

p.d.f of $N(\mu+\sigma^2, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}}$

$$EY = e^{\mu + \frac{\sigma^2}{2}}$$

$$EY^2 = \int_{-\infty}^{\infty} e^{2x} \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2 + 2\mu x - \mu^2 + 4x\sigma^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2 - 2(\mu - 2\sigma^2)x + \mu^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2 - 2(\mu+2\sigma^2)x + (\mu+2\sigma^2)^2 - 4\mu\sigma^2 - 4\sigma^4}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+2\sigma^2))^2 - 4\mu\sigma^2 - 4\sigma^4}{2\sigma^2}} dx$$

$$= e^{\frac{4\mu\sigma^2 + 4\sigma^4}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+2\sigma^2))^2}{2\sigma^2}} dx$$

$$= e^{2\mu + 2\sigma^2}$$

$$VarY = e^{2\mu+2\sigma^2} - \left(e^{\mu + \frac{\sigma^2}{2}}\right)^2$$

$$= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$$

$$= e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

16. $X \sim Exp(2)$ Prove $Y_1 = e^{-2X}$ and $Y_2 = 1-e^{-2X} \sim U(0,1)$

$$f_X(x) = \begin{cases} 2e^{-2x} & x>0 \\ 0 & x\le 0 \end{cases}$$

$F_{Y_1}(y) = P\{Y_1 \le y\} = P\{e^{-2X} \le y\} = P\{X \geq \frac{-\ln y}{2}\} = 1-P\{X < \frac{-\ln y}{2}\}$

$= 1 - F_X(-\frac{\ln y}{2})$

$= y$

$1-e^{-2x}$

$$f_{Y_1}(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{else} \end{cases}$$

$F_{Y_2}(y) = P\{1-e^{-2X} \le y\} = P\{X \le \frac{-\ln(1-y)}{2}\} = F_X(\frac{-\ln(1-y)}{2}) = y$ $\quad f_{Y_2}(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{else} \end{cases}$

$\Rightarrow Y_1, Y_2 \sim U(0,1)$

17. $X \sim LN(\mu, \sigma^2)$ Prove $Y = \ln X \sim N(\mu, \sigma^2)$

$$p_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} & x>0 \\ 0 & \text{else} \end{cases}$$

$y = g(x) = \ln x \uparrow \quad x = e^y \quad h'(x) = e^y$

$x>0 \Rightarrow -\infty < y < \infty \quad p(y) = \frac{1}{\sqrt{2\pi}e^y\sigma} e^{-\frac{(\ln e^y-\mu)^2}{2\sigma^2}} \cdot e^y = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$

$\Rightarrow Y \sim N(\mu, \sigma^2)$

3.1

1. 50件一等，30件二等，20件三等，从中任取5件，X，Y表示取出一等，二等的件数，(X，Y)联合分布列

(1) 不放回：$P\{X=i, Y=j\} = \frac{{}^{50}C_i \, {}^{30}C_j \, {}^{20}C_{5-i-j}}{{}^{100}C_5}$ $\quad i=0,1,2,3,4,5, \; j=0,1,\dots,5-i$

(2) 放回：$P\{X=i, Y=j\} = \frac{5!}{i!\,j!\,(5-i-j)!} (\frac{1}{2})^i (\frac{1}{3})^j (\frac{1}{5})^{5-i-j}$ $\quad i=0,1,2,3,4,5 \quad j=0,1,\dots,5-i$

4. X_i, $i=1,2$，满足 $P\{X_1X_2=0\}=1$，求 $P\{X_1=X_2\}$

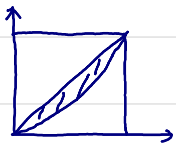
X_i	-1	0	1
P	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$X_1 \backslash X_2$	-1	0	1
-1	0	0.25	0
0	0.25	0	0.25
1	0	0.25	0

$P\{X_1X_2 \neq 0\} = 0$

$P\{X_1 = X_2\} = 0$

9. $p(x,y)=\begin{cases} k & 0<x^2<y<x<1 \\ 0 & \text{else} \end{cases}$ (1) Find k (2) $P\{x>0.5\}$, $P\{Y<0.5\}$



$$\int_0^1\int_{x^2}^{x} k\,dy\,dx=1$$
$$\int_0^1 (kx-kx^2)\,dx=1$$
$$\left[\frac{kx^2}{2}-\frac{kx^3}{3}\right]_0^1=1$$
$$k=6$$

$P\{x>0.5\}$
$$=\int_{0.5}^{1}\int_{x^2}^{x} 6\,dy\,dx$$
$$=\int_{0.5}^{1}(6x-6x^2)\,dx$$
$$=\left[3x^2-2x^3\right]_{0.5}^{1}$$
$$=0.5$$

$P\{Y<0.5\}$
$$=\int_0^{0.5}\int_y^{\sqrt{y}} 6\,dx\,dy$$
$$=\int_0^{\frac{1}{2}}(6\sqrt{y}-6y)\,dy$$
$$=\left[4y^{\frac{3}{2}}-3y^2\right]_0^{0.5}=\sqrt{2}-\frac{3}{4}$$

11. $Y\sim Exp(1)$ $X_k=\begin{cases} 0 & Y\le k \\ 1 & Y>k \end{cases}$ $k=1,2$ X_1, X_2联合分布

$$f_Y(y)=\begin{cases} e^{-y} & y>0 \\ 0 & y\le 0 \end{cases}$$

$$X_1=\begin{cases} 0 & Y\le 1 \\ 1 & Y>1 \end{cases} \qquad X_2=\begin{cases} 0 & Y\le 2 \\ 1 & Y>2 \end{cases}$$

$$P\{Y\le 1\}=\int_0^1 e^{-y}\,dy=\left[-e^{-y}\right]_0^1=1-e^{-1}$$

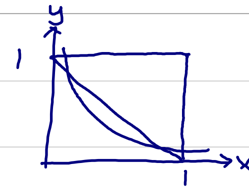
$$P\{Y>1\}=\int_1^{\infty} e^{-y}\,dy=\left[-e^{-y}\right]_1^{\infty}=e^{-1}$$

$$P\{Y\le 2\}=\int_0^2 e^{-y}\,dy=\left[-e^{-y}\right]_0^2=1-e^{-2}$$

$$P\{Y>2\}=e^{-2}$$

$X_1 \backslash X_2$	0	1
0	$1-e^{-1}$	0
1	$e^{-1}-e^{-2}$	e^{-2}

15. (0,1)中抽取2个数，积不小于 $\frac{3}{16}$，和不大于 1 的概率？



$$P\{xy \geq \frac{3}{16},\ x+y \leq 1\}$$

$$= \int_{0.25}^{0.75}\int_{\frac{3}{16x}}^{-x+1} dy\,dx$$

$$= \int_{0.25}^{0.75} (-x+1-\frac{3}{16}x^{-1})\,dx$$

$$= \left[-\frac{x^2}{2}+x-\frac{3}{16}\ln x\right]_{\frac{1}{4}}^{\frac{3}{4}}$$

$$= -\frac{9}{32}+\frac{3}{4}-\frac{3}{16}\ln\frac{3}{4}+\frac{1}{32}-\frac{1}{4}+\frac{3}{16}\ln\frac{1}{4}$$

$$= \frac{1}{4}-\frac{3}{16}(\ln 3-\ln 4+\ln 1+\ln 4)$$

$$= \frac{1}{4}-\frac{3}{16}\ln 3$$

$y=\frac{3}{16x}$

$x+\frac{3}{16x}=1$

$16x^2+3=16x$

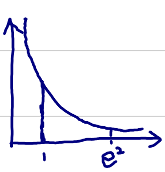
$x=0.75$ or $x=0.25$

3.2

2. $F(x,y) = \begin{cases} 1-e^{-\lambda_1 x}-e^{-\lambda_2 y}-e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max\{x,y\}} & x>0,\ y>0 \\ 0 & \text{else} \end{cases}$ $F_X(x)$, $F_Y(y)$

$F_X(x) = \lim_{y\to\infty} F(x,y) = \begin{cases} 1-e^{-\lambda_1 x} & x>0 \\ 0 & \text{else} \end{cases}$ $F_Y(y) = \lim_{x\to\infty} F(x,y) = \begin{cases} 1-e^{-\lambda_2 y} & y>0 \\ 0 & \text{else} \end{cases}$

4. D由 $y=\frac{1}{x}$, $y=0$, $x=1$, $x=e^2$ 围成，(X, Y)服从均匀分布，$f_X(x)$



$$S(D) = \int_1^{e^2}\int_0^{\frac{1}{x}} dy\,dx = \int_1^{e^2}\frac{1}{x}\,dx = 2$$

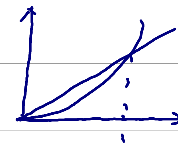
$f(x,y) = \begin{cases} \frac{1}{2} & (x,y)\in D \\ 0 & \text{else} \end{cases}$

$f_X(x) = \int_0^{\frac{1}{x}}\frac{1}{2}\,dy = \begin{cases} \frac{1}{2x} & 1\leq x\leq e^2 \\ 0 & \text{else} \end{cases}$

$$P\{X\leq x,\ Y<\infty\} = F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx = \int_{-\infty}^{x} p_X(x)\,dx$$

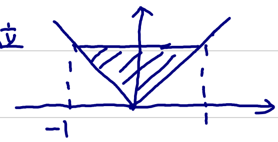
$$\Rightarrow p_X(x) = \int_{-\infty}^{\infty} p(x,y)\,dy$$

6. $p(x,y)=\begin{cases}6 & 0<x^2<y<x<1\\ 0 & \text{else}\end{cases}$ $\quad p_X(x)$, $p_Y(y)$?



$p_X(x)=\int_{x^2}^{x}6\,dy=\begin{cases}6x-6x^2 & 0<x<1\\ 0 & \text{else}\end{cases}$ $\quad p_Y(y)=\int_{y}^{\sqrt{y}}6\,dx=\begin{cases}6\sqrt{y}-6y & 0<y<1\\ 0 & \text{else}\end{cases}$

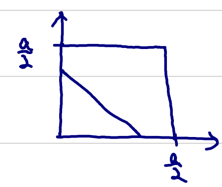
13. $p(x,y)=\begin{cases}1 & |x|<y,\ 0<y<1\\ 0 & \text{else}\end{cases}$ $\quad$ (1) $p_X(x)$, $p_Y(y)$ $\quad$ (2) X与Y是否独立



$p_X(x)=\begin{cases}1+x & -1<x<0\\ 1-x & 0\le x<1\\ 0 & \text{else}\end{cases}$ $\quad p_Y(y)=\begin{cases}2y & 0<y<1\\ 0 & \text{else}\end{cases}$

(2) $p_X(x)p_Y(y)=\begin{cases}2y(1+x) & -1<x<0,\ 0<y<1\\ 2y(1-x) & 0\le x<1,\ 0<y<1\\ 0 & \text{else}\end{cases}$ $\quad p_X(x)p_Y(y)\neq p(x,y)\Rightarrow$ 不独立

15. 在长为 a 的线段中点的两边各取一点，求两点距离小于$\frac{a}{3}$的概率



$X\sim U(0,\frac{a}{2})$, $Y\sim U(0,\frac{a}{2})$, X, Y独立

$p(x,y)=\begin{cases}\frac{4}{a^2} & 0<x<\frac{a}{2},\ 0<y<\frac{a}{2}\\ 0 & \text{else}\end{cases}$

$P\{X+Y<\frac{a}{3}\}=\dfrac{\frac{1}{2}(\frac{a}{3})^2}{(\frac{a}{2})^2}=\frac{2}{9}$

3.3

1.

X\Y	1	2	3
0	0.05	0.15	0.2
1	0.07	0.11	0.22
2	0.04	0.07	0.09

$U=\max\{X,Y\}$ $\quad V=\min\{X,Y\}$ 分布列

X	Y	U	V
0	1	1	0
0	2	2	0
0	3	3	0
1	1	1	1
1	2	2	1
1	3	3	1
2	1	2	1
2	2	2	2
2	3	3	2

U	1	2	3
P	0.12	0.37	0.51

V	0	1	2
P	0.4	0.44	0.16

3.

X	-1	0	1
P	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Y	0	1
P	$\frac{1}{2}$	$\frac{1}{2}$

$P\{XY=0\}=1$ $\quad Z=\max\{X,Y\}$?

$X \backslash Y$	0	1
-1	$\frac{1}{4}$	0
0	0	$\frac{1}{2}$
1	$\frac{1}{4}$	0

X	Y	Z
-1	0	0
-1	1	1
0	0	0
0	1	1
1	0	1
1	1	1

Z	0	1
P	$\frac{1}{4}$	$\frac{3}{4}$

5. $P\{X\geq 0, Y\geq 0\}=\frac{3}{7}$ $\quad P\{X\geq 0\}=P\{Y\geq 0\}=\frac{4}{7}$ $\quad P\{\max\{X,Y\}\geq 0\}$

$= \frac{4}{7}+\frac{4}{7}-\frac{3}{7}=\frac{5}{7}$

6. $p(x,y)=\begin{cases} e^{-(x+y)} & x>0, y>0 \\ 0 & \text{else} \end{cases}$ $\quad$ p.d.f. of (1) $Z=\frac{X+Y}{2}$ $\quad$ (2) $Z=Y-X$

(1) $z\leq 0$, $F_Z(z)=0$ $\quad z>0$, $F_Z(z)=\int_0^{2z}\int_0^{2z-x} e^{-(x+y)}\,dy\,dx = 1-(2z+1)e^{-2z}$

$p_Z(z)=\begin{cases} 4ze^{-2z} & z>0 \\ 0 & z\leq 0 \end{cases}$

(2) $z\leq 0$, $F_Z(z)=\int_{-z}^{\infty}\int_0^{x+z} e^{-(x+y)}\,dy\,dx = \frac{1}{2}e^{z}$

$z>0$, $F_Z(z)=\int_0^{\infty}\int_0^{x+z} e^{-(x+y)}\,dy\,dx = 1-\frac{1}{2}e^{-z}$

$p_Z(z)=\begin{cases} \frac{1}{2}e^{z}, & z\leq 0 \\ \frac{1}{2}e^{-z}, & z>0 \end{cases}$