Influence Functions Derivations

1 Influence Function of Regression Function

Consider the regression function $E(Y|X=x)$. Here we will show that

$$\varphi(z;P) = \frac{1(X=x)}{P(X=x)}\{Y - E_P(Y|X=x)\}$$

We try to show that the pathwise differentiability holds

$$\frac{\partial}{\partial\epsilon}\psi(P_\epsilon)\big|_{\epsilon=0} = \int \varphi(z;P)s_\epsilon(z)dP(z)$$

We denote $Z=(X,Y)$ and $s_\epsilon(z) = \frac{\partial}{\partial\epsilon}\log dP_\epsilon(z)\big|_{\epsilon=0}$.

$$\begin{aligned}
E[s(Z)|X=x] &= \int \frac{\partial}{\partial\epsilon}\log dP_\epsilon(z)\big|_{\epsilon=0}dP(y|x) \\
&= \int \frac{\partial}{\partial\epsilon}\log\{P_\epsilon(X=x)dP_\epsilon(y|x)\}\big|_{\epsilon=0}dP(y|x) \\
&= \int \left\{\frac{\partial}{\partial\epsilon}\log P_\epsilon(X=x)\big|_{\epsilon=0} + \frac{\partial}{\partial\epsilon}\log dP_\epsilon(y|x)\big|_{\epsilon=0}\right\}dP(y|x) \\
&= \frac{\partial}{\partial\epsilon}\log P_\epsilon(X=x)\big|_{\epsilon=0}
\end{aligned}$$

where the last equality uses the facts that $\int dP(y|x) = 1$ and that scores have mean zero

$$\begin{aligned}
\int \frac{\partial}{\partial\epsilon}\log dP_\epsilon(y|x)\big|_{\epsilon=0}dP(y|x) &= \int \frac{\frac{\partial}{\partial\epsilon}dP_\epsilon(y|x)\big|_{\epsilon=0}}{dP(y|x)}dP(y|x) \\
&= \int \frac{\partial}{\partial\epsilon}dP_\epsilon(y|x)\big|_{\epsilon=0} \\
&= \frac{\partial}{\partial\epsilon}\int dP_\epsilon(y|x)\big|_{\epsilon=0} = 0
\end{aligned}$$

In this case the pathwise derivative on the left-hand side is

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}\int y dP_\epsilon(y|x)\big|_{\epsilon=0} &= \int y\left\{\frac{\partial}{\partial\epsilon}\log dP_\epsilon(y|x)\right\}\big|_{\epsilon=0}dP(y|x) \\
&= \int y\left\{\frac{\partial}{\partial\epsilon}\log\frac{dP_\epsilon(z)}{P_\epsilon(X=x)}\right\}\big|_{\epsilon=0}dP(y|x) \\
&= \int y\left\{\frac{\partial}{\partial\epsilon}\log dP_\epsilon(z) - \frac{\partial}{\partial\epsilon}\log P_\epsilon(X=x)\right\}\big|_{\epsilon=0}dP_\epsilon(y|x) \\
&= E\{Ys_\epsilon(Z)|X=x\} - E\{s_\epsilon(Z)|X=x\}E(Y|X=x)
\end{aligned}$$

The first equality holds as long as we can exchange integrals and derivatives since $P_{\epsilon=0}=P$. Now for the right-hand side

$$\begin{aligned}
\int \varphi(z;P)s_\epsilon(z)dP(z) &= E\left[\frac{1(X=x)}{P(X=x)}\{Y - E(Y|X=x)s_\epsilon(Z)\}\right] \\
&= E\{Ys_\epsilon(Z)|X=x\} - E\{s_\epsilon(Z)|X=x\}E(Y|X=x)
\end{aligned}$$

2 Error terms in von Mises Expansion

2.1 Expected Density Functional

We consider the expected density functional $\psi(\mathbb{P}) = \int p_0(z)^2 dz$ as our starting point. We let p_0 denote the density of $\mathbb{P}$. Under regularity conditions, the pathwise derivative is given by

$$\begin{aligned}
\psi'(P_\epsilon)\big|_{\epsilon=0} &= \frac{\partial}{\partial\epsilon}\int p_\epsilon(z)^2 dz\big|_{\epsilon=0} \\
&= \int \frac{\partial}{\partial\epsilon} p_\epsilon(z)^2 dz\big|_{\epsilon=0} \\
&= \int 2p_\epsilon(z)\frac{\partial}{\partial\epsilon} p_\epsilon(z) dz\big|_{\epsilon=0} \\
&= \int 2p_\epsilon(z)\left\{\frac{\partial}{\partial\epsilon}\log p_\epsilon(z)\right\} p_\epsilon(z) dz\big|_{\epsilon=0} \\
&= \int 2\left\{p_0(z) - \psi(\mathbb{P})\right\}\left\{\frac{\partial}{\partial\epsilon}\log p_\epsilon(z)\right\}\big|_{\epsilon=0} p_0(z) dz
\end{aligned}$$

I feel that the above calculation is a bit handwaving, so we try to write out explicitly the submodel to see what's going on. Suppose $p_\epsilon(z) = p_0(z) + \epsilon(\tilde{p_0}(z) - p_0(z))$

$$\begin{aligned}
\psi'(P_\epsilon)\big|_{\epsilon=0} &= \frac{\partial}{\partial\epsilon}\int [p_0(z) + \epsilon\{\tilde{p_0}(z) - p_0(z)\}]^2 dz\big|_{\epsilon=0} \\
&= \int 2[p_0(z) + \epsilon\{\tilde{p_0}(z) - p_0(z)\}][\tilde{p_0}(z) - p_0(z)] dz\big|_{\epsilon=0} \\
&= \int 2[p_0(z) + \epsilon\{\tilde{p_0}(z) - p_0(z)\}]\left\{\frac{\partial}{\partial\epsilon}\log[p_0(z) + \epsilon\{\tilde{p_0}(z) - p_0(z)\}]\right\}[p_0(z) + \epsilon\{\tilde{p_0}(z) - p_0(z)\}] dz\big|_{\epsilon=0} \\
&= \int 2\left\{p_0(z) - \psi(\mathbb{P})\right\}\left\{\frac{\partial}{\partial\epsilon}\log p_\epsilon(z)\right\}\big|_{\epsilon=0} p_0(z) dz
\end{aligned}$$

Now we try to check that the desired influence function has zero mean

$$\begin{aligned}
\int 2\{p_0(z) - \psi(\mathbb{P})\} p_0(z) dz &= 2\int\left(p_0(z)^2 - \int p_0(z)^2 dz p_0(z)\right) dz \\
&= 2\int p_0(z)^2 dz - 2\int p_0(z)^2 dz \int p_0(z) dz \\
&= 2\int p_0(z)^2 dz\left[1 - \int p_0(z) dz\right] \\
&= 0
\end{aligned}$$

Now we try to calculate the error term R_2 in the von Mises expansion.

$$\begin{aligned}
\psi(\bar{P}) - \psi(P) &= \int \varphi(z;\bar{P})d(\bar{P}-P) + R_2(\bar{P},P) \\
&= \int \varphi(z;\bar{P})d\bar{P} - \int \varphi(z;\bar{P})dP + R_2(\bar{P},P) \\
&= -\int \varphi(z;\bar{P})dP + R_2(\bar{P},P) \\
\psi(P) &= \psi(\bar{P}) + \int \varphi(z;\bar{P})dP - R_2(\bar{P},P) \\
R_2(\bar{P},P) &= \psi(\bar{P}) + \int \varphi(z;\bar{P})dP - \psi(P)
\end{aligned}$$

We can also notice that

$$\begin{aligned}
R_2(\bar{P},P) &= E_P(\hat{\psi}_{one}) - \psi(P) \\
&= \left\{\psi(\bar{P}) + \int 2\bar{p}(z) - \psi(\bar{P})dP\right\} - \psi(P) \\
&= \psi(\bar{P}) + \int 2\bar{p}(z)p(z)dz - 2\psi(\bar{P}) - \psi(P) \\
&= -\psi(\bar{P}) + \int 2\bar{p}(z)p(z)dz - \psi(P) \\
&= -\int \left(\bar{p}(z)^2 - 2\bar{p}(z)p(z) + p(z)^2\right) dz \\
&= -\int \{\bar{p}(z) - p(z)\}^2 dz
\end{aligned}$$

2.2 ATE Functional

We can show that the influence function of the ATE functional $\psi(P) = E_P\{E_P(Y|X, A = 1)\}$ is

$$\varphi(Z;P) = \frac{\mathbb{1}(A=1)}{P(A=1|X)}\{Y - E_P(Y|X,A=1)\} + E_P(Y|X,A=1) - \psi(P)$$

To ease the notation, we denote

$$\pi(x) = P(A=1|X=x),\ \ \bar{\pi}(x) = \bar{P}(A=1|X=x) \text{ and } \ \mu(x) = E_P(Y|X=x,A=1)$$

We try to calculate the error term:

$$\begin{aligned}
R_2(\bar{P},P) &= \psi(\bar{P}) + \int \left(\frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)}\{Y - E_{\bar{P}}(Y|X,A=1)\} + E_{\bar{P}}(Y|X,A=1) - \psi(\bar{P})\right) dP - \psi(P) \\
&= \psi(\bar{P}) + \int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} Y dP - \int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} E_{\bar{P}}(Y|X,A=1)dP \\
&+ \int E_{\bar{P}}(Y|X,A=1)dP - \int \psi(\bar{P})dP - \psi(P) \\
&= \int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} Y dP - \int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} E_{\bar{P}}(Y|X,A=1)dP + \int E_{\bar{P}}(Y|X,A=1)dP - \psi(P)
\end{aligned}$$

We consider the three terms one by one

$$
\begin{aligned}
\int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} Y dP &= E_P\left[\frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} Y\right] \\
&= E_P\left[\frac{1}{\bar{P}(A=1|X)} Y\right] \\
&= \int\int \frac{1}{\bar{P}(A=1|X=x)} y f_Y(y|A=1,X=x) P(A=1|X=x) P(X=x) dy dx \\
&= \int \frac{\pi(x)}{\bar{\pi}(x)} y f_Y(y|A=1,X=x) P(X=x) dy dx \\
&= \frac{\pi(x)}{\bar{\pi}(x)} E_P\left[Y|A=1,X=x\right] P(X=x) dx \\
&= \int \frac{\pi(x)}{\bar{\pi}(x)} \mu(x) dP(x)
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} E_{\bar{P}}(Y|X,A=1) dP &= \int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X=x)} E_{\bar{P}}(Y|X=x,A=1) P(X=x) dx \\
&= \int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} \int y \bar{P}(y|X=x,A=1) P(A=1|X=x) dy P(X=x) dx \\
&= \int \frac{\pi(x)}{\bar{\pi}(x)} \bar{\mu}(x) P(X=x) dx \\
&= \int \frac{\pi(x)}{\bar{\pi}(x)} \bar{\mu}(x) dP(x)
\end{aligned}
$$

$$
\begin{aligned}
\int E_{\bar{P}}(Y|X,A=1) dP &= \int E_{\bar{P}}(Y|X=x,A=1) P(X=x) dx \\
&= \int \bar{\mu}(x) dP(x)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
R_2(\bar{P},P) &= \int \frac{\pi(x)}{\bar{\pi}(x)} \mu(x) dP(x) - \int \frac{\pi(x)}{\bar{\pi}(x)} \bar{\mu}(x) dP(x) + \int \bar{\mu}(x) dP(x) - \psi(P) \\
&= \int \left\{\frac{1}{\bar{\pi}(x)} - \frac{1}{\pi(x)}\right\} \left\{\mu(x) - \bar{\mu}(x)\right\} \pi(x) dP(x)
\end{aligned}
$$

2.3 Expected Covariance Functional

Consider the functional $\psi(P) = E_p\{cov_P(A, Y|X)\}$, the influence function is

$$\varphi(Z; P) = \{A - E_P(A|X)\}\{Y - E_P(Y|X)\} - \psi(P)$$

Similarly, we denote $\pi(x) = E_P(A|X = x)$ and $\mu(x) = E_P(Y|X = x)$

$$\begin{aligned}
R_2(\bar{P}, P) &= \psi(\bar{P}) + \int \varphi(z; \bar{P})dP - \psi(P) \\
&= \psi(\bar{P}) + \int \left[\{A - E_{\bar{P}}(A|X)\}\{Y - E_{\bar{P}}(Y|X)\} - \psi(\bar{P})\right] dP - \psi(P) \\
&= \psi(\bar{P}) + \int \left(AY - AE_{\bar{P}}(Y|X) - E_{\bar{P}}(A|X)Y + E_{\bar{P}}(A|X)E_{\bar{P}}(Y|X) - \psi(\bar{P})\right) dP - \psi(P) \\
&= \psi(\bar{P}) + \int AY dP - \int AE_{\bar{P}}(Y|X)dP - \int E_{\bar{P}}(A|X)Y dP \\
&+ \int E_{\bar{P}}(A|X)E_{\bar{P}}(Y|X)dP - \int \psi(\bar{P})dP - \psi(P)
\end{aligned}$$

$$\begin{aligned}
\int AY dP &= E_P[AY] \\
\int AE_{\bar{P}}(Y|X)dP &= E_P[AE_{\bar{P}}(Y|X)] \\
&= \int\int aE_{\bar{P}}(Y|X = x)P(A = a|X = x)f(x)dadx \\
&= \int\int a\bar{\mu}(x)P(A = a|X = x)dadP(x) \\
&= \int \bar{\mu}(x)\pi(x)dP(x) \\
\int E_{\bar{P}}(A|X)Y dP &= \int\int yE_{\bar{P}}(A|X = x)f(y|X = x)P(X = x)dydx \\
&= \int\int y\bar{\pi}(x)f(y|X = x)dydP(x) \\
&= \int \mu(x)\bar{\pi}(x)dP(x) \\
\int E_{\bar{P}}(A|X)E_{\bar{P}}(Y|X)dP &= \int \bar{\pi}(x)\bar{\mu}(x)dP(x)
\end{aligned}$$

$$\begin{aligned}
\psi(P) &= E_P[Cov_P(A, Y|X)] \\
&= E_P[E_P[AY|X] - E_P[A|X]E_P[Y|X]] \\
&= E_P[AY] - E_P[E_P[A|X]E_P[Y|X]] \\
&= E_P[AY] - \int \pi(x)\mu(x)dP(x) \\
\psi(\bar{P}) &= E_{\bar{P}}[Cov_{\bar{P}}(A, Y|X)] \\
&= E_{\bar{P}}[AY] - \int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) \\
\int \psi(\bar{P})dP &= E_P[E_{\bar{P}}\{cov_{\bar{P}}(A, Y|X)\}] \\
&= E_P[E_{\bar{P}}(AY)] - E_P\left[\int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x)\right]
\end{aligned}$$

Based on what we have,

$$\begin{aligned}
R_2(\bar{P}, P) = \psi(\bar{P}) &+ \int AY dP - \int AE_{\bar{P}}(Y|X)dP - \int E_{\bar{P}}(A|X)Y dP \\
&+ \int E_{\bar{P}}(A|X)E_{\bar{P}}(Y|X)dP - \int \psi(\bar{P})dP - \psi(P) \\
&= E_{\bar{P}}[AY] - \int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) + E_P[AY] - \int \bar{\mu}(x)\pi(x)dP(x) \\
&- \int \mu(x)\bar{\pi}(x)dP(x) + \int \bar{\pi}(x)\bar{\mu}(x)dP(x) \\
&- E_{\bar{P}}[AY] + \int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) - E_P[AY] + \int \pi(x)\mu(x)dP(x) \\
&= \int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) - \int \mu(x)\bar{\pi}(x)dP(x) - \int \bar{\mu}(x)\pi(x)dP(x) + \int \pi(x)\mu(x)dP(x) \\
&= \int \{\bar{\pi}(x) - \pi(x)\}\{\bar{\mu}(x) - \mu(x)\}dP(x)
\end{aligned}$$

3 Strategy 2

3.1 ATE Functional

Let $\mu(x) = E(Y|X = x, A = 1)$, $\pi(x) = P(A = 1|X = x)$, and $p(x) = P(X = x)$. Denote $\psi = E\{E(Y|X, A = 1)\}$. The influence function is given by

$$
\begin{aligned}
\mathbb{IF}(\psi) &= \mathbb{IF}\left\{\sum_x \mu(x)p(x)\right\} \\
&= \sum_x \left[\mathbb{IF}\{\mu(x)\}p(x) + \mu(x)\mathbb{IF}\{p(x)\}\right] \\
&= \sum_x \left[\frac{1(x = X, A = 1)}{p(1, x)}\left\{Y - \mu(x)\right\}p(x) + \mu(x)\{1(x = X) - p(x)\}\right] \\
&= \frac{A}{\pi(X)}\left\{Y - \mu(X)\right\} + \mu(X) - \psi
\end{aligned}
$$

3.2 Stochastic Intervention Effect

Let $\psi = \int\int \mu(x, a)dG(a|x)dP(x)$ denote the stochastic intervention effect, where $\mu(x, a) = E(Y|X = x, A = a)$ and $\pi(a|x) = P(A = a|X = x)$

$$
\begin{aligned}
\mathbb{IF}(\psi) &= \left\{\sum_{x,a} \mu(x, a)g(a|x)p(x)\right\} \\
&= \sum_{x,a} \left[\mathbb{IF}\{\mu(x, a)\}g(a|x)p(x) + \mu(x, a)g(a|x)\mathbb{IF}\{p(x)\}\right] \\
&= \sum_{x,a} \left[\frac{1(A = a, X = x)}{\pi(a|x)p(x)}\left\{Y - \mu(x, a)\right\}g(a|x)p(x) + \mu(x, a)g(a|x)\left\{1(X = x) - p(x)\right\}\right] \\
&= \frac{g(A|X)}{\pi(A|X)}\left\{Y - \mu(X, A)\right\} + \sum_a \mu(X, a)g(a|X) - \psi
\end{aligned}
$$

3.3 LATE

Let $\psi = \frac{E\{E(Y|X,R=1)-E(Y|X,R=0)\}}{E\{E(A|X,R=1)-E(A|X,R=0)\}} = \frac{E\{\mu(X,1)-\mu(X,0)\}}{E\{\eta(X,1)-\eta(X,0)\}}$ denote the LATE with instrument R. First note that $\psi = \psi_{iv,num}/\psi_{iv,den}$ where $\psi_{iv,num} = E(Y^{R=1} - Y^{R=0})$ and $\psi_{iv,den} = E(A^{R=1} - A^{R=0})$ Now we try to use the strategy to compute the influence function of ψ.

$$
\begin{aligned}
\mathbb{IF}(\psi_{iv,num}) &= \mathbb{IF}\left(E[E(Y|X,R=1) - E(Y|X,R=0)]\right) \\
&= \mathbb{IF}\left(\sum_x E(Y|X=x,R=1)P(X=x) - E(Y|X=x,R=0)P(X=x)\right) \\
&= \mathbb{IF}\left(\sum_x \mu(x,1)p(x) - \mu(x,0)p(x)\right) \\
&= \sum_x \left(\mathbb{IF}(\mu(x,1))p(x) + \mu(x,1)\mathbb{IF}(p(x)) - \mathbb{IF}(\mu(x,0))p(x) - \mu(x,0)\mathbb{IF}(p(x))\right) \\
&= \sum_x \left(\frac{1(R=1,X=x)}{w(r|x)p(x)}\{Y-\mu(x,1)\}p(x) + \mu(x,1)\mu(x,1)\{1(X=x)-p(x)\}\right. \\
&\left. - \frac{1(R=0,X=x)}{w(r|x)p(x)}\{Y-\mu(x,0)\}p(x) - \mu(x,0)\{1(X=x)-p(x)\}\right. \\
&= \frac{2R-1}{w(R|X)}\{Y-\mu(X,R)\} + \mu(X,1) - \mu(X,0) - \psi_{iv,num}
\end{aligned}
$$

where $w(r|x) = P(R=r|X=x)$. It's exactly the same to derive the denominator.

$$
\mathbb{IF}(\psi_{iv,den}) = \frac{2R-1}{w(R|X)}\{A-\eta(X,R)\} + \eta(X,1) - \eta(X,0) - \psi_{iv,den}
$$

Using the product rule and chain rule,

$$
\begin{aligned}
\mathbb{IF}\left(\frac{\psi_{iv,num}}{\psi_{iv,den}}\right) &= \frac{\mathbb{IF}(\psi_{iv,num})}{\psi_{iv,den}} + \mathbb{IF}(\frac{1}{\psi_{iv,den}})\psi_{iv,num} \\
&= \frac{\mathbb{IF}(\psi_{iv,num})}{\psi_{iv,den}} - \frac{\psi_{iv,num}}{\psi_{iv,den}^2}\mathbb{IF}(\psi_{iv,den}) \\
&= \frac{1}{\psi_{iv,den}}\left(\frac{2R-1}{w(R|X)}\{Y-\mu(X,R)\} + \mu(X,1) - \mu(X,0) - \psi_{iv,num}\right) \\
&- \frac{\psi_{iv,num}}{\psi_{iv,den}^2}\left(\frac{2R-1}{w(R|X)}\{A-\eta(X,R)\} + \eta(X,1) - \eta(X,0) - \psi_{iv,den}\right) \\
&= \frac{1}{\psi_{iv,den}}\left(\frac{2R-1}{w(R|X)}\{Y-\mu(X,R)\} + \mu(X,1) - \mu(X,0)\right) - \frac{\psi_{iv,num}}{\psi_{iv,den}} \\
&- \frac{\psi_{iv,num}}{\psi_{iv,den}^2}\left(\frac{2R-1}{w(R|X)}\{A-\eta(X,R)\} + \eta(X,1) - \eta(X,0)\right) + \frac{\psi_{iv,num}^2}{\psi_{iv,den}^2} \\
&= \frac{1}{\psi_{iv,den}}\left(\frac{2R-1}{w(R|X)}\{Y-\mu(X,R)\} + \mu(X,1) - \mu(X,0)\right. \\
&\left. - \frac{\psi_{iv,num}}{\psi_{iv,den}}\left[\frac{2R-1}{w(R|X)}\{A-\eta(X,R)\} + \eta(X,1) - \eta(X,0)\right]\right)
\end{aligned}
$$

3.4 Time-varying Treatment Effects

Let $\mu_{1,1}(x_2, x_1) = E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1)$, $\pi_t(h_t) = P(A_t = 1|H_t = h_t)$ for $H_t = (\bar{X}_t, \bar{A}_{t-1})$ and let

$$\psi = E(Y^{11}) = \int\int E(Y|A_2 = 1, X_2, A_1 = 1, X_1)dP(X_2|A_1 = 1, X_1)dP(X_1)$$

$$\begin{aligned}
\mathbb{IF}(\psi) &= \mathbb{IF}\left\{\sum_{x_1,x_2} E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1)p(x_2|a_1 = 1, x_1)p(x_1)\right\} \\
&= \sum_{x_1,x_2} [\mathbb{IF}\left\{E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1)\right\} p(x_2|a_1 = 1, x_1)p(x_1) \\
&+ E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1)\mathbb{IF}\left\{p(x_2|a_1 = 1, x_1)\right\} p(x_1) \\
&+ E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1)p(x_2|a_1 = 1, x_1)\mathbb{IF}\{p(x_1)\} \\
&= \sum_{x_1,x_2} \left[\frac{A_2 A_1 1(X_2 = x_2, X_1 = x_1)}{\pi_2(h_2)\pi_1(h_1)} \left\{Y - \mu_{11}(x_2, x_1)\right\}\right. \\
&+ \mu_{11}(x_2, x_1)\frac{A_1 1(X_1 = x_1)}{\pi_1(h_1)} \left\{1(X_2 = x_2) - p(x_2|a_1 = 1, x_1)\right\} \\
&+ \mu_{11}(x_2, x_1)p(x_2|a_1 = 1, x_1) \left\{1(X_1 = x_1) - p(x_1)\right\} \\
&= \frac{A_2 A_1}{\pi_2(H_2)\pi_1(H_1)} \left\{Y - \mu_{11}(X_2, X_1)\right\} + \frac{A_1}{\pi_1(H_1)} [\mu_{11}(X_2, X_1) - E\{\mu(X_2, X_1)|A_1 = 1, X_1\}] \\
&+ E\{\mu(X_2, X_1)|A_1 = 1, X_1\} - \psi
\end{aligned}$$

4 Error Decomposition

Recall the von Mises expansion:

$$\psi(\bar{P}) - \psi(P) = \int \varphi(z;\bar{P})d(\bar{P} - P)(z) + R_2(\bar{P}, P)$$

where $\varphi(z;P)$ is a mean-zero, finite-variance function satisfying $\int \varphi(z;P)dP(z) = 0$ and $\int \varphi(z;P)^2 dP(z) < \infty$, and $R_2(\bar{P}, P)$ is a second-order remainder term. This expansion suggests that generic plug-in estimators of the form $\hat{\psi_{pi}} = \psi(\hat{P})$ have a first-order bias, since evaluating the expansion at $(\hat{P}, P)$ gives

$$\psi(\hat{P}) - \psi(P) = -\int \varphi(z;\hat{P})dP(z) + R_2(\hat{P}, P)$$

It naturally leads to the one-step estimator

$$\hat{\psi} = \psi(\hat{P}) + P_n\{\varphi(Z;\hat{P})\}$$

More examples can be found in the paper:

$$\begin{aligned}
\hat{\psi} - \psi &= \psi(\hat{P}) + P_n\{\varphi(Z;\hat{P})\} - \psi(P) \\
&= (P_n - P)\{\varphi(Z;\hat{P})\} + R_2(\hat{P}, P) \\
&= (P_n - P)\{\varphi(Z;P)\} + (P_n - P)\{\varphi(z;\hat{P}) - \varphi(Z;P)\} + R_2(\hat{P}, P) \\
&= S^* + T_1 + T_2
\end{aligned}$$

S^* is a simple sample average of a fixed function, so by central limit theorem, it behaves as a normally distributed random variable with variance $var(\varphi)/n$, up to error $o_P(1/\sqrt{n})$

The second term

$$(P_n - P)\{\varphi(z;\hat{P}) - \varphi(Z;P)\}$$

is an empirical process term, and is typically of the smallest order since it is a sample average of a term with shrinking variance.

The third term

$$T_2 = R_2(\hat{P}, P) = \psi(\hat{P}) - \psi(P) + \int \varphi(z;\hat{P})dP(z)$$

For non-bias-corrected plug-in estimators, this term will generally dominate, but for one-step estimators it will generally involve second-order products of errors, which can be negligible under nonparametric conditions.

In particular, when the T_1 and T_2 terms are of the order $o_P(1/\sqrt{n})$, then the sample average S^* dominates the decomposition, and so

$$\sqrt{n}(\hat{\psi} - \psi) = \sqrt{n}S^* + o_P(1) \xrightarrow{d} N\left(0, var\{\varphi(Z;P)\}\right)$$