

Influence Functions Derivations

1 Influence Function of Regression Function

Consider the regression function $E(Y|X = x)$. Here we will show that

$$\varphi(z; P) = \frac{1(X = x)}{P(X = x)} \{Y - E_P(Y|X = x)\}$$

We try to show that the pathwise differentiability holds

$$\frac{\partial}{\partial \epsilon} \psi(P_\epsilon) \Big|_{\epsilon=0} = \int \varphi(z; P) s_\epsilon(z) dP(z)$$

We denote $Z = (X, Y)$ and $s_\epsilon(z) = \frac{\partial}{\partial \epsilon} \log dP_\epsilon(z) \Big|_{\epsilon=0}$.

$$\begin{aligned} E[s(Z)|X = x] &= \int \frac{\partial}{\partial \epsilon} \log dP_\epsilon(z) \Big|_{\epsilon=0} dP(y|x) \\ &= \int \frac{\partial}{\partial \epsilon} \log \{P_\epsilon(X = x) dP_\epsilon(y|x)\} \Big|_{\epsilon=0} dP(y|x) \\ &= \int \left\{ \frac{\partial}{\partial \epsilon} \log P_\epsilon(X = x) \Big|_{\epsilon=0} + \frac{\partial}{\partial \epsilon} \log dP_\epsilon(y|x) \Big|_{\epsilon=0} \right\} dP(y|x) \\ &= \frac{\partial}{\partial \epsilon} \log P_\epsilon(X = x) \Big|_{\epsilon=0} \end{aligned}$$

where the last equality uses the facts that $\int dP(y|x) = 1$ and that scores have mean zero

$$\begin{aligned} \int \frac{\partial}{\partial \epsilon} \log dP_\epsilon(y|x) \Big|_{\epsilon=0} dP(y|x) &= \int \frac{\frac{\partial}{\partial \epsilon} dP_\epsilon(y|x) \Big|_{\epsilon=0}}{dP(y|x)} dP(y|x) \\ &= \int \frac{\partial}{\partial \epsilon} dP_\epsilon(y|x) \Big|_{\epsilon=0} \\ &= \frac{\partial}{\partial \epsilon} \int dP_\epsilon(y|x) \Big|_{\epsilon=0} = 0 \end{aligned}$$

In this case the pathwise derivative on the left-hand side is

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \int y dP_\epsilon(y|x) \Big|_{\epsilon=0} &= \int y \left\{ \frac{\partial}{\partial \epsilon} \log dP_\epsilon(y|x) \right\} \Big|_{\epsilon=0} dP(y|x) \\ &= \int y \left\{ \frac{\partial}{\partial \epsilon} \log \frac{dP_\epsilon(z)}{P_\epsilon(X = x)} \right\} \Big|_{\epsilon=0} dP(y|x) \\ &= \int y \left\{ \frac{\partial}{\partial \epsilon} \log dP_\epsilon(z) - \frac{\partial}{\partial \epsilon} \log P_\epsilon(X = x) \right\} \Big|_{\epsilon=0} dP(y|x) \\ &= E\{Y s_\epsilon(Z)|X = x\} - E\{s_\epsilon(Z)|X = x\} E(Y|X = x) \end{aligned}$$

The first equality holds as long as we can exchange integrals and derivatives since $P_{\epsilon=0} = P$. Now for the right-hand side

$$\begin{aligned} \int \varphi(z; P) s_\epsilon(z) dP(z) &= E \left[\frac{1(X = x)}{P(X = x)} \{Y - E(Y|X = x) s_\epsilon(Z)\} \right] \\ &= E\{Y s_\epsilon(Z)|X = x\} - E\{s_\epsilon(Z)|X = x\} E(Y|X = x) \end{aligned}$$

2 Error terms in von Mises Expansion

2.1 Expected Density Functional

We consider the expected density functional $\psi(\mathbb{P}) = \int p_0(z)^2 dz$ as our starting point. We let p_0 denote the density of \mathbb{P} . Under regularity conditions, the pathwise derivative is given by

$$\begin{aligned}
\psi'(P_\epsilon)|_{\epsilon=0} &= \frac{\partial}{\partial \epsilon} \int p_\epsilon(z)^2 dz|_{\epsilon=0} \\
&= \int \frac{\partial}{\partial \epsilon} p_\epsilon(z)^2 dz|_{\epsilon=0} \\
&= \int 2p_\epsilon(z) \frac{\partial}{\partial \epsilon} p_\epsilon(z) dz|_{\epsilon=0} \\
&= \int 2p_\epsilon(z) \left\{ \frac{\partial}{\partial \epsilon} \log p_\epsilon(z) \right\} p_\epsilon(z) dz|_{\epsilon=0} \\
&= \int 2 \{p_0(z) - \psi(\mathbb{P})\} \left\{ \frac{\partial}{\partial \epsilon} \log p_\epsilon(z) \right\} |_{\epsilon=0} p_0(z) dz
\end{aligned}$$

I feel that the above calculation is a bit handwaving, so we try to write out explicitly the submodel to see what's going on. Suppose $p_\epsilon(z) = p_0(z) + \epsilon(\tilde{p}_0(z) - p_0(z))$

$$\begin{aligned}
\psi'(P_\epsilon)|_{\epsilon=0} &= \frac{\partial}{\partial \epsilon} \int [p_0(z) + \epsilon\{\tilde{p}_0(z) - p_0(z)\}]^2 dz|_{\epsilon=0} \\
&= \int 2[p_0(z) + \epsilon\{\tilde{p}_0(z) - p_0(z)\}][\tilde{p}_0(z) - p_0(z)] dz|_{\epsilon=0} \\
&= \int 2[p_0(z) + \epsilon\{\tilde{p}_0(z) - p_0(z)\}] \left\{ \frac{\partial}{\partial \epsilon} \log[p_0(z) + \epsilon\{\tilde{p}_0(z) - p_0(z)\}] \right\} [p_0(z) + \epsilon\{\tilde{p}_0(z) - p_0(z)\}] dz|_{\epsilon=0} \\
&= \int 2 \{p_0(z) - \psi(\mathbb{P})\} \left\{ \frac{\partial}{\partial \epsilon} \log p_\epsilon(z) \right\} |_{\epsilon=0} p_0(z) dz
\end{aligned}$$

Now we try to check that the desired influence function has zero mean

$$\begin{aligned}
\int 2\{p_0(z) - \psi(\mathbb{P})\} p_0(z) dz &= 2 \int \left(p_0(z)^2 - \int p_0(z)^2 dz p_0(z) \right) dz \\
&= 2 \int p_0(z)^2 dz - 2 \int p_0(z)^2 dz \int p_0(z) dz \\
&= 2 \int p_0(z)^2 dz \left[1 - \int p_0(z) dz \right] \\
&= 0
\end{aligned}$$

Now we try to calculate the error term R_2 in the von Mises expansion.

$$\begin{aligned}
\psi(\bar{P}) - \psi(P) &= \int \varphi(z; \bar{P}) d(\bar{P} - P) + R_2(\bar{P}, P) \\
&= \int \varphi(z; \bar{P}) d\bar{P} - \int \varphi(z; \bar{P}) dP + R_2(\bar{P}, P) \\
&= - \int \varphi(z; \bar{P}) dP + R_2(\bar{P}, P) \\
\psi(P) &= \psi(\bar{P}) + \int \varphi(z; \bar{P}) dP - R_2(\bar{P}, P) \\
R_2(\bar{P}, P) &= \psi(\bar{P}) + \int \varphi(z; \bar{P}) dP - \psi(P)
\end{aligned}$$

We can also notice that

$$\begin{aligned}
R_2(\bar{P}, P) &= E_P(\hat{\psi}_{one}) - \psi(P) \\
&= \left\{ \psi(\bar{P}) + \int 2\bar{p}(z) - \psi(\bar{P}) dP \right\} - \psi(P) \\
&= \psi(\bar{P}) + \int 2\bar{p}(z)p(z) dz - 2\psi(\bar{P}) - \psi(P) \\
&= -\psi(\bar{P}) + \int 2\bar{p}(z)p(z) dz - \psi(P) \\
&= - \int (\bar{p}(z)^2 - 2\bar{p}(z)p(z) + p(z)^2) dz \\
&= - \int \{\bar{p}(z) - p(z)\}^2 dz
\end{aligned}$$

2.2 ATE Functional

We can show that the influence function of the ATE functional $\psi(P) = E_P\{E_P(Y|X, A = 1)\}$ is

$$\varphi(Z; P) = \frac{\mathbb{1}(A = 1)}{P(A = 1|X)} \{Y - E_P(Y|X, A = 1)\} + E_P(Y|X, A = 1) - \psi(P)$$

To ease the notation, we denote

$$\pi(x) = P(A = 1|X = x), \quad \bar{\pi}(x) = \bar{P}(A = 1|X = x) \text{ and } \mu(x) = E_P(Y|X = x, A = 1)$$

We try to calculate the error term:

$$\begin{aligned}
R_2(\bar{P}, P) &= \psi(\bar{P}) + \int \left(\frac{\mathbb{1}(A = 1)}{\bar{P}(A = 1|X)} \{Y - E_{\bar{P}}(Y|X, A = 1)\} + E_{\bar{P}}(Y|X, A = 1) - \psi(\bar{P}) \right) dP - \psi(P) \\
&= \psi(\bar{P}) + \int \frac{\mathbb{1}(A = 1)}{\bar{P}(A = 1|X)} Y dP - \int \frac{\mathbb{1}(A = 1)}{\bar{P}(A = 1|X)} E_{\bar{P}}(Y|X, A = 1) dP \\
&\quad + \int E_{\bar{P}}(Y|X, A = 1) dP - \int \psi(\bar{P}) dP - \psi(P) \\
&= \int \frac{\mathbb{1}(A = 1)}{\bar{P}(A = 1|X)} Y dP - \int \frac{\mathbb{1}(A = 1)}{\bar{P}(A = 1|X)} E_{\bar{P}}(Y|X, A = 1) dP + \int E_{\bar{P}}(Y|X, A = 1) dP - \psi(P)
\end{aligned}$$

We consider the three terms one by one

$$\begin{aligned}
\int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} Y dP &= E_P \left[\frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} Y \right] \\
&= E_P \left[\frac{1}{\bar{P}(A=1|X)} Y \right] \\
&= \int \int \frac{1}{\bar{P}(A=1|X=x)} y f_Y(y|A=1, X=x) P(A=1|X=x) P(X=x) dy dx \\
&= \int \frac{\pi(x)}{\bar{\pi}(x)} y f_Y(y|A=1, X=x) P(X=x) dy dx \\
&= \frac{\pi(x)}{\bar{\pi}(x)} E_P [Y|A=1, X=x] P(X=x) dx \\
&= \int \frac{\pi(x)}{\bar{\pi}(x)} \mu(x) dP(x)
\end{aligned}$$

$$\begin{aligned}
\int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} E_{\bar{P}}(Y|X, A=1) dP &= \int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X=x)} E_{\bar{P}}(Y|X=x, A=1) P(X=x) dx \\
&= \int \frac{\mathbb{1}(A=1)}{\bar{P}(A=1|X)} \int y \bar{P}(y|X=x, A=1) P(A=1|X=x) dy P(X=x) dx \\
&= \int \frac{\pi(x)}{\bar{\pi}(x)} \bar{\mu}(x) P(X=x) dx \\
&= \int \frac{\pi(x)}{\bar{\pi}(x)} \bar{\mu}(x) dP(x)
\end{aligned}$$

$$\begin{aligned}
\int E_{\bar{P}}(Y|X, A=1) dP &= \int E_{\bar{P}}(Y|X=x, A=1) P(X=x) dx \\
&= \int \bar{\mu}(x) dP(x)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
R_2(\bar{P}, P) &= \int \frac{\pi(x)}{\bar{\pi}(x)} \mu(x) dP(x) - \int \frac{\pi(x)}{\bar{\pi}(x)} \bar{\mu}(x) dP(x) + \int \bar{\mu}(x) dP(x) - \psi(P) \\
&= \int \left\{ \frac{1}{\bar{\pi}(x)} - \frac{1}{\pi(x)} \right\} \{\mu(x) - \bar{\mu}(x)\} \pi(x) dP(x)
\end{aligned}$$

2.3 Expected Covariance Functional

Consider the functional $\psi(P) = E_P\{cov_P(A, Y|X)\}$, the influence function is

$$\varphi(Z; P) = \{A - E_P(A|X)\} \{Y - E_P(Y|X)\} - \psi(P)$$

Similarly, we denote $\pi(x) = E_P(A|X = x)$ and $\mu(x) = E_P(Y|X = x)$

$$\begin{aligned} R_2(\bar{P}, P) &= \psi(\bar{P}) + \int \varphi(z; \bar{P}) dP - \psi(P) \\ &= \psi(\bar{P}) + \int [\{A - E_{\bar{P}}(A|X)\} \{Y - E_{\bar{P}}(Y|X)\} - \psi(\bar{P})] dP - \psi(P) \\ &= \psi(\bar{P}) + \int (AY - AE_{\bar{P}}(Y|X) - E_{\bar{P}}(A|X)Y + E_{\bar{P}}(A|X)E_{\bar{P}}(Y|X) - \psi(\bar{P})) dP - \psi(P) \\ &= \psi(\bar{P}) + \int AY dP - \int AE_{\bar{P}}(Y|X) dP - \int E_{\bar{P}}(A|X)Y dP \\ &\quad + \int E_{\bar{P}}(A|X)E_{\bar{P}}(Y|X) dP - \int \psi(\bar{P}) dP - \psi(P) \end{aligned}$$

$$\begin{aligned} \int AY dP &= E_P[AY] \\ \int AE_{\bar{P}}(Y|X) dP &= E_P[AE_{\bar{P}}(Y|X)] \\ &= \int \int a E_{\bar{P}}(Y|X = x) P(A = a|X = x) f(x) da dx \\ &= \int \int a \bar{\mu}(x) P(A = a|X = x) da dP(x) \\ &= \int \bar{\mu}(x) \pi(x) dP(x) \\ \int E_{\bar{P}}(A|X)Y dP &= \int \int y E_{\bar{P}}(A|X = x) f(y|X = x) P(X = x) dy dx \\ &= \int \int y \bar{\pi}(x) f(y|X = x) dy dP(x) \\ &= \int \mu(x) \bar{\pi}(x) dP(x) \\ \int E_{\bar{P}}(A|X)E_{\bar{P}}(Y|X) dP &= \int \bar{\pi}(x) \bar{\mu}(x) dP(x) \end{aligned}$$

$$\begin{aligned}
\psi(P) &= E_P[Cov_P(A, Y|X)] \\
&= E_P[E_P[AY|X] - E_P[A|X]E_P[Y|X]] \\
&= E_P[AY] - E_P[E_P[A|X]E_P[Y|X]] \\
&= E_P[AY] - \int \pi(x)\mu(x)dP(x) \\
\psi(\bar{P}) &= E_{\bar{P}}[Cov_{\bar{P}}(A, Y|X)] \\
&= E_{\bar{P}}[AY] - \int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) \\
\int \psi(\bar{P})dP &= E_P[E_{\bar{P}}\{cov_{\bar{P}}(A, Y|X)\}] \\
&= E_P[E_{\bar{P}}(AY)] - E_P \left[\int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) \right]
\end{aligned}$$

Based on what we have,

$$\begin{aligned}
R_2(\bar{P}, P) &= \psi(\bar{P}) + \int AYdP - \int AE_{\bar{P}}(Y|X)dP - \int E_{\bar{P}}(A|X)YdP \\
&\quad + \int E_{\bar{P}}(A|X)E_{\bar{P}}(Y|X)dP - \int \psi(\bar{P})dP - \psi(P) \\
&= E_{\bar{P}}[AY] - \int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) + E_P[AY] - \int \bar{\mu}(x)\pi(x)dP(x) \\
&\quad - \int \mu(x)\bar{\pi}(x)dP(x) + \int \bar{\pi}(x)\bar{\mu}(x)dP(x) \\
&\quad - E_{\bar{P}}[AY] + \int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) - E_P[AY] + \int \pi(x)\mu(x)dP(x) \\
&= \int \bar{\pi}(x)\bar{\mu}(x)d\bar{P}(x) - \int \mu(x)\bar{\pi}(x)dP(x) - \int \bar{\mu}(x)\pi(x)dP(x) + \int \pi(x)\mu(x)dP(x) \\
&= \int \{\bar{\pi}(x) - \pi(x)\} \{\bar{\mu}(x) - \mu(x)\} dP(x)
\end{aligned}$$

3 Strategy 2

3.1 ATE Functional

Let $\mu(x) = E(Y|X = x, A = 1)$, $\pi(x) = P(A = 1|X = x)$, and $p(x) = P(X = x)$. Denote $\psi = E\{E(Y|X, A = 1)\}$. The influence function is given by

$$\begin{aligned}\mathbb{IF}(\psi) &= \mathbb{IF}\left\{\sum_x \mu(x)p(x)\right\} \\ &= \sum_x [\mathbb{IF}\{\mu(x)\}p(x) + \mu(x)\mathbb{IF}\{p(x)\}] \\ &= \sum_x \left[\frac{1(x = X, A = 1)}{p(1, x)} \{Y - \mu(x)\}p(x) + \mu(x)\{1(x = X) - p(x)\}\right] \\ &= \frac{A}{\pi(X)} \{Y - \mu(X)\} + \mu(X) - \psi\end{aligned}$$

3.2 Stochastic Intervention Effect

Let $\psi = \int \int \mu(x, a)dG(a|x)dP(x)$ denote the stochastic intervention effect, where $\mu(x, a) = E(Y|X = x, A = a)$ and $\pi(a|x) = P(A = a|X = x)$

$$\begin{aligned}\mathbb{IF}(\psi) &= \left\{\sum_{x,a} \mu(x, a)g(a|x)p(x)\right\} \\ &= \sum_{x,a} [\mathbb{IF}\{\mu(x, a)\}g(a|x)p(x) + \mu(x, a)g(a|x)\mathbb{IF}\{p(x)\}] \\ &= \sum_{x,a} \left[\frac{1(A = a, X = x)}{\pi(a|x)p(x)} \{Y - \mu(x, a)\}g(a|x)p(x) + \mu(x, a)g(a|x)\{1(X = x) - p(x)\}\right] \\ &= \frac{g(A|X)}{\pi(A|X)} \{Y - \mu(X, A)\} + \sum_a \mu(X, a)g(a|X) - \psi\end{aligned}$$

3.3 LATE

Let $\psi = \frac{E\{E(Y|X, R=1) - E(Y|X, R=0)\}}{E\{E(A|X, R=1) - E(A|X, R=0)\}} = \frac{E\{\mu(X, 1) - \mu(X, 0)\}}{E\{\eta(X, 1) - \eta(X, 0)\}}$ denote the LATE with instrument R. First note that $\psi = \psi_{iv,num}/\psi_{iv,den}$ where $\psi_{iv,num} = E(Y^{R=1} - Y^{R=0})$ and $\psi_{iv,den} = E(A^{R=1} - A^{R=0})$. Now we try to use the strategy to compute the influence function of ψ .

$$\begin{aligned}
\mathbb{IF}(\psi_{iv,num}) &= \mathbb{IF}(E[E(Y|X, R=1) - E(Y|X, R=0)]) \\
&= \mathbb{IF}\left(\sum_x E(Y|X=x, R=1)P(X=x) - E(Y|X=x, R=0)P(X=x)\right) \\
&= \mathbb{IF}\left(\sum_x \mu(x, 1)p(x) - \mu(x, 0)p(x)\right) \\
&= \sum_x (\mathbb{IF}(\mu(x, 1))p(x) + \mu(x, 1)\mathbb{IF}(p(x)) - \mathbb{IF}(\mu(x, 0))p(x) - \mu(x, 0)\mathbb{IF}(p(x))) \\
&= \sum_x \left(\frac{1(R=1, X=x)}{w(r|x)p(x)}\{Y - \mu(x, 1)\}p(x) + \mu(x, 1)\mu(x, 1)\{1(X=x) - p(x)\}\right. \\
&\quad \left.- \frac{1(R=0, X=x)}{w(r|x)p(x)}\{Y - \mu(x, 0)\}p(x) - \mu(x, 0)\{1(X=x) - p(x)\}\right) \\
&= \frac{2R-1}{w(R|X)}\{Y - \mu(X, R)\} + \mu(X, 1) - \mu(X, 0) - \psi_{iv,num}
\end{aligned}$$

where $w(r|x) = P(R=r|X=x)$. It's exactly the same to derive the denominator.

$$\mathbb{IF}(\psi_{iv,den}) = \frac{2R-1}{w(R|X)}\{A - \eta(X, R)\} + \eta(X, 1) - \eta(X, 0) - \psi_{iv,den}$$

Using the product rule and chain rule,

$$\begin{aligned}
\mathbb{IF}\left(\frac{\psi_{iv,num}}{\psi_{iv,den}}\right) &= \frac{\mathbb{IF}(\psi_{iv,num})}{\psi_{iv,den}} + \mathbb{IF}\left(\frac{1}{\psi_{iv,den}}\right)\psi_{iv,num} \\
&= \frac{\mathbb{IF}(\psi_{iv,num})}{\psi_{iv,den}} - \frac{\psi_{iv,num}}{\psi_{iv,den}^2}\mathbb{IF}(\psi_{iv,den}) \\
&= \frac{1}{\psi_{iv,den}}\left(\frac{2R-1}{w(R|X)}\{Y - \mu(X, R)\} + \mu(X, 1) - \mu(X, 0) - \psi_{iv,num}\right) \\
&\quad - \frac{\psi_{iv,num}}{\psi_{iv,den}^2}\left(\frac{2R-1}{w(R|X)}\{A - \eta(X, R)\} + \eta(X, 1) - \eta(X, 0) - \psi_{iv,den}\right) \\
&= \frac{1}{\psi_{iv,den}}\left(\frac{2R-1}{w(R|X)}\{Y - \mu(X, R)\} + \mu(X, 1) - \mu(X, 0)\right) - \frac{\psi_{iv,num}}{\psi_{iv,den}} \\
&\quad - \frac{\psi_{iv,num}}{\psi_{iv,den}^2}\left(\frac{2R-1}{w(R|X)}\{A - \eta(X, R)\} + \eta(X, 1) - \eta(X, 0)\right) + \frac{\psi_{iv,num}^2}{\psi_{iv,den}^2} \\
&= \frac{1}{\psi_{iv,den}}\left(\frac{2R-1}{w(R|X)}\{Y - \mu(X, R)\} + \mu(X, 1) - \mu(X, 0)\right) \\
&\quad - \frac{\psi_{iv,num}}{\psi_{iv,den}}\left[\frac{2R-1}{w(R|X)}\{A - \eta(X, R)\} + \eta(X, 1) - \eta(X, 0)\right]
\end{aligned}$$

3.4 Time-varying Treatment Effects

Let $\mu_{1,1}(x_2, x_1) = E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1)$, $\pi_t(h_t) = P(A_t = 1|H_t = h_t)$ for $H_t = (\bar{X}_t, \bar{A}_{t-1})$ and let

$$\begin{aligned}
\psi &= E(Y^{11}) = \int \int E(Y|A_2 = 1, X_2, A_1 = 1, X_1) dP(X_2|A_1 = 1, X_1) dP(X_1) \\
\mathbb{IF}(\psi) &= \mathbb{IF} \left\{ \sum_{x_1, x_2} E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1) p(x_2|a_1 = 1, x_1) p(x_1) \right\} \\
&= \sum_{x_1, x_2} [\mathbb{IF} \{E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1)\} p(x_2|a_1 = 1, x_1) p(x_1) \\
&\quad + E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1) \mathbb{IF} \{p(x_2|a_1 = 1, x_1)\} p(x_1) \\
&\quad + E(Y|A_2 = 1, X_2 = x_2, A_1 = 1, X_1 = x_1) p(x_2|a_1 = 1, x_1) \mathbb{IF} \{p(x_1)\}] \\
&= \sum_{x_1, x_2} \left[\frac{A_2 A_1 1(X_2 = x_2, X_1 = x_1)}{\pi_2(h_2) \pi_1(h_1)} \{Y - \mu_{11}(x_2, x_1)\} \right. \\
&\quad + \mu_{11}(x_2, x_1) \frac{A_1 1(X_1 = x_1)}{\pi_1(h_1)} \{1(X_2 = x_2) - p(x_2|a_1 = 1, x_1)\} \\
&\quad + \mu_{11}(x_2, x_1) p(x_2|a_1 = 1, x_1) \{1(X_1 = x_1) - p(x_1)\} \\
&= \frac{A_2 A_1}{\pi_2(H_2) \pi_1(H_1)} \{Y - \mu_{11}(X_2, X_1)\} + \frac{A_1}{\pi_1(H_1)} [\mu_{11}(X_2, X_1) - E\{\mu(X_2, X_1)|A_1 = 1, X_1\}] \\
&\quad + E\{\mu(X_2, X_1)|A_1 = 1, X_1\} - \psi
\end{aligned}$$

4 Error Decomposition

Recall the von Mises expansion:

$$\psi(\bar{P}) - \psi(P) = \int \varphi(z; \bar{P}) d(\bar{P} - P)(z) + R_2(\bar{P}, P)$$

where $\varphi(z; P)$ is a mean-zero, finite-variance function satisfying $\int \varphi(z; P) dP(z) = 0$ and $\int \varphi(z; P)^2 dP(z) < \infty$, and $R_2(\bar{P}, P)$ is a second-order remainder term. This expansion suggests that generic plug-in estimators of the form $\hat{\psi}_{pi} = \psi(\hat{P})$ have a first-order bias, since evaluating the expansion at (\hat{P}, P) gives

$$\psi(\hat{P}) - \psi(P) = - \int \varphi(z; \hat{P}) dP(z) + R_2(\hat{P}, P)$$

It naturally leads to the one-step estimator

$$\hat{\psi} = \psi(\hat{P}) + P_n\{\varphi(Z; \hat{P})\}$$

More examples can be found in the paper:

$$\begin{aligned} \hat{\psi} - \psi &= \psi(\hat{P}) + P_n\{\varphi(Z; \hat{P})\} - \psi(P) \\ &= (P_n - P)\{\varphi(Z; \hat{P})\} + R_2(\hat{P}, P) \\ &= (P_n - P)\{\varphi(Z; P)\} + (P_n - P)\{\varphi(z; \hat{P}) - \varphi(Z; P)\} + R_2(\hat{P}, P) \\ &= S^* + T_1 + T_2 \end{aligned}$$

S^* is a simple sample average of a fixed function, so by central limit theorem, it behaves as a normally distributed random variable with variance $\text{var}(\varphi)/n$, up to error $o_P(1/\sqrt{n})$

The second term

$$(P_n - P)\{\varphi(z; \hat{P}) - \varphi(Z; P)\}$$

is an empirical process term, and is typically of the smallest order since it is a sample average of a term with shrinking variance.

The third term

$$T_2 = R_2(\hat{P}, P) = \psi(\hat{P}) - \psi(P) + \int \varphi(z; \hat{P}) dP(z)$$

For non-bias-corrected plug-in estimators, this term will generally dominate, but for one-step estimators it will generally involve second-order products of errors, which can be negligible under nonparametric conditions.

In particular, when the T_1 and T_2 terms are of the order $o_P(1/\sqrt{n})$, then the sample average S^* dominates the decomposition, and so

$$\sqrt{n}(\hat{\psi} - \psi) = \sqrt{n}S^* + o_P(1) \xrightarrow{d} N(0, \text{var}\{\varphi(Z; P)\})$$