Probability spaces, measures and σ-algebras

Statistics

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Definition 1. We say that $F \subset 2^{\Omega}$ is a σ-algebra, if
(a) $\Omega \in F$
(b) If $A \in F$ then $A^c \in F$
(c) If $A_i \in F$ for $i = 1, 2, 3, \ldots$ then $\cup_i A_i \in F$

We construct the definition of sigma-algebra to build a measure on it, so that it rigorously defines what exactly is an 'event'.

Definition 2. (Ω, F) with F a σ-algebra of subsets of Ω is called a measurable space. A measure μ is any countably additive non-negative set function on this space. $\mu : F \to [0, \infty]$:
(a) $\mu(A) \geq \mu(\phi) = 0$ for all $A \in F$
(b) $\mu(\cup_n A_n) = \sum_n \mu(A_n)$ for any countable collection of disjoint sets $A_n \in F$. If $\mu(\Omega) = 1$, we call it a probability measure.
To be honest, I think Stein's method of constructing measures is more natural.

Definition 3. A measure space is a triplet (Ω, F, μ), with μ a measure on the measurable space (Ω, F). A measure space (Ω, F, P) with P a probability measure is called a probability space.

That's the measure theory setting of our naive probability theory. Keep in mind, all we want to do is to rigorously define what an event is.
We can easily check that the intersection of sigma-algebras is also a sigma-algebra. Using the theorem, we can construct another definition.

Definition 4. Given a collection of subsets $A_\alpha \subset \Omega$ we denote the smallest σ-algebra F such that $A_\alpha \in F$ for all $\alpha \in \Gamma$ by $\sigma(A_\alpha, \alpha \in \Gamma)$.

$$\sigma(\{A_\alpha\}) = \cap\{G : G \subset 2^{\Omega} \ \ is \ \ a \ \ sigma - algebra, A_\alpha \in F \ \ \forall \alpha \in \Gamma\}$$

Definition 5. Suppose Ω is a topological space. The Borel σ-algebra on the space is defined as $\sigma(\{O \subset \Omega \ open\})$, we denote it by B_Ω.

This is the most commonly used sigma-algebra that I've seen. Please consult Topology by Munkres if you are not familiar with the standard topology on R.

Now we are trying to construct the Lebesgue measure and Caratheodory's theorem. Here what we are roughly doing is extending the probability measure from an Algebra to a sigma algebra (completion), which takes a lot of time to prove. Also, we can start with a π-class and achieve the same result?

Definition 6. A collection A of subsets of Ω is an algebra if:
(a) $\Omega \in A$
(b) If $E \in A$,$E^c \in A$
(c) If $D, E \in A$, $D \cup E \in A$

Comparing the definition between algebra and sigma-algebra, we can see the only difference is the countably additive proprety, so naturally we get the conclusion that a sigma-algebra is an algebra.

Also we can prove that the algebra generated by a given collection of subsets A, is the intersection of all algebras of subsets of Ω containing A. We can that's equivalent to Stat7609's definition of the generated algebra being the collection of all finite disjoint unions of sets in A.

Theorem 1 (Caratheodory's Extension Theorem)**.** If $\mu_0 : A \to [0, \infty]$ is a countably additive set function on an algebra A then there exists a measure μ on $(\Omega, \sigma(A))$ such that $\mu = \mu_0$ on A. Furthermore, if $\mu_0(\Omega) < \infty$ then such a measure μ is unique.

This is very difficult to prove, but we try to make it clear.

To construct the measure U on $B_(0, 1]$ let $\Omega = (0, 1]$ and the algebra:

$$A = \{(a_1, b_1] \cup \cdots \cup (a_r, b_r] : 0 \le a_1 < b_1 < \cdots < a_r < b_r \le 1, r < \infty\}$$

be a collection of subsets of (0,1]. We can show that $\sigma(A) = B_{(0,1]}$. With U_0 denoting the non-negative set function on A such that:

$$U_0(\cup_{k=1}^r (a_k, b_k]) = \sum_{k=1}^{r} (b_k - a_k)$$

Note that $U_0((0, 1]) = 1$ hence the existence of a unique probability measure U on $((0, 1], B_{(0,1]})$ such that $U(A) = U_0(A)$ for sets $E \in A$ follows by Caratheodory's extension theorem, as soon as we verify that:

Lemma 1. The set function U_0 is countably additive on an Algebra. That is, if A_k is a sequence of disjoint sets in the algebra such that $\cup_k A_k = A$ is in the algebra, then $U_0(A) = \sum_{k=1}^n U_0(A_k)$

The proof of the lemma is based on:

Theorem 2. Show that U_0 is finitely additive on A. That is $U_0(\cup_{k=1}^n A_k) = \sum_{k=1}^n U_0(A_k)$ for any finite collection of disjoint sets $A_1, \ldots, A_n \in A$.

Proof. Let $G_n = \cup_{k=1}^n A_k$ and $H_n = A \backslash G_n$. Then H_n decreases to ϕ and since A_k, A is in the algebra it follows that G_n and hence H_n is also in the algebra. By definition, U_0 is finitely additive on A, so

$$U_0(A) = U_0(H_n) + U_0(G_n) = U_0(H_n) + \sum_{k=1}^{n} U_0(A_k)$$

□

To prove from finitely additive to countably additive, it suffices to show that $U_0(H_n)$ decreases to 0, for then

$$U_0(A) = lim_{n\to\infty} U_0(G_n) = lim_{n\to\infty} \sum_{k=1}^{n} U_0(A_k) = \sum_{k=1}^{\infty} U_0(A_k)$$

Ok, remember what are we trying to show, so now we can start to prove the above assertion. It's like proving the sequential continuity of the function, the easiest way (although not ez at all here) is using contradiction. Assuming that $U_0(H_n) \geq 2\epsilon$ for some $\epsilon > 0$ and all n, where H_n decreases to ϕ are elements of the algebra. By the definition of Algebra and U_0, we can find for each l a set J_l in the algebra whose closure $\bar{J_l}$ is a subset of H_l and $U_0(H_l \backslash J_l) \leq \epsilon 2^{-l}$. With U_0 finitely additive on the algebra this implies that for each n,

$$U_0(\cup_{l=1}^{n}(H_l \backslash J_l)) \leq \sum_{l=1}^{n} U_0(H_l \backslash J_l) \leq \epsilon$$

One can show the inequality easily by plugging in all the numbers. Also finding the J_l things is very similar to Stein's way of proving some measurable sets' properties.
As $H_n \subset H_l$ for all $l \leq n$, we have that

$$H_n \backslash \cap_{l \leq n} J_l = \cup_{l \leq n}(H_n \backslash J_l) \subset \cup_{l \leq n}(H_l \backslash J_l)$$

Hence by finite additivity of U_0 and our assumption that $U_0(H_n) \geq 2\epsilon$, also

$$U_0(\cap_{l \leq n} J_l) = U_0(H_n) - U_0(H_n \backslash \cap_{l \leq n} J_l) \geq U_0(H_n) - U(\cup_{l \leq n}(H_l \backslash J_l)) \geq \epsilon$$

In particular, for every n, the set $\cap_{l \leq n} J_l$ is non-empty and therefore so are the decreasing sets $K_n = \cap_{l \leq n \bar{J_l}}$. Since K_n are compact sets (by Heine-Borel theorem) the set $\cap_l \bar{J_l}$ is then non-empty as well, and since $\bar{J_l}$ is a subset of H_l for all l we arrive at $\cap_l H_l$ is non-empty, contradicting our assumption that H_n decreases to ϕ. Draw some pictures will help you understand the proof.

Definition 7. We say that a measure space (Ω, F, μ) is complete if any subset N of any $B \in F$ with $\mu(B) = 0$ is also in F. If further $\mu = P$ is a probability measure, we say that the probability space is a complete probability space.

Theorem 3. Given a measure space (Ω, F, μ) let $E = \{N : N \subset A \ for \ some \ A \in F \ with \ \mu(A) = 0\}$ denote the collection of $mu-$null sets.(Look like the null space of mu isn't it?). Then, there exists a complete measure space $(\Omega, \bar{F}, \bar{\mu})$, called the completion of the measure space (Ω, F, μ) such that $\bar{F} = \{D \cup E : D \in F, E \in N\}$ and $\bar{\mu} = \mu$ on F.

We skip the proof, if you want, you can go to Durett.

Definition 8. A $\pi-$ system is a collection P of sets closed under finite intersections.

Definition 9. A $\lambda-$ system is a collection L of sets containing Ω and $B \backslash A$ for any $A \subset B$, $A, B \in L$, which is also closed under monotone increasing limits.

Theorem 4. A collection F of sets is a $\sigma-$algebra if and only if it is both a $\pi-$system and a $\lambda-$ system. The backward direction needs a smart decomposition that $A \cup B = \Omega \backslash (A^c \cap B^c) \in F$

The main tool in proving the uniqueness of the extension is the following theorem, which is why we study the systems.

Theorem 5. (Dynkin's $\pi - \lambda$ theorem) If $P \subset L$ with P a $\pi-$system and L a $\lambda-$ system, then $\sigma(P) \subset L$.

We omit the proof also. But we can show next, that the uniqueness part of Caratheodory's theorem is an immediate consequence of the theorem above.

Theorem 6. If two measure μ_1 and μ_2 on $(\Omega, \sigma(P))$ agree on the $\pi-$ system P and are such that $\mu_1(\Omega) =_2 (\Omega) < \infty$, then $\mu_1 = \mu_2$

Proof. Let $L = \{A \in \sigma(P) : \mu_1(A) = \mu_2(A)\}$. Our assumptions imply that $P \subset L$ and that $\Omega \subset L$. Further, $\sigma(P)$ is a $\lambda-$ system (because sigma algebra is both lambda and pi system), and if $A \subset B$, $A, B \in L$, then by additivity of the finite measure μ_1 and μ_2.

$$\mu_1(B \backslash A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \backslash A)$$

, that is $B \backslash A \in L$. Similarly, if A_i increases to A and $A_i \in L$, then by the continuity from below of μ_1 and μ_2,

$$\mu_1(A) = \lim_{n \longrightarrow \infty} \mu_1(A_n) = \lim_{n \longrightarrow \infty} \mu_2(A_n) = \mu_2(A)$$

, so that $A \in L$. We conclude that L is a $\lambda-$ system, hence by Dynkin's $\pi - \lambda$ theorem, $\sigma(P) \subset L$, that is $\mu_1 = \mu_2$ □

Noting that since an algebra A is a $\pi-$ system and $\Omega \in A$, the uniqueness of the extension to $\sigma(A)$ follows from the above theorem. Now, we are trying to outline a proof of Caratheodory's theorem.

Definition 10. An increasing, countably sub-additive, non-negative set function μ^* on a measurable space (Ω, F) is called an outer measure. That is $\mu^* : F \rightarrow [0, \infty]$:
(a) $\mu^*(\phi) = 0$ and $\mu^*(A_1) \leq \mu^*(A_2)$ for any $A_1, A_2 \in F$ with $A_1 \subset A_2$
(b) $\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n)$ for any countable collection of sets $A_n \in F$.

Definition 11. $\mu^*(E) = inf\{\sum_{n=1}^{\infty} \mu_0(A_n) : E \subset \cup_n A_n, A_n \in A\}$

We can show the above-defined μ^* is an outer measure on A, and $\mu^* = \mu_0$ on the algebra A. The third step uses the countable additivity of μ_0 on the algebra to show that for any sets A in the algebra the outer measure μ^* is additive when splitting subsets of Ω by intersections with A and A^c. That is, we show that any element of the algebra is a μ^*-measurable set.

Definition 12. Let λ be a non-negative set function on a measurable space (Ω, F), with $\lambda(\phi) = 0$. We say that $A \in F$ is a $\lambda-$measurable set if $\lambda(E) = \lambda(E \cap A) + \lambda(E \cap A^c)$ for all $E \in F$

The forth step is to prove the following lemma.

Lemma 2 (Caratheodory's Lemma)**.** Let μ^* be an outer measure on a measurable space (Ω, F). Then the μ^* measurable sets in F form a $\sigma-$algebra G on which μ^* is countably additive, so that (Ω, G, μ^*) is a measure space.