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Residues and Duality

based on a seminar on the work of A. Grothendieck

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Preface

In the spring of 1963 I suggested to Grothendieck the possibility of my running a seminar at Harvard on his theory of duality for coherent sheaves – a theory which had been hinted at in his talk to Séminaire Bourbaki in 1957 [1], and in his talk to the International Congress of Mathematicians in 1958 [2], but had never been developed systematically. He agreed, saying that he would provide an outline of the material, if I would fill in the details and write up lecture notes of the seminar. During the summer of 1963, he wrote a series of "prénotes" which were to be the basis for the seminar.

I quote from the preface of the prénotes:

> Les presentes notes donnent une esquisse assez détaillée d'une théorie cohomologique de la dualité des Modules cohérents sur les pré schémas. Les idées principales de la théorie m'etaient connues des 1959, mais le manque de fondements adequats d'Algèbre Homologique m'avait empêché d'aborder une redaction d'ensemble. Cette lacune de fondements est sur le point d'etre comblée par la thèse de VERDIER, ce qui rend en principe possible un exposé satisfaisant. Il est d'ailleurs apparu depuis qu il existe des théories cohomologiques de dualité formellement très analogues a celle développée ici dans toutes sortes d'autres contextes: faisceaux coherents sur les espaces analytiques, faisceaux abéliens sur les espaces topologiques (VERDIER), modules galoisiens (VERDIER, TATE), faisceaux de torsion sur les schémas munis de leur topologie étale, corps de classe en tous genres... Cela me semble une raison assez sérieuse pour se familiariser avec le yoga général de la dualité dans un cas type, comme la théorie cohomologique des residus.
>
> La théorie consiste pour l'essentiel dans des questions de variance: construction d'un foncteur $f^!$ et d'un homomorphisme-trace
>
> $$Rf_*f^! \to id.$$
>
> La construction donnée ici est compliquée et indirecte et n'est pas valable sous des conditions aussi générales qu'on est en droit de s'y attendre. Il faudra sans doute une idée nouvelle pour apporter des simplifications substantielles.

The seminar took place in the fall and winter of 1963-64, with the assistance of David Mumford, John Tate, Stephen Lichtenbaum, John Fogarty, and others, and gave rise to a series of six exposés which were circulated to a limited audience under the title "Séminaire Hartshorne". The present notes are a revised, expanded, and completed version of the prevoius notes.

I would like to take this opportunity to thank all those people who have helped in the course of this work, and in particular A. Grothendieck, who gave continual support and encouragement throughout the whole project.

R. H.
Cambridge, May 1966

Contents

Introduction

The main purpose of these is to prove a duality theorem for the cohomology of quasi-coherent sheaves, with respect to a proper morphism of locally noetherian preschemes. Various such theorems are already known. Typical is the duality theorem for a non-singular complete curve X over an algebraically closed field k, which say that

$$h^0(D) = h^1(K - D),$$

where D is a divisor, K is the canonical divisor, and

$$h^i(D) = \dim_k H^i(X, L(D))$$

for any i, and any divisor D. (See e.g. [3] Ch. II for a proof.)

Various attempts were made to generalize this theorem to varieties of higher dimension, and as Zariski points out in his report [4], his generalization of a lemma of Enriques-Severi [5] is equivalent to the statement that for a normal projective variety X of dimension n over k,

$$h^0(D) = h^n(K - D)$$

for any divisor D. This is also equivalent to a theorem of Serre (See [6] 76 Thm. 4) on the vanishing of the cohomology group $H^1(X, L(-m))$ for m large and L locally free. Using a related theorem (See [6] 75 Thm. 3), Zariski shows how one can deduce on a non-singular projective variety the formula

$$h^i(D) = h^{n-i}(K - D)$$

for $0 \leq i \leq n$. In terms of sheaves, this result corresponds to the fact that the k-vector spaces

$$H^i(X, \mathcal{F}) \quad \text{and} \quad H^{n-i}(X, \mathcal{F}^\vee \otimes \omega)$$

are dual to each other, where $\mathcal{F}$ is locally free sheaf, $F^\vee$ is the dual sheaf $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{O}_X)$, and $\omega = \Omega^n_{X/k}$ is the sheaf of n-differentials on X. Serre gives a proof of this same theorem by analytic methods for a compact complex analytic manifold X.

Grothendieck gave some generalizations of these theorems for non-singular projective varieties, and then in [] announced the general theorem for schemes proper over a field, with arbitrary singularities, which is the subject of the present lecture notes.

To motivate the statement of our main theorem, let us consider the case of projective space $X = \mathbf{P}^n_k$ over an algebraically closed field k. Then there is a canonical isomorphism

$$H^n(X, \omega) \cong k$$

where $\omega^n_{X/k}$ is the sheaf of n-differentials. Combining this with the Yoneda pairing

$$H^i(X, \mathcal{F}) \times \mathrm{Ext}^{n-i}_X(\mathcal{F}, \omega) \to H^n(X, \omega)$$

we obtain a pairing

$$H^i(X, \mathcal{F}) \times \mathrm{Ext}^{n-i}_X(\mathcal{F}, \omega) \to k$$

which one shows easily to be a perfect pairing []. This genarlizes the statements above, becasue for a locally free sheaf $\mathcal{F}$,

$$\mathrm{Ext}^{n-i}(\mathcal{F}, \omega) = \mathrm{Ext}^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega) = H^{n-i}(X, \mathcal{F}^\vee \otimes \omega).$$

Another way of looking at our duality pairing is as an isomorphism

$$\mathrm{Ext}_X^{n-i}(\mathcal{F},\omega) \to \mathrm{Hom}_k(H^i(X,\mathcal{F}),k). \tag{0.1}$$

Since everything is linear over k, we may introduce a k-vector space G, and have an isomorphism

$$\mathrm{Ext}_X^{n-i}(\mathcal{F},G\otimes_k \omega) \to \mathrm{Hom}_k(H^i(X,\mathcal{F}),G). \tag{0.2}$$

Before proceeding further, we must introduce the derived category. It will be discussed in detail in Chapter I, but for the moment it will be sufficient to know the following:

For each abelian category $\mathcal{A}$, there is a category $D(\mathcal{A})$, called the *derived category* of A, whose objects are complexes of objects of A. If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor from one abelian category to another, then under reasonable conditions there is a *right derived functor*

$$RF : D(\mathcal{A}) \to D(\mathcal{B})$$

with the property that for any $X \in Ob(\mathcal{A})$, if X denotes also the complex which is X in degree zero, and zero elsewhere, then $H^i(RF(X)) = R^iF(X)$, where R^iF is the ordinary i-th right derived functor of F. Finally, if $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are two functors then

$$R(G\circ F) = R(G)\circ R(F).$$

This replaces the old-fashioned spectral sequence of a composite functor.

Now we can jazz up our duality for projective space as follows. We replace k by a prescheme Y, so that $X = \mathbf{P}_Y^n$. We consider the derived categories $D(X)$ and $D(Y)$ of the categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules, respectively. Then cohomology H^i becomes Rf_*, the derived functor of the direct image functor f_*, where $f : X \to Y$ is the projection. The functor Ext becomes the derived functor RHom of Hom. We define

$$f^!(G) = f^*(G)\otimes\omega,$$

for $G\in D(Y)$, and we replace F by a complex of sheaves $F\in D(X)$. Then the isomorphism $H^n(X,\omega)\cong k$ gives us an isomorphism

$$Rf_*f^!G{\sim}G$$

which we call the *trace map*. The Yoneda pairing reappears as a natural map

$$R\mathrm{Hom}_X(F,f^!G) \longrightarrow R\mathrm{Hom}_Y(Rf_*F,Rf_*f^!G),$$

which, composed with the trace map gives us the *duality morphism*

$$R\mathrm{Hom}_X(F,f^!G) \longrightarrow R\mathrm{Hom}_Y(Rf_*F,G)$$

which generalize 0.2. This is easily proved to be an isomorphism (III 5.1 below) under the suitable hypotheses on Y, F, G. In fact, the proof is nothing but "general nonsense" once one has the isomorphism 0.1.

Having examined the case of projective space, we can state the following ideal theorem, which is the primum mobile of these notes, although it may never appear explicitly in this form.

References

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