Nonparametric Failure Time Estimation for Degradation Data with Random Effects

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Abstract

Estimating the lifetime of highly reliable products is challenging due to their long operational time. As a result, degradation data are often used to estimate the probability distribution of the first passage time. In practical applications, when unit-to-unit variability or heterogeneity exists, random effects are required to be incorporated into the model. In many applications, the underlying probability distribution of degradation data is often unknown, and using a nonparametric procedure can be an alternative to robustly estimate the first passage time distribution without making any distributional assumptions. Moreover, incorporating random effects for parametric degradation models can often be computationally intensive. This study proposes a novel nonparametric approach using an empirical saddlepoint approximation to estimate the FPT distribution when degradation data consists of random effects. The proposed techniques enable the researchers to do efficient and robust FPT estimation while incorporating random effects without assuming any known probability distribution. Monte Carlo simulation studies and real-life data analysis demonstrate the effectiveness of the proposed nonparametric methods compared to parametric approaches.

Keywords: Degradation data, Empirical saddlepoint approximation, first-passage time distribution, gamma process, inverse-Gaussian process, Monte Carlo simulations, random effect.

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1 Introduction

Degradation data modeling has gained prominence due to its effectiveness in estimating the lifetime of highly reliable products. The failure time or first passage time (FPT) of such products is defined as the time required to reach a predefined performance threshold level. Parametric stochastic models based on Lèvy processes, such as the gamma process, inverse-Gaussian (IG) process, and Wiener process, are commonly applied to model degradation data and estimate the FPT distribution. However, degradation data often exhibit unit-to-unit variability, also known as random effects or heterogeneity, due to factors such as environmental conditions, manufacturing variations, and quality inconsistencies. Accurately incorporating random effects into degradation models is essential for reliable FPT estimation.

The gamma process is widely applied to monotonic degradation data, where degradation increments are independent and follow a gamma distribution (Singpurwalla, 1997; Park and Padgett, 2005; Tsai et al., 2010). Random effects in the gamma process were first introduced by Lawless and Crowder (2004), who modeled these effects by introducing a probability distribution for the scale parameter. Their work also derived a closed-form expression for the cumulative distribution function (CDF) of the FPT distribution in the presence of random effects. Their approach has been applied in various reliability studies (Hao et al., 2015; Pulcini, 2013; Wang et al., 2015; Rodríguez-Picón et al., 2018). Building on this work, Tsai et al. (2012) developed optimal design strategies for gamma process degradation models with random effects and used the expectation-maximization (EM) algorithm to estimate the maximum likelihood estimates (MLEs). More recently, Wang et al. (2021) extended this framework by constructing confidence intervals for parameter estimates in gamma process models with random effects, specifically for small sample sizes. In their paper, Song and Cui (2022) introduced models to estimate the FPT distribution for the bivariate gamma process with random effects. Other recent studies on gamma processes with random effects include Zhou et al. (2023); Zhang et al. (2023); Zheng et al. (2025).

The inverse-Gaussian (IG) process is another monotonic degradation model where degradation increments follow an IG distribution. In their paper, Wang and Xu (2010) incorporated random effects into the volatility parameter of the IG process and used the EM algorithm for parameter estimation via MLEs. Ye and Chen (2014) extended this model by proposing two random effect models for the IG process: a random volatility model and a random drift-volatility model, treating the IG process as the FPT of a Wiener process. Furthermore, Peng (2015) introduced random effects to both the drift and volatility parameters of the IG process, proposing an MLEs-based estimation procedure via the EM algorithm and deriving the corresponding FPT distribution. Further developments in IG processes with random effects include Fang et al. (2022); Sun et al. (2021).

In contrast to the gamma and IG processes, the Wiener process is a non-monotonic parametric degradation model, with degradation increments that follow a normal distribution. FPT distribution of the Wiener process follows an IG distribution, and random effects in Wiener processes have been studied by Wang (2010); Ye et al. (2015); Zhai et al. (2018); Wang et al. (2020).

While parametric degradation models rely on specific probability distributions, real-world applications often involve scenarios where the true distribution is unknown or difficult to estimate, especially in the presence of random effects. Unlike parametric approaches, empirical methods are purely data-driven, avoiding restrictive distributional assumptions and complex parameter estimation procedures such as the expectation-maximization (EM) algorithm. Traditional model selection methods, such as the Akaike Information Criterion (AIC), can compare parametric degradation models but do not identify the true underlying distribution. This limitation makes nonparametric methods particularly valuable when the exact degradation process is uncertain.

Balakrishnan and Qin (2019) introduced an empirical saddlepoint approximation (ESA) method to estimate the FPT distribution in degradation data nonparametrically, incorporating both fixed and random effects. Palayangoda et al. (2020) further improved this method by applying the Lugannani-Rice saddlepoint approximation (Lugannani and Rice, 1980) for the fixed-effect case. The ESA method has since been extended to various degradation applications, including bivariate degradation data (Palayangoda and Ng, 2021b), nonlinear degradation data (Palayangoda et al., 2022), and step-stress degradation testing (Palayangoda et al., 2024). Additionally, Palayangoda et al. (2025) applied the ESA-based FPT distribution to develop a goodness-of-fit testing procedure for nonidentical degradation data with fixed effects.

The ESA method proposed by Balakrishnan and Qin (2019) demonstrates accuracy mainly in the upper percentiles (typically above the 50th percentile) but provides an improper distribution in lower percentiles. In this paper, we propose a novel ESA approach that incorporates random effects to estimate the FPT distribution nonparametrically. The proposed ESA random effect model is compared with the gamma process random effect model of Lawless and Crowder (2004); Tsai et al. (2012) and the IG process random effect model of Peng (2015). Through this novel approach, we aim to enhance the accuracy of nonparametric FPT distribution estimation in the presence of random effects and provide an alternative data-driven computational technique for parametric approaches.

The rest of this paper is organized as follows: In Section 2, we review parametric degradation processes with random effects. Section 3 introduces the proposed ESA method for estimating the FPT distribution with random effects. Furthermore, Monte Carlo simulation studies are conducted to assess the performance of the proposed methods in Section 4. Section 5 applies the proposed random ESA methods to two real-world degradation data sets and compares them with parametric models. Finally, concluding remarks are presented in Section 6.

2 Parametric Degradation Data Modeling with Random Effects

This section summarizes parametric degradation models with random effects used to estimate the FPT distribution for degradation processes. In stochastic approaches to degradation modeling, it is typically assumed that the degradation data follow a Lèvy process. Let the degradation measurement at time t be denoted as $\{X_t, t > 0\}$. If X_t follows a Lèvy process, then the consecutive differences in degradation measurements, $\Delta X_t = X_t - X_{t-1}$, are independent random variables. When $\{\Delta X_t > 0, \forall t\}$, the Lèvy process is classified as monotone process. Commonly used monotone degradation processes include gamma and IG processes. For a monotone Lèvy process with a failure threshold level τ, the FPT, denoted as $F_{T_\tau}(t)$, is given by

$$F_{T_\tau}(t) = \Pr(T_\tau < t) = 1 - \Pr(X_t \leq \tau), \tag{1}$$

where $T_\tau = \inf\{t : X_t \geq \tau\}$ is the failure time.

2.1 Gamma process with random effects

Let X_t follow a gamma process with random effects, then X_t follows a gamma distribution with shape parameter αt and rate parameter ν, denoted as $X_t \sim \mathcal{G}(\alpha t, \nu^{-1})$. For gamma process with random effects, Lawless and Crowder (2004); Tsai et al. (2012) assume heterogeneity in ν, where ν follows a gamma distribution with shape parameter δ and rate parameter η (i.e., $\nu \sim \mathcal{G}(\delta, \eta^{-1})$. Under this model, the FPT distribution for the gamma process random effect is given by

$$F_{T_\tau}(t|\alpha, \eta, \delta) = \int_0^\infty \{1 - \Pr(X_t \leq \tau)\} f(\nu|\delta, \eta) d\nu = 1 - \mathbf{F}_{2\alpha t, 2\delta}\left(\frac{\delta\tau}{\alpha\eta t}\right), \tag{2}$$

where $\mathbf{F}_{2\alpha t,2\delta}$ is F-distribution with $2\alpha t$ and 2δ degrees of freedoms, and $f(\nu|\delta, \eta)$ is the PDF of ν.

2.2 IG process with random effects

Suppose X_t follows an IG process with random effects. The distribution of X_t is then IG with mean parameter μt and shape parameter λt^2, denoted as $X_t \sim \mathcal{IG}(\mu t, \lambda t^2)$. In their paper, Peng (2015) proposed three approaches to model the IG process with random effects. In this study, we focus on their model M_1, where the heterogeneity is applied to both μ and λ. Specifically, this model assumes that the random effects are distributed as

$$\begin{aligned} X_t|\mu, \lambda &\sim \mathcal{IG}(\mu t, \lambda t^2) \\ \gamma|\lambda &\sim \mathcal{N}(\xi, \sigma_\mu^2/\lambda) \\ \lambda &\sim \mathcal{G}(\alpha_\lambda, \beta_\lambda), \end{aligned} \tag{3}$$

where $\gamma = \mu^{-1}$ and $\mathcal{N}(\xi, \sigma_\mu^2/\lambda)$ denotes the normal distribution with mean ξ and variance σ_μ^2/λ. Using the above random effect model, Peng (2015) obtained the FPT distribution for the IG process with random effects by

$$\begin{aligned} F_{T_\tau}(t|\xi, \sigma_\mu, \alpha_\lambda, \beta_\lambda) &= \int_0^\infty \int_{-\infty}^\infty \{1 - \Pr(X_t \leq \tau)\} f(\gamma|\lambda) f(\lambda) d\gamma d\lambda, \\ &= \sqrt{\frac{\beta_\lambda}{2\pi}} \frac{\Gamma(\alpha_\lambda + 0.5)t}{\Gamma(\alpha_\lambda)} \int_\tau^\infty x^{-3/2} (\sigma_\mu^2 x + 1)^{-0.5} \\ &\times \left(1 + \frac{\beta_\lambda(\xi x - t)^2}{2x(\sigma_\mu^2 x + 1)}\right)^{-(\alpha_{\lambda+0.5})} dy \end{aligned} \tag{4}$$

where $f(\gamma|\lambda)$ and $f(\lambda)$ are the PDFs of $\gamma|\lambda$ and λ, respectively.

In this study, we use the two parametric approaches described above to simulate degradation data and compare their performance with the proposed nonparametric random effects model. In other words, we apply the gamma process with random effects model from Lawless and Crowder (2004); Tsai et al. (2012) and the inverse-Gaussian (IG) process with random effects model from Peng (2015) as benchmarks. By comparing these parametric models with the proposed novel nonparametric ESA approach, we aim to evaluate the accuracy and robustness of the proposed method.

3 Nonparametric Degradation Data Modeling with Random Effects

In this section, we introduce a novel nonparametric technique using the ESA method to estimate the FPT distribution when degradation data is associated with random effects. Suppose a degradation experiment is conducted with n units, and each unit is measured m_i times, where $i = 1, 2, \ldots, n$. Let $t_j, j = 1, 2, \ldots, m_i$ represent the time point of measurement j for all units. We assume the initial degradation at time $t = 0$ is zero. Let the degradation data follow a Lèvy process $\{X_t,\ t > 0\}$, and the observed degradation measurement for the i-th unit at time t_j is denoted as $X_j^{(i)} = \{x_j^{(i)}\}, i = 1, 2, \ldots, n,\ j = 1, \ldots, m_i$. The observed difference between two consecutive degradation measurements of unit i is given by $\Delta X_j^{(i)} = X_j^{(i)} - X_{j-1}^{(i)} = \{\Delta x_j^{(i)}\}$ and corresponding time differences be $\Delta t_j = t_j - t_{(j-1)}$.

The ESA approach requires that the degradation data be equally spaced (i.e., Δt_j is a constant). When degradation measurements are not taken at uniform time intervals for a unit i, the conditional random imputation (CRImp) algorithm, a stochastic imputation procedure, proposed by Palayangoda et al. (2020, Section 3.2.2) can be applied to make the degradation differences equally spaced. The effectiveness of the CRImP approach is discussed in Palayangoda et al. (2020); Palayangoda and Ng (2021a). First, we develop the proposed nonparametric procedure assuming the degradation data are equally spaced for all units, and in Section 3.3, we extend the applicability of the proposed method to unequal time intervals.

3.1 Empirical moment generating function and cumulant generating function with random effects for Lèvy process

In this section, we derive the empirical cumulant-generating function (CGF) of the degradation process with random effects. When the degradation data exhibit no heterogeneity (i.e., fixed effects), Palayangoda et al. (2020) proposed an ESA approach to estimate the FPT distribution nonparametrically. In this study, we extend the ESA method by Palayangoda et al. (2020) to account for the heterogeneity of the degradation data. First, we derive the moment-generating function (MGF) for the degradation process to apply the ESA approach with random effects. The MGF for a Lèvy process X_t with random effects is given by

$$\mathcal{M}_{X_t}(s) = \mathbb{E}_W[\mathbb{E}_{X_t}(\exp(sX_t)|W)] = \int f_W(w)\mathcal{M}_{X_t|W}(s)\, dw, \tag{5}$$

where the random variable W incorporate the unit-to-unit variability and $f_W(w)$ is the PDF of W. For empirical estimation, $f_W(w)$ in Eq. (5) can be written as a discrete distribution

$$\mathcal{M}_{X_t}(s) = \sum_{i=1}^{n} f_W(w_i)\mathcal{M}_{X_t^{(i)}}(s), \tag{6}$$

where $f_W(w_i) = \Pr(W = w_i),\ i = 1, 2, \ldots, n,$ is a mass function to incorporate the random effects.

Suppose the degradation process of the unit i follows a Lèvy process and the measurements are taken in equal time intervals, the MGF of the degradation measurement (i.e., $X_t^{(i)}$) at time t can be expressed as (Balakrishnan and Qin, 2019; Palayangoda et al., 2020):

$$\mathcal{M}_{X_t^{(i)}}(s) = [\mathcal{M}_{\Delta X_t^{(i)}}(s)]^t. \tag{7}$$

Thus, the nonparametric approximation of MGF (i.e., empirical MGF) for the Lèvy process for the unit i can be evaluated by:

$$\hat{\mathcal{M}}_{X_t^{(i)}}(s) = \left\{ \frac{1}{m_i} \sum_{j=1}^{m_i} \exp\left(s\Delta x_j^{(i)} \right) \right\}^t. \tag{8}$$

Therefore, the empirical MGF and empirical CGF of the degradation process X_t, incorporating random effects, can be expressed as

$$\hat{\mathcal{M}}_{X_t}(s) = \sum_{i=1}^{n} f_W(w_i) \left[\frac{1}{m_i} \sum_{j=1}^{m_i} \exp\Big(s\Delta x_j^{(i)} \Big) \right]^t, \tag{9}$$

$$\hat{\mathcal{K}}_{X_t}(s) = \log \left\{ \sum_{i=1}^{n} f_W(w_i) \left[\frac{1}{m_i} \sum_{j=1}^{m_i} \exp\Big(s\Delta x_j^{(i)} \Big) \right]^t \right\}, \tag{10}$$

respectively.

3.2 ESA for random effects when degradation data are measured in equally spaced time intervals

Using the empirical CGF with random effects, we extend the ESA approach by Palayangoda et al. (2020) to estimate the FPT distribution for the degradation process with random effects. The FPT distribution for the degradation process with threshold level τ for random effects can be estimated by

$$\hat{F}_{T_\tau}(t) = \Pr(T_\tau < t) = 1 - \Phi(\hat{w}) - \phi(\hat{w}) \left(\frac{1}{\hat{w}} - \frac{1}{\hat{u}} \right) \quad , \quad t \notin [\tilde{t} - \kappa\Delta t_0, \tilde{t} + \kappa\Delta t_0], \tag{11}$$

where Φ and ϕ are the standard normal CDF and PDF, respectively. The quantiles, $\hat{\omega}$ and $\hat{u}$ are defined as

$$\hat{\omega} = \operatorname{sgn}(\hat{s}) \sqrt{2 \left\{ \hat{s}\tau - \hat{\mathcal{K}}_{X_t}(\hat{s}) \right\}} \qquad \text{and} \qquad \hat{u} = \hat{s}\sqrt{\hat{\mathcal{K}}''_{X_t}(\hat{s})}. \tag{12}$$

The saddpointpoint, $\hat{s}$, is determined by solving saddlepoint equation $\hat{\mathcal{K}}'_{X_t}(s) = \tau$. The first, second and third derivatives of $\hat{\mathcal{K}}_{X_t}(s)$ in Eq. (10) with respect to s can be computed as follows:

$$\hat{\mathcal{K}}'_{X_t}(s) = \frac{\sum_{i=1}^{n} a'_i(s)}{\sum_{i=1}^{n} a_i(s)}, \tag{13}$$

$$\hat{\mathcal{K}}''_{X_t}(s) = -\left\{ \frac{\sum_{i=1}^{n} a'_i(s)}{\sum_{i=1}^{n} a_i(s)} \right\}^2 + \frac{\sum_{i=1}^{n} a''_i(s)}{\sum_{i=1}^{n} a_i(s)}, \tag{14}$$

$$\tag{15}$$

where a_i, a'_i, a''_i and a'''_i are given by

$$a_i(s) = \frac{f_W(w_i)}{m_i^t} \left\{ \varphi_i(s)^t \right\}, \tag{16}$$

$$a'_i(s) = \frac{f_W(w_i)}{m_i^t} \left\{ t\varphi_i(s)^{t-1} \varphi'_i(s) \right\}, \tag{17}$$

$$a''_i(s) = \frac{f_W(w_i)}{m_i^t} \{ t(t-1)\varphi_i(s)^{t-2} \varphi'_i(s)^2 + t\varphi_i(s)^{t-1} \varphi''_i(s) \}, \tag{18}$$

where

$$\varphi_i(s) = \sum_{j=1}^{m_i} \exp\Big(s\Delta x_j^{(i)}\Big), \tag{19}$$

$$\varphi_i'(s) = \sum_{j=1}^{m_i} \Delta x_j^{(i)} \exp\Big(s\Delta x_j^{(i)}\Big), \tag{20}$$

$$\varphi_i''(s) = \sum_{j=1}^{m_i} \Delta x_j^{2(i)} \exp\Big(s\Delta x_j^{(i)}\Big). \tag{21}$$

In this study, we explore different approaches to model random effects in degradation data using the random variable W to account for unit-to-unit variability. The flexibility of defining W allows us to capture random effects in various ways, leading to distinct methods for estimating the empirical CGF. We propose using the first two central moments (mean and variance) of the degradation differences in each unit to capture the unit-to-unit variability of the degradation process. Below are four specific methods based on weighting the empirical CGF in Eq. (10):

Equal Weight Method: In this approach, we assume each unit contributes equally to the random effect estimation, where each unit weight is assigned a probability of $1/n$. Therefore, a discrete uniform distribution is applied as the distribution of W, where

$$f_W(w_i) = P(W = w_i) = \frac{1}{n},$$

for all i in Eq. (10). This simplifies the empirical CGF to:

$$\hat{\mathcal{K}}_{X_t}(s) = \log\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{m_i}\sum_{j=1}^{m_i}\exp\Big(s\Delta x_j^{(i)}\Big)\right]^t\right\}.$$

Random Mean Method: This approach models random effects based on the average degradation increment observed within each unit. If a particular unit tends to degrade faster or slower on average than others, this method captures that behavior by assigning a higher or lower probability weight to that unit's contribution. In particular, we define the probability mass function $f_W(w_i)$ in Eq. (10) as,

$$f_W(w_i) = \frac{\bar{y}_i}{\sum_{i=1}^{n}\bar{y}_i}, \tag{22}$$

where $w_i = \bar{y}_i$, and $\bar{y}_i = \sum_{j=1}^{m_i} \Delta x_j{}^{(i)}/m_i,\ i = 1, 2, \ldots, n$. Thus, the ECGF for the random mean method approach is given by

$$\hat{\mathcal{K}}_{X_t}(s) = \log\left\{\frac{1}{\sum_{i=1}^{n} \bar{y}_i} \sum_{i=1}^{n} \left[\bar{y}_i \frac{1}{m_i} \sum_{j=1}^{m_i} \exp\Big(s\Delta x_j^{(i)}\Big)\right]^t\right\}.$$

Assigning higher weights to units with larger average degradation reflects the belief that these units may represent a different (and possibly more extreme) latent degradation mechanism. This method helps capture average-based heterogeneity across the population.

Random Standard Deviation Method: In this method, we assume that the random effect arises from the variability of degradation differences within each unit. Units that show more fluctuation in their degradation behavior over time are considered to contribute more to the random effects. In this case, $f_W(w_i)$ in Eq. (10) is defined as the proportion of the standard deviation of degradation differences of each unit,

$$f_W(w_i) = \frac{\hat{\sigma}_{\Delta x_i}}{\sum_{i=1}^{n} \hat{\sigma}_{\Delta x_i}},$$

where $\hat{\sigma}^2_{\Delta x_i} = \sum_{j=1}^{m_i} \Big((\Delta x_j^{(i)} - \bar{y}_i)^2/(m_i - 1)\Big),\ i = 1, 2, \ldots, n$. The ECGF for the random standard deviation method is then

$$\hat{\mathcal{K}}_{X_t}(s) = \log\left\{\frac{1}{\sum_{i=1}^{n} \hat{\sigma}_{\Delta x_i}} \sum_{i=1}^{n} \left[\hat{\sigma}_{\Delta x_i} \frac{1}{m_i} \sum_{j=1}^{m_i} \exp\Big(s\Delta x_j^{(i)}\Big)\right]^t\right\}.$$

Weighting based on standard deviation allows us to emphasize units that exhibit greater uncertainty in their degradation process, which is an important form of random effect.

Random Standardized Mean Method: This method combines both the average and variability of degradation increments. We standardize the mean degradation differences of each unit by dividing the corresponding standard deviation of the degradation differences and define $f_W(w_i)$ in Eq. (10) such that

$$f_W(w_i) = \frac{\psi_i}{\sum_{i=1}^{n} \psi_i},$$

where $\psi_i = \bar{y}_i/\hat{\sigma}_{\Delta x_i}$, and $\bar{y}_i$ and $\hat{\sigma}_{\Delta x_i}$ are the mean and standard deviation of the degradation differences, respectively. The ECGF for the random standardized mean method is given by

$$\hat{\mathcal{K}}_{X_t}(s) = \log\left\{\frac{1}{\sum_{i=1}^{n} \psi_i} \sum_{i=1}^{n} \left[\psi_i \frac{1}{m_i} \sum_{j=1}^{m_i} \exp\Big(s\Delta x_j^{(i)}\Big)\right]^t\right\}.$$

This method emphasizes the relative variability of degradation differences for each unit by normalizing the mean by its standard deviation.

Saddlepoint approximations with Lugannani and Rice (1980) approach create a singularity (i.e., saddlepoint $\hat{s} = 0$) around the median of the distribution, which may lead to inaccurate probability estimates around the median. For ESA approaches, the median of the FPT can be approximated by $\tilde{t} \approx \tau/\overline{\mathbf{\Delta x}}$, where $\overline{\mathbf{\Delta x}} = \{n \sum_{i=1}^{n} m_i\}^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \Delta x_j^{(i)}$. We define a range of which singularity occurs in FPT as $[\tilde{t} - \kappa \Delta t_0,\ \tilde{t} + \kappa \Delta t_0]$, where κ is a hyperparameter. A polynomial spline interpolation method by Forsythe et al. (1977) is applied to adjust the FPT distribution within the singularity range. For this study, we used $\kappa = 2$ (i.e., four-unit-time points around the median) (see detailed discussion in supplementary materials S1 and Palayangoda et al. (2020)).

3.3 ESA with random effects when degradation data are measured in unequally spaced time intervals

In Sections 3.1 and 3.2, we assumed the degradation data were measured in equally spaced time intervals, where Δt_j is a constant. However, in practical applications, the degradation data may be measured with unequal time intervals. To extend the proposed ESA approach for unequal time measurements, we apply the stochastic imputation with the CRImp algorithm to make the degradation data equally spaced, which allows us to apply the proposed ESA approach in Sections 3.1 and 3.2. The preliminary algorithm for CRImp can be found in Palayangoda et al. (2020, Section 3.2.2) and also included in Section S2 of the supplementary materials.

Suppose a given unit $i,\ i = 1, \dots, n$, the consecutive degradation differences are $\Delta X_j^{(i)} = \{\Delta x_j^{(i)}\},\ j = 1, \dots, m_i$ with corresponding unequal time difference Δt_j. Let the equal time interval (i.e., pseudo-unit-time intervals) with the imputed data be obtained by taking the highest common factor (HCF) of $\{\Delta t_j\}$ (i.e., $\Delta t_0 = HCF(\Delta t_j, j = 1, 2, \dots, m_i)$). The procedure to obtain the FPT distribution for unequally spaced degradation data with random effects is given by:

Step 1: Evaluate the pseudo-unit-time interval, Δt_0, by taking the HCF of $\{\Delta t_j\}$;

Step 2: Select degradation differences of unit i (i.e., $\Delta X_j^{(i)} = \{\Delta x_j^{(i)}\},\ j = 1, \dots, m_i$);

Step 3: Apply CRImp algorithm for unit i and obtain equally spaced degradation differences with Δt_0 time intervals, denoted by $\Delta x_j^{(i)*}$;

Step 4: Repeat Step 2 and Step 3 for all units in the experiment, $i = 1, 2, \ldots, n$, and obtain equally spaced degradation differences for all data, denoted by $\Delta X_j^{(i)*} = \{\Delta x_j^{(i)*}\}, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m_i$;

Step 5: For time t, evaluate the empirical CGF for the random effect using the imputed degradation difference as:

$$\hat{\mathcal{K}}_{X_t}(s) = \log\left\{ f_W(w_i^*) \left[\frac{1}{m_i} \sum_{j=1}^{m_i} \exp\Big(s\Delta x_j^{(i*)}\Big) \right]^t \right\},$$

where w_i^* are the new weights estimated by the imputed degradation differences (except for equal weight method);

Step 6: Estimate the survival probability for time t by Eq. (11) applying the empirical CGF estimated in Step 5.

4 Monte Carlo Simulation Studies

Monte Carlo simulations were conducted to evaluate the performance of the proposed random effects models using the ESA approach. Degradation data with random effects were generated from gamma and IG processes, specifically the gamma process model from Tsai et al. (2012) and the IG process model from Peng (2015, Model 1). For the gamma process, the parameter setting is $\{\alpha = 15, \delta = 30, \eta = 2\}$, and for the IG process, the parameter setting is $\{\epsilon = 1, \sigma = 0.25, \alpha_\lambda = 24, \beta_\lambda = 1.6\}$. These values were chosen to ensure a mean degradation per unit time interval of 1, facilitating direct comparisons. The parameter estimation for the gamma and IG process is performed using an EM algorithm.

To identify the performance of the proposed ESA methods with respect to the sample size, two sample sizes (number of units) are considered: $n = \{15, 30\}$ with two measurement sizes (i.e., number of measurements for each unit): $m = \{15, 30\}$. Without loss of generality, the time between two consecutive measurements is set as $\Delta t_0 = 1$. The FPT distributions are evaluated at two threshold levels: $\tau = \{50, 75\}$.

For each simulated dataset, the following models were applied to estimate the FPT distribution:

1. The true underlying distribution (i.e., gamma or IG random effects model)

2. MLE-based gamma random effects model (Tsai et al., 2012)

3. MLE-based IG random effects model (Peng, 2015)

4. ESA_{BQ} method (Balakrishnan and Qin, 2019), incorporating random effects

5. Four proposed ESA methods, denoted as:

    - ESA_n (Equal Weight Method)
    - ESA_μ (Random Mean Method)
    - ESA_σ (Random Standard Deviation Method)
    - ESA_{cv} (Random Standardized Mean Method)

In addition to the above approaches, to evaluate robustness under unequally spaced measurement times, we reused the same simulated datasets but selectively removed certain measurement points. In other words, for the case when $m = 15$, where original degradation measurement times for all units are $t_j = \{1, 2, 3, \ldots, 15\}$, we only considered simulated degradation data at times $t_j = \{1, 3, 6, 7, 9, 12, 13, 15\}$ (i.e., simulated degradation data, X_j, at $t_j = \{2, 4, 5, 8, 10, 11, 14\}$ are removed for all units). Similarly, when $m = 30$, $t_j = \{1, 3, 6, 7, 9, 12, 13, 15, 18, 19, 2$ Thus, the HCF of these measurement times is $\Delta t_0 = 1$. The proposed ESA approach is applied to these data by imputing them equally spaced using the CRImp method proposed in Section 3.3. We denote the results obtained in this approach by:

- ESA^*_n (Equal Weight Method with unequally spaced time intervals)
- ESA^*_μ (Random Mean Method with unequally spaced time intervals)
- ESA^*_σ (Random Standard Deviation Method with unequally spaced time intervals)
- ESA^*_{cv} (Random Standardized Mean Method with unequally spaced time intervals).

Table 1: Data generated from random Gamma degradation process with parameters $\{\alpha = 15, \delta = 30, \eta = 2\}$ with $n = 15$

	$n=15$ $m=15$ $\tau=50$						$n=15$ $m=15$ $\tau=75$					
	10th percentile			**90th percentile**			**10th percentile**			**90th percentile**		
Model	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**
Gamma	0.26	8.30	**8.56**	0.19	15.82	**16.01**	0.53	19.27	**19.80**	0.34	36.50	**36.84**
IG	0.06	10.42	**10.48**	3.31	14.22	**17.52**	0.11	24.08	**24.19**	7.19	32.70	**39.90**
ESA_{BQ}	-	-	-	2.50	21.43	**23.93**	-	-	-	6.91	49.94	**56.86**
ESA_{n}	0.15	9.31	**9.46**	0.85	18.20	**19.05**	0.39	21.65	**22.04**	2.18	42.23	**44.40**
ESA_{μ}	2.43	12.13	**14.57**	1.29	12.95	**14.24**	5.83	28.23	**34.06**	2.71	30.37	**33.09**
ESA_{σ}	2.31	12.26	**14.57**	1.28	13.50	**14.78**	5.70	28.70	**34.40**	2.77	31.31	**34.08**
ESA_{cv}	0.12	9.52	**9.64**	0.78	18.99	**19.77**	0.31	22.15	**22.46**	2.03	44.09	**46.12**
ESA^*_{n}	0.19	9.23	**9.42**	0.98	18.03	**19.01**	0.43	21.51	**21.95**	2.35	41.91	**44.26**
ESA^*_{μ}	2.57	11.99	**14.56**	1.13	12.86	**14.00**	6.00	27.98	**33.97**	2.51	30.19	**32.70**
ESA^*_{σ}	2.58	12.46	**15.04**	1.05	13.94	**14.99**	5.72	28.79	**34.51**	2.43	32.51	**34.95**
ESA^*_{cv}	0.07	9.41	**9.48**	0.77	19.66	**20.43**	0.18	21.85	**22.04**	1.86	44.91	**46.76**
	$n=15$ $m=30$ $\tau=50$						$n=15$ $m=30$ $\tau=75$					
	10th percentile			**90th percentile**			**10th percentile**			**90th percentile**		
Model	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**
Gamma	0.29	7.57	**7.86**	0.34	14.69	**15.03**	0.62	17.23	**17.86**	0.62	33.05	**33.67**
IG	0.30	9.59	**9.89**	3.57	12.44	**16.01**	0.64	21.89	**22.53**	7.99	28.04	**36.02**
ESA_{BQ}	-	-	-	0.86	19.78	**20.64**	-	-	-	2.53	44.92	**47.45**
ESA_{n}	0.04	8.73	**8.77**	0.11	16.89	**17.01**	0.11	20.13	**20.24**	0.34	38.53	**38.88**
ESA_{μ}	1.79	11.40	**13.19**	2.41	12.16	**14.56**	4.25	26.29	**30.54**	5.09	27.67	**32.75**
ESA_{σ}	1.77	11.47	**13.24**	2.42	12.36	**14.78**	4.19	26.44	**30.63**	5.10	28.13	**33.23**
ESA_{cv}	0.03	8.85	**8.88**	0.10	17.25	**17.35**	0.09	20.38	**20.48**	0.31	39.01	**39.32**
ESA^*_{n}	0.06	8.65	**8.71**	0.16	16.74	**16.90**	0.13	20.00	**20.14**	0.42	38.26	**38.68**
ESA^*_{μ}	1.89	11.27	**13.17**	2.21	12.07	**14.28**	4.37	26.07	**30.44**	4.81	27.51	**32.32**
ESA^*_{σ}	1.85	11.32	**13.18**	2.14	12.55	**14.69**	4.19	26.44	**30.63**	4.64	28.64	**33.29**
ESA^*_{cv}	0.03	8.85	**8.89**	0.13	17.52	**17.65**	0.09	20.32	**20.41**	0.31	39.09	**39.40**

Each approach was evaluated using 10,000 Monte Carlo simulations, where the squared bias (i.e., Bias^2), variance, and mean squared error (MSE) were computed for the 10th and 90th percentiles of the FPT distribution. Note that the 10th percentile estimates for the ESA_{BQ} method were excluded from the analysis, as it can produce survival probabilities exceeding 1 in the lower tail of the distribution (see, for example, Palayangoda et al. (2020)).

Tables 1 and 2 summarize the results for degradation data generated from the gamma process with random effects for $n = 15$ and $n = 30$ cases, respectively. Similarly, simulation results for degradation data generated with the IG process with random effects are presented in Tables 3 and 4 for $n = 15$ and $n = 30$ cases, respectively. As expected, when the sample sizes and the number of measurements increase, we observe lower MSEs, whereas the MSE increases for higher threshold levels due to increased uncertainty.

Table 2: Data generated from random Gamma degradation process with parameters $\{\alpha = 15, \delta = 30, \eta = 2\}$ with $n = 30$

	$n = 30$ $m = 15$ $\tau = 50$						$n = 30$ $m = 15$ $\tau = 75$					
	10th percentile			**90th percentile**			**10th percentile**			**90th percentile**		
Model	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**
Gamma	0.05	4.03	**4.07**	0.01	8.23	**8.24**	0.08	9.40	**9.48**	0.00	18.26	**18.26**
IG	0.31	5.02	**5.32**	2.23	7.20	**9.43**	0.70	11.75	**12.45**	5.01	16.03	**21.04**
ESA$_{BQ}$	-	-	-	2.75	10.48	**13.23**	-	-	-	6.87	23.69	**30.55**
ESA$_n$	0.49	4.71	**5.20**	1.26	9.18	**10.44**	1.22	11.05	**12.27**	3.03	20.62	**23.65**
ESA$_\mu$	4.33	6.63	**10.96**	1.09	6.53	**7.62**	10.07	15.52	**25.59**	2.29	14.77	**17.06**
ESA$_\sigma$	4.29	6.77	**11.06**	1.07	6.73	**7.80**	9.95	15.74	**25.69**	2.26	15.32	**17.58**
ESA$_{cv}$	0.46	4.87	**5.33**	1.20	9.63	**10.83**	1.13	11.39	**12.52**	2.89	21.43	**24.32**
ESA$^*_n$	0.55	4.67	**5.23**	1.43	9.12	**10.55**	1.31	10.99	**12.30**	3.28	20.51	**23.79**
ESA$^*_\mu$	4.50	6.58	**11.08**	0.93	6.50	**7.43**	10.31	15.43	**25.74**	2.07	14.70	**16.78**
ESA$^*_\sigma$	4.68	6.89	**11.57**	0.85	6.93	**7.78**	10.51	16.04	**26.54**	1.94	15.91	**17.84**
ESA$^*_{cv}$	0.36	4.79	**5.14**	1.30	10.19	**11.49**	0.88	11.25	**12.13**	2.99	22.46	**25.45**

	$n = 30$ $m = 30$ $\tau = 50$						$n = 30$ $m = 30$ $\tau = 75$					
	10th percentile			**90th percentile**			**10th percentile**			**90th percentile**		
Model	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**
Gamma	0.06	3.87	**3.93**	0.06	7.21	**7.27**	0.02	2.15	**2.16**	0.00	4.03	**4.03**
IG	0.75	4.88	**5.64**	2.58	6.08	**8.66**	0.04	2.64	**2.68**	0.50	4.25	**4.75**
ESA$_{BQ}$	-	-	-	1.15	9.19	**10.34**	-	-	-	0.79	5.30	**6.09**
ESA$_n$	0.25	4.62	**4.87**	0.33	8.02	**8.35**	0.14	2.67	**2.81**	0.22	4.56	**4.78**
ESA$_\mu$	3.34	6.41	**9.75**	2.05	5.83	**7.88**	1.85	3.71	**5.56**	1.08	3.30	**4.38**
ESA$_\sigma$	3.31	6.50	**9.81**	2.03	5.95	**7.98**	1.81	3.77	**5.58**	1.06	3.44	**4.50**
ESA$_{cv}$	0.24	4.69	**4.93**	0.32	8.17	**8.48**	0.13	2.77	**2.90**	0.20	4.77	**4.97**
ESA$^*_n$	0.29	4.59	**4.88**	0.41	7.97	**8.39**	0.18	2.64	**2.81**	0.30	4.51	**4.80**
ESA$^*_\mu$	3.47	6.36	**9.83**	1.85	5.80	**7.65**	1.98	3.66	**5.64**	0.92	3.26	**4.18**
ESA$^*_\sigma$	3.47	6.53	**10.01**	1.76	6.07	**7.83**	2.20	3.83	**6.03**	1.58	3.24	**4.81**
ESA$^*_{cv}$	0.24	4.68	**4.92**	0.37	8.35	**8.73**	0.11	2.76	**2.87**	0.75	5.46	**6.21**

ESA with equally spaced time intervals: Based on the simulation results, the proposed ESA methods (i.e., ESA$_n$, ESA$_\mu$, ESA$_\sigma$, and ESA$_{cv}$) have outperformed the ESA$_{BQ}$ method in all 90th percentile results with lower bias and variance. For the 10th percentile, ESA$_n$ (i.e., equal weight method) and ESA$_{cv}$ (i.e., random standardized mean method) methods provide MSEs closer to the MSEs of the true degradation process. In addition, the ESA$_{cv}$ approach maintains a relatively lower bias at the 10th percentile. At the 90th percentile, the ESA$_n$, ESA$_\mu$, and ESA$_\sigma$ methods produce MSEs closer to those of the true underlying degradation process; however, the ESA$_{cv}$ approach often has relatively higher MSEs at the 90th percentile.

ESA with unequally spaced time intervals: It is important to note that the MSEs for the proposed ESA approach with unequally spaced time intervals (i.e., ESA$^*_n$, ESA$^*_\mu$, ESA$^*_\sigma$,

Table 3: Data generated from random IG degradation process with parameters $\{\epsilon = 1, \sigma = 0.25$ $\alpha_\lambda = 24, \beta_\lambda = 1$ $\}$ with $n = 15$

	$n=15$ $m=15$ $\tau=50$						$n=15$ $m=15$ $\tau=75$					
	10th percentile			**90th percentile**			**10th percentile**			**90th percentile**		
Model	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**
IG	0.20	4.44	**4.64**	0.14	4.70	**4.85**	0.52	10.14	**10.65**	0.35	10.53	**10.88**
Gamma	0.04	4.10	**4.14**	0.05	5.45	**5.50**	0.09	9.23	**9.32**	0.13	12.41	**12.54**
ESA$_{BQ}$	-	-	-	1.43	5.69	**7.12**	-	-	-	3.71	13.09	**16.80**
ESA$_n$	0.39	4.95	**5.34**	0.84	4.92	**5.76**	0.87	11.23	**12.10**	2.05	11.15	**13.20**
ESA$_\mu$	1.44	6.20	**7.64**	0.08	4.09	**4.17**	3.20	14.08	**17.28**	0.24	9.20	**9.44**
ESA$_\sigma$	2.14	7.12	**9.27**	0.00	4.11	**4.11**	4.88	16.63	**21.50**	0.01	9.23	**9.24**
ESA$_{cv}$	0.05	4.35	**4.40**	1.20	5.41	**6.61**	0.13	9.86	**9.99**	2.93	12.22	**15.15**
ESA$^*_n$	0.47	4.88	**5.35**	0.98	4.85	**5.83**	0.97	11.15	**12.12**	2.25	11.04	**13.29**
ESA$^*_\mu$	1.58	6.12	**7.70**	0.13	4.03	**4.16**	3.39	13.94	**17.33**	0.31	9.12	**9.43**
ESA$^*_\sigma$	2.39	7.10	**9.48**	0.02	4.15	**4.18**	4.99	16.25	**21.25**	0.04	9.45	**9.50**
ESA$^*_{cv}$	0.06	4.44	**4.50**	1.25	5.65	**6.90**	0.12	10.03	**10.16**	2.91	12.61	**15.52**
	$n=15$ $m=30$ $\tau=50$						$n=15$ $m=30$ $\tau=75$					
	10th percentile			**90th percentile**			**10th percentile**			**90th percentile**		
Model	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**
IG	0.14	3.73	**3.87**	0.15	3.87	**4.02**	0.33	8.57	**8.90**	0.22	8.63	**8.86**
Gamma	0.13	3.64	**3.76**	0.00	4.17	**4.18**	0.22	8.18	**8.40**	0.01	9.34	**9.35**
ESA$_{BQ}$	-	-	-	0.38	4.90	**5.27**	-	-	-	1.30	11.02	**12.32**
ESA$_n$	0.10	4.51	**4.61**	0.14	4.28	**4.42**	0.29	10.28	**10.57**	0.50	9.53	**10.03**
ESA$_\mu$	0.69	5.61	**6.30**	0.03	3.60	**3.63**	1.80	12.91	**14.71**	0.02	7.96	**7.98**
ESA$_\sigma$	1.17	6.34	**7.50**	0.16	3.58	**3.73**	3.03	14.98	**18.01**	0.22	7.90	**8.12**
ESA$_{cv}$	0.00	3.93	**3.93**	0.31	4.56	**4.87**	0.00	8.93	**8.93**	0.93	10.21	**11.15**
ESA$^*_n$	0.14	4.45	**4.59**	0.20	4.22	**4.42**	0.35	10.19	**10.54**	0.60	9.43	**10.03**
ESA$^*_\mu$	0.79	5.51	**6.30**	0.01	3.56	**3.57**	1.93	12.78	**14.71**	0.00	7.89	**7.89**
ESA$^*_\sigma$	1.31	6.18	**7.49**	0.08	3.58	**3.66**	3.21	14.73	**17.94**	0.14	7.99	**8.13**
ESA$^*_{cv}$	0.00	4.01	**4.01**	0.35	4.63	**4.98**	0.00	9.04	**9.05**	1.01	10.29	**11.30**

and ESA$^*_{cv}$), have MSEs nearly identical to the proposed ESA approach with equally spaced, validating the robustness of the proposed CRImp method-based ESA approach for unequal time measurement situations.

Model selection based on the simulation results: The simulation results suggest that the empirical FPT distributions obtained with the ESA$_n$ (or ESA$^*_n$) approach perform relatively consistently in both lower and upper percentiles, and estimate the FPT distribution closer to the true underlying distribution. The ESA$_{cv}$ (or ESA$^*_{cv}$) approach has better performance in the lower percentiles, and ESA$_\mu$ (or ESA$^*_\mu$) and ESA$_\sigma$ (or ESA$^*_\sigma$) perform closer to the MLE of the true underlying distribution in the upper percentiles. Based on the simulation results, if we are interested in obtaining the FPT distribution nonparametrically, ESA$_n$ (or ESA$^*_n$) is the

Table 4: Data generated from random IG degradation process with parameters $\{\epsilon = 1, \sigma = 0.25$ $\alpha_\lambda = 24$, $\beta_\lambda = 1$ $\}$ with $n = 30$

	$n = 30$ $m = 15$ $\tau = 50$						$n = 30$ $m = 15$ $\tau = 75$					
	10th percentile			**90th percentile**			**10th percentile**			**90th percentile**		
Model	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**
IG	0.05	2.25	**2.30**	0.03	2.39	**2.42**	0.10	4.91	**5.01**	0.09	5.27	**5.36**
Gamma	0.00	2.07	**2.07**	0.21	2.68	**2.89**	0.01	4.46	**4.47**	0.52	5.95	**6.47**
ESA$_{BQ}$	-	-	-	1.65	2.82	**4.47**	-	-	-	3.94	6.33	**10.27**
ESA$_n$	0.73	2.67	**3.39**	1.11	2.50	**3.61**	1.75	5.89	**7.65**	2.49	5.55	**8.04**
ESA$_\mu$	2.32	3.56	**5.88**	0.14	2.07	**2.21**	5.43	7.97	**13.41**	0.32	4.55	**4.87**
ESA$_\sigma$	3.40	4.38	**7.79**	0.01	2.09	**2.10**	8.16	9.96	**18.13**	0.02	4.58	**4.60**
ESA$_{cv}$	0.18	2.28	**2.46**	1.59	2.75	**4.34**	0.47	5.06	**5.53**	3.64	6.15	**9.78**
ESA$^*_n$	0.82	2.64	**3.46**	1.27	2.47	**3.74**	1.90	5.85	**7.75**	2.72	5.50	**8.22**
ESA$^*_\mu$	2.48	3.52	**6.00**	0.20	2.05	**2.25**	5.66	7.90	**13.56**	0.40	4.52	**4.92**
ESA$^*_\sigma$	3.71	4.34	**8.05**	0.06	2.14	**2.20**	8.42	9.82	**18.24**	0.09	4.72	**4.80**
ESA$^*_{cv}$	0.19	2.36	**2.55**	1.68	2.89	**4.57**	0.49	5.20	**5.69**	3.68	6.45	**10.13**

	$n = 30$ $m = 30$ $\tau = 50$						$n = 30$ $m = 30$ $\tau = 75$					
	10th percentile			**90th percentile**			**10th percentile**			**90th percentile**		
Model	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**	**Bias^2**	**Var**	**MSE**
IG	0.03	1.90	**1.93**	0.03	1.95	**1.98**	0.08	4.16	**4.24**	0.06	4.35	**4.41**
Gamma	0.02	1.87	**1.89**	0.03	2.10	**2.13**	0.03	4.00	**4.03**	0.11	4.74	**4.85**
ESA$_{BQ}$	-	-	-	0.51	2.38	**2.89**	-	-	-	1.45	5.47	**6.92**
ESA$_n$	0.33	2.46	**2.79**	0.27	2.12	**2.39**	0.74	5.36	**6.10**	0.72	4.83	**5.55**
ESA$_\mu$	1.39	3.22	**4.62**	0.01	1.78	**1.79**	3.15	7.08	**10.23**	0.00	4.04	**4.04**
ESA$_\sigma$	2.29	3.90	**6.19**	0.09	1.77	**1.86**	5.30	8.71	**14.01**	0.16	4.02	**4.18**
ESA$_{cv}$	0.03	2.06	**2.09**	0.50	2.28	**2.78**	0.08	4.54	**4.62**	1.30	5.19	**6.50**
ESA$^*_n$	0.39	2.43	**2.82**	0.35	2.09	**2.44**	0.82	5.32	**6.14**	0.84	4.79	**5.63**
ESA$^*_\mu$	1.52	3.19	**4.70**	0.00	1.76	**1.76**	3.33	7.02	**10.35**	0.00	4.01	**4.01**
ESA$^*_\sigma$	2.46	3.87	**6.33**	0.04	1.79	**1.83**	5.46	8.61	**14.07**	0.08	4.04	**4.13**
ESA$^*_{cv}$	0.05	2.10	**2.15**	0.56	2.32	**2.88**	0.11	4.59	**4.70**	1.39	5.29	**6.68**

most suitable approach. In addition, if the objective is to estimate the lower percentiles of the FPT distribution with better precision, then the ESA$_{cv}$ (or ESA$^*_{cv}$) approach can be applied, while for upper percentile estimates, ESA$_\mu$ (or ESA$^*_\mu$) can be considered.

5 Real Data Applications

In this section, we apply the proposed empirical FPT estimation procedures to real-world degradation datasets and compare the results with parametric random effects models. Assuming that the degradation processes exhibit random effects, we analyze two datasets: the GaAs laser degradation data from Meeker and Escobar (1998) and the fatigue crack propagation data from Rodríguez-Picón et al. (2018).

5.1 GaAs laser data set from Meeker and Escobar (1998)

The performance of the proposed ESA methods was evaluated on a practical application using the GaAs laser data set from Meeker and Escobar (1998). In this data set, the degradation variable is the operating current of GaAs laser devices. The FPT is the time when the operating current increases to 10 % from its initial value (i.e., $\tau = 10\%$). The experiment includes 15 GaAs laser devices ($n = 15$), with measurements taken every 250 hours over a total of 4000 hours ($m = 16$). Three of the units reached the threshold level and failed before the experiment termination time.

FPT distributions for devices were estimated using both parametric and nonparametric approaches with random effects. The parametric models considered include the gamma and IG processes, while the nonparametric methods include the ESA_{BQ} method and the proposed ESA methods (ESA_n, ESA_μ, ESA_σ, and ESA_{cv}). Figure 1 illustrates the FPT distributions obtained from the parametric and nonparametric methods, while Table 5 presents the estimated 10th and 90th percentiles. The results indicate that the proposed ESA methods produce estimates closely aligned with those from the parametric random effects models. Notably, the proposed ESA approaches achieve this without relying on any distributional assumptions, demonstrating their robustness and competitiveness compared to traditional parametric models.

Table 5: 10-th percentile and 90th percentile of FPT distribution for GaAs laser degradation data in Meeker and Escobar (1998) with parametric and nonparametric approaches

	Parameteric Random		ESA Random				
	Gamma	IG	ESA_{BQ}	ESA_n	ESA_μ	ESA_σ	ESA_{cv}
10-th	3858	3702	-	3617	3439	3510	3556
90-th	6448	6296	6380	6414	6314	6346	6373

Proposed ESA approach with unequally spaced time intervals: To demonstrate the proposed ESA approaches with random effects for unequally spaced time intervals, we alter the laser degradation data set by removing some degradation measurements to create unequal time intervals. In the data set, we only consider the measurements at time points {250, 500, 1000, 1750, 2000, 2500, 3250, 3500, 4000}, which provides $\Delta t_1 = \Delta t_4 = \Delta t_7 = 250$, $\Delta t_2 = \Delta t_5 = \Delta t_8 = 500$, $\Delta t_3 = \Delta t_6 = 750$.

From this modified laser degradation dataset, we modeled the proposed ESA approaches

Table 6: 10-th percentile and 90th percentile of FPT distribution for modified (i.e., with unequally spaced time intervals) GaAs laser degradation data with proposed ESA approaches

	ESA_n^*	ESA_μ^*	ESA_σ^*	ESA_{cv}^*
10-th	3606	3612	3608	3609
90-th	6462	6464	6460	6479

using the CRImp algorithm to estimate the 10th and 90th percentiles (see Table 6). The results show that there are no substantial changes in the estimates compared to the equally spaced time estimates of parametric or nonparametric approaches in Table 5, further demonstrating the robustness of the proposed nonparametric procedure.

5.2 Fatigue crack data set by Rodríguez-Picón et al. (2018)

Table 7: 10-th percentile and 90th percentile of FPT distribution for fatigue crack data in Rodríguez-Picón et al. (2018) with parametric and nonparametric approaches

	Parameteric Random		ESA Random				
	Gamma	IG	ESA_{BQ}	ESA_n	ESA_μ	ESA_σ	ESA_{cv}
10-th	81.6	74.8	-	72.7	69.6	71.2	70.7
90-th	141.0	141.7	155.8	157.0	152.2	154.2	155.1

The second dataset involves a degradation experiment on crack propagation in an electronic device terminal, as provided by Rodríguez-Picón et al. (2018). The device is considered to have failed when the crack length reaches a critical threshold of 0.4 mm. The experiment includes 10 devices ($n = 10$) and, with crack growth measurements taken every 10 thousand cycles up to 90 thousand cycles. Assuming the presence of random effects in the degradation data, FPT distributions were estimated using parametric approaches, the ESABQ method, and the proposed nonparametric ESA approaches (ESAn, ESAμ, ESAσ, and ESA_{cv}).

Table 7 presents the estimated 10th and 90th percentiles of the FPT distributions, and Figure 2 compares the FPT distributions across all methods. The results indicate that all nonparametric methods produce similar estimates for both percentiles. However, the parametric models yield higher 10th percentile estimates compared to the nonparametric methods, while their 90th percentile estimates are lower. This suggests that the proposed ESA approaches capture greater unit-to-unit variability in the degradation data, leading to broader FPT distributions than the parametric methods.

6 Conclusion

In this study, we proposed novel nonparametric approaches for estimating the FPT distribution using the ESA method in the presence of random effects in degradation data. Specifically, we introduced four ESA-based approaches incorporating different weight structures into the model to capture the random effect. Unlike parametric random effect degradation models, these approaches are data-driven and do not rely on distributional assumptions. Unlike parametric random effects degradation models, proposed ESA methods are data-driven and do not rely on distributional assumptions, making them particularly useful when the underlying degradation process is unknown.

Through Monte Carlo simulations, we demonstrated that the equal weight ESA method (ESA_n) is the most consistent performer across percentiles, providing reliable accuracy and precision in both tails. Among the proposed methods, ESA_{cv} provides more accurate lower percentile estimates, while $\text{ESA}\mu$ and ESA_σ perform better for upper percentiles. Importantly, all ESA methods consistently outperformed the ESA_{BQ} method for the 90th percentile, highlighting their robustness. Furthermore, as sample size and the number of measurements increased, the proposed ESA approaches exhibited lower MSEs, confirming their effectiveness for larger datasets. Moreover, the ESA approaches with imputed degradation data (ESA^*_n, ${\text{ESA}_\mu}^*$, ESA^*_σ, and ESA^*_{cv}) performed comparably to their equally spaced counterparts. This indicates the robustness of the proposed nonparametric ESA methods under irregular sampling schedules, particularly when coupled with CRImp for imputation.

The real data applications further validated the utility of the ESA methods. When applied to the GaAs laser dataset, the ESA-based percentile estimates closely aligned with those from parametric random-effects models. Moreover, in the fatigue crack degradation dataset, the ESA methods captured greater unit-to-unit variability than parametric approaches, offering a more flexible and data-driven alternative for modeling degradation processes. The ESA methods provide a straightforward and robust alternative to parametric estimation methods.

A potential extension is developing empirical FPT estimation procedures for bivariate degradation data. Palayangoda and Ng (2021b) introduced ESA-based methods for estimating the empirical FPT distribution in systems with correlated degradation processes, which could be further extended to incorporate random effects. Furthermore, Palayangoda et al. (2025) proposed a goodness-of-fit testing procedure for degradation processes measured at irregular time intervals (or nonidentical degradation data). The equal weight method (ESA_n) could be extended to perform goodness-of-fit testing for degradation models with random effects. Future research directions also include developing optimal experimental design strategies for degradation studies with random effects. We are currently exploring these extensions and anticipate reporting our findings in future publications.

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References

Balakrishnan, N. and Qin, C. (2019). Nonparametric evaluation of the first passage time of degradation processes. *Applied Stochastic Models in Business and Industry*, 35:571–590.

Fang, G., Pan, R., and Wang, Y. (2022). Inverse gaussian processes with correlated random effects for multivariate degradation modeling. *European Journal of Operational Research*, 300(3):1177–1193.

Forsythe, G., Malcolm, M., and Moler, C. (1977). *Computer Methods for Mathematical Computations*. Prentice Hall, Philadelphia, PA.

Hao, H., Su, C., and Li, C. (2015). Led lighting system reliability modeling and inference via random effects gamma process and copula function. *International Journal of Photoenergy*, 2015:243648.

Lawless, J. and Crowder, M. (2004). Covariates and random effects in a gamma process model with application to degradation and failure. *Lifetime data analysis*, 10:213–227.

Lugannani, R. and Rice, S. (1980). Saddle point approximation for the distribution of the sum of independent random variables. *Advances in Applied Probability*, 12:475–490.

Meeker, W. Q. and Escobar, L. A. (1998). *Statistical Methods for Reliability Data*. John Wiley & Sons, New York, NY.

Palayangoda, L., Ng, H. K. T., and Li, L. (2024). Semiparametric evaluation of first-passage distribution for step-stress accelerated degradation tests. *Applied Stochastic Models in Business and Industry*.

Palayangoda, L. K., Butler, R. W., Ng, H. K. T., Yang, F., and Tsui, K. L. (2022). Evaluation of mean-time-to-failure based on nonlinear degradation data with applications. *IISE Transactions*, 54:286–302.

Palayangoda, L. K. and Ng, H. K. T. (2021a). Nonparametric approximation methods for the first-passage time distribution for degradation data measured with unequal time intervals. In *Advances in Statistics-Theory and Applications*, pages 395–415. Springer.

Palayangoda, L. K. and Ng, H. K. T. (2021b). Semiparametric and nonparametric evaluation of first-passage distribution of bivariate degradation processes. *Reliability Engineering & System Safety*, 205:107230.

Palayangoda, L. K., Ng, H. K. T., and Butler, R. W. (2020). Improved techniques for parametric and nonparametric evaluations of the first-passage time for degradation processes. *Applied Stochastic Models in Business and Industry*, 36:730–753.

Palayangoda, L. K., Ng, H. K. T., and Gafurov, A. (2025). Goodness of fit testing for degradation data. page submitted.

Park, C. and Padgett, W. (2005). Accelerated degradation models for failure based on geometric brownian motion and gamma processes. *Lifetime Data Analysis*, 11:511–527.

Peng, C.-Y. (2015). Inverse Gaussian processes with random effects and explanatory variables for degradation data. *Technometrics*, 57:100–111.

Pulcini, G. (2013). Modeling the mileage accumulation process with random effects. *Communications in Statistics-Theory and Methods*, 42:2661–2683.

Rodríguez-Picón, L. A., Rodríguez-Picón, A. P., Méndez-González, L. C., Rodríguez-Borbón, M. I., and Alvarado-Iniesta, A. (2018). Degradation modeling based on gamma process models with random effects. *Communications in Statistics-Simulation and Computation*, 47:1796–1810.

Singpurwalla, N. (1997). Gamma processes and their generalizations: an overview. *Engineering Probabilistic Design and Maintenance for Flood protection*, pages 67–75.

Song, K. and Cui, L. (2022). A common random effect induced bivariate gamma degradation process with application to remaining useful life prediction. *Reliability engineering & system safety*, 219:108200.

Sun, B., Li, Y., Wang, Z., Ren, Y., Feng, Q., and Yang, D. (2021). An improved inverse gaussian process with random effects and measurement errors for rul prediction of hydraulic piston pump. *Measurement*, 173:108604.

Tsai, C.-C., Tseng, S.-T., and Balakrishnan, N. (2010). Optimal burn-in policy for highly reliable products using gamma degradation process. *IEEE Transactions on Reliability*, 60:234–245.

Tsai, C.-C., Tseng, S.-T., and Balakrishnan, N. (2012). Optimal design for degradation tests based on gamma processes with random effects. *IEEE Transactions on Reliability*, 61:604–613.

Wang, H., Xu, T., and Mi, Q. (2015). Lifetime prediction based on gamma processes from accelerated degradation data. *Chinese Journal of Aeronautics*, 28:172–179.

Wang, X. (2010). Wiener processes with random effects for degradation data. *Journal of Multivariate Analysis*, 101:340–351.

Wang, X., Wang, B. X., Hong, Y., and Jiang, P. H. (2021). Degradation data analysis based on gamma process with random effects. *European Journal of Operational Research*, 292:1200–1208.

Wang, X., Wang, B. X., Jiang, P. H., and Hong, Y. (2020). Accurate reliability inference based on wiener process with random effects for degradation data. *Reliability Engineering & System Safety*, 193:106631.

Wang, X. and Xu, D. (2010). An inverse Gaussian process model for degradation data. *Technometrics*, 52:188–197.

Ye, Z.-S. and Chen, N. (2014). The inverse gaussian process as a degradation model. *Technometrics*, 56(3):302–311.

Ye, Z.-S., Chen, N., and Shen, Y. (2015). A new class of wiener process models for degradation analysis. *Reliability Engineering & System Safety*, 139:58–67.

Zhai, Q., Chen, P., Hong, L., and Shen, L. (2018). A random-effects wiener degradation model based on accelerated failure time. *Reliability Engineering & System Safety*, 180:94–103.

Zhang, Z., Gao, D., Guan, T., Liang, Y., Zhao, J., Wang, L., and Tang, J. (2023). A reliability evaluation method for gamma processes with multiple random effects. *Machines*, 11:905.

Zheng, H., Yang, J., Kang, W., and Zhao, Y. (2025). Accelerated degradation data analysis based on gamma process with random effects. *Quality and Reliability Engineering International*.

Zhou, S., Xu, A., Tang, Y., and Shen, L. (2023). Fast bayesian inference of reparameterized gamma process with random effects. *IEEE Transactions on Reliability*, 73:399–412.

Figure 1: FPT distributions of GaAs laser degradation data in Meeker and Escobar (1998): (1) parametric random effect models (2) nonparametric random effect models

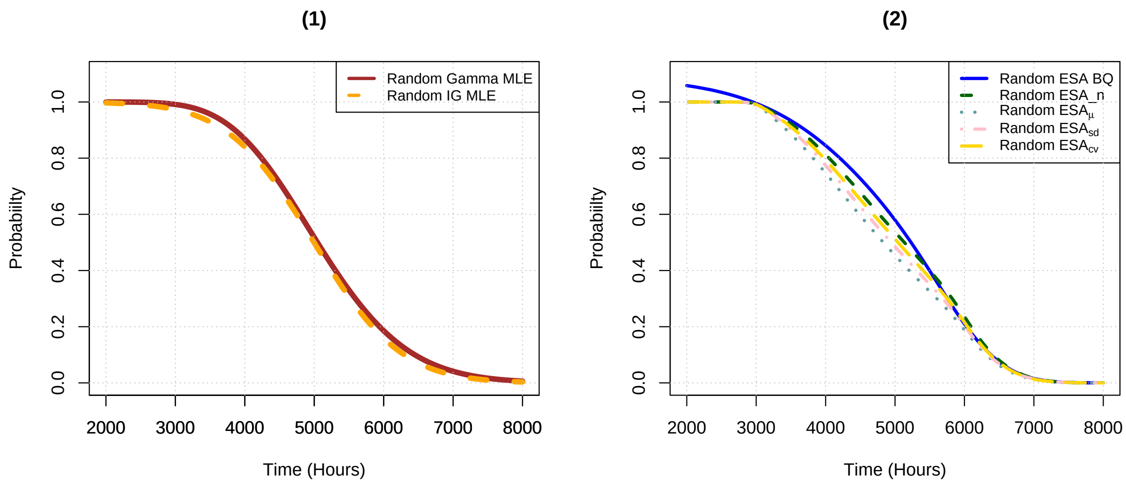

Figure 2: FPT distributions for fatigue degradation data in Rodríguez-Picón et al. (2018): (1) parametric random effect models (2) nonparametric random effect models