Edge Weighted Graph:

Directed Graph: $\sum_i deg(v) = 2|E|$

$\sum_i indegree(v) = \sum_i outdegree = |E|$

Weight Function of an edge:

$w : E \longrightarrow \mathbb{R}$ (or)

$w : V \times V \longrightarrow \mathbb{R}$

$w(u, v)$ = weight of the edge from u to v

if $(u, v) \notin E$, then $w(u, v) = \infty$ } extended weight function

Graph as a Triplet:

$G = (V, E, w)$ → can be a defined function also

Weight matrix: $W \in \mathbb{R}^{|V| \times |V|}$

$W_{ij} \in \mathbb{R}$

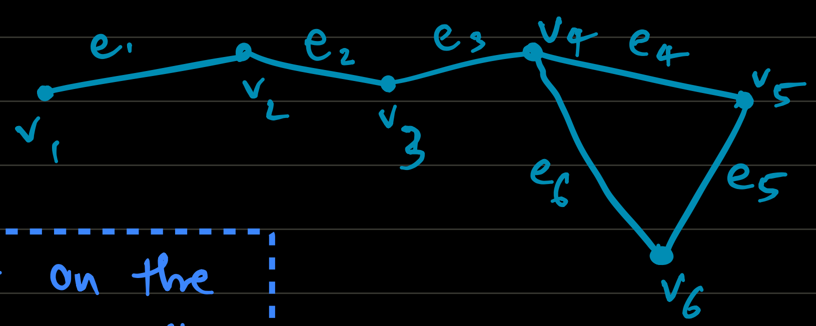
→ $w(i, j)$: weight of the edge from vertex i to vertex j.

Walk:

Sequence of Connected edges

$\langle e_1, e_2, \ldots, e_k \rangle$

Vertices can repeat
edges can repeat



$\langle e_1, e_2, e_3, e_4, e_5, e_6, e_3 \rangle$ is a walk from vertex v_1 to v_3

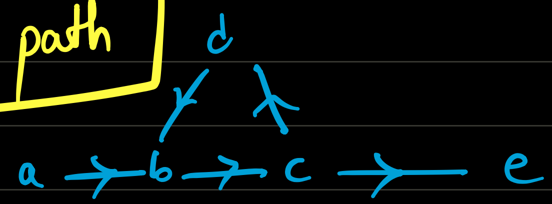
No limit on the Size of a walk

Path: A walk in which all the intermediate vertices are distinct

$\therefore$ $\boxed{\text{Length of any path } \leq n-1}$

$n = |V|$
in $G = (V, E, w)$

Cycle = closed path



$\langle a, b, c, e \rangle$ — path of length 3

$\langle a, b, c, d, b \rangle$ — Not a path; walk of length 4

Weight of a path:

$$w(P) = \sum_{e \in P} w(e)$$

$\downarrow$

Sum of the weights of all the edges in the path

Weight of a walk W:

$$w(W) = \sum_{e \in W} w(e)$$

Weight of a Cycle C:

$$w(C) = \sum_{e \in C} w(e)$$

Negative Cycle:

$$\boxed{w(C) < 0}$$

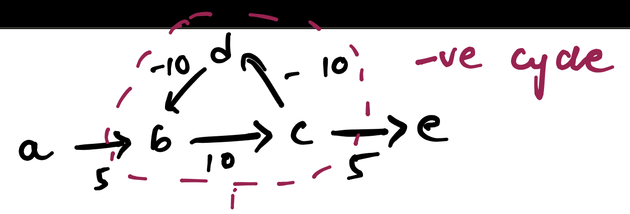
Shortest path : $\delta(u, v)$

$$\delta(u,v) = \min\{ w(P) : P \text{ is a path from } u \text{ to } v \}$$

$\because$ # paths from u to v in any graph is finite, $\delta(u, v)$ is "well-defined"

Shortest Walk : $\alpha(u, v)$

Could be $-\infty$; Shortest walk is not well-defined because the no. of edges can be as many as possible in a walk if the given graph has a -ve cycle in it.



$\langle a, b, c, e \rangle \rightarrow 20$

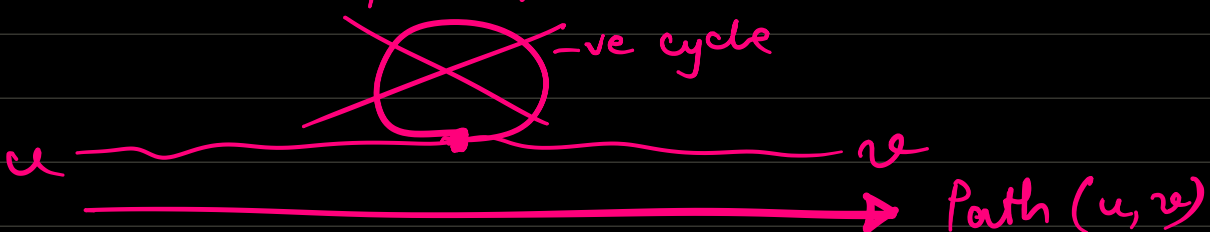
$\langle a, b, c, d, b, c, e \rangle \rightarrow 5 + \cancel{10} - \cancel{10} - 10 + \cancel{10} + 5 = 10$

Reason : -ve cycle exists

$\delta(u, v)$:	Shortest path b/w u and v
$\alpha(u, v)$:	Shortest walk b/w u and v

* The weight of a Shortest walk can be reduced to $-\infty$ if there is a -ve cycle in the walk.

* If W is a walk from u to v, then W Contains a path from u to v



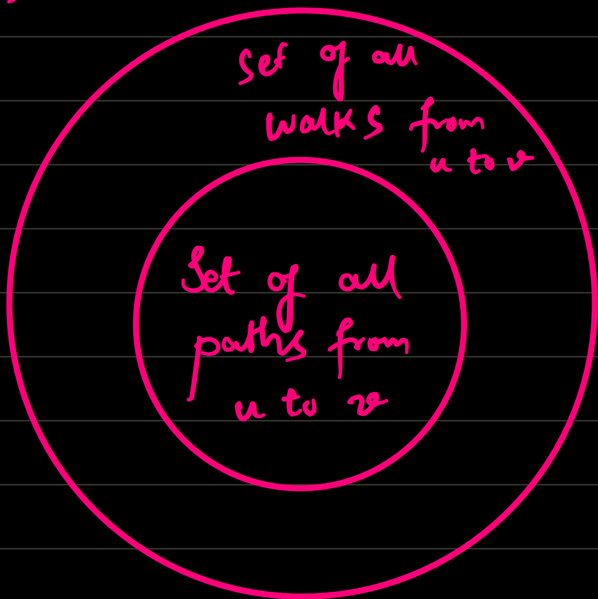
we obtain the path by removing the cycles in the walk

* Assume that G has no -ve cycles. Let W be a walk from u to v. Then W contains a path from u to v such that

$$\boxed{w(P) \le w(W)}$$

no zero cycles also

* If $G = (V, E, w)$ has no -ve cycle, then the shortest walk from u to v is the shortest path from u to v.



$$\alpha(u,v) \le \delta(u,v)$$

Single Source Shortest Path Problem

Redefinition of the Graph in this problem:

$G = (V, E, w, s)$

V: Set of vertices; E: set of edges; w: weight function; s: Source vertex $s \in V$

$$\boxed{\begin{array}{l} \delta(s, v) \quad ; \quad v \in V - \{s\} \\ P_v : \text{Shortest path from } s \text{ to } v \end{array}}$$

$\because$ s is fixed, we rewrite $\delta(s, v)$ as $\delta(v)$

i.e, $\boxed{\delta(v) = \delta(s, v)}$

Problem :- Given : $G = (V, E, w, s)$
Return : $\delta(v)$, P_v $\forall$ $v \in V - \{s\}$

s : Source

Mathematical Foundations of Shortest Path Weights

Theorem 1 : Let $G = (V, E, w, s)$. Let $\delta(v)$ be the weight of the shortest path from s to v.

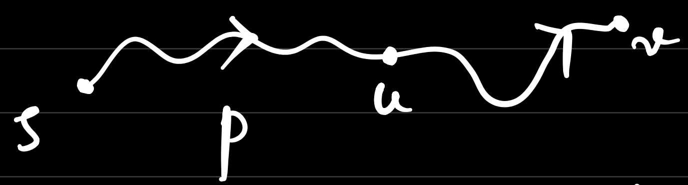
If there exists an edge $(u,v) \in E$ such that

$$\boxed{\delta(u) + w(u,v) < \delta(v)} \quad \text{—— ①}$$

then G has a negative cycle.

Proof:

Let P be a shortest path from s to u.



Case ①: when P doesn't contain v

$\Rightarrow P + (u,v)$ is a path

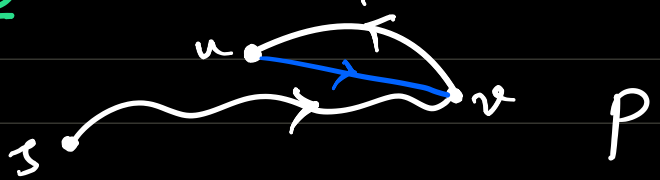
$$\begin{aligned} w(P + (u,v)) &= w(P) + w(u,v) \\ &= \delta(u) + w(u,v) \\ &< \delta(v) \end{aligned}$$

↪ by ①

This is impossible!

$\because$ A path from s to v with weight smaller than $\delta(v)$ cannot exist.

Case ②: when P contains v



P can be broken down into two paths:

$$P[s,u] = P[s,v] + P[v,u]$$

$$\boxed{\delta(u) = X + Y} \quad \text{—— ②}$$

$X = w(P[s,v])$

$Y = w(P[v,u])$

$\because$ $P[s, v]$ is a path from s to v,
$w(P[s, v]) \geq \delta(v)$

$\Rightarrow \quad X \geq \delta(v)$ ——— ③

$\because \quad \delta(u) + w(u, v) < \delta(v)$

$\Rightarrow X + Y + w(u, v) < \delta(v)$ (from ②)

$\Rightarrow \cancel{X} + Y + w(u, v) < \cancel{X}$ (from ③)

$\Rightarrow \quad Y + w(u, v) < 0$

$\Rightarrow$ weight of the cycle $P[v, u] + (u, v)$,
$Y + w(u, v) < 0$

$\Rightarrow$ –ve cycle !

$\therefore$ we have shown that if there is an edge (u, v), such that $\delta(u) + w(u, v) < \delta(v)$, then G has a negative cycle.

Contrapositive: $p \rightarrow q \equiv \neg q \rightarrow \neg p$ //

So, by contrapositioning the above theorem, we can also say that
If G has no negative cycle, then for every edge (u, v):

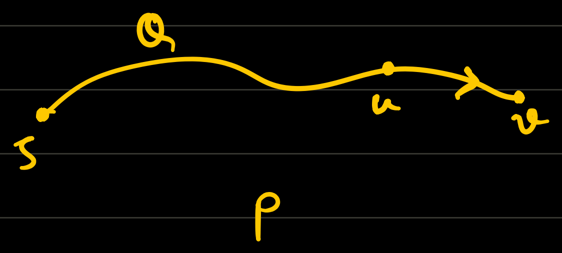
$$\boxed{\delta(u) + w(u, v) \geq \delta(v)}$$

(or)

$$\boxed{\delta(v) \leq \delta(u) + w(u, v)}$$

Theorem-2 :

Let P be a shortest path from s to v and let (u, v) be its last edge.



Let Q be the part of P from s to u.

$P = Q + (u, v)$

$w(P) = w(Q) + w(u, v)$

If we prove Q is a shortest path from s to u, then $w(Q) = \delta(u)$ and

$\delta(v) = w(P) = w(Q) + w(u, v)$

$\Rightarrow \delta(v) = \delta(u) + w(u, v)$

proof:

By Contradiction !

Assume Q is not a shortest path from s to u and there exists another path Q' which is a shortest path from s to u

So, $w(Q') < w(Q)$

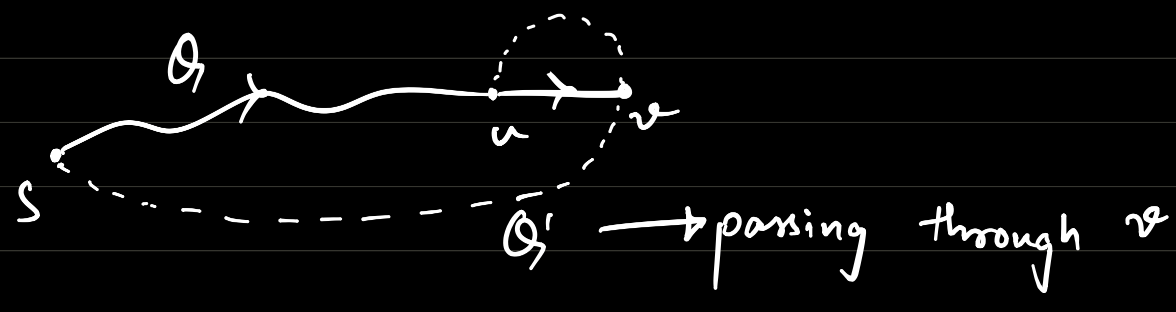
$Q' + (u, v)$ is a path from s to v

$$
\begin{aligned}
w(Q' + (u, v)) &= w(Q') + w(u, v) \\
&< w(Q) + w(u, v) \\
&= w(P) \\
&= \delta(v)
\end{aligned}
$$

$\Rightarrow$ $Q' + (u, v)$ is a shortest path from s to v with weight smaller than weight of P.

But P was the shortest path from s to v

This is a contradiction !



Thus Q' must pass through v.

$\therefore$ $Q' + (u, v)$ is a walk from s to v

$\because$ vertex v is occuring twice

$$w(Q') + w(u, v) < w(Q) + w(u, v) = w(P)$$

$$Q' + (u, v) = Q'[s, v] + \left(Q'[v, u] + (u, v) \right)$$

Q' is passing through v. Split Q' at v.

$$Q' = Q'[s, v] + Q'[v, u]$$

$$\Rightarrow Q' + (u, v) = Q'[s, v] + Q'[v, u] + (u, v)$$

$$\Rightarrow w(Q' + (u, v)) = w(Q'[s, v] + Q'[v, u] + (u, v))$$

$$\Rightarrow \underbrace{w(Q'(s, v))}_{\text{path with smaller weight}} \leq w(Q' + (u, v)) < w(P)$$

// This contradicts the minimality of P.

This implies such a path Q' from s to u with weight smaller than the weight of Q cannot exist.

$\Rightarrow$ Q is a shortest path from s to u

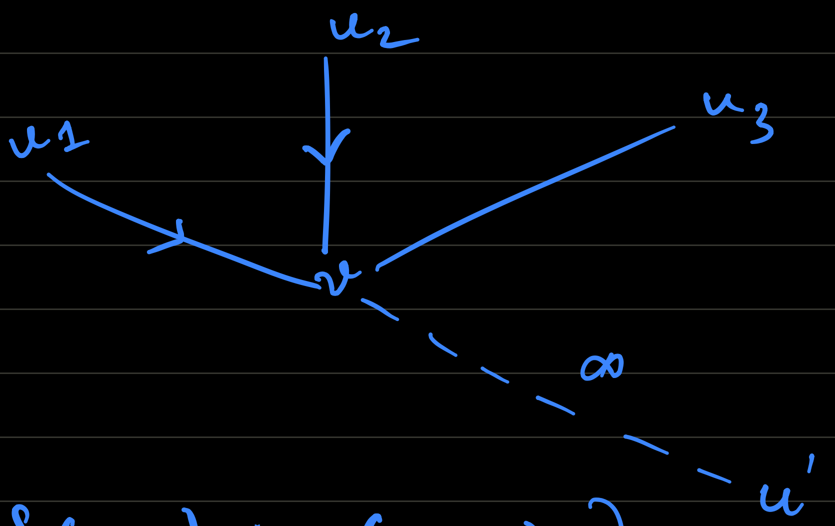
$\Rightarrow$ $w(Q) = \delta(u)$

$$\Rightarrow \quad \delta(v) = w(Q + (u,v)) = w(Q) + w(u,v)$$

$$\boxed{\begin{aligned} \delta(v) &= \min_{(u,v)\in E} \{\delta(u) + w(u,v)\} \\ &= \min_{\substack{u \in V \\ (u,v)\in E}} \{\delta(u) + w(u,v)\} \end{aligned}}$$

Extended weight function

$$\boxed{\begin{aligned} w(u,v) &= \infty \quad \text{if } (u,v) \notin E \text{ and } (u \neq v) \\ w(u,u) &= 0 \quad \forall\, u \in V \end{aligned}}$$



$\rightarrow \delta(u') + w(u',v) = \infty$

$$\left.\begin{aligned} &\delta(u_1) + w(u_1, v), \\ &\delta(u_2) + w(u_2, v), \\ &\delta(u_2) + w(u_3, v) \\ &\vdots \end{aligned}\right\} \text{find min.} = \delta(v)$$

we set $\delta(s) = 0$ (source) as all cycles are non-negative

In Summary,

Bellman Equations for Shortest Path weights.

$$\boxed{\begin{aligned} \delta(s) &= 0 \\ \delta(v) &= \min_{u \neq v} \{\delta(u) + w(u,v)\} \end{aligned}}$$

We have shown that

$x_v = \delta(v)$ is a Solution for the Bellman Equations.

<u>Muse</u>: what if Bellman Equations has multiple Solutions?

$\therefore$ Condition for Bellman Equation to have unique Solution:

1. No negative cycles
2. No zero cycles