Solving Bellman Equations

Bellman Equations

$G = (V, E, W, s)$

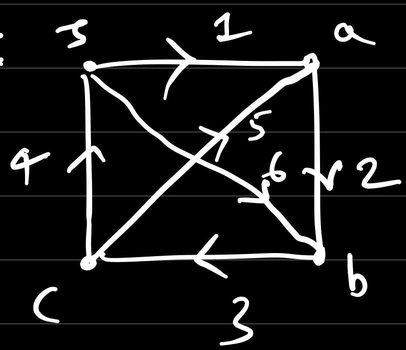
$\delta(v)$ = weight of the shortest path from s to v

G has no -ve cycles

x_v : variable associated with vertex $v \in V$

$$\boxed{\begin{aligned} x_s &= 0 \\ x_v &= \min_{u \neq v} \{ x_u + w(u,v) \} \end{aligned}}$$

Example:



$$x_v = \min_{u \neq v} \underbrace{\{ x_u + w(u,v) \}}_{\downarrow}$$

no. of terms here = Indegree (v)

Recursively Enumerated

$$x_v = \min \{ x_u + w(u,v) \}$$

$$\begin{aligned} x_a &= \min \{ x_s + w(s,a), \; x_c + w(c,a) \} \\ &= \min \{ W(s,a), \; x_c + w(c,a) \} \\ &= \min \{ 1, \; x_c + 5 \} \end{aligned}$$

$$x_c = \min \{ x_b + w(b,c) \} = 3 + x_b$$

$$\begin{aligned} x_b &= \min \{ x_s + w(s,b), \; x_a + w(a,b) \} \\ &= \min \{ 6, \; x_a + 2 \} \end{aligned}$$

$$x_s = 0$$
$$x_a = \min\{1,\ x_c + 5\}$$
$$x_b = \min\{6,\ x_a + 2\}$$
$$x_c = x_b + 3$$

$\rightarrow$ = "depends on"

$$x_a \rightleftarrows x_b$$

$$\left.\begin{aligned} \Rightarrow x_a &= \min\{1,\ x_b + 8\} \\ x_b &= \min\{6,\ x_a + 2\} \end{aligned}\right\} \quad \boxed{\text{Circular dependency}}$$

So lets Solve by cases,

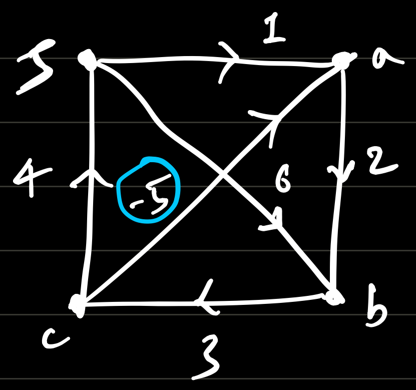
Decision Tree:

- x_a
  - 1
    - $\boxed{x_b = 3}$ ✓ final answer
  - $x_b + 8$
    - $x_b = \min\{6,\ x_b + 2\}$
      - 6
        - $x_a = \min\{1, 14\}$
        - $x_a = 1$
        - but $x_a = x_b + 8$
        - $= 6 + 8 = 14$
        - $\Rightarrow$ CONTRADICTION!
        - So, $x_b \neq 6$
      - $x_b = x_b + 2$
        - $x_a = x_b + 10$
        - but $x_a = x_b + 8$
        - CONTRADICTION!
        - So $x_b \neq x_b + 2$

$$\boxed{\begin{aligned} x_a &= 1 \\ x_b &= 3 \\ x_c &= 6 \\ x_s &= 0 \end{aligned}}$$

is Solution to the Bellman Equations of the given graph

Unique Solution!



Same graph with $w(c, a)$ changed to -5.

Now, the Bellman Equations will change:

$$x_s = 0$$
$$x_v = \min_{u \neq v} \{ x_u + w(u, v) \}$$

$$x_a = \min \{ 1, x_c - 5 \}$$
$$x_b = \min \{ 6, x_a + 2 \}$$
$$x_c = \min \{ x_b + 3 \} = x_b + 3$$

$$\Rightarrow \quad x_a = \min \{ 1, x_b + 3 - 5 \} = \min \{ 1, x_b - 2 \}$$
$$x_b = \min \{ 6, x_a + 2 \}$$

Circular Dependency

x_a

$\to 1 \to \boxed{x_b = 3}$ ✓

Consistent

$\Downarrow$

One Solution

$$\boxed{\begin{aligned} x_a &= 1 \\ x_b &= 3 \\ x_c &= 6 \end{aligned}}$$ ✓

$\to x_b - 2 \to x_b = \min \{ 6, x_b \}$

$\to 6 \to x_a = \min \{ 1, 4 \}$

$$\boxed{\begin{aligned} x_a &= 1 \\ x_b &= 6 \\ x_c &= 9 \end{aligned}}$$ ✓

$\to 0 \to x_a = \min \{ 1, 2 \}$

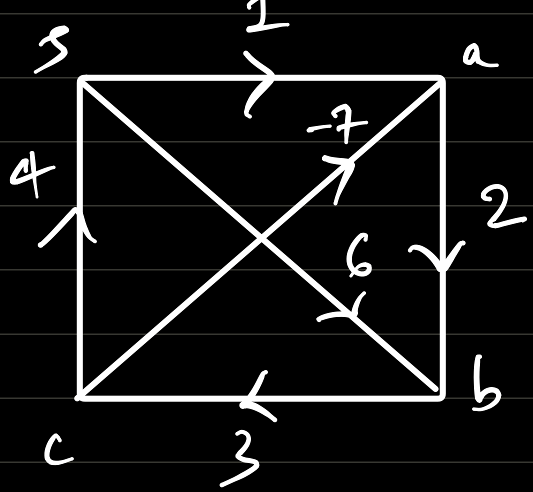
$$\boxed{\begin{aligned} x_a &= 1 \\ x_b &= 0 \\ x_c &= 3 \end{aligned}}$$ ✓

3 Solutions

$\therefore$ any value $x_a \le 1$ will have a solution

$\Rightarrow$ Infinitely many solutions are possible if there is a zero cycle

The example here taken has zero cycle: $\underline{abca}$ //



Now, the graph has a -ve cycle: abca with weight -2

Now, the Bellman Equations will be

$$x_s = 0$$
$$x_v = \min_{u \to v} \{ x_u + w(u, v) \}$$

$$x_a = \min \{ 1, x_c - 7 \}$$
$$x_b = \min \{ 6, x_a + 2 \}$$
$$x_c = \min \{ x_b + 3 \} = x_b + 3$$

$\Rightarrow$ $\left.\begin{aligned} x_a &= \min \{ 1, x_b - 4 \} \\ x_b &= \min \{ 6, x_a + 2 \} \end{aligned}\right\}$ Circular Dependency

x_a

1 → $x_b = 3$ → $x_a = -1$

CONTRADICTION!

So $x_a \ne 1$

$x_b - 4$ → $x_b = \min \{ 6, x_b - 2 \}$

6 → $x_a = 2$, $x_b = 4$ ✗

$x_b - 2$ → $x_a = x_b - 6$ but $x_a = x_b - 4$ ✗

$\therefore$ No Solution in the case of a negative cycle.

Summary:

1. See if the given graph has -ve cycle
2. If it has a -ve cycle, then there is no Solution for Bellman Equations

$\Downarrow$

Single source Shortest path

3. If the Graph has a zero cycle, then there are infinitely many solutions for Bellman equations.
4. If there is no negative cycle in the graph, then there definitely is a Solution to the Bellman Equations.

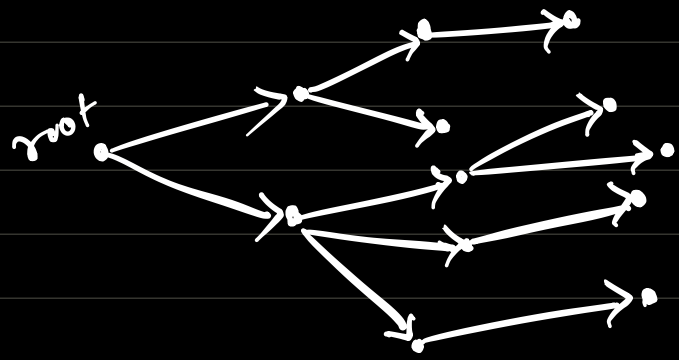
$G = (V, E, W, S)$

- no negative or zero cycle $\rightarrow$ Unique Solution
- zero cycles only $\rightarrow$ Infinitely many Solutions
- -ve cycles only $\rightarrow$ No Solution

Out-Tree

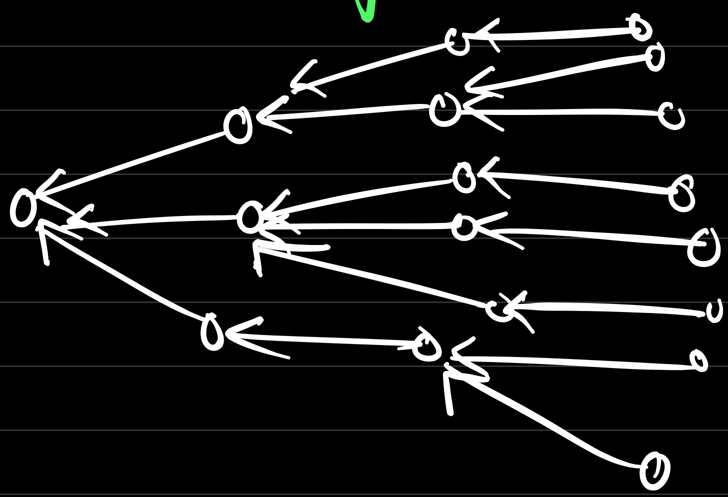
DAG from a root node to other nodes

i.e, all the edges are directed away from the root node.



In-Tree

DAG in which all the edges are directed toward a single node



Bellman Edge

$$x_v = x_u + w(u, v)$$

$\Rightarrow (u, v)$ is a Bellman edge Corresponding to v.

where x_u, x_v are Solutions to the Bellman equations.

$$\boxed{\begin{aligned} x_s &= 0 \\ x_v &= \min_{u \neq v} \{ x_u + w(u, v) \} \end{aligned}}$$

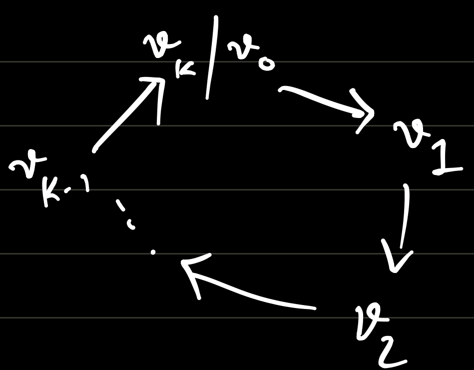
if $|V| = n$, then there are $(n-1)$ Bellman edges

$B(G) = (V, E')$

E' → set of all Bellman edges

$B(G)$ → Out-tree

$B_G = B(G)$ has no cycles.



$v_k = v_0$

$$\delta(v_1) = \delta(v_0) + w(v_0, v_1)$$
$$\delta(v_2) = \delta(v_1) + w(v_1, v_2)$$
$$\vdots$$
$$\delta(v_k) = \delta(v_{k-1}) + w(v_{k-1}, v_k)$$

$$\sum_0^{k} \cancel{\delta(v_i)} = \sum_0^{k-1} \cancel{\delta(v_i)} + w(C)$$

$w(C) = 0$ but we assumed no zero cycle or no -ve cycle.

This is a contradiction.

$\therefore$ B_G has no cycles

* you'll never encounter a vertex that was already visited. If so, it forms a cycle.

* In B_G, every set of edges from root to leaves are shortest paths from root to leaf

$\therefore$ (v_{i-1}, v_i) : Bellman edge

$$\Downarrow$$

$$\delta(v_i) = \delta(v_{i-1}) + w(v_{i-1}, v_i)$$

$$w(v_{i-1}, v_i) = \delta(v_i) - \delta(v_{i-1})$$

$$\Downarrow$$

"telescoping sum" $\Rightarrow$ cancels terms

$$\sum_{i=1}^{n} w(v_{i-1}, v_i) = \sum_{i=1}^{n} \left(\delta(v_i) - \delta(v_{i-1})\right)$$

$$= \cancel{\delta(v_1)} - \delta(v_0) + \cancel{\delta(v_2)} - \cancel{\delta(v_1)} + \delta(v_3) - \cancel{\delta(v_2)} + \cdots + \delta(v_n) - \cancel{\delta(v_{n-1})}$$

$$\boxed{\underbrace{\sum_{i=1}^{K} w(v_{i-1}, v_i)}_{\text{weight of the path } v_1 \to v_2 \to \cdots \to v_K} = \delta(v_K) - \delta(v_0)}$$

$v_0 \rightarrow$ Source : S

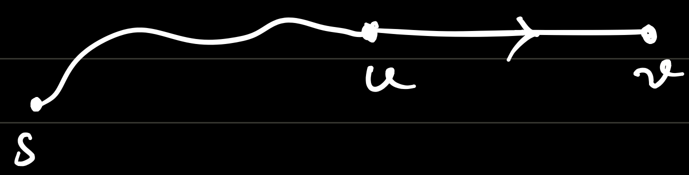
$\delta(s) = 0$

$\Rightarrow$ if $v_k = v$, then

$$w(P_v) = \delta(v)$$

↳ Shortest path from s to v

$$\delta(v) = \delta(u) + w(u, v)$$



$$\boxed{\begin{array}{l} u = p(v) \quad \text{↳ parent/previous} \\ p(s) = \text{undefined} \end{array}}$$

order of generation

$\therefore$ Bellman path $= (s, \dots, p(p(v)), p(v), v)$ Sequence

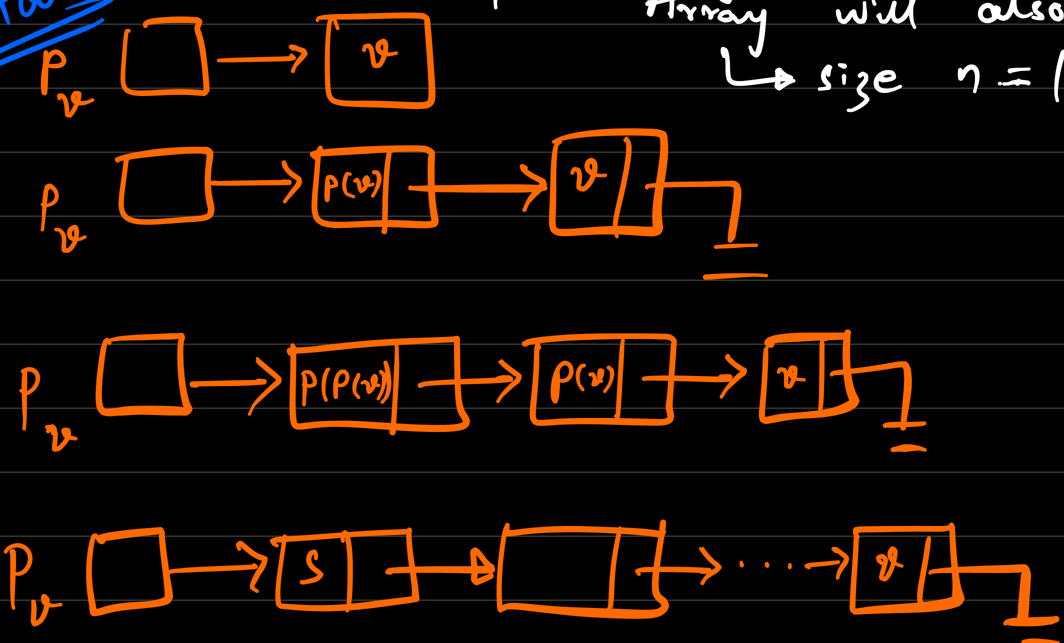
↓

Can be a linked list with $p(x)$ being added in the front.

Array will also work

↳ size $n = |V|$

Linked List Implementation



Bellman Tree is the Shortest path tree (implicitly). We are not storing the weights explicitly to show that it is the shortest path. We're storing the parent/previous nodes based on the weights.

NOTE: Bellman Tree evaluation problem assumes no zero cycle and/no -ve cycle.

Bounded Walk

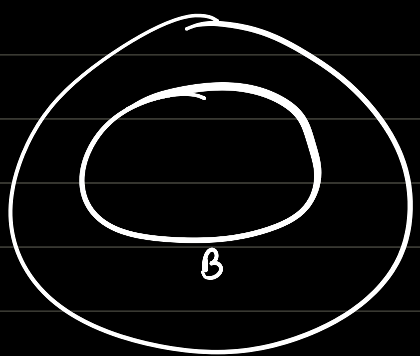
$W_K(v)$: Set of all walks from s to v containing atmost K edges $(K \geq 0)$

$\therefore$ $W_K(v)$ is finite set.

$\alpha_K(v)$: weight of the shortest walk in $W_K(v)$

$$W_K(v) \subseteq W_{K+1}(v)$$

$\because$ A walk with atmost K edges is automatically a walk with atmost $K+1$ edges.



A

$B \subseteq A \implies \min(B) \geq \min(A)$

$\because$ every element of B is in A and $\min(A)$ is smaller than every other element of A. So, $\min(A) \leq \min(B)$

$$\implies \alpha_K(v) \geq \alpha_{K+1}(v)$$

not a finite sequence $\left\{ \alpha_0(v) \geq \alpha_1(v) \geq \dots \geq \alpha_K(v) \geq \alpha_{K+1}(v) \geq \dots \right.$

for a vertex v.

$P_K(v) = \{ P \mid P \text{ is a path from } s \text{ to } v \text{ with atmost } K \text{ edges} \}$

$$P_K(v) \subseteq P_{K+1}(v)$$

$$\Rightarrow \underbrace{\delta_0(v) \ge \delta_1(v) \ge \dots \ge \delta_{n-1}(v)}_{\text{finite sequence}} = \delta_n(v) = \delta_{n+1}(v) = \dots$$

$n = |V|$

$\delta_{n-1}(v) = \delta_n(v) = \dots$ is due to the fact that any path in G can have atmost $n-1$ edges.

$\delta_K \longrightarrow$ finite always ✓

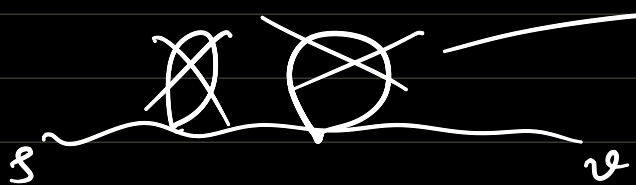
$\alpha_K \longrightarrow$ not finite always ✗

$$P_K(v) \subseteq W_K(v)$$

$$\Rightarrow \alpha_K(v) \le \delta_K(v)$$

* G has no $-$ve cycle $\Rightarrow \underbrace{\alpha_K(v)}_{\substack{\text{weight of} \\ \text{Shortest walk} \\ \text{with atmost } K \text{ edges}}} = \underbrace{\delta_K(v)}_{\substack{\text{weight of} \\ \text{Shortest path} \\ \text{with atmost} \\ K \text{ edges}}}$

i.e, every Shortest walk is indeed a Shortest path



$\longrightarrow$ after removing the cycles also, one can have a walk from s to v

Shortest walk cannot contain any cycles.

Cycle = closed path.

$\alpha_K(s) \longrightarrow$ closed walk

$\delta_K(s) \longrightarrow$ cycle

What is $\delta_0(s)$, $\alpha_0(s)$, $\delta_0(v)$, $\alpha_0(v)$?

$$\delta_0(s) \downarrow 0, \quad \alpha_0(s) \downarrow 0, \quad \delta_0(v) \downarrow \infty, \quad \alpha_0(v) \downarrow \infty$$

$\because$ no path or walk of length 0 exists from s to v

If G has no -ve cycles,

$$\begin{array}{ccccccccc} \alpha_0(v) & \geq & \alpha_1(v) & \geq & \alpha_2(v) & \geq & \cdots & \geq & \alpha_{n-1}(v) \\ \| & & \| & & \| & & & & \| \\ \delta_0(v) & & \delta_1(v) & & \delta_2(v) & & & & \delta_{n-1}(v) \end{array}$$

$$\delta_{n-1}(v) = \underbrace{\alpha_{n-1}}_{\text{atmost } n-1 \text{ edges b/w } s \text{ and } v}(v) = \delta(v)$$

Theorem: $\alpha_{i+1}(v) \leq \alpha_i(v) + w(u,v) \quad \forall (u,v) \in E$

So, $\exists (u,v) \in E$ such that

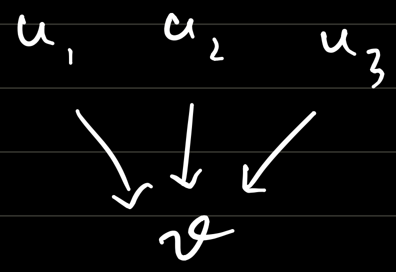
$$\alpha_{i+1}(v) = \alpha_i(u) + w(u,v)$$

$$\alpha_{i+1}(v) = \min_{(u,v) \in E} \{ \delta_i(u) + w(u,v) \}$$

In Bellman equns. for Shortest paths, there was non-linearity and circular dependency.
But here, there is no such things because

$\alpha_{i+1}(v)$ is dependent on a completely different different set of values: $\alpha_i(v)$

$$\underbrace{\alpha_i[s],\ \alpha_i[\,],\ \ldots\ldots}$$
$$\downarrow$$
$$\alpha_{i+1}[s],\ \alpha_i[\,],\ \ldots\ldots$$



$\alpha_{i+1}[v]$ needs $\alpha_i[u_1]$, $\alpha_i[u_2]$ and $\alpha_i[u_3]$, which are computed in the previous stage (i^{th} stage)

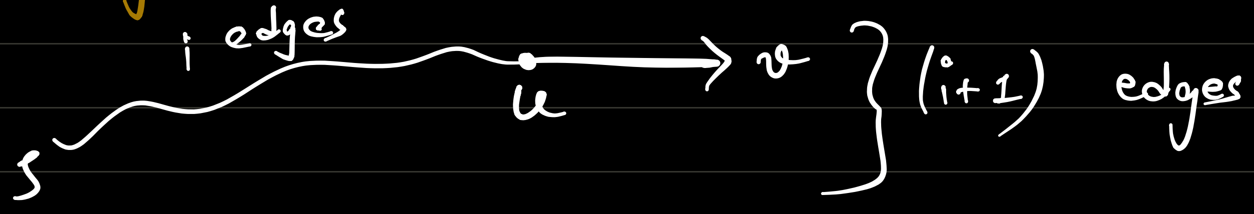
array $\alpha_{i+1}[\,]$ is computed using $\alpha_i[\,]$ and this is called <u>Improvement</u>.

So, $\alpha_{i+1}[v]$ doesn't depend on any value in the current stage $(i+1)$

* If the shortest walk in $W_{i+1}(v)$ has atmost i edges, then it will be a shortest walk even in $W_i(v)$

Hence, $\alpha_{i+1}[v] = \alpha_i[v]$

Assume that shortest walk has $i+1$ edges exactly.



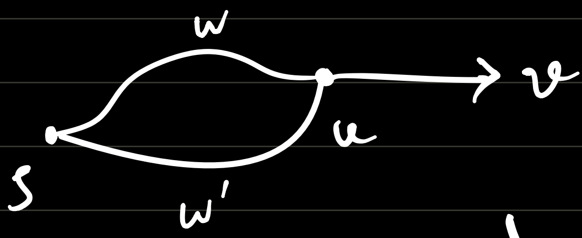
$W[s,v] = W[s,u] + (u,v)$

$\downarrow$

This is a shortest walk.

"Cut and Paste" Argument.

Proof:



$w(W'[s,u]) < w(W[s,u])$

$|W'| \leq i$ edges

$\Rightarrow w(W'[s,u]) + w(u,v) < w(W[s,u]) + w(u,v)$

$= w[W[s,v]]$

$= \alpha_{i+1}[v]$

$\Rightarrow \underbrace{w(W'[s,u] + (u,v))}_{\leq (i+1) \text{ edges}} < \alpha_{i+1}[v]$

But we started with the assumption that $W[s,v]$ is the shortest walk from s to v.

This contradicts the minimality of W. (Shorter than the shortest is not possible)

$\therefore$ Such a W' cannot exist.

$\Rightarrow W[s,u]$ is a shortest walk with $\leq i$ edges

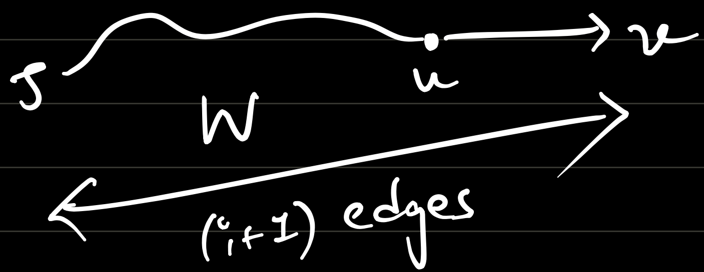
$\Rightarrow w(W[s,u]) = \alpha_{i+1}(v)$

$= w(W[s,u] + (u,v))$

$= \alpha_i(u) + w(u,v)$

$\therefore \alpha_{i+1}(v) = \alpha_i(u) + w(u,v)$ for some (u,v)

$$\alpha_{i+1}(v) \leq \alpha_i(u) + w(u,v)$$



$$\alpha_{i+1}(v) \leq \alpha_i(u) + w(u,v)$$

We have proved that

$$\alpha_{i+1}(v) = \min_{(u,v)\in E} \{ \alpha_i(u) + w(u,v) \}$$

$$\alpha_0(s) = 0$$
$$\alpha_0(v) = \infty \quad \forall\ v \neq s$$

$\alpha_0[\ ]$: array of values
$\downarrow$
$\alpha_1[\ ]$: array of values
$\downarrow$
$\alpha_2[\ ]$: array of values
$\downarrow$
$\vdots$
$\alpha_{n-1}[\ ]$: array of values

applying $(n-1)$ improvement steps.

NOTE: The above Procedure is not applicable to any arbitrary graph. The input graph G must have no -ve cycle. Zero-cycle won't make any difference, though.