Properties of shortest path Distances.

Let $G = (V, E, w, s)$.

$\delta(v)$ = weight of shortest path from s to v.

u $\qquad\qquad$ v

$\delta(u)$ $\qquad\qquad$ $\delta(v)$.

if $\delta(u) + w(u,v) < \delta(v)$

Then G has a negative cycle.

Theorem. If G has a negative cycle, then G has an edge (u,v) $\ni$

$$\delta(u) + w(u,v) < \delta(v).$$

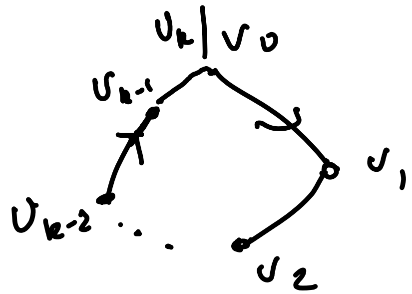
Proof: (by Contradiction).

Suppose $\forall (u,v)$ we have

$$\delta(u) + w(u,v) \geq \delta(v).$$

Let C be a negative cycle.

$$\langle v_0, v_1, v_2 \dots, v_k \rangle. \quad v_k = v_0.$$



$\xrightarrow{\text{negative cycle } C}$

$v_k = v_0$.

$$\delta(v_0) + w(v_0, v_1) \geq \delta(v_1)$$
$$\delta(v_1) + w(v_1, v_2) \geq \delta(v_2)$$
$$\vdots$$
$$\delta(v_{k-1}) + w(v_{k-1}, v_k) \geq \delta(v_k)$$
$$\underline{\sum_{i=0}^{k-1} \delta(v_i)} + w(C) \geq \underline{\sum_{i=1}^{k} \delta(v_i)}.$$

$$\delta(v_0) + \delta(v_1) + \cdots + \delta(v_{k-1})$$
$$= \delta(v_k) + \delta(v_1) + \cdots + \delta(v_{k-1})$$
$$= \delta(v_1) + \delta(v_2) + \cdots + \delta(v_{k-1}) + \delta(v_k)$$
$$\Rightarrow \sum_{i=0}^{k-1} \delta(v_i) = \sum_{i=1}^{k} \delta(v_i)$$

$$\Rightarrow \quad w(C) \geq 0$$

this contradicts the fact that C is a negative cycle.

$\Rightarrow \exists$ an edge $(u, v) \ni \delta(u) + w(u, v) < \delta(v)$.

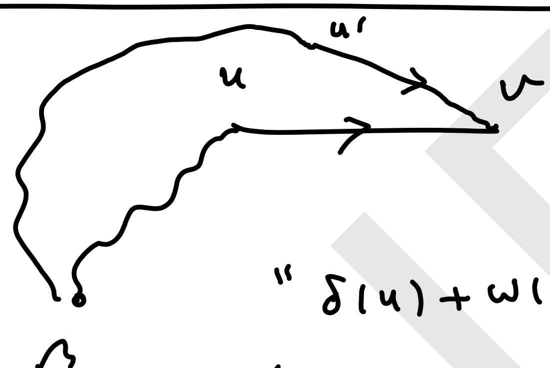
Theorem 1 & 2 imply:

G has a negative cycle

<u>iff</u>

$\exists\ (u,v) \ni$

$\delta(u) + w(u,v) < \delta(v)$.



" $\delta(u) + w(u,v) = \delta(v)$ "

" $\delta(u') + w(u,v) \geq \delta(v)$ "

Intuitively, we want, something like

$\delta(u) + w(u,v) \geq \delta(v)$

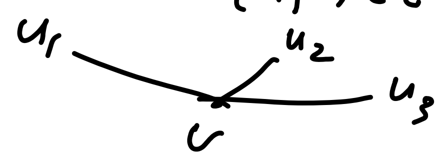
$\forall\ (u,v) \in E$.

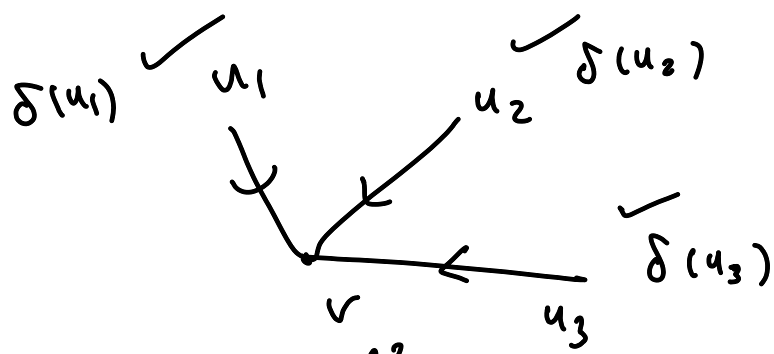
<u>AND</u>

$\exists\ (u,v) \ni$

$\delta(u) + w(u,v) = \delta(v)$.

$$\delta(v) = \min_{(u,v)\in E} \{ \underline{\delta(u) + w(u,v)} \}$$





$\delta(v)$??

We assume that G has no negative cycles.

If G has no negative cycles,

(1) Contrapositive of Th. 1 is True:

$\forall (u,v) \in E.$

$\delta(u) + w(u,v) \geq \delta(v).$

If G has no negative cycle,

$\exists$ an edge $(u,v) \ni$

$\delta(v) = \delta(u) + w(u,v).$

Theorem 3.