Properties of shortest path Distances.

Let $G = (V, E, w, s)$.

$\delta(v)$ = weight of shortest path from s to v.

if $\delta(u) + w(u,v) < \delta(v)$

Then G has a negative cycle.

Theorem. If G has a negative cycle, then G has an edge (u,v) $\ni$

$$\delta(u) + w(u,v) < \delta(v).$$

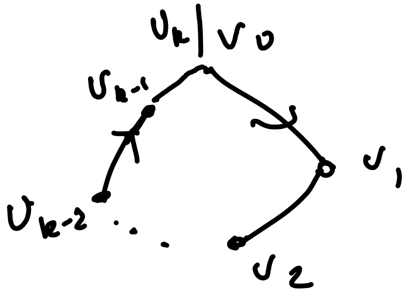
Proof: (by contradiction).

Suppose $\forall (u,v)$ we have

$$\delta(u) + w(u,v) \geq \delta(v).$$

Let C be a negative cycle.

$$\langle v_0, v_1, v_2 \cdots, v_k \rangle. \quad v_k = v_0 .$$



negative cycle C

$$v_k = v_0 .$$

$$\delta(v_0) + w(v_0, v_1) \geq \delta(v_1)$$

$$\delta(v_1) + w(v_1, v_2) \geq \delta(v_2)$$

$$\vdots$$

$$\delta(v_{k-1}) + w(v_{k-1}, v_k) \geq \delta(v_k)$$

$$\underline{\sum_{i=0}^{k-1}} \delta(v_i) + w(C) \geq \underline{\sum_{i=1}^{k}} \delta(v_i) .$$

$$\delta(v_0) + \delta(v_1) + \cdots + \delta(v_{k-1})$$

$$= \delta(v_k) + \delta(v_1) + \cdots + \delta(v_{k-1})$$

$$= \delta(v_1) + \delta(v_2) + \cdots + \delta(v_{k-1}) + \delta(v_k)$$

$$\Rightarrow \sum_{i=0}^{k-1} \delta(v_i) = \sum_{i=1}^{k} \delta(v_i)$$

$$\Rightarrow \quad w(C) \geq 0$$

This contradicts the fact that C is a negative cycle.

$\Rightarrow \exists$ an edge $(u, v) \ni$
$\delta(u) + w(u, v) < \delta(v)$.

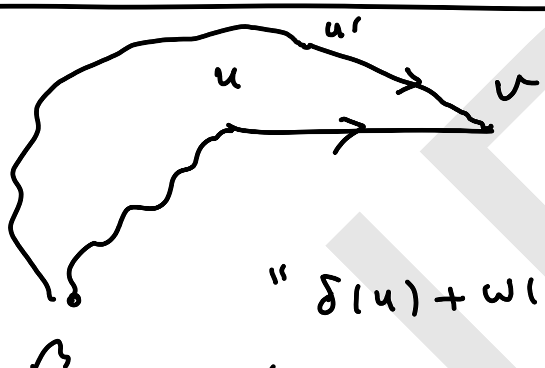
Theorem 1 & 2 imply:

G has a negative cycle

<u>iff</u>

$\exists\ (u,v) \ni$

$\delta(u) + w(u,v) < \delta(v)$.



" $\delta(u) + w(u,v) = \delta(v)$ "

" $\delta(u') + w(u,v) \geq \delta(v)$ "

Intuitively, we want, something like

$\delta(u) + w(u,v) \geq \delta(v)$

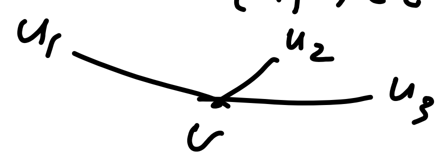
$\forall\ (u,v) \in E$.

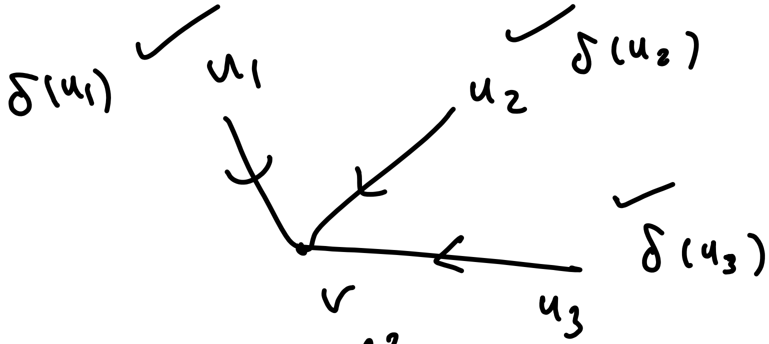
<u>AND</u>

$\exists\ (u,v) \ni$

$\delta(u) + w(u,v) = \delta(v)$.

$$\delta(v) = \min_{(u,v) \in E} \left\{ \underline{\delta(u) + w(u,v)} \right\}$$





$\delta(v)$??

We assume that G has no negative cycles.

If G has no negative cycles,

(1) Contrapositive of Th. 1 is True:

$\forall (u,v) \in E$.

$\delta(u) + w(u,v) \geq \delta(v)$. ||

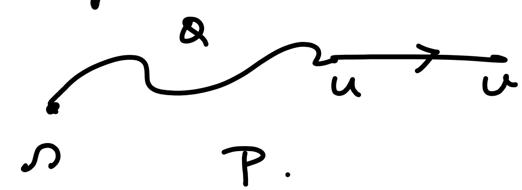
If G has no negative cycle,

$\exists$ an edge $(u,v) \ni$

$\delta(v) = \delta(u) + w(u,v)$.

Theorem 3.

Let P be a shortest path from s to v and let (u,v) be its last edge.



Let Q be the part of P from s to u.

$P = Q + (u,v)$

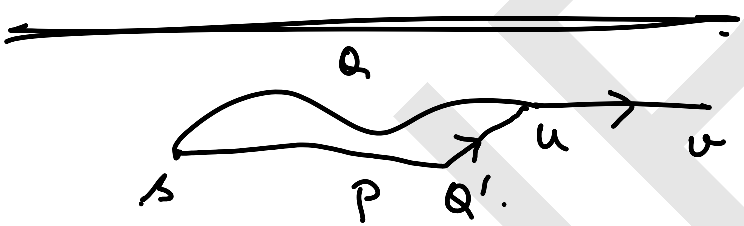
$w(P) = w(Q) + w(u,v)$.

$$w(P) = w(Q) + w(u,v)$$

if we prove Q is a shortest path from s to u.

Then, $w(Q) = \delta(u)$ &

$$\delta(v) = w(P)$$
$$= w(Q) + w(u,v)$$
$$= \delta(u) + w(u,v).$$



If Q is not a shortest path, Let Q' be a shortest path from s to u.

$$w(Q') < w(Q)$$

Q' does not contain v.

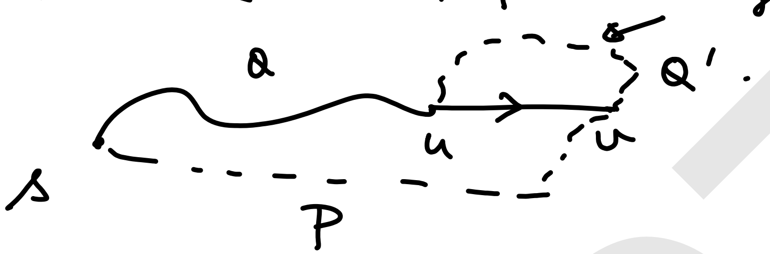
Then $Q' + (u,v)$ is a path from s to v.

$$w(Q' + (u,v)) = w(Q') + w(u,v)$$
$$< w(Q) + w(u,v)$$
$$= w(P)$$
$$= \delta(v).$$

That is $Q' + (u,v)$ is a path from s to v with weight smaller than weight of P.

This is a contradiction.

Thus Q' must pass through v.



$Q' + (u,v)$ is a WALK from s to v.

$$w(Q') + w(u,v) < w(Q) + w(u,v) = w(P).$$

$$Q' + (u,v) = Q'[s,v] + \left(Q'[v,u] + (u,v)\right)$$

Q' is passing through v.

Split Q' at v.

$$Q' = Q'[s,v] + Q'[v,u]$$

$$Q' + (u,v) = Q'[s,v] + \underbrace{Q'[v,u] + (u,v)}_{\text{cycle}}$$

$$w\left(Q' + (u,v)\right) \geq \underline{Q'(s,v)}.$$

cycle → non negative wt.

$$w\left(\underline{Q'(s,v)}\right) \leq w\left(Q' + (u,v)\right) < w(P)$$

$Q'(s,v)$ → path with smaller weight.

This contradicts the minimality of P.

This implies such a path Q'
from s to u with weight
smaller than Q can not exist.

$\Rightarrow$ Q is a shortest path from
s to u.

$\Rightarrow \quad w(Q) = \delta(u).$

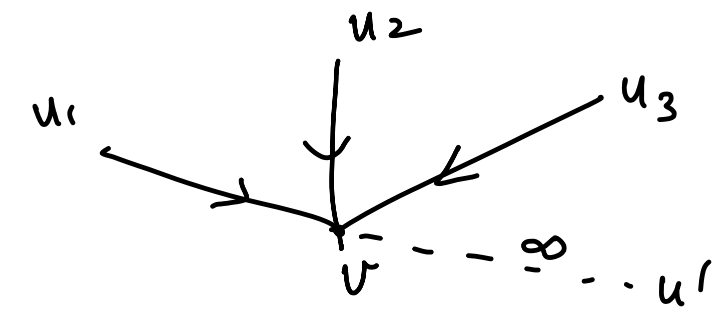
$$
\begin{aligned}
\Rightarrow \quad \delta(v) &= w(P) \\
&= w(Q + (u,v)) \\
&= w(Q) + w(u,v) \\
&= \delta(u) + w(u,v)
\end{aligned}
$$

$$
\begin{aligned}
\delta(v) &= \min_{(u,v)\in E} \{\delta(u) + w(u,v)\}. \\
&= \min_{\substack{u \in V \\ (u,v)\in E}} \{\delta(u) + \underline{w(u,v)}\}
\end{aligned}
$$

$$
\left\{
\begin{array}{l}
w(u,v) = \infty \text{ if } (u,v) \notin E, \\
\qquad\qquad\qquad (u \neq v). \\
w(u,u) = 0 \ \forall u.
\end{array}
\right.
$$

extention of wt function,

$$\underline{\underline{\delta(v)}} = \min_{\substack{u \\ u\neq v}} \{\delta(u) + w(u,v)\}.$$



$\delta(u_1) + w(u, v)$,

$\delta(u_2) + w(u_2, v)$,

$\delta(u_3) + w(u_3, v)$

$\vdots$

$\delta(u') + w(u', v) = \infty$

find min

$\delta(v)$

we set $\delta(s) = 0$ as all cycles are non negative.

In sum,

$$\delta(s) = 0\,;$$

$$\delta(v) = \min_{u \neq v} \{\delta(u) + w(u, v)\}$$

Bellman Equations for shortest path weights.

x_v — variable associated with the vertex v

AND

form the equation

$$x_s = 0$$

$$x_v = \min_{u \neq v} \{ x_u + w(u,v) \}$$

Bellman Equations.

We have shown that

$x_v = \delta(v)$ is a solution for the Bellman equations.

If G has No negative cycles or No zero cycles, then Bellman Equations have a UNIQUE Solution.