Strongly Connected Components of a Directed Graph.

Note Title

Strongly connected graph.

Let $G = (V, E)$ be a directed graph.

Two vertices $u, v \in V$ are said to be mutually reachable if there is a path from u to v and there is a path from v to u in G.

G is a strongly connected graph iff every pair of vertices is mutually reachable.

perform $n(n-1)$ DFS.

& confirm that from every vertex there is a path to all other vertices.

$$O(n^2(n+m))$$

$$m = O(n^2) \longrightarrow \underline{\underline{O(n^4)}}$$

Let $G = (V, E)$ be a directed graph. Let s be a vertex in V.

If there is a path from s to v and a path from v to s $\forall$ $v \in V - \{s\}$, then G is strongly connected.

$$\left\{ \begin{array}{ll} (s, v) & — \ (n-1) \\ (v, s) & — \ (n-1) \end{array} \right.$$

$(2n-2)$ DFS calls.

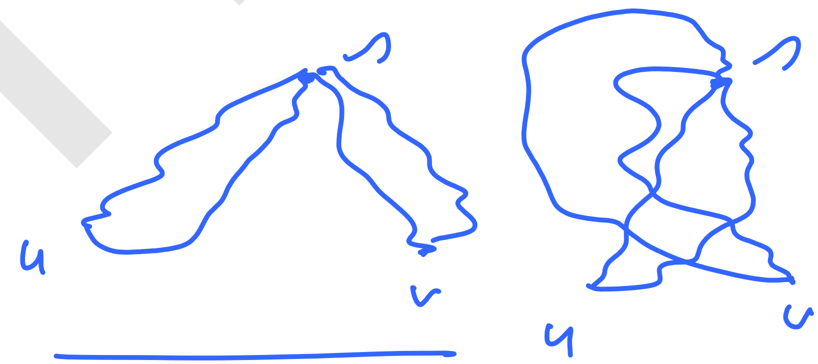
$n(n-1)$ DFS calls.

Proof: ~~$(u, v) \in$~~

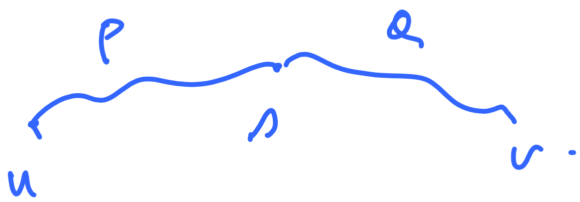
$u, v \in V$.

? Path from u to v ?

Path from v to u ?

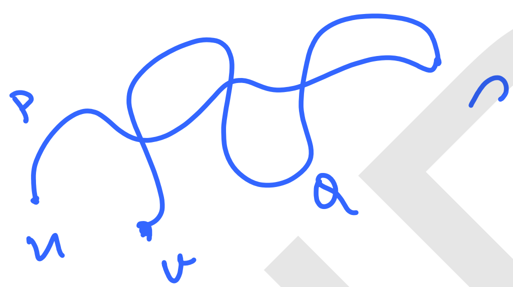


If there is a walk from u to v then there is a path from u to v.



$P(u,s) + Q(s,v)$ is a path.

P & Q are disjoint.



P + Q defines a walk from u to v. But $\Rightarrow$ a path from u to v.

That is,

if there is a path from u to s and a path from s to v, then there is a path from u to v in G. (Similar proof for path from v to u)

$\Rightarrow$ u & v are mutually reachable.

Fix a vertex s.

Perform one DFS starting from s.

If all vertices are black

$\Rightarrow$ all vertices are reachable from s. ✓

Consider now G^R

reverse of G.

each edge is reversed:

Perform 1 DFS starting from s. in G^R ✓

if all vertices are reachable

$\Rightarrow$ there is a path from s to v in G^R $\forall v$.

$\Rightarrow$ there is a path from v to s in G $\forall v$.

Let $s \in V$.

1. Perform DFS from s in G.
2. if there is a white vertex
   return " G is NOT strongly connected.
3. construct G^R
4. Perform DFS from s in G^R
5. if there is a white vertex
   return " G is NOT strongly connected

else return " G is strongly connected ".

Complexity is

$O(n+m)$

2 DFSs

The next natural Question is if G is NOT strongly connected,

How to find the strongly Connected components of G?