Note Title

Single Source shortest Paths Problem.

Walks, Paths,
shortest Walks, shortest paths.

$G = (V, E, w, s)$ $\quad$ $s \in V$.

$\delta(s, v)$ $\quad$ $v \in V - \{s\}$.

Let P_v denote a shortest path from s to v.

Since s is fixed,
we denote $\delta(s, v)$ by $\delta(v)$.

Given, $G = (V, E, w, s)$

output, $\delta(v)$, P_v $\forall\, v \in V - \{s\}$.

Single source shortest path Problem.

(SSSP) problem

s - source.

How to construct equations whose solutions are $\delta(v)$ values..? ||

Mathematical foundations of shortest-path weights.

Theorem 1 : Let $G = (V, E, w, s)$

Let $\delta(v)$ be the weight of shortest path from s to v.

If there exists an edge $(u, v) \in E$ such that

$$\delta(u) + w(u, v) < \delta(v) \quad \text{---- } (1)$$

Then G has a negative cycle.

Proof : Let P be a shortest path from s to u.

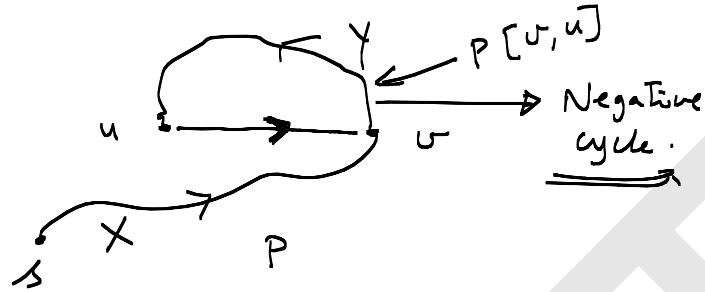
Case 1 . P does not contain v.

In this case $P + (u, v)$ is a path and

$$\begin{aligned} w(P + (u,v)) &= w(P) + w(u,v) \\ &= \delta(u) + w(u,v) \\ &< \delta(v) \quad \text{by } (1) \end{aligned}$$

This is impossible.
A path from s to v with weight smaller than $\delta(v)$ ~~is NOT possible~~ can not exist.

Hence P must contain v as shown below.



$$P[s,u] = P[s,v] + P[v,u]$$

$\delta(u)$ $\quad X \quad Y$.

$$w(P[s,v]) = X$$

$$w(P[v,u]) = Y$$

$$\delta(u) = X + Y \quad \text{- - - -} \quad (2)$$

Since $P[s,v]$ is a path from s to v,

$$w(P[s,v]) \geq \delta(v)$$

$$\Rightarrow \quad X \geq \delta(v) \quad \text{- - - -} \quad (3)$$

From (1)

$$\delta(u) + w(u,v) < \delta(v)$$

$$\Rightarrow \quad X + Y + w(u,v) < \delta(v) \quad \text{by (2)}$$

$$\Rightarrow \quad X + Y + w(u,v) < X \quad \text{by (3)}$$

$$\Rightarrow \quad Y + w(u,v) < 0.$$

$\Rightarrow$ weight of the cycle $P[v,u] + (u,v)$

$Y + w(u,v) < 0$.

Thus, we have shown that,
if there is an edge (u, v)
$\ni \delta(u) + w(u, v) < \delta(v)$
then G has a negative cycle.

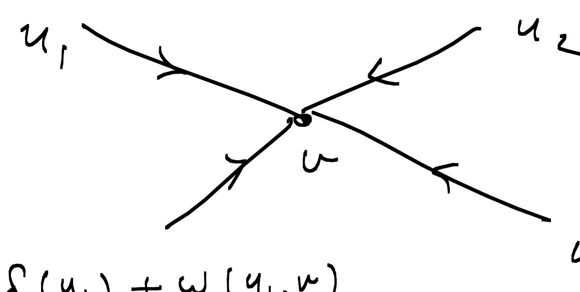
($P[v, u] + (u, v)$ is the negative cycle)

$$\boxed{\dfrac{p \Rightarrow q}{\neg q \Rightarrow \neg p}}$$

Contra positive

If G has no negative cycle.
then for every edge (u, v)
$\delta(u) + w(u, v) \geq \delta(v)$

$\delta(v) \leq \delta(u) + w(u, v)$



$\leq \delta(u_1) + w(u_1, v)$
$\leq \delta(u_2) + w(u_2, v)$
$\vdots$