Note Title

Bellman Equations

$G = (V, E, w, s)$

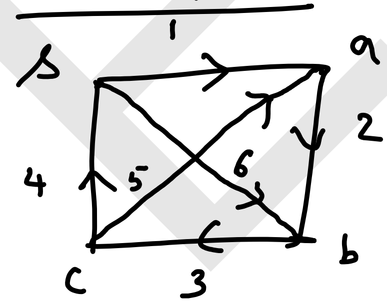
$\delta(v)$ = weight of shortest path from s to v.

G has no negative cycles.

x_v – variable associated with vertex $v \in V$.

$$\begin{cases} x_s = 0 \\ x_v = \min\limits_{u \neq v} \{ x_u + w(u,v) \} \end{cases}$$

Example:



$x_b = \min\{x_a + 2, x_s + 6\}$

$= \min\{6, x_a + 2\}$

$x_c = x_b + 3$;

$$x_v = \min_{(u,v)} \{ x_u + w(u,v) \}.$$

No of terms here is same as in-degree of v.

$x_s = 0$; $x_a = \min\{ x_s + w(s,a), x_c + w(c,a) \}$

$= \min\{1, x_c + 5\}$.

$$x_s = 0;$$

$$x_a = \text{Min}\{1, x_c + 5\}$$

$$x_b = \text{Min}\{6, x_a + 2\}$$

$$x_c = x_b + 3.$$

$$x_a = \text{Min}\{1, x_b + 3 + 5\}$$

$$= \text{Min}\{1, x_b + 8\}$$

$$x_b = \text{Min}\{6, x_a + 2\}.$$

$\rightarrow$ if $x_a = x_b + 8$,

$$x_b = \text{Min}\{6, x_b + 8 + 2\}$$

$$= \text{Min}\{6, x_b + 10\}.$$

$$\Rightarrow x_b = 6.$$

Since $x_a = x_b + 8$

$$\Rightarrow x_a = 6 + 8 = 14.$$

Since $x_a = \text{Min}\{1, x_b + 8\}$

$$x_a \leq 1$$

This contradicts $x_a = 14$ X.

$\Rightarrow x_a$ must be $= 1$.

$$\Rightarrow x_b = \text{Min}\{6, 1 + 2\}$$

$$= 3$$

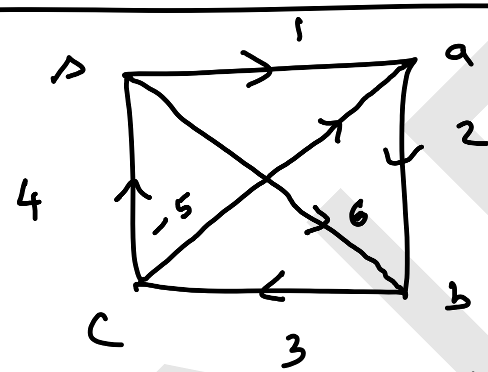
$$\Rightarrow x_c = 3 + 3 = 6.$$

$$\left\{ \begin{array}{l} x_s = 0, \ x_a = 1, \ x_b = 3, \ x_c = 6 \\ \delta(s) = 0; \ \delta(a) = 1, \ \delta(b) = 3, \ \delta(c) = 6 \end{array} \right\}$$

The solution of BE is

$$x_v = \delta(v) \quad \forall\, v \in V$$

This solution is unique, because other possibilities did not result in any values for x_v.



ascra – is a zero cycle.

$x_s = 0$; $x_a = \text{Min}\{1, x_c - 5\}$.

$x_b = \text{Min}\{6, x_a + 2\}$.

$x_c = x_b + 3$.

$$x_a = \text{Min}\{1, x_b + 3 - 5\}$$

$$= \text{Min}\{1, x_b - 2\}.$$

$$x_b = \text{Min}\{6, x_a + 2\}.$$

if $x_a = 1 \Rightarrow x_b = 3$, $\Rightarrow x_c = 6$ ✓

$x_a = \text{Min}\{1, 3-2\}$
$= \text{Min}\{1, 1\}$
$= 1$

if $x_b = 2$ ✓, $x_a = 0$, $x_c = 5$.

$x_b = \text{Min}\{6, 2\}$
$= 2$ ✓

$\rightarrow x_s = 0,\ x_a = 1,\ x_b = 3,\ x_c = 6.$

$\rightarrow x_s = 0,\ x_a = 0,\ x_b = 2,\ x_c = 5$

$$x_a = \text{Min}\{1, x_b - 2\}$$

$$x_b = \text{Min}\{6, x_a + 2\}.$$

if $x_b = 0$, $x_a = \text{Min}\{1, -2\}$

$$= -2$$

$$x_b = \text{Min}\{6, -2+2\}$$

$$= \text{Min}\{6, 0\}$$

$$= 0$$

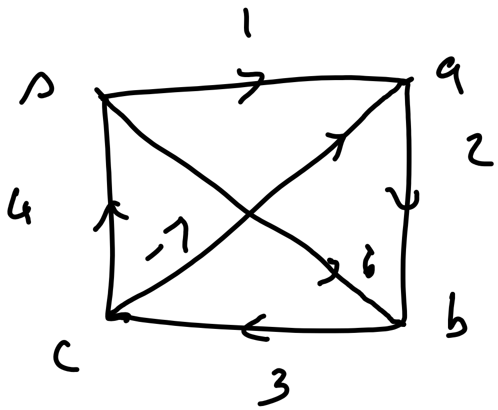
$x_s = 0,\ x_a = -2,\ x_b = 0,\ x_c = 3.$

$$x_a = \text{Min}\{1, x_a\}.$$

if $x_a \leq 1$

$$\left\{\begin{aligned} x_a &= -7 \\ x_b &= -5, \\ x_c &= -2 \end{aligned}\right\} \text{ another solution}$$

This has infinitely many solutions?



$abca$ — is a Negative cycle.

$x_s = 0; \quad x_a = \min\{x_s + w(s,a),\ x_c + w(c,a)\}$

$= \min\{1,\ x_c - 7\}$

$x_b = \min\{6,\ x_a + 2\}, \quad x_c = x_b + 3.$

$x_a = \min\{1,\ x_b + 3 - 7\}$

$= \min\{1,\ x_b - 4\}.$

$x_b = \min\{6,\ x_a + 2\}.$

if $x_a = 1; \Rightarrow 1 \le x_b - 4.$

$\Rightarrow \underline{x_b \geq 5}.$

$x_b = \min\{6, 3\}$

$= \underline{3}$

if $x_a = x_b - 4$? $\Rightarrow x_b - 4 \le 1$

$\Rightarrow x_b \le 5.$

$x_b = \min\{6,\ x_b - 4 + 2\}$

$= \min\{6,\ x_b - 2\}. \Rightarrow x_b = 6$

if G has a negative cycle,
BE has no solutions.

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if G has a Zero cycle,
BE may have infinitely many solution,
Solution May not be Unique.

We assume that
G has no negative cycles
no zero cycles.

It can be shown that,
if G has no $-$ve, zero cycles.
then the solution for BE is Unique & the solution is infact $x_v = \delta(v)$.

We solve SSSP problem on a $\underline{\underline{G}}$
(No Negative cycles, No zero cycles) ||.

We form the BE
& attempt to solve the same.

HOWEVER.

No Direct Method available
to solve BE.

How to circumvent this
Difficulty?

We see in next session:
