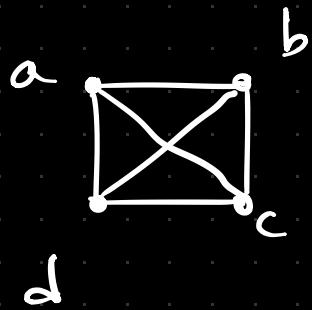


Depth - First Search

Depth First Search (DFS)

- * Systematic exploration of a graph
- * Undirected graph
- * Directed graph



There's gonna be 2 categories of vertices :

① Visited

② Unvisited

We'll always choose an unvisited vertex at any point of traversal.

- * Dead End : No more vertices to explore that are unvisited from the current vertex.
- * If you reach a dead end during your traversal, the Control goes back to the previous vertex which called this dead end. This is called Backtracking.

DFS (G_i, u)

mark u as visited

For each $v \in \text{Adj}(u)$

if v is NOT visited

DFS (G_i, v)

} Recursive
Backtracking

Terminology

Tree

Edge

$(u, v) \in E$ is a tree edge iff
 u is visited and
 v is not visited.

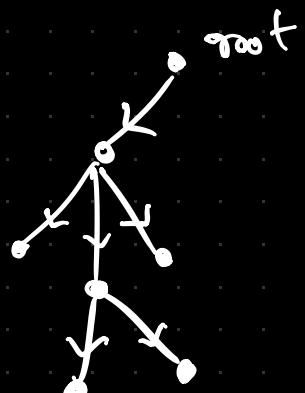
Non - Tree Edge : $(u, v) \in E$

↓
visited

visited

* Always go from visited to unvisited vertex.

DFS Tree :



An out-tree

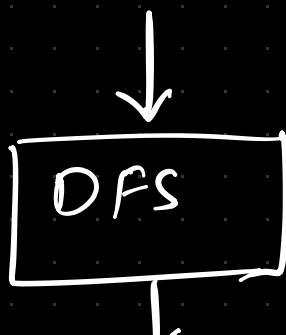
all the edges in the tree orient away from the root toward leaf.

The non-tree edges are oriented toward the root and are called Back edges.

This view of "Back Edges" is important.

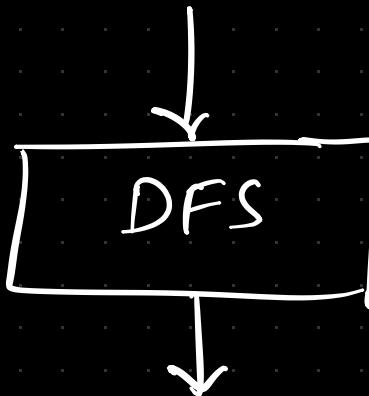
- * when the input graph G is disconnected, then DFS will produce a **forest**
This is called a **DFS Forest**.

G : connected



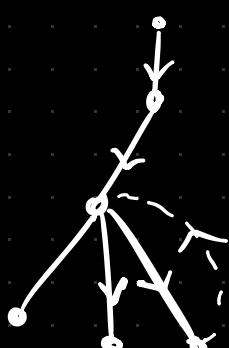
Tree

G : disconnected



Forest
(Set of trees)

State of a Node in DFS



node :

visited

unvisited

Live

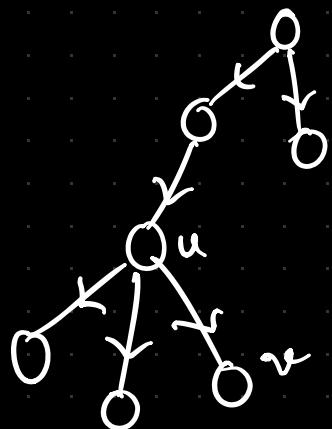
This is ←
when the node's adjacent vertices are
being explored.

Coloring The Vertices To
know the State of the Node

white : NOT visited

Gray : Visited and Live

Black : Completed the visit & left for
its parent



u is the parent of v

$$u = p(v)$$



Notation

$$v.p = u$$



$\therefore (v.p, v)$ is a Tree Edge.

* We've 4 attributes for a node:

$\rightarrow v.\text{color}$

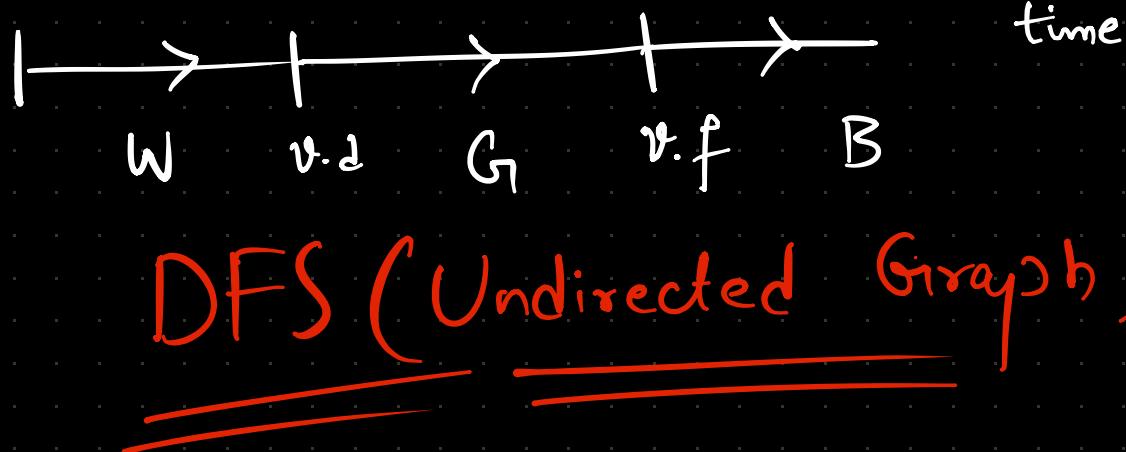
$\rightarrow v.f$: finish time

$\rightarrow v.p$

$\rightarrow v.d$: discovery time (first time when you enter)

* we are going to keep track of the Order in
which the graph was explored (which vertex
was visited first). TIME : global variable
will tell us at what time the node's visited

a node and at what time the computation on the current node is finished.



DFS (Undirected Graph)

DFS (G, u):

Time = Time + 1 ;

v.d = Time ;

u.color = Gray ;

For each $v \in \text{Adj}(u)$:

if ($v.$ color == white) :

$v.p = u$;

DFS (G, v) ;

Time = Time + 1 ;

v.f = Time ;

u.color = Black ;

This is called DFS Skeleton .

We may add more Computational Steps at various "Control points" of the above Code

and generate more information about the graph.

Extended view of Computations on the graph with DFS :

DFS (G, u) :

Time = Time + 1 ;

$u.d = \text{Time}$;

|| Discovery Time.

|| First time u is visited

|| ① process at u when u is encountered

|| for the first time

|| (like preorder)

$u.\text{color} = \text{Gray}$; || u is active now

For each $v \in \text{Adj}(u)$:

|| ② include the computation to be

|| done for every edge here.

|| The current edge is (u, v)

if ($v.\text{color} == \text{white}$)



// ③ Include the computations to be done
// on the tree edges here. The current
// tree edge is (u, v)

$v.p = u;$

DFS(G, v);

// ④ Include the computations to be
// done while backing from v to u .

else if ($v == u.p$)

// (u, v) is the second copy of the

// tree edge (v, u)

else

// (u, v) is a back edge

// ⑤ include here the computations
// to be done for the back edge.

Time = Time + 1;

$u.f = \text{Time};$

// ⑥ Include the computations to be done
// while leaving u . (like post order)

$u.\text{Color} = \text{Black};$

end;

There are 6 control points in the DFS
Skeleton.

One can perform additional computational
steps and expand the DFS code to obtain
more information about the graph "on the
go."

One straight application of this approach
is finding the cut vertex in an
undirected graph, a canonical example.

Cut Vertex Search using

DFS

Cut Vertex: A v is a cut vertex

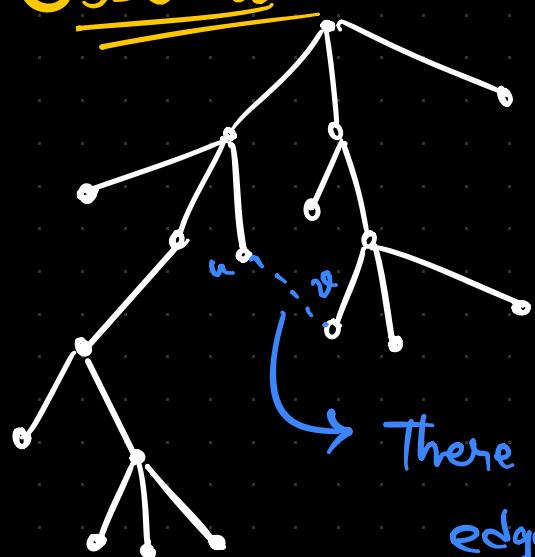
in a graph G iff \exists a pair of vertices $x, y \ni$ every path b/w $x \& y$

Contains v .

(Computationally Infeasible)

* Lets view G in terms of tree edges and non-tree edges.

Observation:

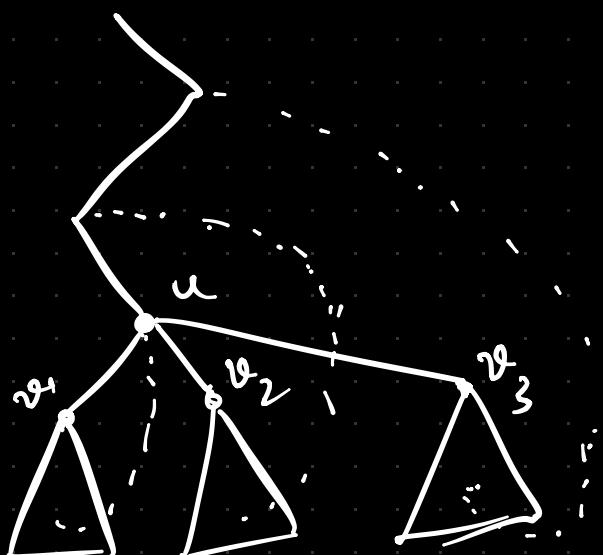


new attribute for each vertex

$v.cv = 0$ if v is not a cut vertex
 $= 1$ if v is a cut vertex

There won't be a non-tree edge like this because while performing DFS from u , we must have included (u, v) as a tree edge.

Problem: Label all the vertices in the graph as cut vertex or not.



\Rightarrow Removal of u disconnects v_1 but not v_2 and v_3 . //

Descendants of v_1 are max. connected to u only, and thus removing u will disconnect v_1 .

* Low(u) : = $\min \{ v.d \mid v \text{ is a vertex reachable from } u \text{ by using zero or more tree edges and zero or one back edge} \}$

\downarrow

$\min \{ \text{discovery time of the back edges emanating from } u \text{ descendants of } u \}$

$$= \min \{ u.d, \min \{ v.d \mid (u, v) \text{ is a back edge} \},$$

$$\min \{ v.d \mid v \text{ is reachable from a descendant of } u \text{ by a back edge} \}$$

Observe that

$$= \min \{ v.d \mid v \text{ is reachable from } u \text{ by a back edge} \}$$

$$= \min \{ \text{Low}(v) \mid v \text{ is a child of } u \text{ in the DFS tree} \}$$

recursive
computation of
 $\text{Low}(u)$

NOTE: All values needed to compute $\text{Low}(u)$ are not available in one place to work on them.

The values mentioned in the above equations are generated at various times during the execution of the program.

Hence, we update $\text{Low}(u)$ as and when a relevant value for computing $\text{Low}(u)$ is generated.

Specifically, $\text{Low}(u)$ may be initialized as $u.d$ when $u.d$ was assigned a value.

★ $\text{Low}(u)$ may be updated as

$\text{Low}(u) = \min \{\text{Low}(v), \text{re.d}\}$
where (u, v) is a **back edge** and this may be done when the back edge was encountered in the DFS.

★ $\text{Low}(v)$ for a child v of u can be recursively computed and $\text{Low}(u)$ is updated as $\min \{\text{Low}(u), \text{Low}(v)\}$ when the control returns from v to u after finishing the DFS at the child v .

In order to carry out these steps, we add

$$u.\text{low} = u.\text{d} \text{ at control point } ①$$

$$u.\text{low} = \min \{u.\text{low}, v.\text{low}\} \text{ at control point } ④,$$

where we're backing from v to u

and finally, we add

$$u.\text{low} = \min \{u.\text{low}, v.\text{d}\} \text{ at control point } ⑤ \text{ where}$$

we know that (u, v) is a back edge.

Cut Vertex - detailed Pseudocode

$\text{DFS_LOW}(G) \quad // G \text{ is connected}$

For each $u \in V$

$u.\text{color} = \text{white};$

$u.P = \text{NIL};$

$u.d = 0;$

$u.\text{low} = 0;$

Time = 0

$\text{DFS_low}(G, u)$

$\text{DFS_low}(G, u)$

Time = Time + 1

$u.d = \text{Time}$

① $\rightarrow u.\text{low} = u.d$

$u.\text{color} = \text{Gray} \quad // u \text{ is in active mode}$

For each $v \in \text{Adj}(u) \quad // \text{children of } u$
if ($v.\text{color} = \text{white}$)

$\text{DFS_low}(G, v)$

④ $\rightarrow u.\text{low} = \min \{ u.\text{low}, v.\text{low} \}$
 else if ($v \neq \text{e.p.}$) $\Downarrow v$ is not parent of u
 in DFS Tree
⑤ $\rightarrow u.\text{low} = \min \{ u.\text{low}, v.d \}$

$\text{Time} = \text{Time} + 1$

$u.f = \text{Time}$

$u.\text{color} = \text{Black}$

END //

v in $\text{neighbor}(u)$
 and v was discovered
 earlier than $u(d)$
 not discovered
 at all.

* The 'low' values are useful in arriving at a new characterization for cut vertices.

Let $u \neq \text{root (DFS Tree)}$

$\Rightarrow u$ is a cut vertex iff \exists a child v of u in DFS Tree such that $\text{low}(v) \geq u$.

Root of the DFS Tree is a cut vertex iff it has more than 1 child in the DFS Tree.

Thus, at control point ④, we do the following minor change:

$u.\text{cre} = 1$ if u is a cut vertex
 0 else

Initially, $u.\text{cr} = 0 \quad \forall u \in V$

All we need to do now is add the

following at Control point ④ :

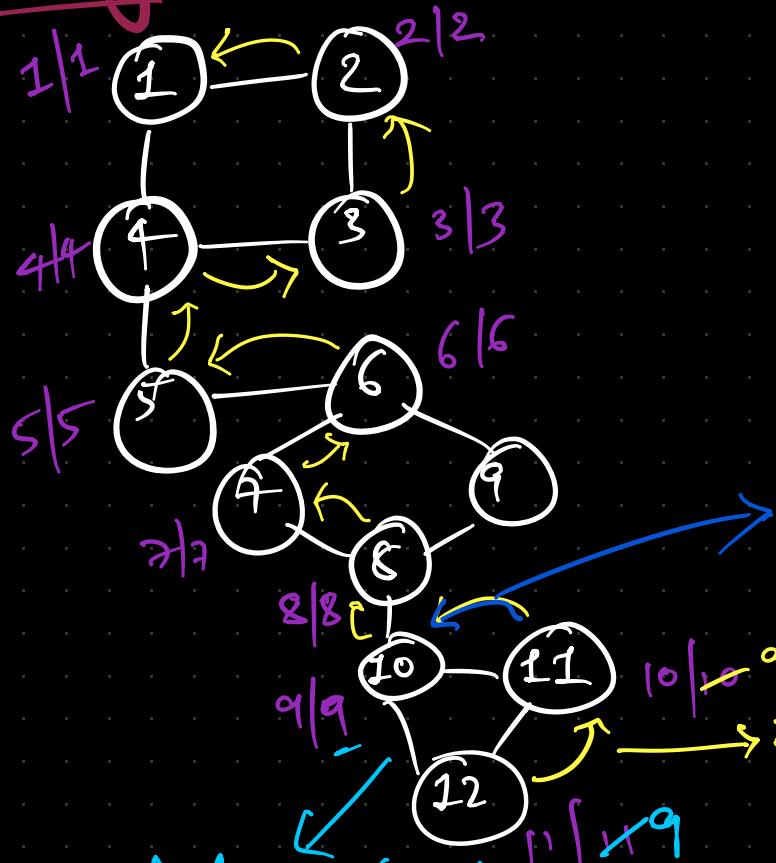
(backing from v to u)

if ((u is not the root) & & ($v.\text{low} \geq u.\text{cr}$))

$u.\text{cr} = 1$

* For the root node, we check if its degree is ≥ 2 in DFS tree to determine if it is a cut vertex.

Dry-Run Example:



low | d

$\text{low}(10) \geq u.d$
 $\Rightarrow 10$
we can't reach before 10
 $\Rightarrow 10$ is a cut vertex

backtrack \Rightarrow
 $\text{low}(u) = \min(\text{low}(u), \text{low}(v))$

$\text{back edge } (12, 11)$
(visited, visited) $\Rightarrow \text{low}(u) = \min\{\text{low}(u), v.d\}$

Iterative DFS

* Recursion is quite intuitive and meaningful.

* Iterative version of recursion is not so intuitive and straightforward as recursion.

Recursive DFS :

$\text{DFS}(G, u)$

mark u as visited ;

for all $v \in \text{adj}(u)$

if v is not visited

$\text{DFS}(G, v)$;

Vanilla DFS

Intuitive

Easy

Simple

Straightforward

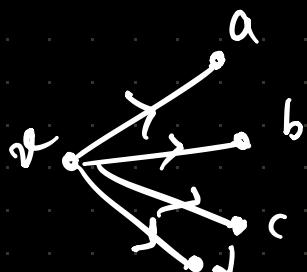
* Let G be represented as Adjacency List

$v \rightarrow \text{Adj}(v)$



head of the linked list

Containing the out neighbors of v .

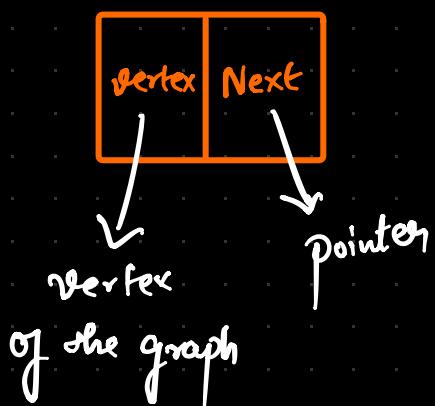


ptr : An independent pointer to indicate a specific position in the adjacency list.

It's more of like a running pointer in the list.

(v, ptr) : A position indicated by the pointer ptr in the adjacency list of v $\text{Adj}(v)$.

A Box in the Linked List



* Any recursive function in Computer maintains implicitly a system stack, in which the System stores the references related to all the references of the future computations.

* For this to be done explicitly, one has to maintain a stack for the same purposes.

Non-Recursive DFS (G, u)

$S = \emptyset$; // S is a stack containing
 // pairs, (v, ptr) that correspond
 // to the position of the neighbor
 // of v to be explored.

Time = Time + 1;

$u.d = \text{Time}$; // discovery time of
 // the vertex u .

$u.\text{color} = \text{Gray}$; // u is active now

push $((u, u.\text{first}), S)$; // push the
 // position indicator
 // for u onto the stack

while ($S \neq \emptyset$)

$(v, \text{ptr}) = \text{pop}(S)$;

if ($\text{ptr} \neq \text{NULL}$)

push $((v, \text{ptr.next}), S)$;

$w = \text{ptr.vertex}$;

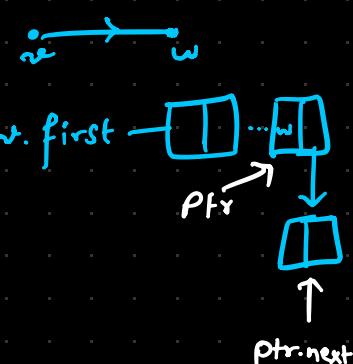
if ($w.\text{color} == \text{White}$)

$w.p = v$;

Time = Time + 1;

$w.\text{color} = \text{Gray}$;

push $((w, w.\text{first}), S)$;



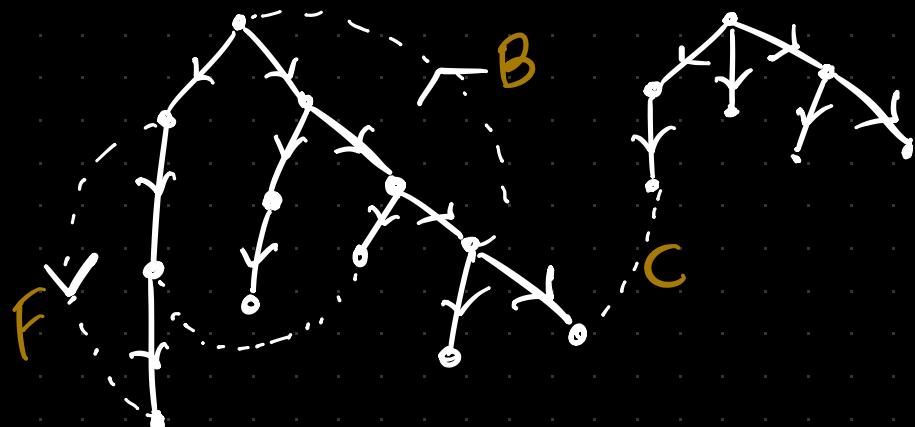
else // (v, w) is a non-tree edge
// So, do computations related to
// the non-tree edge (v, w) .

else // ($\text{ptr} == \text{NULL}$)
// \Rightarrow exploration related to v is complete
 $v.\text{color} = \text{Black};$
 $\text{Time} = \text{Time} + 1;$
 $v.f = \text{Time};$

// $(v, \text{ptr}) == (v, \text{NULL})$ indicates that all
// neighbors of v were visited. Hence visited
// v may be finished.

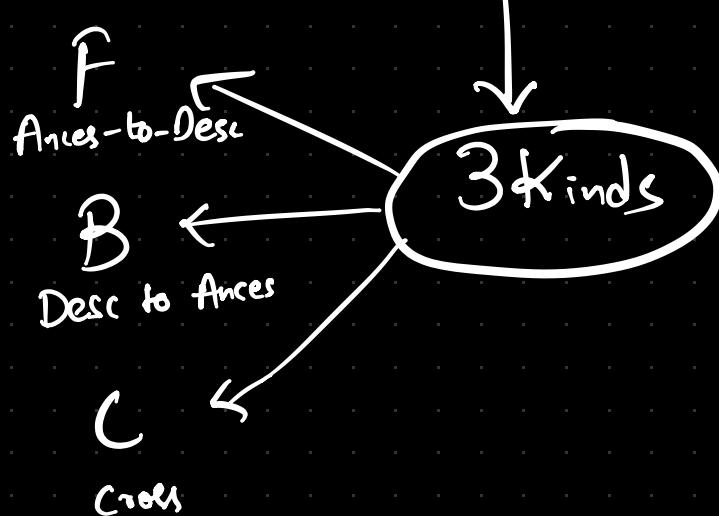
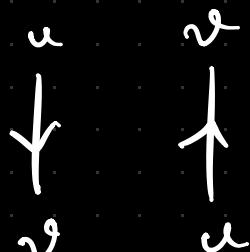
- * Keep pushing the positions, for which exploration are still to be done, onto the stack.
- * After finishing the visit corresponding to the position (v, ptr) , we must continue with $(v, \text{ptr.next})$. That's why we push $(v, \text{ptr.next})$ onto the stack.

DFS (Directed Graph)



(visited, unvisited) — Tree Edge
 (visited, visited) — Non-Tree Edge

$$e = (u, v) \in E$$



F B will take values $1, 2, \dots, 2^n$

* Time, the global variable takes the values
 1 to 2^n consecutive integers.

* Various vertices are active in various sub-intervals

$[l.d, u.f]$
 ↓
 discovery time → finish time



These time intervals have some significance and also have interesting properties.

- ① Disjoint Time intervals (or)
- ② Time intervals containing the other

BUT They never overlap

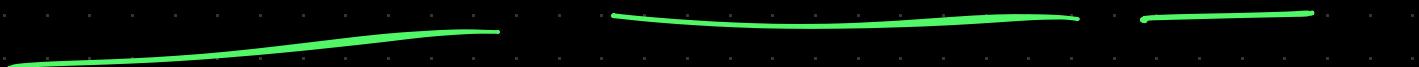


~~not possible~~



possible ✓

$$u.d < v.d < v.f < u.f$$



disjoint ✓

* One has to keep track of the parent node for every node in directed graph also
 $u.p \quad \forall u \in V \text{ in } G = (V, E)$

DFS (G)

For each $u \in V$
 $u.\text{color} = \text{white};$
 $u.p = \text{NULL};$

} Initialization

Time = 0 ;

For each $u \in V$
if ($u.\text{color} == \text{white}$)

DFS (G, u);

This is
another sub-
routine.

DFS (G, u)

Time = Time + 1 ;

$u.d = \text{Time};$

$u.\text{color} = \text{Gray};$ // u is active now

For each $v \in \text{Adj}(u)$

if ($v.\text{color} == \text{white}$)

$v.p = u;$

// (u, v) is a tree edge

DFS(G_i, v);

else

// (u, v) is a non-tree edge.

// The color of v can be Gray or Black

if ($v.\text{color} == \text{Gray}$)

// (u, v) is a back edge (B)

if ($v.f < u.d$)

// (u, v) is a cross edge (C)

else

// (u, v) is a forward edge (F)

elaboration
of
Control
point related
to
non-tree
edges

