



Indian Institute of Science Bangalore
Department of Computational and Data Sciences (CDS)
DS284: Numerical Linear Algebra
Assignment 4 [Posted Oct 15, 2023]

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Notations: Vectors and matrices are denoted below by bold faced lower case and upper case alphabets respectively.

Problem 1

In this problem you will test different algorithms for the least squares problem to approximate the function $f(t) = \sin(10t)$ for $t \in [0, 1]$ using a polynomial fit. To this end, first generate $m = 100$ data points using the above function which forms your given data i.e $(t_i, f(t_i))$ for $i = 1 \dots m$. Using this data, we would like to construct a 14th degree least squares polynomial fit to $f(t)$. Determine its least square fit using the following methods.

- (a) Use QR Factorization with your implementation of Modified Gram Schmidt. You should write your own back substitution code for solving the resulting triangular system.
- (b) Using QR Factorization with your implementation of Householder factorization.
- (c) Using SVD (Computed with any inbuilt libraries in MATLAB/Python/Octave)
- (d) Using normal equations, you can use backslash command in MATLAB to solve this system.

Accept the MATLAB/Octave/Python least squares solution (given by backslash "\ " in MATLAB) as the truth. Display and plot the approximation given by this "true" solution and compare it with $f(t)$. Compare with the solution given by 4 methods described above. Explain the results.

Problem 2

Given below is the matrix \mathbf{A} and vector \mathbf{b}

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 2 & 8 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Find the least square solution for the above. Now do the same for the below matrix also.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

Were you able to find the least square solution? If not explain why? Must there be a restriction on \mathbf{A} for a least square solution to exist or will it always exist?

Problem 3

- (a) Let \mathbf{A} be a non-singular square matrix and let $\mathbf{A} = \mathbf{QR}$ be its QR factorization. Let also $\mathbf{A}^T \mathbf{A} = \mathbf{U}^T \mathbf{U}$ be the Cholesky factorization of $\mathbf{A}^T \mathbf{A}$. Can you conclude that $\mathbf{R} = \mathbf{U}$? If yes, prove it; if not, why not?

- (b) Recall that by $\mathbf{A} \in \mathbb{R}^{m \times m}$, being symmetric and strictly positive definite, we mean $\mathbf{A} = \mathbf{A}^T$ and $\forall \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \neq 0$, we have $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ is positive semi-definite if $\forall \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \neq 0$, we have $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$

If $\{\phi_i(x)\}_{i=1 \dots m}$ denote m linearly independent basis functions (non-zero) defined over $[-1, 1]$ in an m -dimensional vector space then show that the matrix $\mathbf{M} = \int_{-1}^1 \phi_i(x) \phi_j(x) dx$ for $i, j = 1, 2 \dots m$ is a symmetric positive definite matrix.

Similarly show that the matrix $\mathbf{K} = \int_{-1}^1 \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx} dx$ for $i, j = 1, 2 \dots m$ is a symmetric positive semi-definite matrix.

- (c) Show that a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric, strictly positive definite if and only if there exists a matrix $\mathbf{B} \in \mathbb{R}^{m \times n}$ of rank n , where $n \leq m$, such that $\mathbf{A} = \mathbf{B}^T \mathbf{B}$. Assuming that \mathbf{A} is of this form, is there a unique such \mathbf{B} ?

Problem 4

For each of the following statements prove that it is true or give an example to show that it is false. Assume $\mathbf{A} \in \mathbb{C}^{m \times m}$ unless otherwise indicated.

- (a) If λ is an eigenvalue of \mathbf{A} and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$.
- (b) If \mathbf{A} is real and λ is an eigenvalue of \mathbf{A} then so is $\bar{\lambda}$.
- (c) If \mathbf{A} is real and λ is an eigenvalue of \mathbf{A} , then so is λ^* . (λ^* is the complex conjugate of λ).
- (d) If λ is an eigenvalue of \mathbf{A} and \mathbf{A} is non-singular, then λ^{-1} is the eigenvalue of \mathbf{A}^{-1} .
- (e) If all the eigenvalues of \mathbf{A} are zero, then $\mathbf{A} = \mathbf{0}$.
- (f) If \mathbf{A} is diagonalizable and all its eigenvalues are equal, then \mathbf{A} is diagonal.

Problem 5

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ with entries a_{ij} for $i, j = 1, 2, \dots, n$ and define the closed disks $D(a_{ii}, r_i)$ centered at the diagonal entries a_{ii} of \mathbf{A} of radius $r_i = \sum_{j=1}^n (1 - \delta_{ij})|a_{ij}|$ for $i = 1, 2, \dots, n$. Note that δ_{ij} represents Kronecker delta i.e. $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The above disks are called Greshgorin's disks.

- (a) Prove that every eigenvalue of \mathbf{A} lies in a Greshgorin disk.
(*Hint*: Let λ be any eigenvalue of \mathbf{A} and \mathbf{x} be the corresponding eigenvector with largest entry 1.)
- (b) Suppose that \mathbf{A} is diagonally dominant i.e. $|a_{ii}| > \sum_{j=1}^n (1 - \delta_{ij})|a_{ij}|$ for all $i = 1, 2, \dots, n$. Prove that \mathbf{A} is invertible.
- (c) Give estimates based on (a), for the eigenvalues of:

$$\mathbf{A} = \begin{bmatrix} 8 & 2 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{bmatrix} \quad \text{where } |\epsilon| < 1$$