Université Paris 1 Panthéon Sorbonne

UFR27 - MMMEF - Yiel Curve Models

Project Explicit Finite Differencies

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1 Black-Scholes Equation

1.1 The equation

In a B&S framework:

$$dX(t) = X(t)rdt + X(t)\sigma dW(t) \tag{1}$$

$$X(0) = X_0$$

Infinitesimal linear generator

$$f(t,x) \to \mathcal{L}^X f(t,x) = \frac{\partial f}{\partial t} + rx\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}$$

Initial PDE :

$$P(L,G) = \begin{cases} Lf = 0 \\ f(T,x) = G(x) \end{cases}$$

where $Lf \equiv \mathcal{L}^X f - rf$.

1.2 Change of variable

Let $Y(t) = \ln(X(t))$, the logarithm of the price of the underlying, and let $h(t, Y(t)) \equiv f(t, \ln(X(t)))$, the price of the option expressed as a function of the logarithm of the price. **Show that h is the solution of:**

$$\begin{cases} \frac{\partial h}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial h}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 h}{\partial x^2} = rh \\ h(T,x) = G(e^x) \end{cases}$$

We rewrite our three partial differentials (and we recall that $(U+V)' = U'V + UV'$):

$$\frac{\partial f(t,\ln(X))}{\partial t} = \frac{\partial h(t,Y)}{\partial t}$$
$$\frac{\partial f(t,\ln(X))}{\partial x} = \frac{1}{X}\frac{\partial f(t,\ln(X))}{\partial x} = \frac{1}{X}\cdot\frac{\partial h(t,Y)}{\partial x}$$
$$\frac{\partial^2 f(t,\ln(X))}{\partial x^2} = -\frac{1}{X^2}\frac{\partial f(t,\ln(X))}{\partial x} + \frac{1}{X^2}\frac{\partial^2 f(t,\ln(X))}{\partial x^2} \quad = -\frac{1}{X^2}\frac{\partial h(t,Y)}{\partial x} + \frac{1}{X^2}\frac{\partial^2 h(t,Y)}{\partial x^2}$$

Initially, we have this equation $\frac{\partial f}{\partial t} + rx\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - rf = 0$
and $h(t,Y(t)) = f(t,\ln(X(t)))$ where $Y(t) = \ln(X(t)) \iff X(t) = e^{Y(t)}$
So we can replace the partial derivatives by the new one inside the initial equation:

$$
\begin{aligned}
&\implies \frac{\partial h}{\partial t} + r\frac{\partial h}{\partial x} + \frac{1}{2}\sigma^2 x^2 \left(-\frac{1}{x^2}\frac{\partial h}{\partial x} + \frac{1}{x^2}\frac{\partial^2 h}{\partial x^2}\right) - rh = 0, \\
&\implies \frac{\partial h}{\partial t} + r\frac{\partial h}{\partial x} + \left(\frac{1}{2}\sigma^2\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\sigma^2\frac{\partial h}{\partial x}\right) - rh = 0, \\
&\implies \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\sigma^2\frac{\partial^2 h}{\partial x^2} = rh.
\end{aligned}
$$

with $h(T,x) = G(e^x)$

1.3 Euler scheme and discretization

The domain of f is $[0,T] \times \mathbb{R}^+$
We consider the interval $[X_{\min}, X_{\max}]$
For discretization :

- Time: $\tau = \{t_0 = 0, \dots, t_j = j\delta t, \dots, t_{\text{ntime}} = T\}$ with $\delta t = T/\text{ntime}$,
- Space: $\chi = \{x_0 = X_{\min}, \dots, x_i = x_{\min} + i\delta x, \dots, x_{\text{nspace}} = X_{\max}\}$ with $\delta x = \frac{X_{\max} - X_{\min}}{\text{nspace}}$. We denote $\hat{\chi}$ the set χ without its boundaries $X_{\min}$ and $X_{\max}$.

Define $h(j,i) = h(t_j, x_i)$. **Approximate the partial derivatives using $h(j,i)$ values on $\hat{\chi}$ and show that**

$$
\begin{aligned}
h(j,i) = h(j+1,i) + \delta t \Bigg[&\frac{1}{2}\sigma^2 \frac{h(j+1,i+1) - 2h(j+1,i) + h(j+1,i-1)}{(\delta x)^2} \\
&+ \left(r - \frac{1}{2}\sigma^2\right)\frac{h(j+1,i+1) - h(j+1,i-1)}{2\delta x} - rh(j+1,i)\Bigg] + o(1).
\end{aligned}
$$

Let us consider $h(j,i) = h(t_j, x_i) = h_i^j$.
We compute the three partial derivatives for the FDM:
for the **forward method** we use :

$$
\begin{aligned}
\frac{\partial h_i^j}{\partial t} &\approx \frac{h_i^{j+1} - h_i^j}{\delta t} + o(\delta t) \\
\frac{\partial h_i^j}{\partial x} &\approx \frac{h_{i+1}^j - h_{i-1}^j}{2\delta x} + o((\delta x)^2) \\
\frac{\partial^2 h_i^j}{\partial x^2} &\approx \frac{h_{i+1}^j - 2h_i^j + h_{i-1}^j}{(\delta x)^2} + o((\delta x)^2)
\end{aligned}
$$

but we prefer the **backward method**, so we use :

$$\frac{\partial h_i^j}{\partial t} \approx \frac{h_i^{j+1} - h_i^j}{\delta t} + o(\delta t)$$
$$\frac{\partial h_i^j}{\partial x} \approx \frac{h_{i+1}^{j+1} - h_{i-1}^{j+1}}{2\delta x} + o((\delta x)^2)$$
$$\frac{\partial^2 h_i^j}{\partial x^2} \approx \frac{h_{i+1}^{j+1} - 2h_i^{j+1} + h_{i-1}^{j+1}}{(\delta x)^2} + o((\delta x)^2)$$

The PDE is the following:

$$\frac{\partial h}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial h}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 h}{\partial x^2} - rh = 0$$

and by replacing the partial differentials, we get:

$$\frac{h_i^{j+1} - h_i^j}{\delta t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{h_{i+1}^{j+1} - h_{i-1}^{j+1}}{2\delta x} + \frac{1}{2}\sigma^2 \frac{h_{i+1}^{j+1} - 2h_i^{j+1} + h_{i-1}^{j+1}}{(\delta x)^2} - rh_i^j + o(\delta t) = 0$$

$$\iff h_i^{j+1} - h_i^j + \delta t\left[(r - \frac{1}{2}\sigma^2)\frac{h_{i+1}^{j+1} - h_{i-1}^{j+1}}{2\delta x} + \frac{1}{2}\sigma^2 \frac{h_{i+1}^{j+1} - 2h_i^{j+1} + h_{i-1}^{j+1}}{(\delta x)^2} - rh_i^j\right] + o(1) = 0$$

We can take the derivative of h in j or $j+1$, it does not matter.

$$\iff h_i^j = h_i^{j+1} + \delta t\left[\frac{1}{2}\sigma^2 \frac{h_{i+1}^{j+1} - 2h_i^{j+1} + h_{i-1}^{j+1}}{(\delta x)^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{h_{i+1}^{j+1} - h_{i-1}^{j+1}}{2\delta x} - rh_i^{j+1}\right] + o(1)$$

1.4 Boundary conditions

The previous equation holds for $\mathbf{i = 1, \dots, nspace}$ as $\mathbf{h_{-1}^j}$ and $\mathbf{h_{nspace+1}^j}$ are not defined. Then we have $\mathbf{nspace - 1}$ equations for $\mathbf{nspace + 1}$ unknowns. The 2 remaining equations come from the 2 boundary conditions $\mathbf{i = 0}$ ($\mathbf{x_i = x_0}$) and $\mathbf{i = nspace}$ ($\mathbf{x_i = x_{nspace}}$).
We assume that $\forall j : \frac{\partial^2 h}{\partial x^2}(t_j, x_0) = \frac{\partial^2 h}{\partial x^2}(t_j, x_{nspace}) = 0$
Use noncentered estimators of the first derivative and **write the discretisation equations for** $i = 0$ **and** $i = nspace$.

$$\frac{\partial^2 h}{\partial x^2}(t_j, x_0) = \frac{\partial^2 h}{\partial x^2}(t_j, x_{nspace}) = 0 \iff \frac{\partial h_0^j}{\partial x} = \frac{h_{nspace}^j}{\partial x^2} = 0$$

For $i = 0$ ($x_i = x_0 = X_{\min}$):

$$\text{we get} \quad h_0^j = h_0^{j+1} + \delta t\left(r - \frac{1}{2}\sigma^2\right)\frac{h_1^{j+1} - h_0^{j+1}}{\delta x} - rh_0^{j+1}$$

For $i = nspace$ $(x_i = x_{nspace} = X_{\max})$:

$$\text{we get} \quad h^j_{nspace} = h^{j+1}_{nspace} + \delta t \left(r - \frac{1}{2}\sigma^2 \right) \frac{h^{j+1}_{nspace} - h^{j+1}_{nspace-1}}{\delta x} - r h^{j+1}_{nspace}$$

2 Implementation

Python.