CALCULUS I
GIẢI TÍCH I

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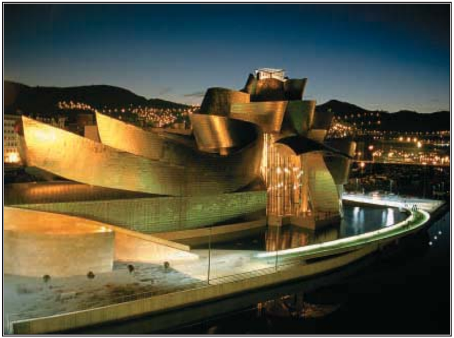


2 Limits and Derivatives

Contents

1.1. Average rates of change.

One of the main application of calculus if to determine how one variable changes in relation to another.

Average speed (of a motion) of the time interval from $t = a$ to tb is:

$$\frac{s(b) - s(a)}{b - a}.$$

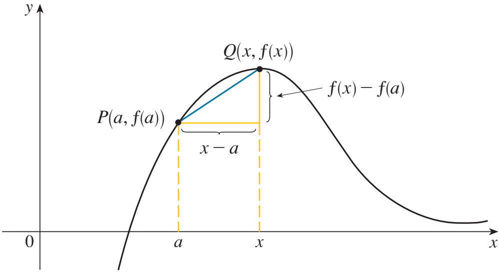
(This quotient is often called: difference quotient, Newton quotient, or average speed)

Definition (Average rate of change)

The average rate of change of a function $y = f(x)$ with respect to x as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a}.$$

Example 1.



The average rate of change in this case is: ......

Instantaneous rates of change

Example 2. (Instantaneous speed) A car is moving on a straight line (from the origin at $t = 0$) with $f(t)$ is the distance from the car to the origin at time t.

Average speed

Instantaneous speed of the car at t_0 is defined by

$$\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

Note that the limit can be rewritten in the form:

$$\lim_{t \to t_0} \frac{f(t_0 + h) - f(t_0)}{h}.$$

Example 3. (The slope of tangent line)

Definition 2.1.

The derivative of f at x_0, denote d by $f'(x_0)$, is

$$f'(x_0)) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if this limit exist (as a finite number). If f has a derivative at x_0 then we say that f is differentiable at x_0.

- Differentiable on (a, b)
- Geometrical meaning
- Mechanical meaning

Note: Pay attention to the speed and velocity of a motion.

Example 4. A motion described by a function:

$$s = f(t) = \frac{1}{t+1}.$$

(t: in seconds and s: in meters). Find the velocity and the speed of the motion after 2 seconds.

Definition 2.2. (right and left derivatives)

- Right/left derivatives
- Differentiable on $[a, b]$.

Remark. f is differentiable at x_0 iff there exists $f'(x_0^+)$, $f'(x_0^-)$ and $f'(x_0^+) = f'(x_0^-)$.

Example 5. $f(x) = |x|$. Find left and right derivative of f at $x_0 = 0$.

Differentiability and continuity

Theorem 2.1.

f is differentiable at x_0 then f is continuous at x_0.

Proof.

Remark.

(i) The converse of Theorem 2.1 is not true.

(ii) How can a function fails to be differentiable?

(iii) Notations (alternative).

Exercises of Chapter 2, Part 1. [From the book of James Stewart]

Section 2.6.. 11, 13, 14, 16, 24, 25.

Section 2.7. 21, 22, 28, 34, 36.

Section 2.8. 30, 32 35.

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

3.2. Derivative of a composite function

The Chain Rule If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Examples Find the derivatives of the following functions.
1. Find the derivative of $h(x) = \sqrt{x^2+1}$
$h(x) = f(g(x))$ where $f(x) = \sqrt{x}$ and $g(x) = x^2$.
Since $f'(x) = \frac{1}{2\sqrt{x}}$ and $g'(x) = 2x$, we have

$$h'(x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

2. $h(x) = e^{tanx}$; $\quad h(x) = \cos(\sin(x^2+1x)$.

3.4. Derivative of an inverse function.

Theorem (derivative of inverse function)

Assume that the function $y = f(x)$ has the derivative $y'(x_0) \neq 0$ and that the inverse function $x = x(y)$ exists and is continuous at $y_0 = y(x_0)$. Then the inverse function $x = x(y)$ has the derivative at y_0 which is given by

$$x'_y(y_0) = \frac{1}{y'_x(x_0)}.$$

Example. Find the derivative of $y = \tan^{-1} x$. Let $y = \tan^{-1} x$. This is the inverse function of $x = \tan y$, $-\pi/2 < y < \pi/2$. One has

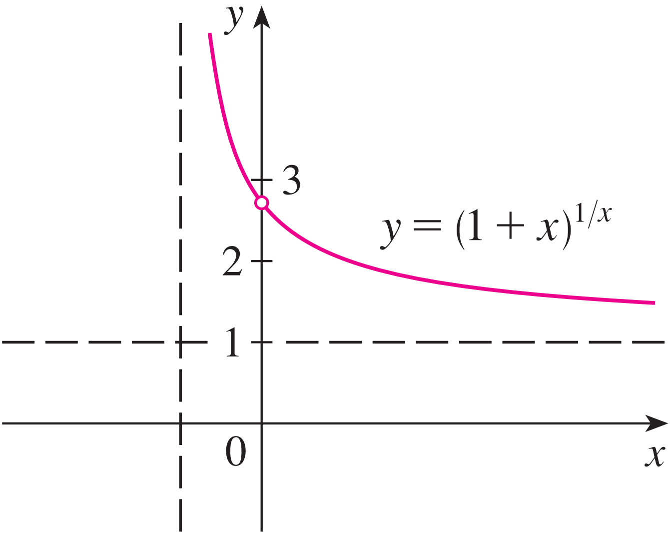
$$x'_y = \frac{1}{\cos^2 y} = 1 + \tan^2 y = 1 + x^2.$$

Using the formula of the derivative of the converse function we get $(\tan^{-1} x)' = \frac{1}{x^2+1}$.

3.5. The number *e* and hyperbolic functions

$$\boxed{\lim_{x\longrightarrow 0}(1+x)^{1/x}=e}$$

[Students read the book of Robert A. Adams, pages 213 to 218]


y
3
2
1
0
x
$y = (1+x)^{1/x}$

Like π, the number e is irrational (π and e are the two most important numbers in mathematics). In Section 2.4.2 below we will discover some formulas for calculating e to any desired degree of accuracy. Using them we can get the decimal expansion of e as

$$e = 2.718281828459045...$$

Certain combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and applications that they deserve to be given special names: *hyperbolic functions*. These are:

- the *hyperbolic sine function* is $\sinh x := \frac{e^x - e^{-x}}{2}$,
- the *hyperbolic cosine function* is $\cosh x := \frac{e^x + e^{-x}}{2}$,
- the *hyperbolic tangent function* is $\tanh x := \frac{\sinh x}{\cosh x}$,
- the *hyperbolic secant function* is $\text{sech } x := \frac{1}{\cosh x}$.

The reason for the names of these functions is that they are related to the hyperbola in much the same way that the trigonometric functions are related to the circle. We have also properties similar to that of trigonometric functions:

$$\begin{aligned}
&\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y, \\
&\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y, \\
&\cosh^2 x - \sinh^2 x = 1, \\
&\sinh 2x = 2 \sinh x \cosh x, \\
&\cosh 2x = \cosh^2 x + \sinh^2 x.
\end{aligned}$$

The derivatives of the hyperbolic functions are easily obtained by Theorem 2.3.1

$$\begin{aligned}
&(\sinh x)' = \cosh x, \\
&(\cosh x)' = \sinh x, \\
&(tanhx)' = 1 - \tanh^2 x.
\end{aligned}$$

4. Growth and decay problems

Many changing quantities in practice increase or or decrease at a rate proportional to their size. For instance, a certain cell culture grows a a rate proportional to the number of cells present. All these phenomena can be modeled mathematically in the same way.
If $y(t)$ denote the value of a quantity at time t and $y(t)$ changes at a rate proportional to its size, then

$$\frac{dy}{dt} = k \cdot y.$$

(k is called the constant of proportionality).

• It is easy to see that

$$y(t) = Ce^{kt},\ t \in \mathbb{R}$$

for any $C \in \mathbb{R}$ is a solution of the previous equation (called differential equation).

• Conversely, ...

Examples.

1/ Newton's cooling law. According to the Newton's law of cooling, a hot object introduced into a cooler environment will cool at a rate proportional to to the excess of its its temperature above that of its environment.

If a cup of tea sitting in a room maintained at a temperature of $20^o C$ cools from $80^o C$ to $50^o C$ in five minutes, how much longer will it take t cool to $40^o C$?

The equation:

$$\frac{dy}{dt} = k \cdot (y - 20).$$

Changing variable: $u(t) = y(t) - 20$ ...
ANS. $t = 5\frac{\ln 3}{\ln 2} \approx 7.92$

2/ Growth of a certain cell culture.

In most of the cases that we encountered, the functions have been given in an explicit form

$$y = f(x)$$

where y is given explicitly in terms of x, e.g.,
$y = 3x - 2, \; y = \sin x$, ...

However, in some situation, some equation can not be solved for y, such as,

$$y^5 + 8y^3 + 6x^2y^2 + 2x^3y + 6 = 0.$$

In this case, y is said to be given (as a function of x but) implicitly in terms of x.

In such a case, it may be possible to find $\frac{dy}{dx}$ (derivative of y) by implicit differentiation method.
Keep in mind that y is a function of x and assume that $\frac{dy}{dx}$ exists.

Example 1. Find $\frac{dy}{dx}$ if

$$5y^2 + \sin y = x^2.$$

Differentiate both sides of the equation, considering y is a function of x, we have

$$\begin{aligned}
& \frac{d}{dx}\left[5y^2 + \sin y\right] = \frac{d}{dx}[x^2] \\
\Longleftrightarrow \quad & 5\frac{d}{dx}(y^2) + \cos y \frac{dy}{dx} = 2x \\
\Longleftrightarrow \quad & 10y\frac{dy}{dx} + \cos y \frac{dy}{dx} = 2x.
\end{aligned}$$

Solving the last equation for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}.$$

Note: the right-hand side still contains y and x.

Example 2. Find the slope of the tangent line to the curve defined by the equation

$$5y^2 + \sin y = x^2$$

at the point $P(\frac{1}{2}\sqrt{5\pi^2 + 4}, \frac{\pi}{2})$.

We first note that P lies in this curve and the slope of the tangent:

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y} = \frac{\sqrt{5\pi^2 + 4}}{5\pi}$$

(substitute $x = \frac{1}{2}\sqrt{5\pi^2 + 4}$ and $y = \frac{\pi}{2}$ in the formula). Students write the equation of the tangent line!

Example 3. Find the equation of the tangent line at the point $P(2, 4)$ from the curve (C) which has the equation $x^3 + y^3 = 9xy$.

Implicit differentiation technique

(to find $\frac{dy}{dx}$ for an equation containing y and x)

Step 1. Differentiate both sides of the equation, treating y as a function of x, we get another equation with the unknown $\frac{dy}{dx}$.

Step 2. Solve the obtained equation for $\frac{dy}{dx}$.

EXAMPLE 2

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.
(c) At what points on the curve is the tangent line horizontal or vertical?

SOLUTION

(a) Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the y^3 term and the Product Rule on the $6xy$ term, we get

$$3x^2 + 3y^2y' = 6y + 6xy'$$

or

$$x^2 + y^2y' = 2y + 2xy'$$

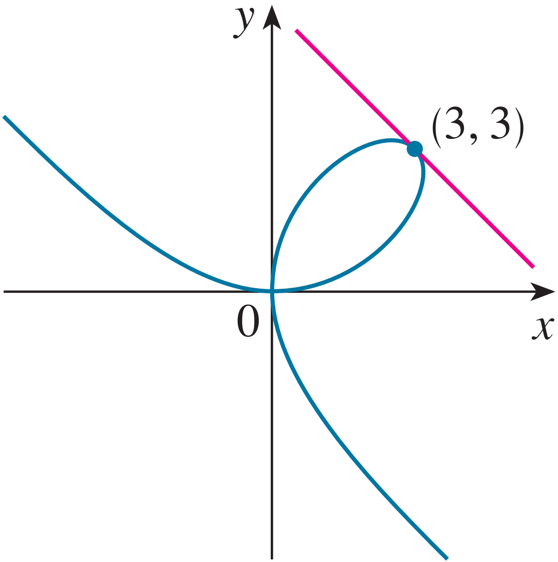
We now solve for y':

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

5. Implicit differentiation



EXAMPLE 3 Find y' if $\sin(x + y) = y^2 \cos x$.

SOLUTION Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = 2yy' \cos x + y^2(-\sin x)$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

So
$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

Let f be a function. Assume that f has its derivative $f'(x_0)$ at x_0. Let

$$\Delta x = x - x_0, \quad \Delta y = f(x) - f(x_0).$$

Then by definition of derivative

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

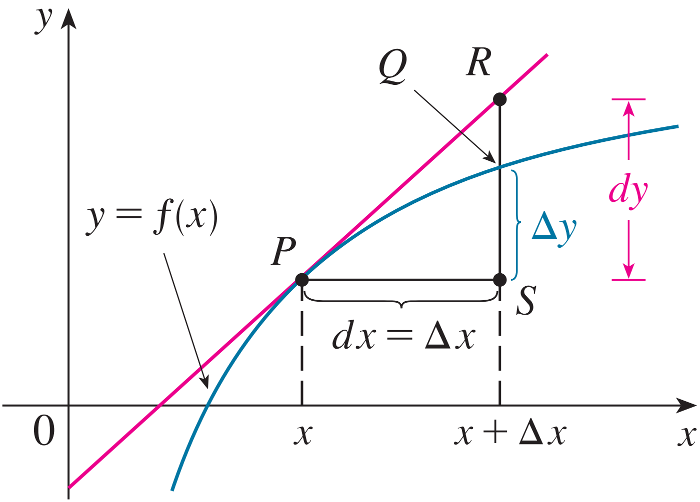
Therefore, as long as Δx is close to 0 (but $\Delta x \neq 0$),

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}, \text{ or, } \Delta y \approx \frac{dy}{dx} \cdot \Delta x = f'(x_0) \cdot \Delta x. \tag{1}$$

The expression $f'(x_0) \cdot \Delta x$ is called the differential of f at x_0 and denoted by dy or df. Thus,

$$dy \equiv df := f'(x_0) \cdot \Delta x.$$

6. Local linear approximation. Differentials



If $f(x) = x$ for all x then $df = dx = 1 \cdot \Delta x$ and hence, $\Delta x = dx$. For this reason, differential of f is often written in the form

$$dy = f'(x_0) \cdot dx.$$

Now, the formula (1) can be rewritten as

$$\Delta y \approx dy, \quad \text{or,}$$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x. \tag{2}$$

The formula (2) is called the linear approximation of f at x_0. It is useful to evaluate approximately the value $f(x_0 + \Delta x)$ when Δx is small and $f(x_0)$, $f'(x_0)$ are known (often much easier than evaluate directly from the formula of f, see examples below).

Approximate $\sqrt{50}$.

Note that $\sqrt{49} = 7$. So we set $y = f(x) = \sqrt{x}$ $(x > 0)$. Let $x_0 = 49$, $\Delta x = 1$. We have $f'(x) = \frac{1}{2\sqrt{x}}$. By (2),

$$f(50) = f(49+1) \approx f(49) + f'(49) \cdot 1 \approx 7 + \frac{1}{14} \approx 7.07107.$$

Note: Differentials are also used to estimate errors that might enter into measurements of a physical quantity.

Example 2. In a precision manufacturing process, ball bearings must be made with a radius of 0.6 millimeter, with a maximum error in the radius of ± 0.015 millimeter. Estimate the maximum error in the volume of the ball bearing. (Answer: $\Delta V \approx dV \approx \pm 0.0679$).

6. Local linear approximation. Differentials

Differential formulas

From the formulas of differentiation, we get the formulas for differentials:

$$
\begin{aligned}
d[C] &= 0,\\
d[f \pm g] &= d[f \pm d[g],\\
d[fg] &= fd[g] + gd[f],\\
d[\frac{f}{g}] &= \frac{gd[f] - fd[g]}{g^2}.
\end{aligned}
$$

To understand the behavior of a function on an interval, it is important to know the rate at which the function is increasing or decreasing. The second derivative of the function gives us information on this matter.

If f is a function then f', its derivative, is also a function, e.g., $f(x) = \sin x$ then $f'(x) = \cos x$; $f(x) = \sin x^2$ then $f'(x) = 2x \cos x^2$; ...
The derivative of f', if it exists, is called the second derivative of f, denoted by f''.

The derivative of f'', if it exists, is called the third derivative of f, denoted by f'''.

By such a way, we can define the n^{th} derivative of f being the derivative of the $(n-1)^{th}$ derivative of f, denoted by $f^{(n)}$. That is.

$$f^{(n)} := \left(f^{(n-1)}\right)'.$$

Notations: For a function $y = f(x)$, we often write

$$\begin{array}{lllll} f''(x) & \text{or} & \dfrac{d^2y}{dx^2} & \text{or} & \dfrac{d^2f(x)}{dx^2} \\ f'''(x) & \text{or} & \dfrac{d^3y}{dx^3} & \text{or} & \dfrac{d^3f(x)}{dx^3} \\ f^{(4)}(x) & \text{or} & \dfrac{d^4y}{dx^4} & \text{or} & \dfrac{d^4f(x)}{dx^4}, \dots \end{array}$$

Consider s(t) which describes the position of a vehicle along a straight line at time t. The first derivative of s, s'(t) = v(t) gives the velocity of the vehicle.

The instantaneous rate of change of the velocity of the vehicle is called the acceleration. Since this is the derivative of the velocity, acceleration is the second derivative of the position function.

Note:

• $s'(t) > 0$: the car is moving forward; if $s'(t) < 0$: the car is moving backward.
[Distinguish: velocity from speed].

• Think of the practical meaning of the cases:
(a) $s'(t) > 0$ and $a(t) > 0$,
(b) $s'(t) > 0$ and $a(t) < 0$.

If we consider the differential dy,

$$dy = y'(x)dx,$$

as a function of the (only) variable x. Then dy may has a differential:

$$d(dy) := d^2y = d(y'dx) = d(y')dx = (y''dx)dx = y''(dx)^2 =: dx^2,$$

which will be called **second differential** of the function y. Similarly, in general we can define the n^{th}-order differential $d^ny := d(d^{n-1}y)$. We have:

$$d^3y = y'''dx^3, ..., \ \ d^ny = y^{(n)}dx^n.$$

All exercises are taken from the book of James Stewart (version 2001)
Section 3.6 (page243-244) 7, 9, 11, 13, 1424, 25.
Section 3.8. (256) 5, 7, 9, 16, 17.