Big-O notation

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Motivation

- compare functions in terms of their growth
  - including functions describing algorithm steps
- ignore irrelevant details:
  - lower-order terms
  - constant factors
- basis for informal engineering designs
  - how big will my data grow?
  - will my existing solution still work adequately at scale?

Big-O

$$f(x) = O(g(x)) \text{ or } f(x) \in O(g(x))$$

Informally:

- $f(x)$ grows no faster than $g(x)$

Heuristically:

- as $x \longrightarrow \infty$, $f(x)$ is bounded above by some constant times $g(x)$

Formally:

$$\exists(C \in \mathbb{R}^+) : \exists(x_0 \in \mathbb{R}) : \forall(x > x_0) : f(x) < Cg(x)$$

Big-O

Examples:

- $x^2 - 3x + 6 = O(x^2)$ (*e.g.* choose $x_0 = 1$, $C = 5$)
- $x^2 - 3x + 6 = O(x^4 + 3)$ (*e.g.* choose $x_0 = 1$, $C = 2$)
- $x + 2x\log(x) + 3(\log(x))^2 = O(x\log(x))$ (*e.g.* choose $x_0 = 20$, $C = 3$)

Big-Ω

$$f(x) = \Omega(g(x)) \text{ or } f(x) \in \Omega(g(x))$$

Informally:

- $f(x)$ grows no slower than $g(x)$

Heuristically:

- as $x \longrightarrow \infty$, $f(x)$ is bounded below by some constant times $g(x)$

Formally:

$$\exists(C \in \mathbb{R}^+) \; : \; \exists(x_0 \in \mathbb{R}) \; : \; \forall(x > x_0) \; : \; f(x) > Cg(x)$$

Big-Ω

Examples:

- $x^2 - 3x + 6 = \Omega(x^2)$ (*e.g.* choose $x_0 = 3$, $C = \frac{1}{2}$)
- $x^2 - 3x + 6 = \Omega(x)$ (*e.g.* choose $x_0 = 3$, $C = 1$)
- $x + 2x\log(x) + 3(\log(x))^2 = \Omega(\log(x)^2)$ (*e.g.* choose $x_0 = 1$, $C = 1$)

Big-Θ

$$f(x) = \Theta(g(x)) \text{ or } f(x) \in \Theta(g(x))$$

Informally:

- $f(x)$ grows like $g(x)$

Heuristically:

- as $x \longrightarrow \infty$, $f(x)$ is bounded above and below by constants times $g(x)$

Formally:

$$\exists (C_1, C_2 \in \mathbb{R}^+) : \exists (x_0 \in \mathbb{R}) : \forall (x > x_0) : C_1 g(x) < f(x) < C_2 g(x)$$

Big-Θ

Examples:

- $x^2 - 3x + 6 = \Theta(x^2)$ (*e.g.* choose $x_0 = 3$, $C_1 = \frac{1}{2}$, $C_2 = 5$)

Little-o

$$f(x) = o(g(x)) \text{ or } f(x) \in o(g(x))$$

Informally:

- $f(x)$ grows much slower than $g(x)$

Heuristically:

- as $x \longrightarrow \infty$, $\frac{f(x)}{g(x)} \longrightarrow 0$

Formally:

$$\forall(\varepsilon \in \mathbb{R}^{+}) : \exists(x_0 \in \mathbb{R}) : \forall(x > x_0) : f(x) < \varepsilon g(x)$$

Common complexity classes

$\Theta(1)$	slowest growth
$\Theta(\log(n))$	
$\Theta((\log(n))^{1+c})$	
$\Theta(n^c)$	
$\Theta(n)$	
$\Theta(n \log(n))$	
$\Theta(n^{1+c})$	
$\Theta((1+c)^n)$	
$\Theta(n!)$	
$\Theta(n^n)$	fastest growth

for $0 < c < 1$

Work

1. Reading
    - CLRS, chapter 3
    - DPV, section 0.3
2. Problems from CLRS:
    - 1-1 Comparison of running times
    - 3-2 Relative asymptotic growths
    - 3-3 Ordering by asymptotic growth rates
3. Exercises from DPV: 0.1, 0.2
4. do the big-O quiz on the VLE