Implicit heaps

Goldsmiths Computing

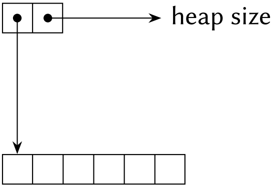
Implicit representation

implicit representations, previously:

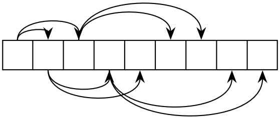
- dope vector (multidimensional array)
- sorted sequence (binary search tree)
- partially-sorted sequence (insertion sort)

Implicit heap

- an array
- a heap size (must be ≤ array length)



Parents and children



Parents and children

For zero-based arrays

function LEFT(i)
    return 2×i+1
end function
function RIGHT(i)
    return 2×i+2
end function
function PARENT(i)
    return $\left\lfloor \frac{i-1}{2} \right\rfloor$
end function

(one-based arrays have simpler calculations, but generalise less well)

Heapify

Given a root with two (max-)heaps as children, make the root be a valid max heap.

```
function MAX-HEAPIFY(a,i)
    l ← LEFT(i)
    r ← RIGHT(i)
    largest ← i
    if l < a.heapsize ∧ a[l] > a[largest] then
        largest ← l
    end if
    if r < a.heapsize ∧ a[r] > a[largest] then
        largest ← r
    end if
    if largest ≠ i then
        SWAP(a[i],a[largest])
        MAX-HEAPIFY(a,largest)
    end if
end function
```

(Also called siftDown)

Complexity analysis

Time complexity

$$T(N) \leq T\left(\frac{2N}{3}\right) + \Theta(1)$$

$$\implies \Theta(\log(N)) \text{ or } \Theta(h)$$

Constructing a heap in one go

Half of the nodes are already heaps!

function BUILD-MAX-HEAP(a)
  a.heapsize ← a.length
  for $\left\lfloor \frac{\text{a.length}}{2} \right\rfloor < j \leq 0$ **do**
    MAX-HEAPIFY(a,j)
  end for
end function

Complexity analysis

First analysis

- $\frac{N}{2}$ calls to MAX-HEAPIFY
- each takes time $O(\log(N))$

$$\implies O(N \log(N))$$

Improved bound

- most calls to MAX-HEAPIFY are near the leaves
- height of most trees is small

$$T(h) \leq O\left(1 \times \frac{N}{2} + 2 \times \frac{N}{2^2} + 3 \times \frac{N}{2^3} + \ldots + h \times \frac{N}{2^h}\right)$$

But $\sum_{k=0}^{\infty} \frac{k}{2^k} = 2$ (proof?)

$$\implies O(N)$$

Operations

insert!

function INSERT!(heap,k)
    heap[heap.heapsize] $\leftarrow$ k
    i $\leftarrow$ heap.heapsize
    heap.heapsize $\leftarrow$ heap.heapsize + 1
    while i > 0 $\wedge$ heap[PARENT(i)] < heap[i] **do**
        SWAP(heap[i],heap[PARENT(i)])
        i $\leftarrow$ PARENT(i)
    end while
end function

Operations

extract-max!

function EXTRACT-MAX!(heap)
    max ← heap[0]
    heap[0] ← heap[heap.heapsize-1]
    heap.heapsize ← heap.heapsize - 1
    MAX-HEAPIFY(heap,0)
    return max
end function

Complexity analysis

insert!

- at most h calls to SWAP

$$\implies \Theta(\log(N))$$

extract-max!

- same as MAX-HEAPIFY

$$\implies \Theta(\log(N))$$

Heapsort

```
function HEAPSORT(array)
    BUILD-MAX-HEAP(array)
    while array.heapsize > 0 do
        i ← array.heapsize
        array[i] ← EXTRACT-MAX!(array)
    end while
    return array
end function
```

Complexity analysis

- N calls to EXTRACT-MAX!
- each call takes $O(\log N)$ time

$$\implies O(N \log N)$$

- worst case, the first $\frac{N}{2}$ calls to EXTRACT-MAX! each do $\lfloor \log N \rfloor$ work

$$\implies \Theta(N \log N)$$

Priority queues

A priority queue tracks items along with priorities, and provides access to the highest-priority item.

maximum	return the highest-priority item
extract-max!	remove and return the highest-priority item
insert![o]	insert an item into the priority queue

(exactly the same as the heap operations)

Work

1. Reading
   - CLRS, chapter 6
2. Questions from CLRS

   6-1 Building a heap using insertion
3. Lab work

   3.1 (week of 28th January) implement an implicit heap class, with methods for:
   - computing the parent and children indices from a given index
   - constructing a heap in-place from a provided array input
   - inserting items into the heap (maintaining the heap property)
   - removing and returning the maximum element from the heap (maintaining the heap property)
   - performing heapsort

   3.2 (week of 28th January) measure the difference in operations between constructing a heap in-place and by repeated insertions. When (if ever) does the difference in scaling become noticeable?