Multiplication

Goldsmiths Computing

January 13, 2019

Motivation

- working with numbers as a data structure
- everyone knows how to multiply
- almost no-one knows how to multiply efficiently

Previously

Numbers as array of digits (binary: bits)

- numbers have a *width* w, at least $1 + \log_b(n)$

Logical operations

and(x,y)	return the bitwise logical and of x and y
xor(x,y)	return the bitwise exclusive-or of x and y

Arithmetic operations

add(x,y)	return the sum of x and y
sub(x,y)	return the difference between x and y
shift(x,n)	return x multiplied by the base n times

Complexity

- until now, $\Theta(1)$
- in fact, $\Theta(w) \sim \Theta(log(n))$

(logarithmic factor is usually irrelevant, or width is taken as constant)

Problem

Given these basic operations:

- how do we implement multiplication?
- how efficient is it?

Example

123×135:

Primary (old-)school multiplication

$$
\begin{array}{cccccc}
 & & & 1 & 2 & 3 \\
\times & & & 1 & 3 & 5 \\
\hline
 & & & & 1 & 5 \\
 & & & 1 & 0 & \\
 & & & 5 & & \\
 & & & & 9 & \\
 & & & 6 & & \\
 & & 3 & & & \\
 & & & 3 & & \\
 & & 2 & & & \\
 & 1 & & & & \\
\hline
 & 1 & 6 & 6 & 0 & 5
\end{array}
$$

Example

123×135:

Primary school multiplication

×	100	20	3
100	10000	2000	300
30	3000	600	90
5	500	100	15

Complexity analysis

time

For each digit in x

- multiply with each digit in y.

Assume x and y are each of width w

$$\implies \Theta(w^2)$$

Divide and conquer

Write

- $x = x_{hi} \times b^{w/2} + x_{lo}$
- $y = y_{hi} \times b^{w/2} + y_{lo}$

Then $x \times y = x_{hi}y_{hi}\, b^{w} + (x_{hi}y_{lo} + x_{lo}y_{hi})b^{w/2} + x_{lo}y_{lo}$

Example

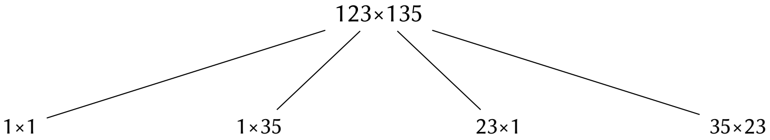
$$x \times y = x_{hi} y_{hi}\, b^{w} + (x_{hi} y_{lo} + x_{lo} y_{hi}) b^{w/2} + x_{lo} y_{lo}$$

$$\begin{aligned}
123 \times 135 &= (1 \times 100 + 23) \times (1 \times 100 + 35) \\
&= 1 \times 1 \times 10000 + (23 \times 1 + 1 \times 35) \times 100 + 23 \times 35 \\
&= 10000 + 5800 + 805
\end{aligned}$$

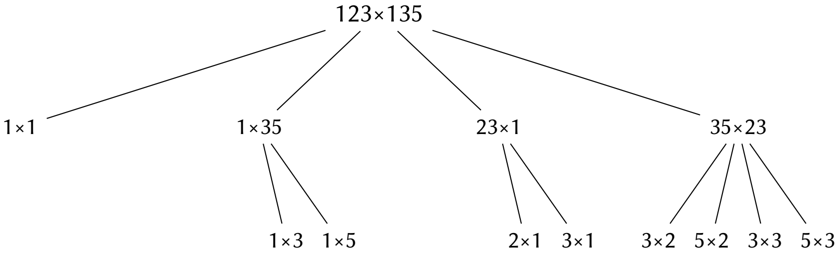
Recursion tree

123×135

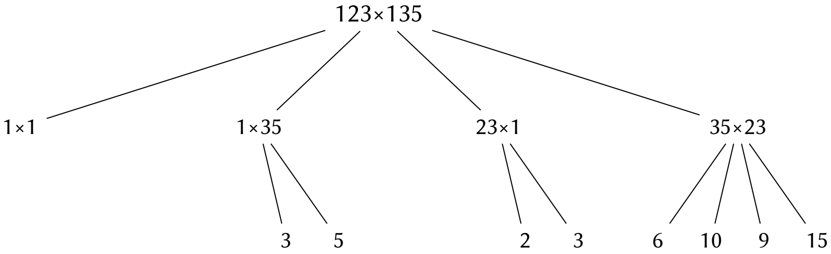
Recursion tree



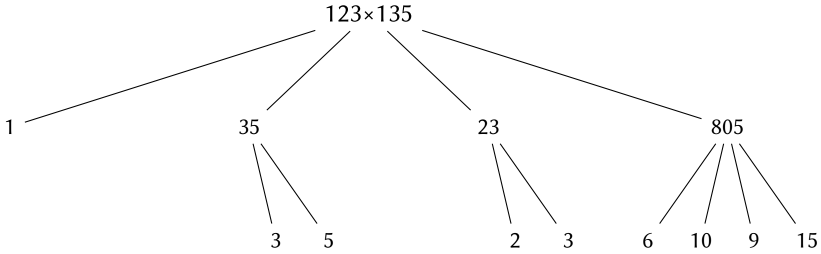
Recursion tree



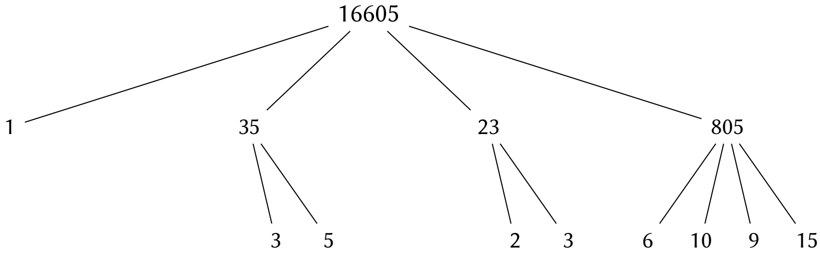
Recursion tree



Recursion tree



Recursion tree



Complexity analysis

time

$x \times y = x_{hi} y_{hi}\, b^{w} + (x_{hi} y_{lo} + x_{lo} y_{hi}) b^{w/2} + x_{lo} y_{lo}$

- $T_w = 4T_{w/2} + O(w)$

solution

Use the master theorem:

- $a = 4,\ b = 2,\ c = 1 \implies$ case 1
- $\log_b a = \log_2(4) = 2$

$$\implies T(w) \in \Theta(w^2)$$

Divide and conquer, Karatsuba

Still with

- $x = x_{hi} \times b^{w/2} + x_{lo}$
- $y = y_{hi} \times b^{w/2} + y_{lo}$

Calculate

- $z_{hi} = x_{hi} \times y_{hi}$
- $z_{lo} = x_{lo} \times y_{lo}$
- $z_c = (x_{hi} + x_{lo}) \times (y_{hi} + y_{lo})$

Then $x \times y = z_{hi}\, b^{w} + (z_c - z_{hi} - z_{lo}) b^{w/2} + z_{lo}$

Example

123×135:

- $z_{hi} = 1$; $z_{lo} = 35 \times 23 = 805$; $z_c = 36 \times 24 = 864$

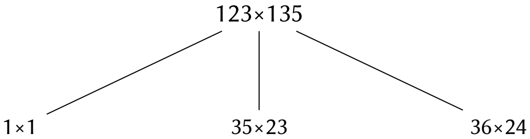
so:

$$\begin{aligned} 123 \times 135 &= 1 \times 10000 + (864 - 805 - 1) \times 100 + 805 \\ &= 10000 + 5800 + 805 \\ &= 16605 \end{aligned}$$

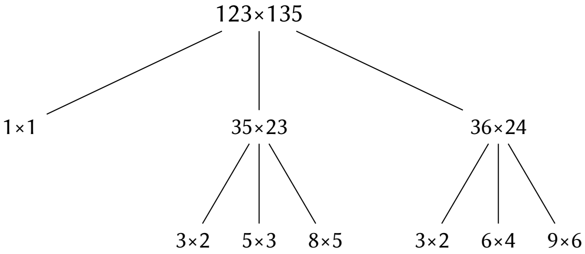
Recursion tree

123×135

Recursion tree



Recursion tree



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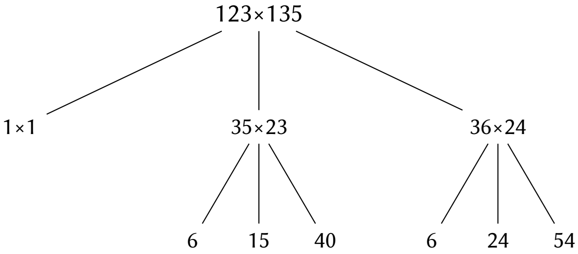

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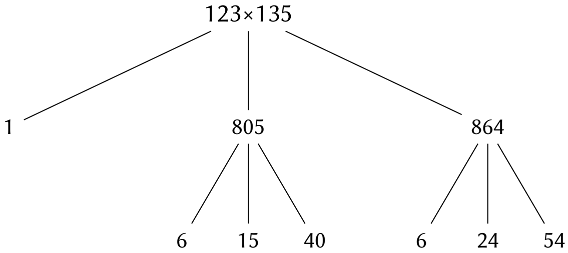

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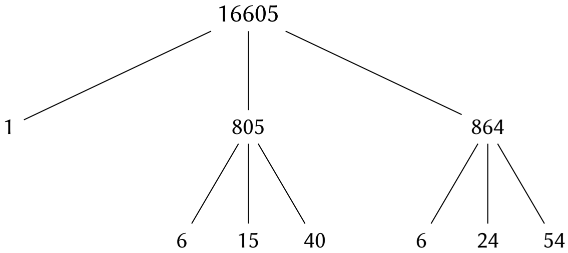

Complexity analysis

time

$x \times y = z_{hi}\, b^{w} + (z_c - z_{hi} - z_{lo}) b^{w/2} + z_{lo}$

- $T_w = 3T_{w/2} + O(w)$

solution

Use the master theorem:

- a = 3, b = 2, c = 1 $\Longrightarrow$ case 1
- $\log_b a = \log_2(3) = 1.58$

$$\Longrightarrow T(w) \in \Theta(w^{1.58})$$

Work

1. Reading
    - CLRS, section 4.2: matrix multiplication