James R. Munkres - Topology (2nd edition)
Answers to Selected Exercises

Marc-André Brochu

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Connectedness and Compactness

3.23 Connected Spaces

$\boxed{\textbf{3.23.2}}$ Suppose C, D are two open sets forming a separation of $A = \bigcup A_n$. By Lemma 23.2, we know that for each n, the subspace A_n is either completely contained in C or completely contained in D.

Let $J = \{i \in \mathbb{N} \mid A_i \subseteq D\}$. Because the natural numbers are well-ordered, there exists a least element of J, say k. Without loss of generality, $A_0 \subseteq C$; therefore $k > 0$. By minimality of k, we have $A_k \subseteq D$ while $A_{k-1} \subseteq C$. However, $A_{k-1} \cap A_k \neq \varnothing$ by hypothesis! This contradicts the fact that $C \cap D \neq \varnothing$. From this we conclude that A is in fact connected.

$\boxed{\textbf{3.23.6}}$ Let $S = C \cap A$ and $T = C \cap (X - A)$. From the exercise statement we know that S and T are nonempty; moreover $C = S \cup T$. If neither S nor T contains a limit point of the other, then C is not connected (according to Lemma 23.1). Therefore, since C is actually connected, either (1) T contains a limit point of S, or (2) S contains a limit point of T.

In the first case, the set $\overline{S} \cap T$ is nonempty. Because

$$\overline{S} \cap T = \overline{C \cap A} \cap C \cap (X - A) \subseteq C \cap \overline{A} \cap \overline{X - A} = C \cap \mathrm{Bd}(A),$$

we have that C intersects $\mathrm{Bd}(A)$.

In the second case, the set $S \cap \overline{T}$ is nonempty. Since

$$S \cap \overline{T} = C \cap A \cap \overline{C \cap (X - A)} \subseteq C \cap \overline{A} \cap \overline{X - A} = C \cap \mathrm{Bd}(A),$$

we also have that C intersects $\mathrm{Bd}(A)$.

$\boxed{\textbf{3.23.11}}$ Suppose C, D are two open sets forming a separation of X. Take some $y \in Y$ such that $p^{-1}(\{y\})$ intersects C. By hypothesis, $p^{-1}(\{y\})$ is a connected subspace of X. In this case, Lemma 23.2 tells us that $p^{-1}(\{y\})$ is completely contained in C. Thus C is a saturated open set of X (i.e. it is the preimage of a subset of Y). A similar argument shows that D is also saturated.

Since p is a quotient map, it maps saturated open sets of X to open sets of Y. Hence $p(C)$ and $p(D)$ are (disjoint*, nonempty) open sets of Y. Because p is surjective and X is the union of C and D, we have $Y = p(C) \cup p(D)$. This is in contradiction with the hypothesis that Y is connected! Therefore X must be connected.

*Maybe it is not clear why $p(C)$ and $p(D)$ are disjoint. The reason is this: suppose they are not and take some y in their intersection. Then there exists $c \in C$ and $d \in D$ such that $p(c) = p(d) = y$. As a consequence $p^{-1}(\{y\})$ contains both c and d, that is, it intersects both C and D. But this is in contradiction with C and D being saturated sets! Hence there cannot be anything in common between $p(C)$ and $p(D)$.