

Image

Combinatoire

Author: CatMono

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Preface

This is the preface of the book...

Chapter 1 Basic Counting Principles

1.1 Addition and Multiplication Principles

1.2 Bijection Principle

1.3 Permutations and Combinations

Definition 1.1 (Permutation and Combination)

Let n be a non-negative integer, and k be an integer such that $0 \leq k \leq n$. The number of ways to choose k elements from a set of n distinct elements and arrange them in a specific order is called the number of permutations of n elements taken k at a time, denoted as $P(n, k)$ (or ${}_n P_k$, P_n^k or A_n^k). It is given by the formula:

$$P(n, k) = \frac{n!}{(n - k)!}.$$

The number of ways to choose k elements from a set of n distinct elements without regard to the order of selection is called the number of combinations of n elements taken k at a time, denoted as $C(n, k)$ (or ${}_n C_k$, C_n^k or $\binom{n}{k}$). It is given by the formula:

$$C(n, k) = \frac{n!}{k!(n - k)!} = \binom{n}{k}.$$



Property The following properties hold for permutations and combinations:

1. $A_n^0 = 1$ and $A_n^n = n!$.
2. $C_n^0 = 1$ and $C_n^n = 1$.
3. $C_n^k = C_n^{n-k}$.
4. $A_n^k = k! \cdot C_n^k$.
5. $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$ (**Pascal's triangle/YangHui's triangle**)¹.

In Pascal's triangle, each element is equal to the sum of the two elements directly above it, as shown in Figure 1.1.

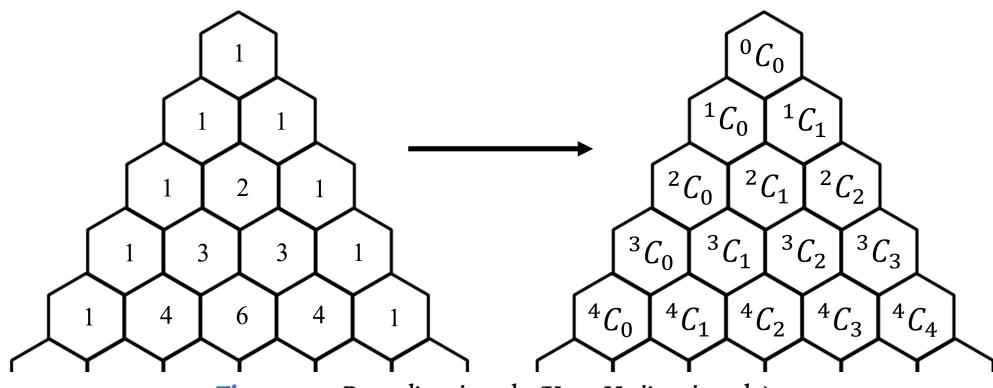


Figure 1.1: Pascal's triangle (YangHui's triangle).

¹This property can also be understood that to choose k elements from $n + 1$, you can first take one element A :

- The number of ways that include A is C_n^{k-1} ;
- The number of ways that does not include A is C_n^k .

$$6. (a+b)^n = \sum_{k=0}^n C_n^k a^k b^{n-k} \text{ (**Binomial theorem**)}$$

Therefore, we can see the relationship between Pascal's triangle and the Binomial theorem, as shown in Figure 1.2. Here, C_n^k is the element in the n -th row and k -th column of Pascal's triangle.

Exponent	Pascal's Triangle	Binomial Expansion
0	1	$(a+b)^0 = 1$
1	1 1	$(a+b)^1 = 1a + 1b$
2	1 2 1	$(a+b)^2 = 1a^2 + 2ab + 1b^2$
3	1 3 3 1	$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$
4	1 4 6 4 1	$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$
5	1 5 10 10 5 1	$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

Figure 1.2: Pascal's triangle and Binomial theorem.

Chapter 2 Binomial Coefficients

Chapter 3 Recurrence Relations and Generating Functions

Chapter 4 Inclusion-Exclusion Principle

4.1 Inclusion-Exclusion Principle

Theorem 4.1 (Inclusion-Exclusion Principle)

Let A_1, A_2, \dots, A_n be finite sets. Then the number of elements in the union of these sets is given by:

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|.$$

Denote $S_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$ for $k = 1, 2, \dots, n$. Then the formula can be rewritten as:

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{k=1}^n (-1)^{k+1} S_k.$$



Specially, when $n = 2$, we have:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

When $n = 3$, we have:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

The inclusion-exclusion principle also has a more commonly used form, **the complement form (property counting method)**:

Let U be the universal set that contains all the elements under consideration, and $\overline{A_i} = U \setminus A_i$ be the complement of A_i in U . Then we have:

$$\left| \bigcup_{i=1}^n A_i \right| = |U| - \left| \bigcap_{i=1}^n \overline{A_i} \right| = |U| - \sum_{k=0}^n (-1)^k S_k.$$

4.2 Möbius Inversion

4.3 Generalizations of Inclusion-Exclusion

Chapter 5 Special Counting Sequences

5.1 Catalan Numbers

Definition 5.1 (Catalan Numbers)

The n -th Catalan number C_n is given by:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \binom{2n}{n} - \binom{2n}{n+1}.$$

The first ten Catalan numbers are:

$$\begin{aligned} C_0 &= 1, & C_1 &= 1, & C_2 &= 2, & C_3 &= 5, & C_4 &= 14, \\ C_5 &= 42, & C_6 &= 132, & C_7 &= 429, & C_8 &= 1430, & C_9 &= 4862. \end{aligned}$$



Property The Catalan numbers satisfy multiple recurrence relations:

1.

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad (n \geq 1), \quad C_0 = 1.$$

This recurrence relation reflects the self-similarity of Catalan numbers.

2.

$$C_n = \frac{2(2n-1)}{n+1} C_{n-1} \quad (n \geq 1), \quad C_0 = 1.$$

This recurrence relation can be derived from the closed-form expression of Catalan numbers.

3. Let $G(x) = \sum_{n=0}^{\infty} C_n x^n$ be the generating function of Catalan numbers. Then $G(x)$ satisfies the functional equation:

$$G(x) = 1 + xG(x)^2,$$

id est,

$$G(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

This functional equation can be used to derive the closed-form expression of Catalan numbers using the Lagrange inversion formula.

Catalan numbers is the answer to many combinatorial problems,

Ballot problem There is an $n \times n$ grid graph, with the bottom-left corner at $(0, 0)$ and the top-right corner at (n, n) . Starting from the bottom-left corner, and *moving only right or up one unit at each step*, the total number of paths to reach the top-right corner without going above the diagonal $y = x$ (but allowing touching it) is denoted as C_n .

Dyck path counting problem A Dyck path of semilength n is a lattice path from $(0, 0)$ to $(2n, 0)$ that never dips below the x -axis and consists of steps $(1, 1)$ (up step) and $(1, -1)$ (down step). The number of Dyck paths of semilength n is C_n .

Counting non-intersecting chords in a circle There are $2n$ points on a circle. The number of ways to pair these points with n chords such that no two chords intersect is the Catalan number C_n .

Triangulation counting problem The number of ways to divide a convex $(n+2)$ -sided region into triangular

regions without intersecting diagonals is C_n .

Binary Tree Counting Problem The number of structurally different binary trees with n nodes is C_n . Equivalently, the number of structurally different full binary trees with n non-leaf nodes is C_n .

Counting problem of parenthesis sequences The number of valid parenthesis sequences consisting of n pairs of parentheses is C_n .

Stack popping sequence counting problem The push sequence of a stack (of infinite size) is $1, 2, \dots, n$, and the number of valid popping sequences is C_n .

Sequence Counting Problem The number of sequences a_1, a_2, \dots, a_{2n} consisting of $n+1$'s and $n-1$'s such that the partial sums satisfy $a_1 + a_2 + \dots + a_k \geq 0$ ($k = 1, 2, 3, \dots, 2n$) is C_n .

5.2 Stirling Numbers

5.3 Pigeonhole Principle

Chapter 6 Extremal Principle

Chapter 7 Ramsey Theory

Chapter 8 Design Theory

Chapter 9 Pólya Counting

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