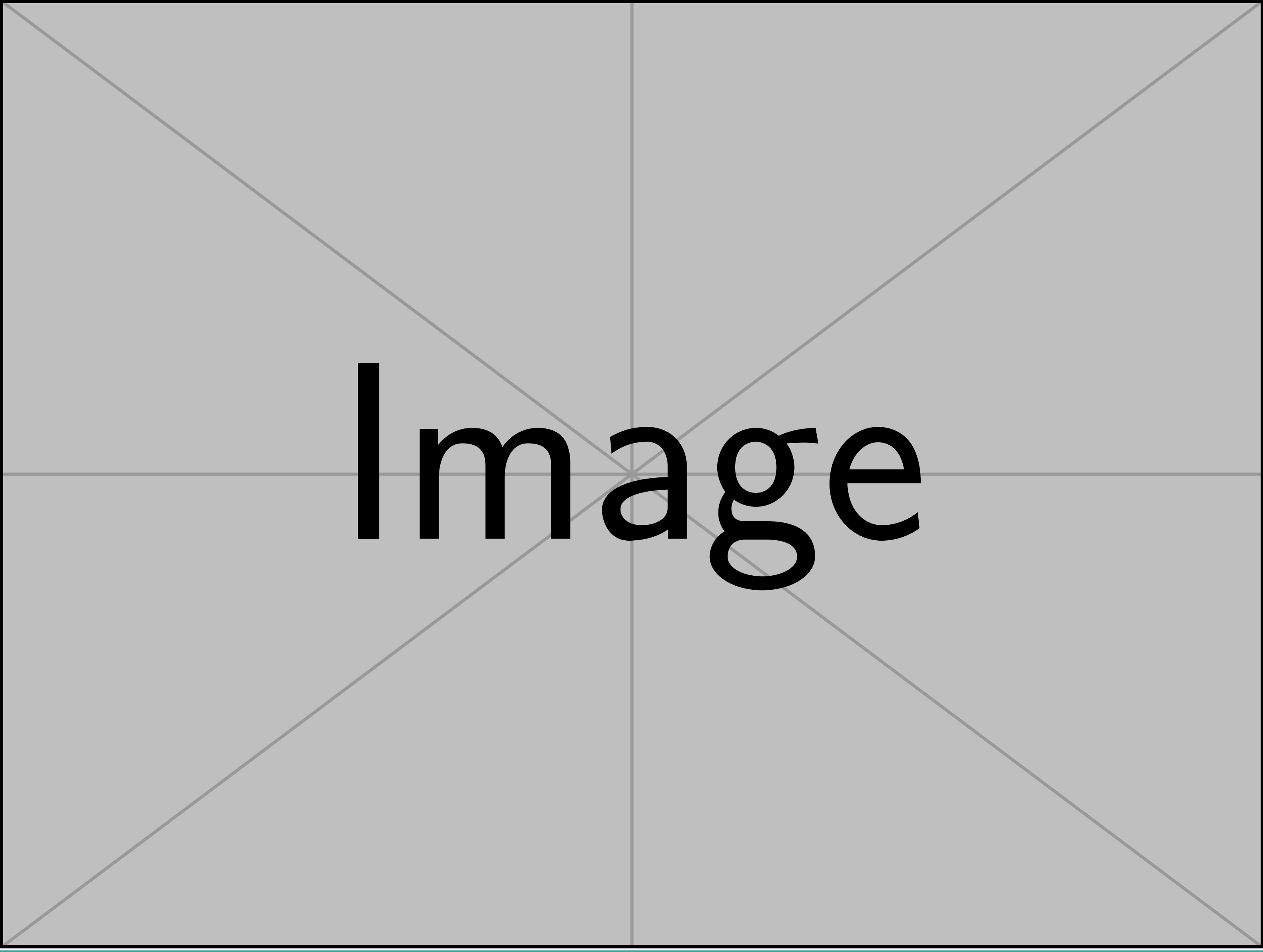

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Contents

Preface

This is the preface of the book…

Chapter 1 Lebesgue Measure

Definition 1.1 (Measurable Space)

A **measurable space** is a pair (X, Σ_X) where X is a set and Σ_X is a σ-algebra of X. ♣

Definition 1.2 (Measure)

Let (X, Σ_X) be a measurable space. A **measure** on (X, Σ_X) is a function $\mu : \Sigma_X \to [0, +\infty]$ that satisfies the following properties:

(i) $\mu(\emptyset) = 0$ (Null empty set);

(ii) (Countable Additivity/σ-additivity) For any countable collection $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in Σ_X,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

♣

Definition 1.3 (Measure Space)

A **measure space** is a triple (X, Σ_X, μ) where (X, Σ_X) is a measurable space and μ is a measure on (X, Σ_X). ♣

Caution *Note the distinction between measurable space and measure space: A measurable space is simply a pair of a set and its σ-algebra, while a measure space adds the concept of measure on top of that.*

1.1 Outer Measure and Measurable Sets

¶ `Outer Measure`

Definition 1.4 (Lebesgue Outer Measure)

Let $E \subseteq \mathbb{R}^{\varkappa}$. If $\{I_k\}$ is a countable collection of open rectangles such that $E \subseteq \bigcup_k I_k$, then $\{I_k\}$ is called a L-**cover** of E.

The **Lebesgue outer measure** (simplified outer measure) of E is defined as

$$m^*(E) = \inf\left\{\sum_k |I_k| : \{I_k\} \text{ is an } L\text{-cover of } E\right\},$$

where $|I_k|$ denotes the volume of the rectangle I_k.

If for any L-cover $\{I_k\}$ of E,

$$\sum_k |I_k| = +\infty,$$

we define $m^*(E) = +\infty$. Otherwise, $m^*(E)$ is a non-negative real number. ♣

Sets with outer measure zero are called **null sets**.

Remark In modern real analysis education, Lebesgue outer measure is the foundational tool for defining Lebesgue measure. Although the concept of Lebesgue inner measure also exists, it is used less frequently.

Inner measure can aid intuition in finite measure spaces or those with topological structures (e.g., calculating Lebesgue measure), but due to its dependence on external conditions, inability to naturally define σ-algebras, and lack of general measurability criteria, it has been supplanted by the Carathéodory method in foundational theory.

Property

Non-negativity *For any* $E \subseteq \mathbb{R}^\kappa$, $m^*(E) \geq 0$.

Monotonicity *If* $A \subseteq B \subseteq \mathbb{R}^\kappa$, *then* $m^*(A) \leq m^*(B)$.

Countable Subadditivity *For any countable collection* $\{A_i\}_{i=1}^\infty$ *of subsets of* $\mathbb{R}^\kappa$,

$$m^*\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty m^*(A_i).$$

Theorem 1.1 (Translation Invariance)

For any $E \subseteq \mathbb{R}^\kappa$ and any vector $x_0 \in \mathbb{R}^\kappa$,

$$m^*(E + x_0) = m^*(E),$$

where $E + x = \{x + x_0 : x \in E\}$.

¶ Carathéodory Measurability Criterion

Theorem 1.2 (Carathéodory Measurability Criterion)

A set $E \subseteq \mathbb{R}^\kappa$ is **Carathéodory measurable** (or simply **measurable**) if for every set $T \subseteq \mathbb{R}^\kappa$,

$$m^*(T) = m^*(T \cap E) + m^*(T \cap E^c).$$

The collection of all measurable sets forms a σ-algebra, denoted by $\mathscr{M}$. Thus, $(\mathbb{R}^\kappa, \mathscr{M}, m)$ constitutes a measure space, where m is the Lebesgue measure.

Definition 1.5 (Continuity of Measure for Increasing Sequences)

Let $\{E_i\}_{i=1}^\infty$ be a sequence of measurable sets such that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$. Then,

$$m\left(\bigcup_{i=1}^\infty E_i\right) = \lim_{i\to\infty} m(E_i).$$

Similarly, for a sequence of measurable sets $\{F_i\}_{i=1}^\infty$ such that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ and $m(F_1) < +\infty$,

$$m\left(\bigcap_{i=1}^\infty F_i\right) = \lim_{i\to\infty} m(F_i).$$

1.2 Measurable Sets and Borel Sets

Lemma 1.1 (Carathéodory Lemma)

Let $G \neq \mathbb{R}^n$ be an open set, $E \subset G$. Let

$$E_k = \{x \in E : d(x, G^c) \geq 1/k\}, \quad k = 1, 2, \ldots$$

Then each E_k is closed and

$$m^*(E) = \lim_{k\to\infty} m^*(E_k).$$

Chapter 2 Measurable Functions

Chapter 3 Lebesgue Integral

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