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Géométrie Analytique

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Preface

This is the preface of the book...

Chapter 1 Preliminaries

Chapter 2 Coordinates and Vectors

2.1 Coordinate Systems

Definition 2.1 (Coordinate Frame)

A fixed point O in \mathbb{R}^3 space, together with three non-coplanar ordered vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, is called a **coordinate frame** (or **reference frame**) in space, denoted by $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors, then the frame is called a **Cartesian frame**. Furthermore, if $\mathbf{e}_1 \perp \mathbf{e}_2, \mathbf{e}_2 \perp \mathbf{e}_3, \mathbf{e}_3 \perp \mathbf{e}_1$, then the frame is called a **rectangular Cartesian frame**, or simply a **rectangular frame**.

Generally, $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called **affine frame**.



2.2 Theorems about Vectors

2.3 Products of Vectors

- ¶ Inner Product (Dot Product)
- ¶ Outer Product (Cross Product)
- ¶ Mixed Product
- ¶ Double Cross Product

2.4 Linear Independence

Chapter 3 Locus and Equation

3.1 Parametric Equations

3.2 Common Curves and Surfaces

Chapter 4 Planes and Space Lines

4.1 Equations of Planes

Point-Vector Form

In space, fix a point $M_0 = (X_0, Y_0, Z_0)$ and two non-collinear vectors $\mathbf{a} = (X_1, Y_1, Z_1)$ and $\mathbf{b} = (X_2, Y_2, Z_2)$. The equation of the plane passing through the point M_0 and parallel to the vectors \mathbf{a} and \mathbf{b} is given by:

$$\mathbf{r} = O\vec{M} + \lambda\mathbf{a} + \mu\mathbf{b},$$

or in coordinate form:

$$\begin{cases} x = X_0 + \lambda X_1 + \mu X_2 \\ y = Y_0 + \lambda Y_1 + \mu Y_2 \\ z = Z_0 + \lambda Z_1 + \mu Z_2 \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$.

Taking the dot product of both sides of the parametric vector equation with $\mathbf{a} \times \mathbf{b}$, we eliminate λ and μ to obtain $(\mathbf{r} - O\vec{M}_0, \mathbf{a}, \mathbf{b}) = 0$, that is,

$$\begin{vmatrix} x - X_0 & y - Y_0 & z - Z_0 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0. \quad (4.1)$$

All above forms are called the **point-vector form** of the plane equation.

Given three non-collinear points $M_1(X_1, Y_1, Z_1)$, $M_2(X_2, Y_2, Z_2)$ and $M_3(X_3, Y_3, Z_3)$, the equation of the plane passing through these three points is given by:

$$\mathbf{r} = O\vec{M}_1 + \lambda M_1\vec{M}_2 + \mu M_1\vec{M}_3,$$

or in coordinate form:

$$\begin{cases} x = X_1 + \lambda(X_2 - X_1) + \mu(X_3 - X_1) \\ y = Y_1 + \lambda(Y_2 - Y_1) + \mu(Y_3 - Y_1) \\ z = Z_1 + \lambda(Z_2 - Z_1) + \mu(Z_3 - Z_1) \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$. And the determinant form is:

$$\begin{vmatrix} x - X_1 & y - Y_1 & z - Z_1 \\ X_2 - X_1 & Y_2 - Y_1 & Z_2 - Z_1 \\ X_3 - X_1 & Y_3 - Y_1 & Z_3 - Z_1 \end{vmatrix} = 0,$$

or equivalently,

$$\begin{vmatrix} x & y & z & 1 \\ X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \end{vmatrix} = 0.$$

All above forms are also called the **three-point form** of the plane equation.

If plane intersects the three coordinate axes at $M_1(X_1, 0, 0)$, $M_2(0, Y_2, 0)$, $M_3(0, 0, Z_3)$ (where $X_1, Y_2, Z_3 \neq 0$),

0), then the equation of the plane can be expressed in the form:

$$\frac{x}{X_1} + \frac{y}{Y_2} + \frac{z}{Z_3} = 1,$$

which is called the **intercept form** of the plane equation.

General Form

The general equation is obtained by expanding the determinant form of the parametric equation 4.1 of a plane:

$$Ax + By + Cz + D = 0,$$

where

$$A = \begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}, \quad B = \begin{vmatrix} Z_1 & X_1 \\ Z_2 & X_2 \end{vmatrix}, \quad C = \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix}, \quad D = - \begin{vmatrix} X_0 & Y_0 & Z_0 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}.$$

Special cases include:

Theorem 4.1

Any plane in space can be represented by a linear equation in three variables x , y , and z , and conversely, every such equation represents a plane in space.



Point-Normal Form

Given a point $M_0(X_0, Y_0, Z_0)$ on the plane and a normal vector $\mathbf{n} = (A, B, C)$ of the plane, the equation of the plane can be expressed as:

$$\mathbf{n} \cdot (\mathbf{r} - O\vec{M}_0) = 0,$$

or in coordinate form:

$$A(x - X_0) + B(y - Y_0) + C(z - Z_0) = 0.$$

If a perpendicular is drawn from the origin to the plane, with the foot of the perpendicular being $M_0(X_0, Y_0, Z_0)$, and the unit normal vector of the plane being $\mathbf{n}_0 = (\cos \alpha, \cos \beta, \cos \gamma)$, then

$$\mathbf{n} \cdot \mathbf{r} - O\vec{M}_0 = 0,$$

or in coordinate form:

$$x \cos \alpha + y \cos \beta + z \cos \gamma - |O\vec{M}_0| = 0. \quad (4.2)$$

For the general equation of a plane, it can be converted into the form 4.2 by simply multiplying by a **normalization factor** λ , where:

$$|\lambda| = \frac{1}{|\mathbf{n}|} = \frac{1}{\sqrt{A^2 + B^2 + C^2}},$$

and λ has the opposite sign as D .

4.2 Linear Equations

Point-Vector Form

Given a point $M_0(X_0, Y_0, Z_0)$ in space and a direction vector $\mathbf{v} = (X, Y, Z)$ of the line, then

$$\mathbf{r} = O\vec{M}_0 + \lambda \mathbf{v},$$

and the parametric equations of the line can be expressed as:

$$\begin{cases} x = X_0 + \lambda X \\ y = Y_0 + \lambda Y \\ z = Z_0 + \lambda Z \end{cases}$$

where $\lambda \in \mathbb{R}$. Eliminate the parameter λ to obtain the symmetric equation (standard equation):

$$\frac{x - X_0}{X} = \frac{y - Y_0}{Y} = \frac{z - Z_0}{Z}. \quad (4.3)$$

Given two points $M_1(X_1, Y_1, Z_1)$ and $M_2(X_2, Y_2, Z_2)$ in space, the equation of the line passing through these two points is given by:

$$\mathbf{r} = O\vec{M}_1 + \lambda M_1\vec{M}_2,$$

or in coordinate form:

$$\begin{cases} x = X_1 + \lambda(X_2 - X_1) \\ y = Y_1 + \lambda(Y_2 - Y_1) \\ z = Z_1 + \lambda(Z_2 - Z_1) \end{cases}$$

It can also be expressed in symmetric form:

$$\frac{x - X_1}{X_2 - X_1} = \frac{y - Y_1}{Y_2 - Y_1} = \frac{z - Z_1}{Z_2 - Z_1}.$$

The coordinates of the direction vector of a line, X, Y, Z , or a set of numbers proportional to it, l, m, n ($l : m : n = X : Y : Z$), are called the **direction numbers** of the line.

General Form

The intersection of two planes determines a line:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases}$$

where $A_1 : B_1 : C_1 \neq A_2 : B_2 : C_2$

Theorem 4.2

Any line in space can be represented by a system of two linear equations in three variables x, y , and z , and conversely, every such system represents a line in space.



Projection Form

In the symmetric equation 4.3 of a line, X, Y , and Z are not all zero. Without loss of generality, let us assume Z is not zero. Then, we have:

$$\begin{cases} x = az + c \\ y = bz + d, \end{cases}$$

where $a = \frac{X}{Z}$, $b = \frac{Y}{Z}$, $c = X_0 - \frac{X}{Z}Z_0$, and $d = Y_0 - \frac{Y}{Z}Z_0$. This form is called the **projection form** of the line equation. This line can be regarded as the intersection line of the two planes represented by these two equations. These planes are respectively parallel to the y -axis and x -axis, and perpendicular to the xOz and yOz coordinate planes.

4.3 Relative Positions of Points, Lines and Planes

4.4 Pencil of Planes and Lines

Chapter 5 Common Surfaces

5.1 Cylinder Surfaces

Definition 5.1 (Cylinder Surface)

In space, the surface generated by a family of parallel lines, which are parallel to a fixed direction (the direction of the cylinder

$$\mathbf{s} = (X : Y : Z)$$

) and intersect a fixed curve (the directrix

$$\Gamma : \begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0, \end{cases}$$

), is called a **cylinder**.



Cylinder can be expressed as:

$$\begin{aligned} & \{(x, y, z) | F(x, y, z) = 0\} \\ &= \bigcup_{(x_1, y_1, z_1) \in \Gamma} \{(x, y, z) | \frac{x - x_1}{X} = \frac{y - y_1}{Y} = \frac{z - z_1}{Z}\} \\ & \xrightarrow{\text{directrix } \Gamma: \mathbf{r}(u) = (x(u), y(u), z(u))} \{\mathbf{r} | \mathbf{r} = \mathbf{r}(u) + v\mathbf{s}, \quad u, v \in \mathbb{R}\} \\ & \xrightarrow{\text{parametric form}} \left\{ (x, y, z) | \begin{cases} x = x(u) + vX, \\ y = y(u) + vY, \\ z = z(u) + vZ, \end{cases} \quad u, v \in \mathbb{R} \right\}. \end{aligned}$$

To solve the equation of a cylinder,

$$\begin{cases} \frac{x - x_1}{X} = \frac{y - y_1}{Y} = \frac{z - z_1}{Z}, \\ F_1(x_1, y_1, z_1) = 0, \\ F_2(x_1, y_1, z_1) = 0. \end{cases}$$

Some special cases of cylinders (in 5.1):

Elliptical Cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hyperbolic Cylinder $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parabolic Cylinder $y^2 = 2px$

Their equations are all quadratic, so they are collectively called **quadratic cylinders**.

When a plane intersects a elliptical cylinder to form an ellipse (or a circle), the following rules apply:

1. The center of the resulting ellipse lies on the axis of the cylindrical surface.
2. $b = R$, meaning the length of the semi-minor axis of the ellipse is equal to the radius of the cylindrical surface.
3. $\sin \theta = \frac{R}{a}$, where a represents the length of the semi-major axis of the ellipse, and θ represents the angle between the axis of the cylinder and the plane.

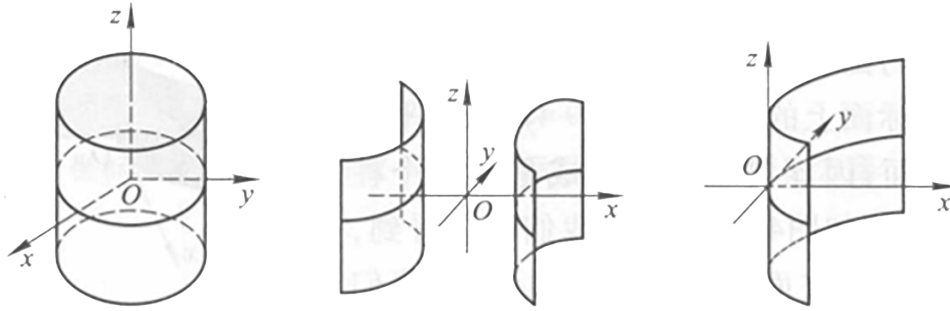



Figure 5.1: Cylinder Surface.

Theorem 5.1

In a spatial Cartesian coordinate system (as well as in an affine coordinate system), a surface represented by a ternary equation containing only two variables (coordinates) is a cylinder whose generatrices are parallel to the coordinate axis corresponding to the missing variable (coordinate).

**Example 5.1**

 **Note** Two methods to prove that an equation is a cylindrical surface:

1. Rewrite as a product equation using determinants and linear systems.
2. Take the directrix, set the direction, solve, and compare.

5.2 Cone Surfaces

Definition 5.2 (Cone Surface)

In space, the surface generated by a family of lines passing through a fixed point (the vertex

$$A : \mathbf{r}_0 = (x_0, y_0, z_0)$$

) and intersecting a fixed curve (the directrix

$$\Gamma : \begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0, \end{cases}$$

), is called a **cone**.



Cone can be expressed as:

$$\begin{aligned} & \{(x, y, z) | F(x, y, z) = 0\} \\ &= \bigcup_{(x_1, y_1, z_1) \in \Gamma} \{(x, y, z) | \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}\} \\ & \underline{\underline{\text{directrix } \Gamma: \mathbf{r}(u) = (x(u), y(u), z(u))}} \{ \mathbf{r} | \mathbf{r} = \mathbf{r}_0 + v(\mathbf{r}(u) - \mathbf{r}_0), \quad u, v \in \mathbb{R} \} \\ & \underline{\underline{\text{parametric form}}} \left\{ (x, y, z) | \begin{cases} x = x_0 + v(x(u) - x_0), \\ y = y_0 + v(y(u) - y_0), \\ z = z_0 + v(z(u) - z_0), \end{cases} \quad u, v \in \mathbb{R} \right\}. \end{aligned}$$

To solve the equation of a cone,

$$\begin{cases} \frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}, \\ F_1(x_0, y_0, z_0) = 0, \\ F_2(x_0, y_0, z_0) = 0. \end{cases}$$

Theorem 5.2

A homogeneous equation in x, y, z always represents a cone with its vertex at the origin. That is, a homogeneous equation in $x - x_0, y - y_0, z - z_0$ always represents a cone with its vertex at (x_0, y_0, z_0) .



5.3 Surfaces of Revolution

Definition 5.3 (Surface of Revolution)

In space, the surface generated by rotating a curve (the generatrix

$$\Gamma : \begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0, \end{cases}$$

) around a fixed straight line (the axis of revolution

$$l : \frac{x - x_0}{X} = \frac{y - y_0}{Y} = \frac{z - z_0}{Z}$$

) is called a **surface of revolution**.

Any point $M_1(x_1, y_1, z_1)$ on the generatrix Γ of a surface of revolution generates a circle upon rotation, which is called a **parallel**; the intersection of the surface with each half-plane bounded by l is called a **meridian**.



Surface of revolution can be expressed as:

$$\begin{aligned} & \{(x, y, z) | F(x, y, z) = 0\} \\ &= \bigcup_{(x_1, y_1, z_1) \in \Gamma} \left\{ (x, y, z) \mid \begin{cases} X(x - x_1) + Y(y - y_1) + Z(z - z_1) = 0, \\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \end{cases} \right\} \end{aligned}$$

Taking the plane of the directrix as the coordinate plane and the axis of rotation as the coordinate axis, the equation of the surface of revolution assumes a special form (see 5.2). As shown in the figure, the generatrix is

$$\Gamma : \begin{cases} F(y, z) = 0, \\ x = 0. \end{cases}$$

The equation obtained by rotating around the y -axis is

$$F(y, \pm \sqrt{x^2 + z^2}) = 0.$$

Similarly, the equation obtained by rotating around the z -axis is

$$F(\pm \sqrt{x^2 + y^2}, z) = 0.$$

That is: retain the coordinate that shares the name with the axis of rotation, and express the other coordinate in the equation as the square root of the sum of the squares of the other two coordinates. Based on this pattern

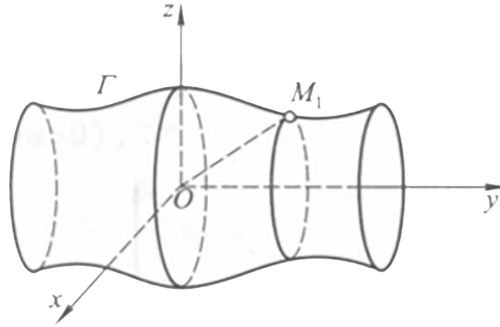


Figure 5.2: Surface of Revolution.

of the equation for a surface of revolution, it is also possible to determine in reverse whether an equation represents a surface of revolution.

Some special cases of surfaces of revolution:

Rotate ellipse

$$\Gamma : \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, & (a > b) \\ z = 0, \end{cases}$$

- around x -axis (long axis): $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$ (prolate spheroid)
- around y -axis (short axis): $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$ (oblate spheroid)

(see 5.3).

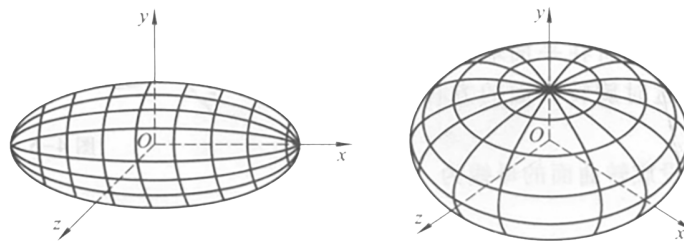


Figure 5.3: Ellipsoids.

Rotate hyperbola

$$\Gamma : \begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, & (b > c) \\ x = 0, \end{cases}$$

- around y -axis (real axis): $\frac{y^2}{b^2} - \frac{x^2}{c^2} - \frac{z^2}{b^2} = 1$ (two-sheet hyperboloid)
- around z -axis (unreal axis): $\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (one-sheet hyperboloid)

(see 5.4).

Rotate parabola

$$\Gamma : \begin{cases} y^2 = 2pz \\ x = 0, \end{cases}$$

around z -axis (axis of symmetry): $x^2 + y^2 = 2pz$ (paraboloid) (see 5.5).

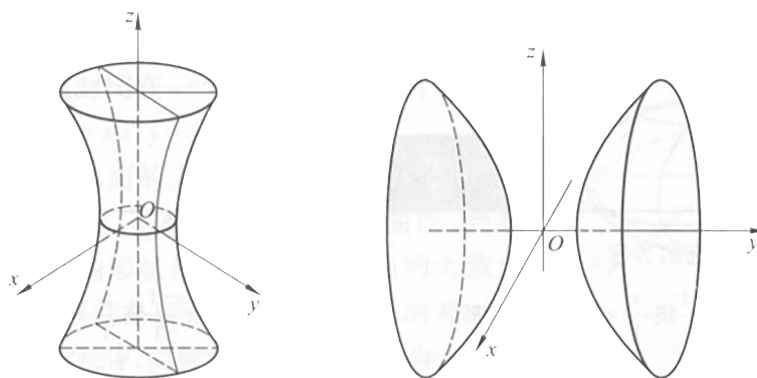


Figure 5.4: Hyperboloids.

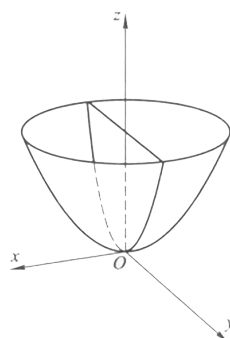


Figure 5.5: Paraboloid.

Rotate circle

$$\Gamma : \begin{cases} (y - b)^2 + z^2 = a^2 & (b > a > 0), \\ x = 0, \end{cases}$$

around z -axis: $(x^2 + y^2 + z^2 + b^2 - a^2)^2 = 4b^2(x^2 + y^2)$ (torus) (see 5.6).

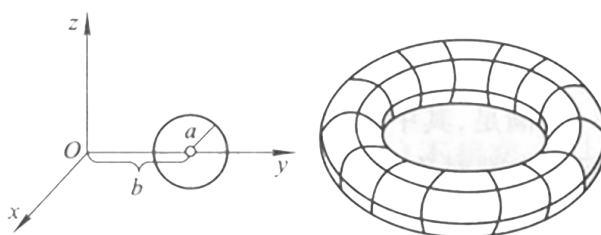


Figure 5.6: Torus.

5.4 Quadric Surfaces

5.5 Ruled Surfaces

Chapter 6 Conic Sections

6.1 General Equation of Conic Sections

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$F_1(x, y) \equiv a_{11}x + a_{12}y + a_{13}$
 $F_2(x, y) \equiv a_{12}x + a_{22}y + a_{23}$
 $F_3(x, y) \equiv a_{13}x + a_{23}y + a_{33}$
 $\Phi(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2$

6.2 Conic Sections and Lines

6.3 Simplification of Conic Equations

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