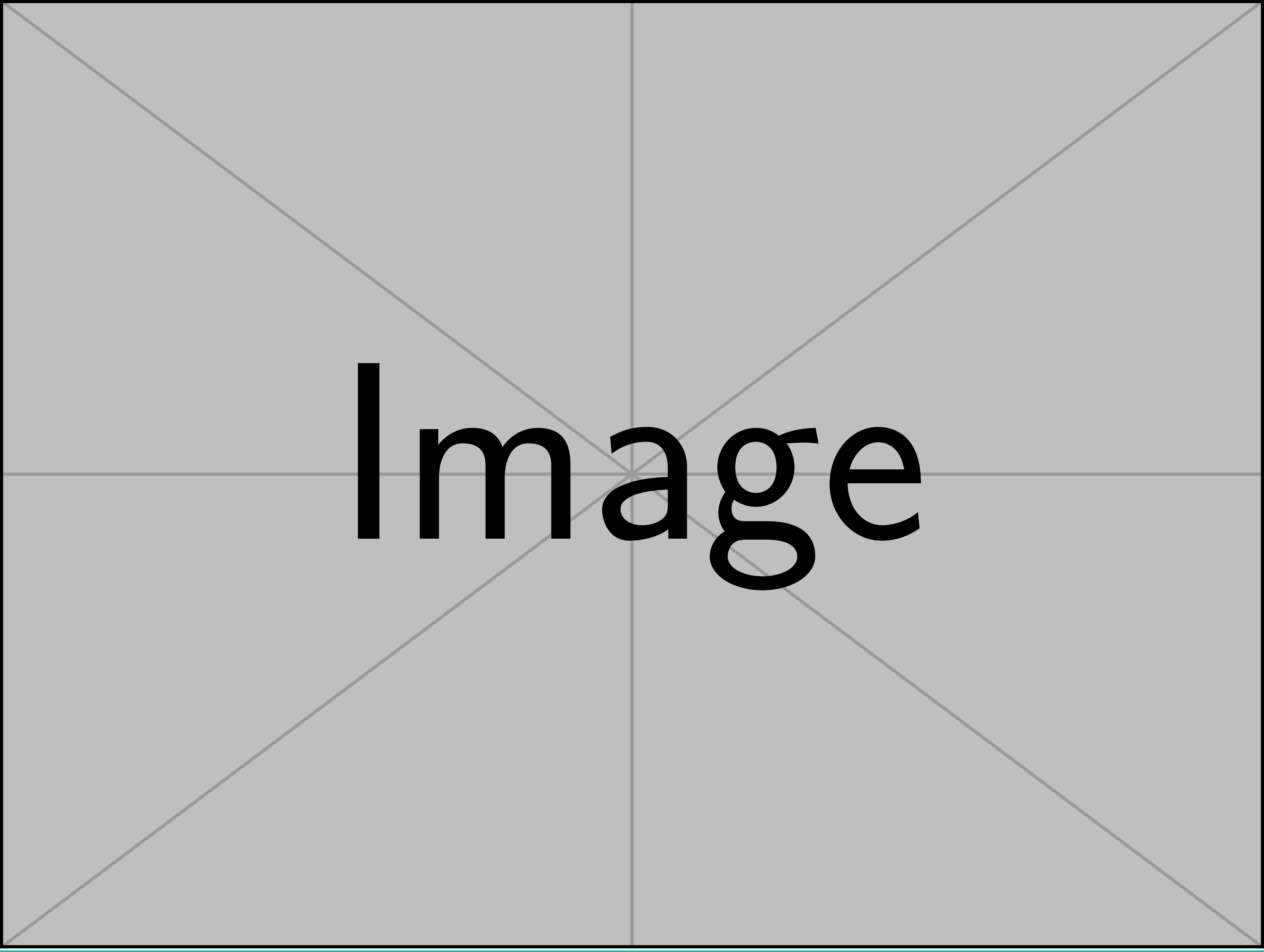

Polynôme

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Preface

This is the preface of the book...

Chapter 1 Preliminaries

Chapter 2 Univariate Polynomial Ring

2.1 Univariate Polynomials

2.2 Division

Theorem 2.1 (Euclidean Division (Division with Remainder))

Let $f(x), g(x) \in P[x]$ with $g(x) \neq 0$. Then there exist unique polynomials $q(x), r(x) \in P[x]$ such that

$$f(x) = g(x) \cdot q(x) + r(x)$$

where $r(x) = 0$ or $\deg(r) < \deg(g)$.

♡

Definition 2.1 (Exact Division)

If there exists $h(x) \in P[x]$ such that $f(x) = g(x) \cdot h(x)$, we say that $g(x)$ divides $f(x)$ and write $g(x) \mid f(x)$. (In other words, the remainder $r(x) = 0$.)

♣

Property

Caution *In Euclidean division, $g(x) \neq 0$ is required. However, in the case of $g(x) \mid f(x)$, $g(x)$ can equal 0. In this situation, $f(x) = g(x)h(x) = 0 \cdot g(x) = 0$, meaning that the* ***zero polynomial can only divide the zero polynomial****.*

2.3 Greatest Common Divisor and Relatively Prime

¶ `Greatest Common Divisor`

Definition 2.2 (Greatest Common Divisor (GCD))

Let $f(x), g(x) \in P[x]$. A polynomial $d(x) \in P[x]$ is called a greatest common divisor of $f(x)$ and $g(x)$ if:

1. $d(x) \mid f(x)$ and $d(x) \mid g(x)$;
2. For any polynomial $h(x) \in P[x]$, if $h(x) \mid f(x)$ and $h(x) \mid g(x)$, then $h(x) \mid d(x)$.

The greatest common divisor of $f(x)$ and $g(x)$, whose leading coefficient is 1 (also called **monic**), is denoted as $(f(x), g(x))$.

♣

Property

Theorem 2.2 (Euclidean Algorithm)

For all $f(x), g(x) \in P[x]$, there exists $d(x) \in P[x]$, where $d(x)$ is a greatest common divisor of $f(x)$ and $g(x)$, and $d(x)$ can be expressed as a linear combination of $f(x)$ and $g(x)$, i.e., there exist $u(x), v(x) \in P[x]$ such that

$$d(x) = u(x)f(x) + v(x)g(x).$$

The converse proposition does not hold in general.

♡

¶ `Relatively Prime`

Definition 2.3 (Relatively Prime)

Two polynomials $f(x)$ and $g(x)$ in $P[x]$ are called relatively prime if $(f(x), g(x)) = 1$, meaning they have no common divisor other than the zero-degree polynomial (nonzero constant). ♣

2.4 Least Common Multiple

2.5 Opposite Polynomials

Chapter 3 Factorization and Roots

3.1 Irreducible Polynomials

Definition 3.1 (Irreducible Polynomial)

A polynomial $p(x)$ of degree ≥ 1 over a field P is called an irreducible polynomial over the field P if it cannot be expressed as the product of two polynomials of lower degree than $p(x)$ over the field P. ♣

Proposition 3.1

For all $f(x), g(x) \in P[x]$, $p(x)$ is an irreducible polynomial in $P[x]$, which is equivalent to the following two propositions:

1. Either $p(x) \mid f(x)$ or $(p(x), f(x)) = 1$;
2. If $p(x) \mid f(x)g(x)$, then either $p(x) \mid f(x)$ or $p(x) \mid g(x)$.

Similarly, monic polynomial $p(x)$, with degree greater than 0, is a power of an irreducible polynomial over the field P if and only if for all $f(x), g(x) \in P[x]$,

1. Either $p(x) \mid f^m(x)$ $(m \in \mathbb{N}^*)$ or $(p(x), f(x)) = 1$;
2. If $p(x) \mid f(x)g(x)$, then either $p(x) \mid f^m(x)$ $(m \in \mathbb{N}^*)$ or $p(x) \mid g(x)$. ♠

3.2 Polynomials with Rational Coefficients

Definition 3.2 (Primitive Polynomial)

A polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients is called a **primitive polynomial** if the greatest common divisor of its coefficients is ± 1, i.e., $(a_n, a_{n-1}, \ldots, a_1, a_0) = \pm 1$. ♣

Lemma 3.1 (Gauß's Lemma)

The product of two primitive polynomials is also a primitive polynomial. ♡

With the help of Gauß's lemma, we can establish the following important theorem:

Theorem 3.1

If a polynomial $f(x)$ with integer coefficients is reducible over the field of rational numbers $\mathbb{Q}$, then it is also reducible over the ring of integers $\mathbb{Z}$. ♡

A corollary can be derived from this theorem:

Corollary 3.1

Let $f(x), g(x) \in \mathbb{Z}[x]$ be two polynomials, and $g(x)$ is primitive. If $f(x) = g(x)h(x)$, where $h(x) \in \mathbb{Q}[x]$, then $h(x) \in \mathbb{Z}[x]$. ♡

¶ `Searching and Judging of Rational Roots`

Now we can use the following theorem to search for rational roots of polynomials with integer coefficients:

Theorem 3.2 (Rational Root Theorem)

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. If $\frac{r}{s}$ (in lowest terms) is a rational root of $f(x)$, then $r \mid a_0$ and $s \mid a_n$.
Obviously, if $f(x)$ is monic, then any rational root must be an integer divisor of a_0.

Next, we can use the following theorem to judge whether a polynomial with integer coefficients is irreducible over the field of rational numbers:

Theorem 3.3 (Eisenstein's Criterion)

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. If there exists a prime number p such that:

1. $p \nmid a_n$;
2. $p \mid a_i$ for all $i = 0, 1, \ldots, n-1$;
3. $p^2 \nmid a_0$;

then $f(x)$ is irreducible over the field of rational numbers $\mathbb{Q}$.

3.3 Relation between Roots and Coefficients

Theorem 3.4 (Vièta's Formulas)

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial of degree n over field P, and let its n roots (counting multiplicities) be $r_1, r_2, \ldots, r_n$ in an extension field of P. Then the following relations hold:

$$\begin{aligned} &r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}, \\ &r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n = \frac{a_{n-2}}{a_n}, \\ &\vdots \\ &r_1r_2\cdots r_n = (-1)^n\frac{a_0}{a_n}. \end{aligned}$$

Using symmetric polynomial notation (5.1), Vièta's formulas can be expressed as:

$$\begin{aligned} \sigma_1(r_1, r_2, \ldots, r_n) &= -\frac{a_{n-1}}{a_n}, \\ \sigma_2(r_1, r_2, \ldots, r_n) &= \frac{a_{n-2}}{a_n}, \\ &\vdots \\ \sigma_n(r_1, r_2, \ldots, r_n) &= (-1)^n\frac{a_0}{a_n}, \end{aligned}$$

that is,

$$\sigma_i(r_1, r_2, \ldots, r_n) = (-1)^i\frac{a_{n-i}}{a_n}, \quad i = 1, 2, \ldots, n.$$

3.4 Root of Unity

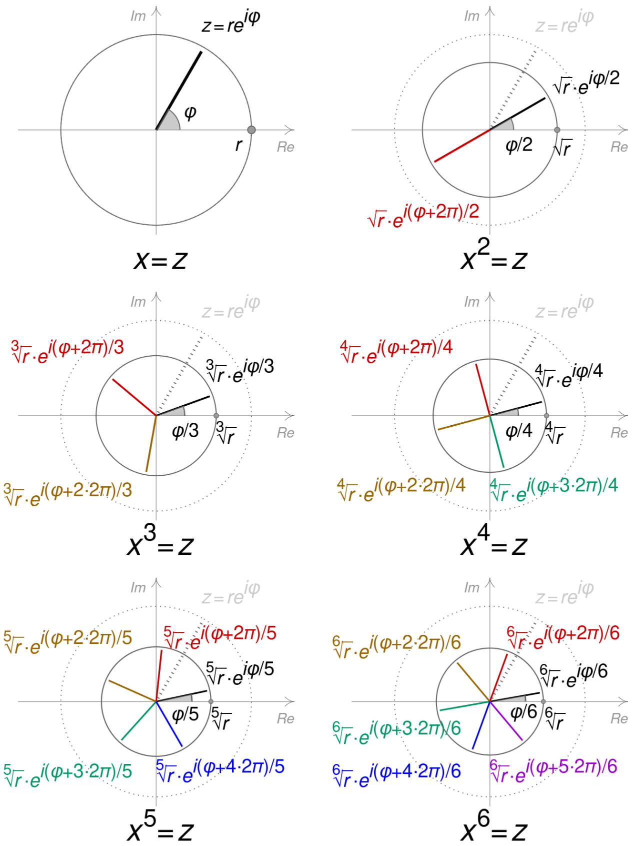
Definition 3.3 (Root of Unity)

Let P be a number field and $n \in \mathbb{N}^*$. An element $\omega \in P$ is called an n-th root of unity if it satisfies the equation $x^n - 1 = 0$, i.e., $\omega^n = 1$. ♣

Unless otherwise specified, the roots of unity may be taken to be complex numbers, and in this case, the n-th roots of unity are

$$\omega_k = \exp \frac{2k\pi i}{n} = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

Obviously, the modulus of each n-th root of unity is 1, i.e., $|\omega_k| = 1$, and they are evenly distributed on the unit circle in the complex plane, with an angle of $\frac{2\pi}{n}$ between adjacent roots.



Property

1. *The n-th roots of unity form a cyclic group under multiplication, with $\omega = \exp \frac{2\pi i}{n}$ as a generator.*

Proposition 3.2 (Formulas for Sums and Differences of Powers)

For $n \in \mathbb{N}^+$ and n being odd:

$$a^n + b^n = (a+b)\left(a^{n-1}b^0 - a^{n-2}b^1 + a^{n-3}b^2 - \cdots - a^1b^{n-2} + a^0b^{n-1}\right).$$

When n is even, there is no general formula for the n-th power sum.

For $n \in \mathbb{N}^+$:

$$a^n - b^n = (a-b)\left(a^{n-1}b^0 + a^{n-2}b^1 + a^{n-3}b^2 + \cdots + a^0b^{n-1}\right).$$

Commonly used special cases:

$$a^2 - b^2 = (a+b)(a-b).$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2), \quad a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

$$\begin{aligned} a^4 - b^4 &= (a^2+b^2)(a^2-b^2) = (a^2+b^2)(a+b)(a-b), \\ &= (a-b)(a^3 + a^2 b + ab^2 + b^3). \end{aligned}$$

When $b = 1$,

$$x^n + 1 = (x+1)\big(x^{n-1} - x^{n-2} + x^{n-3} - \cdots + x - 1\big), \quad n \in \mathbb{N}^+,\ n \text{ is odd}.$$

$$x^n - 1 = (x-1)\big(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1\big), \quad n \in \mathbb{N}^+.$$

Chapter 4 Integral Valued Polynomials

4.1 Lagrange Interpolation Polynomial

Chapter 5 Multivariate Polynomial

5.1 Symmetric Polynomial

Definition 5.1 (Symmetric Polynomial)

A polynomial $f(x_1, x_2, \ldots, x_n)$ in n variables is called a **symmetric polynomial** if it remains unchanged under any permutation of its variables. In other words, for any permutation σ of the set $\{1, 2, \ldots, n\}$, the following holds:

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n).$$

♣

Some common symmetric polynomials include:

Elementary Symmetric Polynomials

$$\sigma_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad k = 1, 2, \ldots, n.$$

That is,

$$\begin{aligned} \sigma_0 &= 1, \\ \sigma_1 &= x_1 + x_2 + \cdots + x_n, \\ \sigma_2 &= \sum_{1 \le i < j \le n} x_i x_j, \\ &\vdots \\ \sigma_n &= x_1 x_2 \cdots x_n, \\ \sigma_k &= 0, \quad k > n. \end{aligned}$$

Any symmetric polynomial can be expressed as a polynomial in elementary symmetric polynomials.

Power Sum Symmetric Polynomials

$$p_k(x_1, x_2, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k, \quad k = 1, 2, \ldots.$$

Complete Homogeneous Symmetric Polynomials

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{i_1 + i_2 + \cdots + i_n = k} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \quad k = 1, 2, \ldots.$$

Theorem 5.1 (Newton's Identities)

For $k \ge 1$, the following relations hold between the elementary symmetric polynomials σ_k and the power sum symmetric polynomials p_k:

$$k\sigma_k = \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-i} p_i.$$

♡

We introduce some notations for convenience, where f can be any function of n variables, not necessarily be polynomials:

Cyclic Sum Perform a cyclic shift on all variables in an expression, then sum the resulting terms:

$$\sum_{\text{cyc}} f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} f(x_i, x_{i+1}, \ldots, x_{i+n-1}),$$

where the indices are taken modulo n.

For example,

$$\sum_{\text{cyc}} \frac{a}{b+c} = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

which is called Nesbitt's inequality, to be proved later.

Symmetric Sum Sum over all distinct permutations of the variables in an expression:

$$\sum_{\text{sym}} f(x_1, x_2, \ldots, x_n) = \sum_{\sigma \in S_n} f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}),$$

where S_n is the set of all permutations of n elements.

For example,

$$\sum_{\text{sym}} a^3 = a^3 + b^3 + c^3, \quad \sum_{\text{sym}} a^2b = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$$

5.2 Symmetric Inequalities

Definition 5.2 (Symmetric Inequality)

An inequality $f(x_1, x_2, \ldots, x_n) \geq g(x_1, x_2, \ldots, x_n)$ is called a **symmetric inequality** if the polynomial $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ are symmetric polynomials. ♣

¶ Power Mean Inequality

Theorem 5.2 (Power Mean Inequality)

For positive real numbers $a_1, a_2, \ldots, a_n > 0$, define the power mean of order p as:

$$M_p(a_1, a_2, \ldots, a_n) = \begin{cases} \left(\frac{a_1^p + a_2^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}}, & p \neq 0 \\ \lim_{p \to 0} M_p(a_1, a_2, \ldots, a_n) = \sqrt[n]{a_1 a_2 \cdots a_n}, & p = 0. \end{cases}$$

Specially, when $p \to 0$, it is the **geometric mean** (G)

$$G = \sqrt[n]{a_1 a_2 \cdots a_n};$$

when $p = 1$, it is the **arithmetic mean** (A)

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n};$$

when $p = 2$, it is the **quadratic mean** (Q)

$$Q = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}};$$

when $p = -1$, it is the **harmonic mean** (H)

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

The following inequalities hold:

$$\cdots \leqslant M_{-2} \leqslant M_{-1} \leqslant M_0 \leqslant M_1 \leqslant M_2 \leqslant \cdots.$$

Thus, we have:

$$H \leqslant G \leqslant A \leqslant Q.$$

♡

When $n = 2$, in the chain of power mean inequalities, we can insert the **logarithmic mean**: logarithmic mean of a and b is defined as:

$$L(a,b) = \frac{a-b}{\ln a - \ln b} \quad (a \neq b, a, b > 0),$$

then we have:

$$G(a,b) \leqslant L(a,b) \leqslant A(a,b).$$

¶ Schur Inequality

Theorem 5.3 (Schur Inequality)

For non-negative real numbers $a, b, c \geq 0$ and a real number $r \geq 0$, the following inequality holds:

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0.$$

When $r = 1$, the following well-known special case can be derived:

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a).$$

¶ Muirhead Inequality

This part mainly references [3].

Definition 5.3 (Convex Hull)

Let V be a linear space over the field $\mathbb{R}$, for a set X, the **convex hull** of X is defined as the intersection of all convex sets containing X:

$$S := \bigcap_{X \subseteq K \subseteq V} K, \quad \text{where } K \text{ is a convex set.}$$

For an n-dimensional vector $\alpha = (a_1, a_2, \ldots, a_n)$, define $\alpha_{[j]}, 1 \leqslant j \leqslant n$ is the j-th item of α after sorting $a_1, a_2, \ldots, a_n$ in descending order, i.e.,

$$a_{[1]} \geqslant a_{[2]} \geqslant \cdots \geqslant a_{[n]}.$$

Then we can obtain

$$a_{\downarrow} = (a_{[1]}, a_{[2]}, \cdots, a_{[n]}).$$

Define S_n be the set of all permutations of the set $\{1, 2, \ldots, n\}$, then we define the convex hull of α as:

$$H(\alpha) = \{b_{\tau(1)}, b_{\tau(2)}, \ldots, b_{\tau(n)} | \tau \in S_n\}.$$

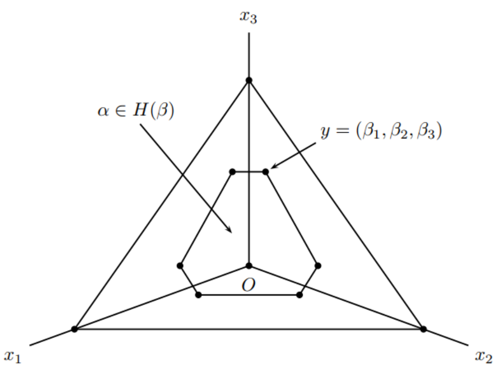


Figure 5.1: Relation graph of convex hull in $\mathbb{R}^3$.

Theorem 5.4 (Muirhead Inequality)

For $\alpha = (a_1, a_2, \ldots, a_n), \beta = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, and $\alpha \in H(\beta)$[a], then for positive numbers $x_1, x_2, \ldots, x_n > 0$, the following inequality holds:

$$\sum_{\sigma \in S_n} x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdots x_{\sigma(n)}^{a_n} \leqslant \sum_{\sigma \in S_n} x_{\sigma(1)}^{b_1} x_{\sigma(2)}^{b_2} \cdots x_{\sigma(n)}^{b_n},$$

where $\sum_{\sigma \in S_n}$ denotes summation over all permutations σ in S_n.
The equality holds if and only if $x_1 = x_2 = \cdots = x_n$ or $\alpha_\downarrow = \beta_\downarrow$.

[a] It is often called **Muirhead's condition** that $\alpha \in H(\beta)$.

Since Muirhead's condition is difficult to verify directly, we derive the following conditions.

Definition 5.4 (Majorization)

For $\alpha = (a_1, a_2, \ldots, a_n), \beta = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, if the following conditions hold:

1. $a_{[1]} + a_{[2]} + \cdots + a_{[n]} = b_{[1]} + b_{[2]} + \cdots + b_{[n]}$;
2. For all $k = 1, 2, \ldots, n-1$,

$$a_{[1]} + a_{[2]} + \cdots + a_{[k]} \leqslant b_{[1]} + b_{[2]} + \cdots + b_{[k]};$$

then we say that β majorizes α, denoted as $\alpha \prec \beta$.

Definition 5.5 (Doubly Stochastic Matrix)

An $n \times n$ matrix $P = (p_{ij})$ is called a **doubly stochastic matrix** if the sum of each row and the sum of each column both equal 1.

Now we can give two equivalent conditions:

Theorem 5.5

In the same conditions as Muirhead's inequality, the following two statements are equivalent to $\alpha \prec \beta$:

1. There exists a doubly stochastic matrix D such that $\alpha = D\beta$.
2. $\alpha \in H(\beta)$.

Example 5.1 Nesbitt's Inequality For positive real numbers $a, b, c > 0$, the following inequality holds:

$$\sum_{\text{cyc}} \frac{a}{b+c} = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof By finding a common denominator and cross-multiplying, we have:

$$2\sum_{\text{cyc}} a(a+b)(a+c) \geqslant 3(a+b)(b+c)(c+a),$$

which is equivalent to

$$\sum_{\text{sym}} a^3 \geqslant \sum_{\text{sym}} a^2 b.$$

Note that $(3,0,0) \succ (2,1,0)$, thus by Muirhead's inequality, the above inequality holds. ■

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