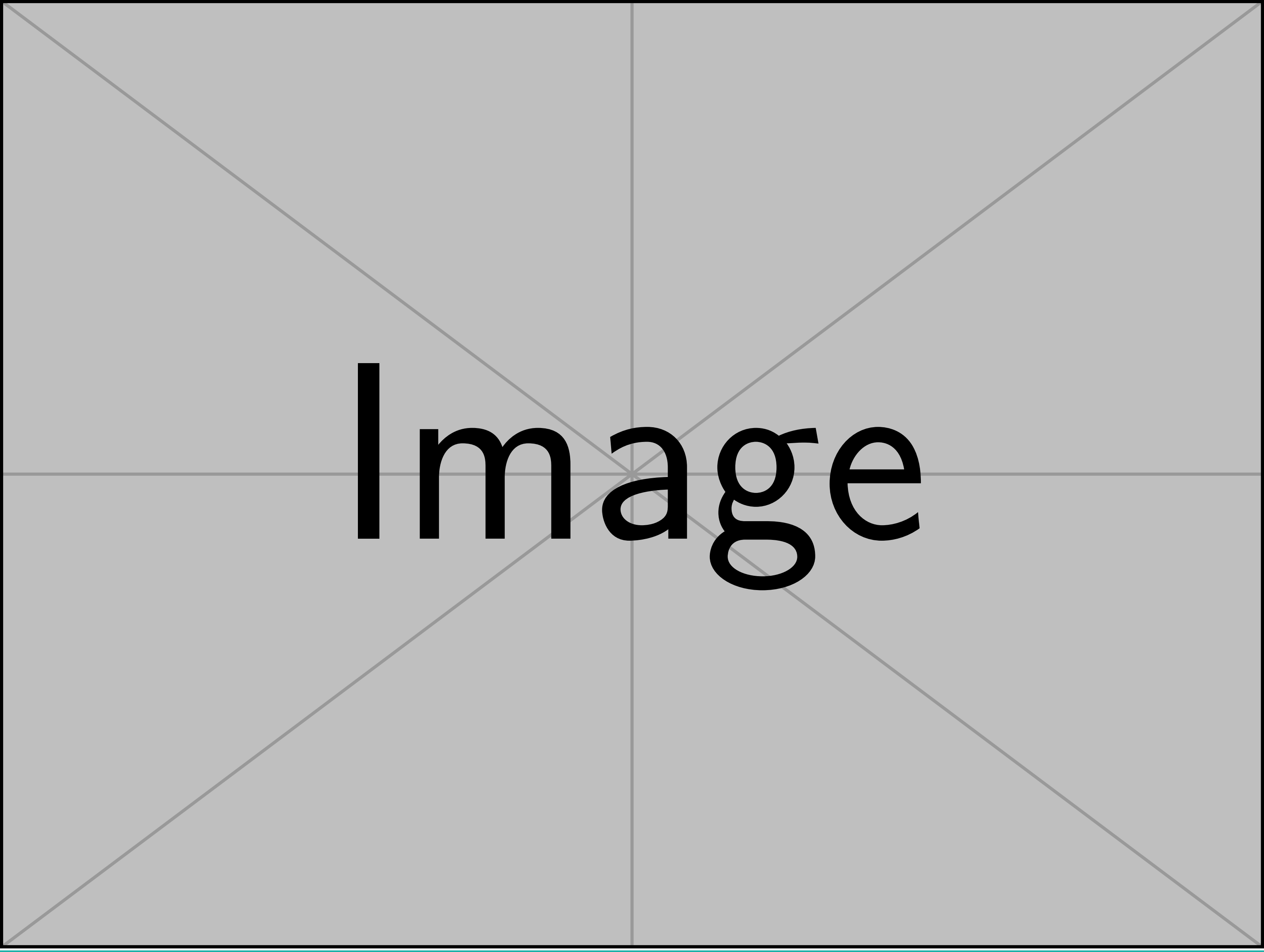


# Analyse Mathématique

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# Contents

# Preface

This is the preface of the book…

# Chapter 1 Preliminaries

## 1.1 Trigonometric Formulas

**Product-to-Sum Formulas:**

$$\begin{aligned}
\sin\alpha\cos\beta &= \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right] \\
\cos\alpha\sin\beta &= \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right] \\
\cos\alpha\cos\beta &= \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right] \\
\sin\alpha\sin\beta &= -\frac{1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right]
\end{aligned}$$

**Sum and Difference Formulas:**

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta
\end{aligned}$$

**Sum-to-Product Formulas:**

$$\begin{aligned}
\sin\alpha+\sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
\sin\alpha-\sin\beta &= 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \\
\cos\alpha+\cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
\cos\alpha-\cos\beta &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}$$

**Double Angle Formulas:**

$$\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha-\sin^2\alpha = 2\cos^2\alpha-1 = 1-2\sin^2\alpha \\
\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

**Half Angle Formulas:**

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
\tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}
\end{aligned}$$

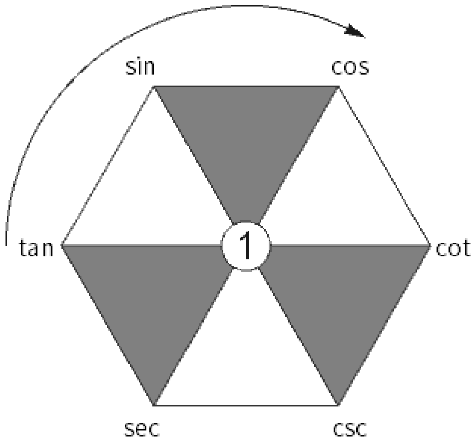
**Power-Reducing Formulas:**

$$\begin{aligned}
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2} \\
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2}
\end{aligned}$$

**Angle Decomposition Formulas:**

$$\sin^2\alpha - \sin^2\beta = \sin(\alpha+\beta)\sin(\alpha-\beta)$$

$$\cos^2\alpha - \sin^2\beta = \cos(\alpha+\beta)\cos(\alpha-\beta)$$




## Remark

- On the gray triangle, the sum of the squares of the two numbers above is equal to the square of the number below, for instance, $\tan^2 x + 1 = \sec^2 x$
- The three trigonometric functions in the clockwise direction have the following properties: $\tan x = \frac{\sin x}{\cos x}$, etc.

# Chapter 2 Limits of Sequences and Continuity of Real Number System

## 2.1 Convergent Sequences

¶ Convergent Sequences

¶ Properties of Convergent Sequences

¶ Cauchy Proposition and Fitting Method

*Proposition 2.1 (Cauchy Proposition)*

Let $\lim_{n\to\infty} x_n = l$, then:

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = l.$$

**Note**

1. *In the proposition, $l$ can be $+\infty$ or $-\infty$.*
2. *Let $\lim_{n\to\infty} x_n = l$, then:*

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = \lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = \lim_{n\to\infty} \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} = l.$$

It can be proved directly by Stolz theorem 2.1. On top of that, it can also be proved by the **fitting method**.

*Proof*

■

**Remark** To prove $\lim_{n\to\infty} x_n = A$, the key is to show that $|x_n - A|$ can be arbitrarily small. For this purpose, it is generally recommended to simplify the expression of $x_n$ as much as possible. However, in some cases, $A$ can also be transformed into a form similar to $x_n$. This method is called the fitting method. The core idea behind the method of fitting is to appropriately divide into units of 1 for analysis.

## 2.2 Indeterminate Form

¶ Infinitely Large Quantities and Infinitesimal Quantities

¶ Indeterminate Forms

*Theorem 2.1 (Stolz-Cesàro theorem)*

**Type $\frac{0}{0}$** Let $\{a_n\}$, $\{b_n\}$ be two infinitesimal sequences, where $\{a_n\}$ is also a strictly monotonic decreasing sequence. If

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l \text{ (finite or } \pm\infty),$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

**Type $\frac{*}{\infty}$** Let $\{a_n\}$ be a strictly monotonic increasing sequence of divergent large quantities. If

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l \text{ (finite or } \pm\infty),$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

**Note**

1. *The inverse proposition of Stolz's Theorem does not hold.*
2. *If $a_1$ is an undefined infinite quantity $\infty$, Stolz Theorem does not hold.*

*Theorem 2.2 (Silverman-Toeplitz Theorem)*

Let

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{bmatrix},$$

where the infinite triangular matrix satisfies:

1. $\forall j, \lim_{n\to\infty} a_{nj} = 0$.(Every column sequence converges to 0.)
2. $\sup_{i\in\mathbb{N}} \sum_{j=1}^{i} |a_{ij}| < \infty$.(The absolute row sums are bounded.)

And $\lim_{n\to\infty} x_n = l$. We denote $y_n$ as the weighted sum sequence: $y_n = \sum_{j=1}^{n} a_{nj}x_j$. Then the following results hold:

1. If $l = 0$, then $\lim_{n\to\infty} y_n = 0$.
2. If $l \neq 0$ and $\lim_{n\to\infty} \sum_{j=1}^{n} a_{ij} = 1$, then $\lim_{n\to\infty} y_n = l$.

## 2.3 Subsequences

¶ Subsequences

¶ Upper Limits and Lower Limits

## 2.4 Completeness of The Real Numbers

¶ Dedkind Completeness

¶ Least Upper Bound Property

¶ Monotone Convergence Theorem

¶ Bolzano-Weierstrass Theorem

¶ Nested Interval Theorem

¶ Cauchy Completeness

*Definition 2.1 (Cauchy Sequence)*

A sequence $\{x_n\}$ is called a **Cauchy sequence** if for any $\varepsilon > 0$, there exists a positive integer $N$ such that when $m, n > N$,

$$|x_n - x_m| < \varepsilon.$$

*Theorem 2.3 (Cauchy Convergence Criterion for Sequences)*

A sequence $\{x_n\}$ converges if and only if it is a Cauchy sequence. ♡

¶ Heine-Borel Theorem

## 2.5 Iterative Sequences

Formally, $x_0$ is a **fixed point** of the function $f$ if $f(x_0) = x_0$.

*Theorem 2.4 (Banach Fixed-Point Theorem (Contraction Mapping Theorem))*

There exists a contraction mapping (in 3.2) $f$ on an interval $I$, which admits a unique fixed point $x^* \in I$. Furthermore, $x^*$ can be found as follows: start with an arbitrary point $x_0 \in I$ and define the iterative sequence $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \cdots$. Then $\lim_{n\to\infty} x_n = x^*$. ♡

**Remark** The following inequalities are equivalent and describe the speed of convergence:

$$
\begin{aligned}
|x_n - x^*| &\leqslant \frac{L^n}{1-L} |x_1 - x_0|, \\
|x_{n+1} - x^*| &\leqslant \frac{L}{1-L} |x_{n+1} - x_n|, \\
|x_{n+1} - x^*| &\leqslant L |x_n - x^*|.
\end{aligned}
$$

Any such value of $L < 1$ is the Lipschitz constant for $f$, and the smallest one is sometimes called **the best Lipschitz constant** of $L$.

# Chapter 3 Limits and Continuity of Functions

## 3.1 Limits of Functions

¶ **Definition of Limit**

¶ **Limits of Functions and Sequences**

*Theorem 3.1 (Heine Theorem)*

Let $f$ be a function defined on a deleted neighborhood $\mathring{U}(x_0)$ of $x_0$. The following two statements are equivalent:

1. $\lim_{x\to x_0} f(x) = A$.
2. For any sequence $\{x_n\} \subset \mathring{U}(x_0)$ with $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = A$ for the sequence $\{f(x_n)\}$.

♡

## 3.2 Continuous Functions

## 3.3 Infinitesimal and Infinite Quantities

## 3.4 Continuous Functions on Closed Intervals

¶ **Concerning Theorems**

*Theorem 3.2 (The Bolzano-Cauchy Intermediate-Value Theorem)*

♡

*Theorem 3.3 (Zero Point Existence Theorem)*

♡

¶ **Uniform Continuity and Lipschitz Continuity**

*Definition 3.1 (Uniform Continuity)*

♣

*Theorem 3.4 (Uniform Continuity Theorem)*

♡

*Theorem 3.5 (Cantor's Theorem)*

♡

*Definition 3.2 (Lipschitz Continuity)*

If there exists a constant $L > 0$ such that for any $x_1, x_2 \in I$,

$$|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|\,,$$

then $f$ is called **Lipschitz continuous** on $I$.

Specially, if $L < 1$, then $f$ is called a **contraction mapping** on $I$.

♣

**Remark**

- If $f$ is Lipschitz continuous on $I$, then $f$ is uniformly continuous on $I$. ($\forall \varepsilon > 0$, just let $\delta = \frac{\varepsilon}{L}$)
- If $f$ is uniformly continuous on $I$, then $f$ is continuous on $I$.
- The converse of the above two statements does not hold.

## 3.5 Period Three Implies Chaos

## 3.6 Functional Equations

# Chapter 4 Differential

## 4.1 Differential and Derivative

## 4.2 Higher-Order Derivatives

## 4.3 Differential Mean Value Theorems

*Definition 4.1 (Extremum)*

Let $f(x)$ is defined on $(a,b)$, $x_0 \in (a,b)$. If there exists $U(x_0,\delta) \subset (a,b)$ such that $f(x) \leqslant f(x_0)$ on it, then $x_0$ is called a local maximum point of $f$, and $f(x_0)$ is referred to as the corresponding local maximum value. The definition of the minimum value is analogous.

*Lemma 4.1 (Fermat's Lemma)*

If $f$ is differentiable at $x_0$ which is a local extremum, then $f'(x_0) = 0$.

*Theorem 4.1 (Rolle's Theorem)*

If $f \in C[a,b]$, $f \in D(a,b)$ and $f(a) = f(b)$, then there exists $\xi \in (a,b)$ such that $f'(\xi) = 0$.

**Enhanced Version:**If $f \in D(a,b)$(finite or infinite interval), and $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$ , then there exists $\xi \in (a,b)$ such that $f'(\xi) = 0$.

*Theorem 4.2 (Lagrange's Mean Value Theorem)*

If $f \in C[a,b]$, $f \in D(a,b)$, then there exists $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

*Theorem 4.3 (Cauchy's Mean Value Theorem)*

If $f, g \in C[a,b]$, $f, g \in D(a,b)$ and $g'(x) \neq 0$ for all $x \in (a,b)$, then there exists $\xi \in (a,b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

## 4.4 Theorems and Applications concerning Derivatives

*Theorem 4.4 (Darboux's Intermediate Value Theorem for Derivatives)*

If $f(x) \in D[a,b]$, and $f'_+(a) \cdot f'_-(b) < 0$, then there at least exists $\xi \in (a,b)$ such that $f'(\xi) = 0$.

*Theorem 4.5 (Theorem on the Limit of Derivatives)*

If $f(x) \in C(U(x_0))$, $\mathring{D}(U(x_0))$, and $\lim_{x\to x_0} f'(x) = A$, then $f$ is differentiable at $x_0$ and $f'(x_0) = A$.

**Remark** In fact, $\lim_{x\to x_0} f'(x) = A$ has already been shown to imply that $f \in \mathring{D}(U(x_0))$.

## 4.5 Taylor Theorem

## 4.6 Applications of Taylor Theorem

# Chapter 5 Integral

# Chapter 6 Numerical Series

## 6.1 Convergence of Numerical Series

## 6.2 Positive Term Series

## 6.3 General Term Series

¶ `Cauchy Convergence Criterion for Series`

¶ `Alternative Series`

¶ `Abel-Dirichlet Test`

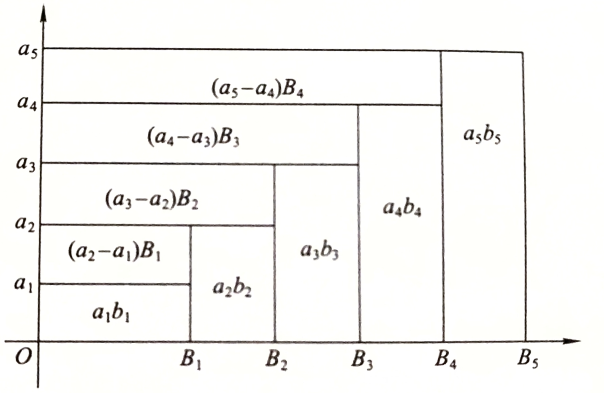
*Theorem 6.1 (Abel Transform (Discrete Integration by Parts/Summation by Parts))*

Let $\{a_n\}$, $\{b_n\}$ be two sequences, then for any $n \in \mathbb{N}^+$,

$$\sum_{k=1}^{n} a_k b_k = a_n B_n + \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k,$$

where $B_n = \sum_{k=1}^{n} b_k$.

$$\sum_{k=1}^{5} a_k b_k = a_5 B_5 - \sum_{k=1}^{4} (a_{k+1} - a_k) B_k.$$



# Chapter 7 Series of Functions

# Chapter 8 Power Series

# Chapter 9 Limits and Continuity in Euclidean Spaces

## 9.1 Continuous Mappings

¶ Continuous Mappings on Compact Sets

¶ Continuous Mappings on Connected Sets

*Definition 9.1 (Connected Set)*

Let $S$ be a set of points in $\mathbb{R}^n$. If a continuous mapping

$$\gamma : [0, 1] \to \mathbb{R}^n$$

satisfies that the range of $\gamma([0, 1])$ lies entirely within $S$, we call $\gamma$ a **path** in $S$, where $\gamma(0)$ and $\gamma(1)$ are referred to as the starting point and ending point of the path, respectively.

If for any two points $\mathbf{x}, \mathbf{y} \in S$, there exists a path in $S$ with $\mathbf{x}$ as the starting point and $\mathbf{y}$ as the ending point, $S$ is called path-connected, or equivalently, $S$ is called a **connected set**.

A connected open set is called an **(open) region**. The closure of an (open) region is referred to as a closed region. ♣

**Remark** Intuitively, this means that any two points in $S$ can be connected by a curve lying entirely within $S$. Clearly, a connected subset of $\mathbb{R}$ is an interval, and a connected subset of $\mathbb{R}$ is compact if and only if it is a closed interval.

# Chapter 10 Multi-variable Differential Calculus

## 10.1 Directional Derivatives and Total Differential

¶ Directional Derivative

**Definition 10.1 (Directional Derivative)**

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^1$, $\mathbf{e}$ is a unit vector in $\mathbb{R}^n$, $\mathbf{x}^0 \in U$. Define

$$u(t) = f(\mathbf{x}^0 + t\mathbf{e}).$$

If the derivative of $u$ at $t = 0$

$$u'(0) = \lim_{t\to 0} \frac{u(t) - u(0)}{t} = \lim_{t\to 0} \frac{f(\mathbf{x}^0 + t\mathbf{e}) - f(\mathbf{x}^0)}{t}$$

exists and is finite, it is called the **directional derivative** of $f$ at $\mathbf{x}^0$ in the direction $\mathbf{e}$, denoted by $\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0)$. It is the rate of change of $f$ at $\mathbf{x}^0$ in the direction $\mathbf{e}$. ♣

Consider the following set of unit coordinate vectors: $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. Let $\mathbf{e}_i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$ denote the standard orthonormal basis in $\mathbb{R}^n$, where the 1 appears in the $i$-th position. That is,

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For a function $f$, the directional derivative of $f$ at the point $\mathbf{x}_0$ in the direction of $\mathbf{e}_i$ is called the $i$th first-order **partial derivative** of $f$ at $\mathbf{x}^0$, denoted by

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^0) \quad \text{or} \quad \mathrm{D}_i f(\mathbf{x}^0) \quad \text{or} \quad f_{x_i}(\mathbf{x}^0) \quad (i = 1, 2, \cdots, n).$$

$\mathrm{D}_i = \frac{\partial}{\partial x_i}$ is called the $i$th partial differential operator ($i = 1, 2, \cdots, n$).

Let $\mathbf{e}_i = \sum_{i=0}^n \mathbf{e}_i \cos\alpha$ be a unit vector, where $\sum_{i=0}^n \cos^2\alpha = 1$. If $\frac{\partial f}{\partial x_i}$ is continuous at $\mathbf{x}^0$, then the directional derivative of $f$ at $\mathbf{x}^0$ along the direction $\mathbf{e}$ is given by:

$$\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}^0)\cos\alpha_i.$$

This is the formula for **expressing a directional derivative using partial derivatives**.

**Note** *Let $\mathbf{e}$ be a direction, then $\|-\mathbf{e}\| = \|\mathbf{e}\| = 1$, which implies that $-\mathbf{e}$ is also a direction. At this point, we have:*

$$\frac{\partial f}{\partial(-\mathbf{e})}(\mathbf{x}^0) = -\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0).$$

**Definition 10.2 (Jacobian Matrix (Gradient))**

Let

$$Jf(\mathbf{x}) = (\mathrm{D}_1 f(\mathbf{x}), \mathrm{D}_2 f(\mathbf{x}), \ldots, \mathrm{D}_n f(\mathbf{x})),$$

which is called the **Jacobian matrix** of the function $f$ at the point $\mathbf{x}$, (a $1 \times n$ matrix) whose counterpart is the first-order derivative of a single-variable function.

Henceforth, we represent the point $\mathbf{x}$ in $\mathbb{R}^n$ and its increments $\mathbf{h}$ as column vectors. In this way, the differential of the function can be expressed using matrix multiplication as follows:

$$\mathrm{d}f(\mathbf{x}^0)(\boldsymbol{\Delta}\mathbf{x}) = Jf(\mathbf{x}^0)\boldsymbol{\Delta}\mathbf{x}.$$

The Jacobian matrix of the function $f$ is also frequently denoted as grad $f$ (or $\nabla f$), that is,

$$\operatorname{grad} f(\mathbf{x}) = Jf(\mathbf{x}),$$

which is called the **gradient** of the scalar function $f$.

**Note** *Let $U \subset \mathbb{R}^n$ be an open set, and $\mathbf{f} : U \to \mathbb{R}^m$ be a $C^k$ mapping:*

- *$k = 0$, $\mathbf{f}$ is a continuous mapping;*
- *$0 < k < +\infty$, $f_i$ has continuous partial derivatives up to order $k$, $i = 1, 2, \dots, m$;*
- *$k = +\infty$, $f_i$ has continuous partial derivatives of all orders, $i = 1, 2, \dots, m$;*
- *$k = \omega$, $f_i$ is really analytic, i.e., in the neighborhood of any point $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in U$, $f_i$ can be expanded into a convergent ($n$-dimensional) power series, $i = 1, 2, \dots, m$.*

*Let $C^k(U, \mathbb{R}^m)$ denote the totality of $C^k$ mappings from $U$ to $\mathbb{R}^m$.*

## Total Differential

**Definition 10.3 (Total Differential)**

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^1, \mathbf{x}^0 \in U, \mathbf{\Delta x} = (\Delta x_1, \Delta x_2, \cdots, \Delta x_n) \in \mathbb{R}^n$. If

$$f(\mathbf{x}^0 + \mathbf{\Delta x}) - f(\mathbf{x}^0) = \sum_{i=1}^{n} A_i \Delta x_i + o(\|\mathbf{\Delta x}\|) \qquad (\|\mathbf{\Delta x}\| \to 0),$$

where $A_1, A_2, \dots, A_n$ are constants independent of $\mathbf{\Delta x}$, then the function $f$ is said to be **differentiable** at the point $\mathbf{x}^0$, and the linear main part $\sum_{i=1}^{n} A_i \Delta x_i$ is called the **total differential** of $f$ at $\mathbf{x}^0$, denoted as

$$df(\mathbf{x}^0)(\mathbf{\Delta x}) = \sum_{i=1}^{n} A_i \Delta x_i.$$

If $f$ is differentiable at every point in the open set $U$, then $f$ is called a differentiable function on $U$.

**Theorem 10.1 (Conditions of Differentiability)**

**Necessary Condition** If an $n$-variable function $f$ is differentiable at the point $\mathbf{x}_0$, then $f$ is continuous at $\mathbf{x}^0$ and possesses first-order partial derivatives $\frac{\partial f}{\partial x_i}(\mathbf{x}^0)$ at $\mathbf{x}^0$ for $i = 1, 2, \dots, n$, and

$$\mathbf{A} = (A_1, A_2, \dots, A_n) = Jf(\mathbf{x}^0) = \left(\mathrm{D}_1 f(\mathbf{x}^0), \mathrm{D}_2 f(\mathbf{x}^0), \dots, \mathrm{D}_n f(\mathbf{x}^0)\right).$$

[a] However, the converse is not true.

**Sufficient Condition** Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^1$ be an $n$-variable function. If $Jf = (\mathrm{D}_1 f, \mathrm{D}_2 f, \dots, \mathrm{D}_n f)$ is continuous at $\mathbf{x}^0$ (i.e., $\frac{\partial f}{\partial x_i}$ is continuous at $\mathbf{x}^0$ for $i = 1, 2, \dots, n$), then $f$ is differentiable at $\mathbf{x}^0$. However, the converse is not necessarily true.

[a] It is referred to as the total differential formula, and the more common form is

$$\mathrm{d}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\,\mathrm{d}x + \frac{\partial f}{\partial y}(x_0, y_0)\,\mathrm{d}y.$$

**Note**

- *The continuity of the derivative function at $\mathbf{x}^0$ implies that the original function $f$ is differentiable in some neighborhood of $\mathbf{x}^0$.*
- *In fact, this condition can be relaxed to require that one partial derivative exists at the point, while the remaining $n-1$ partial derivative functions are continuous at that point.*

*Proof* Taking a function of three variables as an example.

Assume the 3-ary function $f : \mathbb{R}^3 \to \mathbb{R}$ meets:

1. There exists $f_z(x_0, y_0, z_0)$.
2. The partial derivative functions $f_x(x, y, z)$ and $f_y(x, y, z)$ are continuous at $(x_0, y_0, z_0)$, i.e. there are partial derivatives in some neighborhood of $(x_0, y_0, z_0)$.

Consider the total increment of $f$ at the point $(x_0, y_0, z_0)$:

$$\begin{aligned}\Delta f = & \underbrace{[f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) - f(x_0, y_0+\Delta y, z_0+\Delta z)]}_{I_1} \\ & + \underbrace{[f(x_0, y_0+\Delta y, z_0+\Delta z) - f(x_0, y_0, z_0+\Delta z)]}_{I_2} \\ & + \underbrace{[f(x_0, y_0, z_0+\Delta z) - f(x_0, y_0, z_0)]}_{I_3}.\end{aligned}$$

For $I_1, I_2$, by the Lagrange's Mean Value Theorem of unary functions, there exist $\theta_1, \theta_2 \in (0, 1)$ such that

$$I_1 = f_x(x_0+\theta_1\Delta x, y_0+\Delta y, z_0+\Delta z)\Delta x,$$
$$I_2 = f_y(x_0, y_0+\theta_2\Delta y, z_0+\Delta z)\Delta y.$$

Then by the continuity of the their partial derivatives at $(x_0, y_0, z_0)$, we have

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} I_1 = f_x(x_0, y_0, z_0)\Delta x, \qquad \lim_{\Delta x, \Delta y, \Delta z \to 0} I_2 = f_y(x_0, y_0, z_0)\Delta y.$$

They can be expressed in terms of infinitesimals($\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$):

$$I_1 = f_x(x_0, y_0, z_0)\Delta x + \alpha_1\Delta x, \quad \alpha_1 \to 0(\rho \to 0),$$
$$I_2 = f_y(x_0, y_0, z_0)\Delta y + \alpha_2\Delta y, \quad \alpha_2 \to 0(\rho \to 0).$$

For $I_3$, by the definition of the partial derivative $f_z(x, y, z)$ at $(x_0, y_0, z_0)$, we have

$$I_3 = f_z(x_0, y_0, z_0)\Delta z + \alpha_3\Delta z, \quad \alpha_3 \to 0(\rho \to 0).$$

Accordingly,

$$\begin{aligned}\Delta f &= I_1 + I_2 + I_3 \\ &= [f_x(x_0, y_0, z_0)\Delta x + \alpha_1\Delta x] + [f_y(x_0, y_0, z_0)\Delta y + \alpha_2\Delta y] + [f_z(x_0, y_0, z_0)\Delta z + \alpha_3\Delta z] \\ &= f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z + [\alpha_1\Delta x + \alpha_2\Delta y + \alpha_3\Delta z].\end{aligned}$$

Apparently,

$$\lim_{\rho \to 0} \frac{\alpha_1\Delta x + \alpha_2\Delta y + \alpha_3\Delta z}{\rho} = 0,$$

i.e. $\alpha_1\Delta x + \alpha_2\Delta y + \alpha_3\Delta z = o(\rho)$. Therefore, $f(x, y, z)$ is differentiable at $(x_0, y_0, z_0)$, which completes the proof. ∎

**Note** *(At some point)*

1. *Differentiable*
   - $\Longrightarrow$ *Continuous*
   - $\Longrightarrow$ *Partial derivatives exist:* $\mathrm{D}_{\vec{u}} = \nabla f \cdot \vec{u}$
2. *Directional Derivative*
   - *All directional derivatives exist* $\not\Longrightarrow$ *differentiable or continuous.*
   - *All directional derivatives exist and are equal* $\not\Longrightarrow$ *differentiable.*
3. *Partial Derivative*
   - *The continuity and existence of directional/partial derivatives are mutually exclusive.*

## ¶ Higher-Order Partial Derivatives and Differential

If the first-order partial derivative of $f$, $\frac{\partial f}{\partial x_i}$, itself possesses partial derivatives, then the second-order partial derivative of $f$ is defined, and is denoted as follows(the first is also called the mixed partial derivative):

$$f_{x_ix_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right), \quad f_{x_ix_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right), \quad i,j = 1,2,\ldots,n.$$

Similarly, higher-order partial derivatives of order $3, 4, \cdots m, \cdots$ can be defined.

The following theorem provides the conditions under which mixed partial derivatives are equal.

*Theorem 10.2 (Conditions for Equality of Mixed Partial Derivatives)*

1. Let $U \subset \mathbb{R}^2$ be an open set, and $f : U \to \mathbb{R}$ be a function of two variables. If $f_{xy}$ and $f_{yx}$ are continuous at $(x_0, y_0) \in U$, then
$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$
2. Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}$ be a function of $n$ variables. If $f$ has partial derivatives up to order $k$ in $D$, and all of them are continuous at $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in U$, then
$$\frac{\partial^l f}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_l}}(\mathbf{x}^0) = \frac{\partial^l f}{\partial x_{i_2}\partial x_{i_1}\cdots\partial x_{i_l}}(\mathbf{x}^0) = \cdots = \frac{\partial^l f}{\partial x_{i_l}\partial x_{i_{l-1}}\cdots\partial x_{i_1}}(\mathbf{x}^0),$$
that is, the order of taking partial derivatives $l(\leq k)$ does not affect the result.[a]

[a] If the condition "$f_{xy}$ and $f_{yx}$ are continuous at $(x_0, y_0)$", is not satisfied, then the conclusion "$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$" does not necessarily hold.

*Proof* When $k \neq 0, h \neq 0$, define

$$\varphi(y) = f(x_0 + h, y) - f(x_0, y),$$

and

$$\psi(x) = f(x, y_0 + k) - f(x, y_0).$$

Applying the Lagrange Mean Value Theorem, we have

$$\begin{aligned}
&[f(x_0+h, y_0+k) - f(x_0, y_0+k)] - [f(x_0+h, y_0) - f(x_0, y_0)] \\
=&\varphi(y_0+k) - \varphi(y_0) \\
=&\varphi'(y_0+\theta_1 k)k \\
=&[f_y(x_0+h, y_0+\theta_1 k) - f_y(x_0, y_0+\theta_1 k)]k \\
=&f_{yx}(x_0+\theta_2 h, y_0+\theta_1 k)hk, \quad 0<\theta_1,\theta_2<1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&[f(x_0+h, y_0+k) - f(x_0, y_0+k)] - [f(x_0+h, y_0) - f(x_0, y_0)] \\
=&[f(x_0+h, y_0+k) - f(x_0+h, y_0)] - [f(x_0, y_0+k) - f(x_0, y_0)] \\
=&\psi(x_0+h) - \psi(x_0) \\
=&\psi'(x_0+\theta_3 h)h \\
=&[f_x(x_0+\theta_3 h, y_0+k) - f_x(x_0+\theta_3 h, y_0)]h \\
=&f_{xy}(x_0+\theta_3 h, y_0+\theta_4 k)hk, \quad 0<\theta_3,\theta_4<1.
\end{aligned}$$

Therefore,

$$f_{xy}(x_0+\theta_3 h, y_0+\theta_4 k) = f_{yx}(x_0+\theta_2 h, y_0+\theta_1 k).$$

Since $f_{xy}$ and $f_{yx}$ are continuous at $(x_0, y_0)$, letting $h \to 0, k \to 0$, we obtain

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

By applying 10.1 and the principle of mathematical induction, one can immediately derive the following result. ■

Suppose $z = f(x, y)$ has continuous partial derivatives in the domain $U \subset \mathbb{R}^2$. Then $z$ is differentiable, and

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y.$$

If $z$ also has continuous second-order partial derivatives, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are also differentiable, and thus $\mathrm{d}z$ is differentiable. We call the differential of $\mathrm{d}z$ the second-order differential of $z$, denoted as

$$\mathrm{d}^2 z = \mathrm{d}(\mathrm{d}z).$$

In general, based on the $k$-th order differential $\mathrm{d}^k z$ of $z$, its $(k+1)$-th order differential (if it exists) is defined as

$$\mathrm{d}^{k+1} z = \mathrm{d}(\mathrm{d}^k z), \quad k = 1, 2, \cdots.$$

Due to the fact that for the independent variables $x$ and $y$, we always have

$$\mathrm{d}^2 x = \mathrm{d}(\mathrm{d}x) = 0, \qquad \mathrm{d}^2 y = \mathrm{d}(\mathrm{d}y) = 0,$$

the second-order differential of $z = f(x, y)$ is given by

$$\begin{aligned}
\mathrm{d}^2 z = \mathrm{d}(\mathrm{d}z) &= \mathrm{d}\left(\frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y\right) \\
&= \mathrm{d}\left(\frac{\partial z}{\partial x}\right)\mathrm{d}x + \frac{\partial z}{\partial x}\mathrm{d}^2 x + \mathrm{d}\left(\frac{\partial z}{\partial y}\right)\mathrm{d}y + \frac{\partial z}{\partial y}\mathrm{d}^2 y \\
&= \left(\frac{\partial^2 z}{\partial x^2}\mathrm{d}x + \frac{\partial^2 z}{\partial x \partial y}\mathrm{d}y\right)\mathrm{d}x + \left(\frac{\partial^2 z}{\partial y \partial x}\mathrm{d}x + \frac{\partial^2 z}{\partial y^2}\mathrm{d}y\right)\mathrm{d}y \\
&= \frac{\partial^2 z}{\partial x^2}(\mathrm{d}x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\mathrm{d}x\mathrm{d}y + \frac{\partial^2 z}{\partial y^2}(\mathrm{d}y)^2,
\end{aligned}$$

where $(\mathrm{d}x)^2$ and $(\mathrm{d}y)^2$ denote $\mathrm{d}^2 x$ and $\mathrm{d}^2 y$ respectively. If we treat $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ as operators for partial differentiation and define

$$\left(\frac{\partial}{\partial x}\right)^2 = \frac{\partial^2}{\partial x^2}, \quad \left(\frac{\partial}{\partial y}\right)^2 = \frac{\partial^2}{\partial y^2}, \quad \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x \partial y},$$

then the formulas for the first and second differentials can be written as

$$\begin{aligned}
\mathrm{d}z &= \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right)z, \\
\mathrm{d}^2 z &= \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right)^2 z.
\end{aligned}$$

Similarly, we define

$$\left(\frac{\partial}{\partial x}\right)^p \left(\frac{\partial}{\partial y}\right)^q = \frac{\partial^{p+q}}{\partial x^p \partial y^q} = \frac{\partial^q}{\partial y^q}\left(\frac{\partial}{\partial x}\right)^p, \quad (p, q = 1, 2, \dots)$$

It is easy to use mathematical induction to prove the formula for higher-order differentials:

$$\mathrm{d}^k z = \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right)^k z, \quad k = 1, 2, \cdots.$$

For an $n$-variable function $u = f(x_1, x_2, \dots, x_n)$, higher-order differentials can be similarly defined, and the following holds:

$$\mathrm{d}^k u = \left(\mathrm{d}x_1 \frac{\partial}{\partial x_1} + \mathrm{d}x_2 \frac{\partial}{\partial x_2} + \cdots + \mathrm{d}x_n \frac{\partial}{\partial x_n}\right)^k u, \quad k = 1, 2, \ldots.$$

## 10.2 Differential of Vector-Valued Functions

Consider an $n$-dimensional vector-valued function defined on a domain $U \subset \mathbb{R}^n$:

$$f : U \to \mathbb{R}^m,$$
$$\mathbf{x} \mapsto \mathbf{y} = f(\mathbf{x})$$

Expressed in coordinate vector form:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in U$$

1. If each component function $f_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$ is partially differentiable at $\mathbf{x}^0$, then the vector-valued function **f** is differentiable at $\mathbf{x}^0$, and we define the matrix

$$\left(\frac{\partial f}{\partial x_j}(\mathbf{x}^0)\right)_{m\times n} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}^0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}^0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}^0) \end{pmatrix}$$

   This matrix is called the Jacobian matrix of **f** at $\mathbf{x}^0$, denoted by $f'(\mathbf{x}^0)$ (or $\mathrm{D}f(\mathbf{x}^0)$, $J_f(\mathbf{x}^0)$).
   For the special case $m = 1$, i.e., $n$-variable scalar function $z = f(x_1, x_2, \ldots, x_n)$, the derivative at $\mathbf{x}^0$ is

$$f'(\mathbf{x}^0) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}^0), \frac{\partial f}{\partial x_2}(\mathbf{x}^0), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x}^0)\right)$$

   If the vector-valued function **f** is differentiable at every point in $U$, then **f** is said to be differentiable on $U$, and the corresponding relationship is

$$\mathbf{x} \in U \mapsto f'(\mathbf{x}) = J_f(\mathbf{x})$$

   where $f'(\mathbf{x})$ (or $\mathrm{D}f(\mathbf{x})$, $J_f(\mathbf{x})$) denotes the derivative of **f** at **x** in $U$.
2. If every component function $f_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$ of **f** has continuous partial derivatives at $\mathbf{x}^0$, then every element of the Jacobian matrix of **f** is continuous at $\mathbf{x}^0$. In this case, **f** is said to have a continuous derivative at $\mathbf{x}^0$ as a vector-valued function.
   If the derivative of a vector-valued function **f** is continuous at every point in $U$, then **f** is said to have a continuous derivative on $U$.
3. If there exists an $m \times n$ matrix $A$ that depends only on $\mathbf{x}^0$ (and not on $\Delta\mathbf{x}$), such that in the neighborhood of $\mathbf{x}^0$,

$$\Delta\mathbf{y} = f(\mathbf{x}^0 + \Delta\mathbf{x}) - f(\mathbf{x}^0) = A\Delta\mathbf{x} + o(\|\Delta\mathbf{x}\|)$$

   (where $\Delta\mathbf{x} = (\Delta x_1, \Delta x_2, \ldots, \Delta x_n)^T$ is a column vector and $\|\Delta\mathbf{x}\|$ denotes its norm), then $f$ is said to be differentiable at $\mathbf{x}^0$ as a vector-valued function, and $A\Delta\mathbf{x}$ is called the differential of $f$ at $\mathbf{x}^0$, denoted as $\mathrm{d}\mathbf{y}$. If we denote $\Delta\mathbf{x}$ by $\mathrm{d}\mathbf{x}$ ($\mathrm{d}\mathbf{x} = (\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n)^T$), then

$$\mathrm{d}\mathbf{y} = A\,\mathrm{d}\mathbf{x}.$$

If the vector-valued function $\mathbf{f}$ is differentiable at every point in $U$, then $\mathbf{f}$ is said to be differentiable on $U$.

Combining the above three points, we obtain the following unified statement:

A vector-valued function $\mathbf{f}$ is continuous, differentiable, and has derivatives if and only if each of its coordinate component functions $f_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$ is continuous, differentiable, and has derivatives.

## 10.3 Derivatives of Composite Mappings (Chain Rule)

Let $U \subset \mathbb{R}^l$ and $V \subset \mathbb{R}^n$ be open sets, and let

$$\mathbf{g} : U \to V \quad \text{and} \quad \mathbf{f} : V \to \mathbb{R}^m$$

be mappings. If $\mathbf{g}$ is derivative at $\mathbf{u}^0 \in U$ and $\mathbf{f}$ is differentiable at $\mathbf{x}^0 = \mathbf{g}(\mathbf{u}^0)$, then the composite mapping $\mathbf{f} \circ \mathbf{g}$ is differentiable at $\mathbf{u}^0$, and:

$$J(\mathbf{f} \circ \mathbf{g})(\mathbf{u}^0) = J\mathbf{f}(\mathbf{x}^0) J\mathbf{g}(\mathbf{u}^0).$$

**Note**

1. *outer differentiable + inner derivative = total derivative*
2. *outer differentiable + inner differentiable = total differentiable*
3. 

Specially, define $z = f(x, y), (x, y) \subset D_f \subset \mathbb{R}^2$, $\mathbf{g} : D_g \to \mathbb{R}^2, (u, v) \mapsto (x(u, v), y(u, v))$, and $g(D_g) \subset D_f$, then we have composite function

$$z = f \circ \mathbf{g} = f\left[x(u, v), y(u, v)\right], \quad (u, v) \in D_g.$$

$$\mathbb{R}^2 \xrightarrow{\mathbf{g}:\text{derivative}} \mathbb{R}^2 \xrightarrow{f:\text{differentiable}} \mathbb{R}$$

If $\mathbf{g}$ is derivative at $(u_0, v_0) \in D_g$, and $f$ is differentiable at $(x_0, y_0) = \mathbf{g}(u_0, v_0)$, then $z = f \circ \mathbf{g}$ is differentiable at $(u_0, v_0)$, and at the point,

$$\begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

*Proof*

■

## 10.4 Mean Value Theorem and Taylor's Formula

*Definition 10.4 (Convex Region)*

Let $D \subseteq \mathbb{R}^n$ be a region. If every line segment connecting any two points $\mathbf{x}_0, \mathbf{x}_1 \in D$ (denoted by $\overline{\mathbf{x}_0\mathbf{x}_1}$) is entirely contained in $D$, i.e., for any $\lambda \in [0, 1]$, we have

$$\mathbf{x}_0 + \lambda(\mathbf{x}_1 - \mathbf{x}_0) \in D,$$

then $D$ is called a convex region. ♣

**Theorem 10.3 (Lagrange's Mean Value Theorem)**

Let $f$ be differentiable on a convex region $D \subseteq \mathbb{R}^n$. For any two points $\mathbf{a}, \mathbf{b} \in D$, there exists a point $\xi \in \overline{\mathbf{ab}}$ such that:

$$f(\mathbf{b}) - f(\mathbf{a}) = Jf(\xi)(\mathbf{b} - \mathbf{a}).$$

*Proof*

■

**Theorem 10.4**

Let $D$ be a region in $\mathbb{R}^n$. If for any $\mathbf{x} \in D$, we have

$$Jf(\mathbf{x}) = 0,$$

then $f$ is constant on $D$.

*Proof*

■

**Theorem 10.5 (Taylor's Formula)**

**Lagrange's Remainder** Let $D \subseteq \mathbb{R}^n$ be a convex region, and let $f : D \to \mathbb{R}$ have $m+1$ continuous partial derivatives. For $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in D$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in D$, there exists $\xi \in \overline{\mathbf{x}^0\mathbf{x}}$ such that:

$$f(\mathbf{x}) = f(\mathbf{x}^0) + \sum_{k=1}^{m} \frac{1}{k!} \left( \sum_{i=1}^{n} (x_i - x_i^0) \frac{\partial}{\partial x_i} \right)^k f(\mathbf{x}^0) + \frac{1}{(m+1)!} \left( \sum_{i=1}^{n} (x_i - x_i^0) \frac{\partial}{\partial x_i} \right)^{m+1} f(\xi).$$

**Peano's Remainder** Let $D \subseteq \mathbb{R}^n$ be a convex region, and let $f : D \to \mathbb{R}$ have $m$ continuous partial derivatives. Then:

$$f(\mathbf{x}) = f(\mathbf{x}^0) + \sum_{k=1}^{m} \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}} (\mathbf{x}^0) \prod_{j=1}^{k} (x_{i_j} - x_{i_j}^0) + R_m(\mathbf{x} - \mathbf{x}^0),$$

where $R_m(\mathbf{x} - \mathbf{x}^0) = O(\|\mathbf{x} - \mathbf{x}^0\|^{m+1})$ or $o(\|\mathbf{x} - \mathbf{x}^0\|^m)$, as $\|\mathbf{x} - \mathbf{x}^0\| \to 0$.

In applications, particularly important is the expression of the first three terms in Taylor's formula, which is given as (let $x_1 - x_1^0$ be denoted by $\Delta x_1$, and similarly for other variables; $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \ldots, \Delta x_n)$):

$$f(\mathbf{x}) = f(\mathbf{x}^0) + Jf(\mathbf{x}^0)(\Delta \mathbf{x}) + \frac{1}{2!}(\Delta \mathbf{x}) Hf(\mathbf{x}^0)(\Delta \mathbf{x})^{\mathrm{T}} + \cdots,$$

where the matrix

$$Hf(\mathbf{x}^0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{\mathbf{x}^0}$$

is called the **Hessian matrix** of the function $f$. It is an $n \times n$ symmetric matrix.

## 10.5 Implicit Function Theorem

*Theorem 10.6 (Implicit Function Theorem)*

Let $U \subset \mathbb{R}^{n+1}$ be an open set, and $F : U \to \mathbb{R}$ be an $n+1$-variable function. If:

1. $F \in C^k(U, \mathbb{R})$, where $1 \leq k \leq +\infty$;
2. $F(\mathbf{x}^0, y^0) = 0$, where $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \mathbb{R}^n$, $y^0 \in \mathbb{R}$, and $(\mathbf{x}^0, y^0) \in U$ (i.e., the equation $F(\mathbf{x}, y) = 0$ has a solution $(\mathbf{x}^0, y^0)$);
3. $F'_y(\mathbf{x}^0, y^0) \neq 0$.

Then there exists an open interval $I \times J$ containing $(\mathbf{x}^0, y^0)$ ($I$ being an open interval in $\mathbb{R}^n$ containing $\mathbf{x}^0$, and $J$ being an open interval in $\mathbb{R}$ containing $y^0$), as shown in Fig. 10.1, such that:

1. $\forall x \in I$, the equation $F(\mathbf{x}, y) = 0$ has a unique solution $y = f(\mathbf{x})$, where $f : I \to J$ is an $n$-variable function (called the **implicit function** $f$, hidden within the equation $F(\mathbf{x}, f(\mathbf{x})) = 0$, though not necessarily explicitly expressed);
2. $y^0 = f(\mathbf{x}^0)$;
3. $f \in C^k(I, \mathbb{R})$;
4. When $x \in I$, $\frac{\partial f}{\partial x_i} = \frac{\partial y}{\partial x_i} = -\frac{F_x(\mathbf{x},y)}{F_y(\mathbf{x},y)}$, $i = 1, 2, \ldots, n$, where $y = f(x)$.

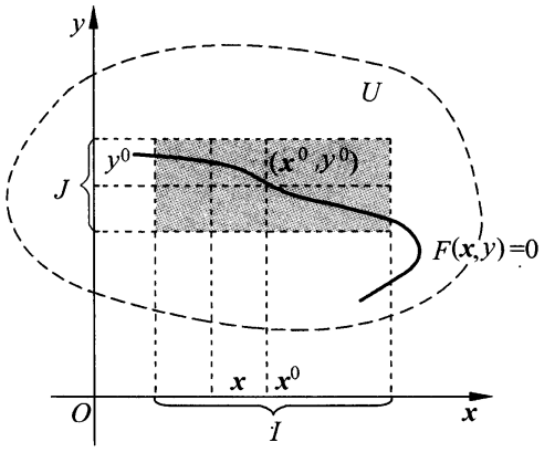



**Figure 10.1:** Implicit Function

*Proof* Only the single-variable implicit function theorem is proved; the multi-variable case can be derived using mathematical induction.

Without loss of generality, assume $F_y(x^0, y^0) > 0$.

First, prove the existence of the implicit function. From the continuity of $F_y(x^0, y^0) > 0$ and $F_y(x, y)$, it is known that there exist closed rectangle:

$$D^* = \{(x, y) \mid |x - x_0| \leq \alpha, |y - y_0| \leq \beta\} \subset U,$$

where the following holds:

$$F_y(x, y) > 0.$$

Thus, for fixed $x_0$, the function $F(x^0, y)$ is strictly monotonically increasing within $[y^0 - \beta, y^0 + \beta]$. Furthermore, since,

$$F(x^0, y^0) = 0,$$

it follows that:

$$F(x^0, y^0 - \beta) < 0, \quad F(x^0, y^0 + \beta) > 0.$$

Due to the continuity of $F(x, y)$ within $D^*$, there exists $\rho > 0$ such that along the line segment:

$$x = x^0 + \rho,\ y = y^0 + \beta,$$

we have $F(x, y) > 0$, and along the line segment:

$$x = x^0 + \rho,\ y = y^0 - \beta,$$

we have $F(x, y) < 0$. Therefore, for any point $\bar{x} \in (x^0 - \rho, x^0 + \rho)$, treat $F(x, y)$ as a single-variable function of $y$. Within $[y^0 - \beta, y^0 + \beta]$, this function is continuous. From the previous discussion, we know:

$$F(\bar{x}, y^0 - \beta) < 0, \quad F(\bar{x}, y^0 + \beta) > 0.$$

According to the zero point existence theorem 3.3, there must exist a unique $\bar{y} \in [y^0 - \beta, y^0 + \beta]$ such that $F(\bar{x}, \bar{y}) = 0$. Furthermore, because $F_y(x, y) > 0$ within $D^*$, this $\bar{y}$ is unique. Denote the corresponding relationship as $\bar{y} = f(\bar{x})$, then the function $y = f(x)$ is defined within $(x^0 - \rho, x^0 + \rho)$, satisfying $F(x, f(x)) = 0$, and clearly:

$$y^0 = f(x^0).$$

Further proving the continuity of the implicit function $y = f(x)$ on $(x^0 - \rho, x^0 + \rho)$: Let $\bar{x} \in (x^0 - \rho, x^0 + \rho)$ be any point. For any given $\varepsilon > 0$ ($\varepsilon$ being sufficiently small), since $F(\bar{x}, \bar{y}) = 0$ ($\bar{y} = f(\bar{x})$), from the previous discussion we know:

$$F(\bar{x}, \bar{y} - \varepsilon) < 0, \quad F(\bar{x}, \bar{y} + \varepsilon) > 0.$$

Furthermore, due to the continuity of $F(x, y)$ on $D^*$, there exists $\delta > 0$ such that:

$$F(x, \bar{y} - \varepsilon) < 0, \quad F(x, \bar{y} + \varepsilon) > 0, \quad \text{when} \quad x \in O(x^0, \delta).$$

By reasoning similar to the previous discussion, it can be obtained that when $x \in O(x^0, \delta)$, the corresponding implicit function value must satisfy $f(x) \in (\bar{y} - \varepsilon, \bar{y} + \varepsilon)$, i.e.,

$$\left| f(x) - f(x^0) \right| < \varepsilon.$$

This implies that $y = f(x)$ is continuous on $(x^0 - \rho, x^0 + \rho)$.

Finally, prove the differentiability of $y = f(x)$ on $(x^0 - \rho, x^0 + \rho)$: Let $\bar{x} \in (x^0 - \rho, x^0 + \rho)$ be any point. Take $\Delta x$ sufficiently small such that $\bar{x} = x + \Delta x \in (x^0 - \rho, x^0 + \rho)$. Denote $\bar{y} = f(\bar{x})$ and $\bar{y} + \Delta y = f(\bar{x})$. Clearly,

$$F(\bar{x}, \bar{y}) = 0 \quad \text{and} \quad F(\bar{x}, \bar{y} + \Delta y) = 0.$$

Using the multi-variable function's mean value theorem 10.3, we obtain:

$$\begin{aligned} 0 &= F(\bar{x}, \bar{y} + \Delta y) - F(\bar{x}, \bar{y}) \\ &= F_x(\bar{x} + \theta\Delta x, \bar{y} + \theta\Delta y)\Delta x + F_y(\bar{x} + \theta\Delta x, \bar{y} + \theta\Delta y)\Delta y, \end{aligned}$$

where $0 < \theta < 1$. Note that $F_y \neq 0$ on $D^*$, hence:

$$\frac{\Delta y}{\Delta x} = -\frac{F_x(\bar{x} + \theta\Delta x, \bar{y} + \theta\Delta y)}{F_y(\bar{x} + \theta\Delta x, \bar{y} + \theta\Delta y)}.$$

Let $\Delta x \to 0$. Considering the continuity of $F_x$ and $F_y$, we obtain:

$$\left.\frac{dy}{dx}\right|_{x=\bar{x}} = -\frac{F_x(\bar{x}, \bar{y})}{F_y(\bar{x}, \bar{y})}.$$

Thus:

$$f'(\bar{x}) = -\frac{F_x(\bar{x}, \bar{y})}{F_y(\bar{x}, \bar{y})}.$$

The proof is complete. ■

# Chapter 11 Multiple Integrals

# Bibliography


[1] 徐森林, 薛春华. 数学分析. 第一版. 清华大学出版社, 2005.

[2] 陈纪修, 於崇华. 数学分析. 第三版. 高等教育出版社, 2019.

[3] 常庚哲, 史济怀. 数学分析教程. 第三版. 中国科学技术大学出版社, 2012.

[4] 裴礼文. 数学分析中的典型问题与方法. 第三版. 高等教育出版社, 2021.

[5] 汪林. 数学分析中的问题与反例. 第一版. 高等教育出版社, 2015.

[6] 谢惠民, 恽自求, 易法槐, 钱定边. 数学分析习题课讲义. 第二版. 高等教育出版社, 2019.

[7] Walter Rudin. *Principles of Mathematical Analysis*. Third Edition. McGraw-Hill, 1976.

[8] 菲赫金哥尔茨. 微积分学教程. 第八版. 高等教育出版社, 2006.

[9] Wikipedia. `https://en.wikipedia.org/wiki/`.