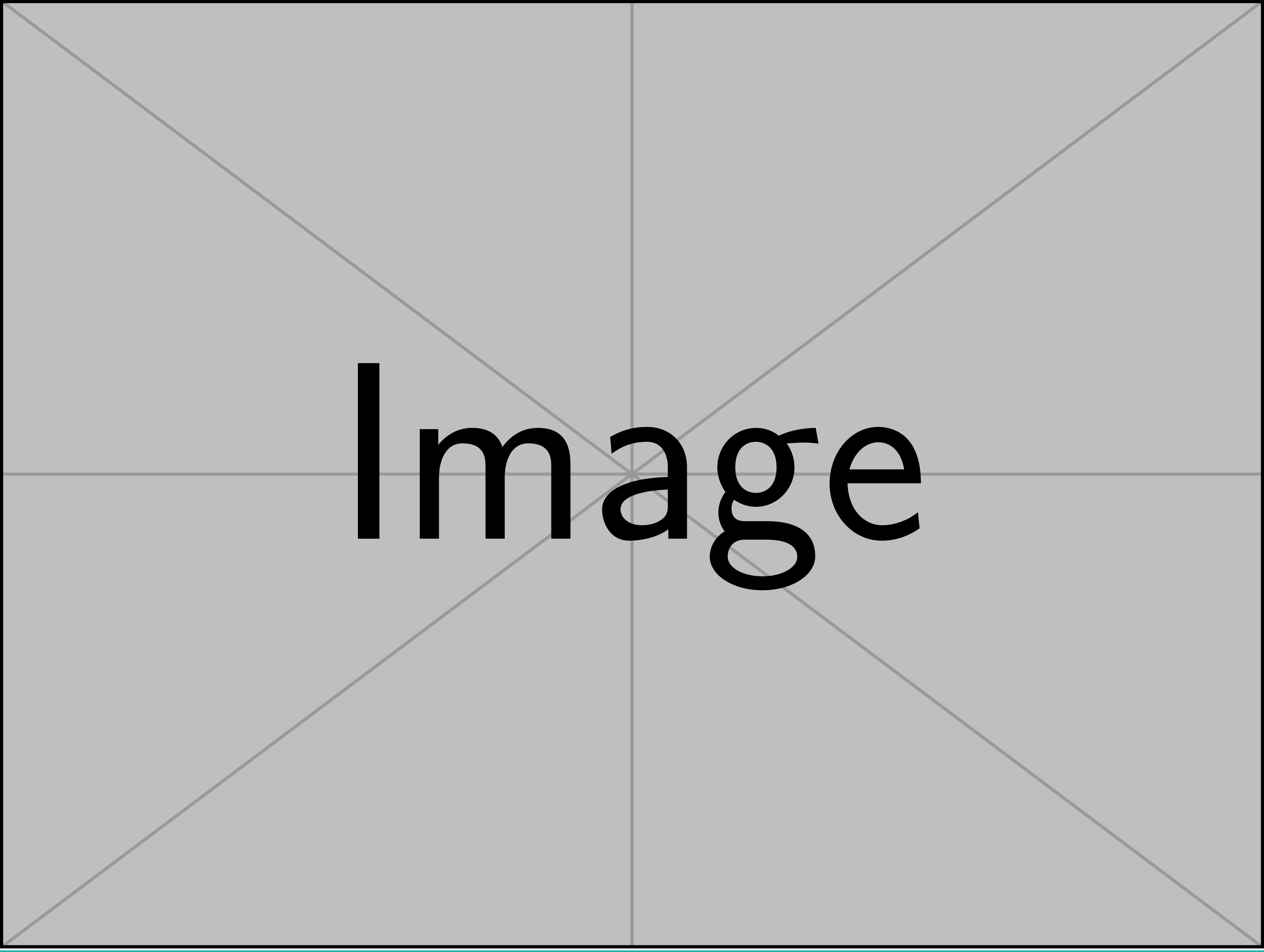

Équation Différentielle Ordinaire

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Contents

Preface

This is the preface of the book…

Alert Through this text, we focus exclusively on real-valued differential equations, where all quantities are assumed to be real unless stated otherwise.

Chapter 1 Introduction

1.1 Classification of Differential Equations

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a **differential equation**. Differential equations can be classified according to the following criteria:

¶ `Number of Independent Variables`

An **ordinary differential equation(ODE)** is defined as an equation of the following form:

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \dots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n}\right) = 0, \quad n \in \mathbb{N}, \tag{1.1}$$

or, using the prime notation for derivatives,

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0, \quad n \in \mathbb{N}.$$

If there are two or more independent variables, the equation is called a **partial differential equation(PDE)**.

¶ `Order`

The order of a differential equation is the order of the highest derivative present in the equation.

- A first-order equation has the form $F(x, y, y') = 0$.
- A second-order equation has the form $F(x, y, y', y'') = 0$.
- Higher-order equations involve derivatives of order three or more.

Note *Crucially, the order tells you how many initial conditions are needed to find a unique solution.*

¶ `Linearity`

An n-th order differential equation is linear if it can be written in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where the coefficients $a_i(x)$ and the term $g(x)$ depend only on the independent variable x. Otherwise, it is nonlinear.

Note *Specially, for the aforementioned equation, if $g(x) = 0$, it is called **homogeneous**, and **non-homogeneous** otherwise.*

1.2 Solution to a Ordinary Differential Equation

¶ `Particular and General Solutions`

Let J be an interval in $\mathbb{R}$. A function $y = \phi(x)$ defined on the interval J is called a solution to equation (1.1) if it satisfies:

$$F(x, \phi(x), \phi'(x), \phi''(x), \dots, \phi^{(n)}(x)) = 0 \quad x \in J.$$

The interval J is then called the interval of existence of the solution $y = \phi(x)$.

Generally speaking, the solution to equation (1.1) contains one or more arbitrary constants, the determination of which depends on other conditions that the solution must satisfy. If a solution to a differential equation does not contain any arbitrary constants, it is called a **particular solution** of the differential equation.

Suppose $y = \phi(x; c_1, c_2, \cdots, c_n)$ is a solution to equation (1.1), where $c_1, c_2, \dots, c_n$ are arbitrary constants. If $c_1, c_2, \dots, c_n$ are mutually independent, then $y = \phi(x; c_1, c_2, \dots, c_n)$ is called the **general solution** to equation (1.1). Here, "mutually independent" means that the Jacobian determinant is non-zero:

$$\det \frac{\partial(\phi, \phi', \dots, \phi^{(n-1)})}{\partial(c_1, c_2, \dots, c_n)} \neq 0, \quad x \in J.$$

When all the arbitrary constants in the general solution are determined, one obtains a particular solution to the differential equation.

¶ **Initial Conditions, Explicit and Implicit Solutions**

Let $y = \phi(x)$ be a solution to equation (1.1) that also satisfies

$$\phi(x_0) = y_0, \quad \phi'(x_0) = y_0', \dots, \quad \phi^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{1.2}$$

The conditions (1.2) are called the **initial conditions** for equation (1.1), and $y = \phi(x)$ is called the solution to equation (1.1) satisfying the initial conditions (1.2). Such initial value problems are often referred to as **Cauchy problems**.

A function $y = \phi(x)$ that turns the differential equation (1.1) into an identity is called an **(explicit) solution** to the equation. If a solution $y = \phi(x)$ to the differential equation (1.1) is determined by the relation $\Phi(x, y) = 0$, then $\Phi(x, y) = 0$ is called an **implicit solution** to the differential equation (1.1). An implicit solution is also called an "integral".

¶ **Integral Curve and Direction Field**

Consider the first-order differential equation:

$$\frac{dy}{dx} = f(x, y), \tag{1.3}$$

where f is continuous in a planar region G. Suppose

$$y = \phi(x), \quad x \in J$$

is a solution to this equation, where $J \subset \mathbb{R}$ is an interval. Then the set of points in the plane

$$\Gamma = (x, y)|y = \phi(x), x \in J$$

is a differentiable curve in the plane. This curve is called a solution curve or an **integral curve**.

Let $(x_0, y_0) \in \Gamma$. The slope of the tangent line to the curve Γ at this point is

$$\phi'(x_0) = f(x_0, y_0).$$

Therefore, the equation of the tangent line is

$$y - y_0 = f(x_0, y_0)(x - x_0).$$

This implies that even without knowing the explicit expression for ϕ, we can obtain the slope and equation of the tangent line to the solution curve at a given point from equation (1.3).

Remark Note that in a small neighborhood of a point on a differentiable curve, the tangent line can be seen as a first-order approximation of the curve. Utilizing this viewpoint, one can obtain an approximate solution to the differential equation. In fact, this is the fundamental idea behind Euler's method.

At each point P in the region G, we can draw a short line segment $l(P)$ with slope $f(P)$. We call $l(P)$ the line element of equation (1.3) at point P. The region G together with the entire collection of these line elements is called the lineal **linear element field** or **direction field** for equation (1.3).

Theorem 1.1

A necessary and sufficient condition for a continuously differentiable curve $\Gamma = \{(x, y)|y = \psi(x), x \in J\}$ in the plane to be an integral curve of equation (1.3) is that for every point (x, y) on the curve Γ, its tangent line at that point coincides with the line element determined by equation (1.3) at that point.

Proof The necessity follows from the preceding discussion. We now prove the sufficiency. For any point $(x, y) = (x, \psi(x))$ on the curve Γ, the slope of the tangent line to Γ at this point is $\psi'(x)$. By the condition of the theorem, we have $\psi'(x) = f(x, y)$. Since (x, y) is an arbitrary point on the curve, it follows that $y = \psi(x)$ is a solution to equation (1.3). ■

Chapter 2 First Order Equations

2.1 Exact Equations

Definition 2.1 (Exact Equations)

An equation of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{2.1}$$

is called the symmetric form of a first-order differential equation.

If there exists a continuously differentiable function $u(x,y)$ such that

$$\mathrm{d}U(x,y) = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y,$$

then equation (2.1) is said to be an **exact equation** or a **total differential equation**.

It follows that, when equation (2.1) is exact, it can be rewritten as

$$\mathrm{d}(U(x,y)) = 0,$$

which implies

$$U(x,y) = c, \tag{2.2}$$

where c is an arbitrary constant. Equation (2.2) is called the **general integral** of equation (2.1). ♣

Remark It should be noted that, strictly speaking, equation (2.1) is not a differential equation. However, expressing a first-order differential equation in the form of (2.1) is extremely convenient for analysis. This formulation does not necessarily require y to be expressed as a function of x. For the sake of simplicity in description, we often refer to the symmetric form (2.1) as a differential equation, too.

Theorem 2.1

Let the functions $M(x,y)$ and $N(x,y)$ be continuous in a simply connected domain $D \subset \mathbb{R}^2$, and suppose their first-order partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are also continuous. Then a necessary and sufficient condition for equation (2.1) to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

in the domain D. When this condition holds, for any $(x_0, y_0), (x,y) \in D$, a general integral of equation (2.1) is given by

$$\int_\gamma M(x,y)dx + N(x,y)dy = c,$$

where γ is any curve composed of finitely many smooth segments within D connecting (x_0,y_0) and (x,y), and c is an arbitrary constant. ♡

Proof

■

The aforementioned proof also serves as a method for determining the bivariate function $U(x,y)$ that satisfies specific conditions. In addition to this approach, there exist two simpler methods for solving $U(x,y)$.

Utilizing Curve Integrals to Solve $U(x,y)$

Term Combination Method Utilizing the properties of bivariate differential functions, we combine the terms of the differential equation into a full differential form. This method requires familiarity with some simple bivariate differential functions, such as:

$$
\begin{aligned}
y\mathrm{d}x + x\mathrm{d}y &= \mathrm{d}(xy), \\
\frac{y\mathrm{d}x - x\mathrm{d}y}{y^2} &= \mathrm{d}\left(\frac{x}{y}\right), \\
\frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2} &= \mathrm{d}\left(\frac{y}{x}\right), \\
\frac{1}{x}\mathrm{d}x + \frac{1}{y}\mathrm{d}y = \frac{y\mathrm{d}x + x\mathrm{d}y}{xy} &= \mathrm{d}(\ln|xy|), \\
\frac{1}{x}\mathrm{d}x - \frac{1}{y}\mathrm{d}y = \frac{y\mathrm{d}x - x\mathrm{d}y}{xy} &= \mathrm{d}(\ln|\frac{x}{y}|), \\
\frac{y\mathrm{d}x - x\mathrm{d}y}{x^2 - y^2} &= \frac{1}{2}\mathrm{d}\left(\ln\left|\frac{x-y}{x+y}\right|\right), \\
\frac{y\mathrm{d}x + x\mathrm{d}y}{x^2 + y^2} &= \mathrm{d}\left(\arctan\frac{y}{x}\right), \\
\frac{y\mathrm{d}x - x\mathrm{d}y}{x^2 + y^2} &= \mathrm{d}\left(\operatorname{arccot}\frac{y}{x}\right).
\end{aligned}
$$

The theory above can also be rewritten in differential form:

Let:

$$\omega^1 = M(x,y)\,dx + N(x,y)\,dy.$$

The differential form ω^1 is said to be **closed** if $\mathrm{d}\omega^1 = 0$. It is called **exact** if there exists a function $U(x,y)$ such that $\omega^1 = \mathrm{d}U(x,y)$. By the Poincaré theorem, it can be concluded that on $\mathbb{R}^2$, a first-order differential form is exact if and only if it is closed. Note that:

$$\mathrm{d}\omega^1 = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y.$$

Clearly, $\mathrm{d}\omega^1 = 0$ holds if and only if:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Under this condition, the expression for the function $U(x,y)$ is:

$$U(x,y) = \int \omega^1.$$

2.2 Separable Equations

Definition 2.2 (Separable Equations)

If the functions $M(x,y)$ and $N(x,y)$ in Equation (2.1) can both be written as the product of a function of x and a function of y, that is,

$$M(x,y) = M_1(x)M_2(y), \quad N(x,y) = N_1(x)N_2(y),$$

then equation (2.1) is called a separable equation.

When equation (2.1) is a separable equation, it can be written as

$$M_1(x)M_2(y)\,\mathrm{d}x + N_1(x)N_2(y)\,\mathrm{d}y = 0, \tag{2.3}$$

or more conveniently as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \left(= -\frac{M_1(x)}{N_1(x)} \cdot \frac{N_2(y)}{M_2(y)} \right). \tag{2.4}$$

♣

Theorem 2.2 (Solutions to Separable Equations)

All the solutions to the separable equation (2.3) are given by:

$$\int_{x_0}^{x} \frac{M_1(t)}{N_1(t)} \mathrm{d}t + \int_{y_0}^{y} \frac{N_2(s)}{M_2(s)} \mathrm{d}s = c,$$

and

$$y \equiv b_i, \quad i = 1, 2, \ldots, m, x \equiv a_j, \quad j = 1, 2, \ldots, n,$$

where $M_2(b_i) = 0 \quad (i = 1, 2, \ldots, m)$ and $N_1(a_j) = 0 \quad (j = 1, 2, \ldots, n)$, c is arbitrary constant. ♡

2.3 Homogeneous Equations

Definition 2.3

A first-order differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

is called a **homogeneous equation** if both M and N are homogeneous functions [a] of the same degree n. In other words, for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

$f(x, y)$ can be rewritten as $g\left(\frac{y}{x}\right)$.

[a] A function $f(x, y)$ is called a homogeneous function of degree n if it satisfies the condition $f(tx, ty) = t^n f(x, y)$ for all $t > 0$. ♣

The equation

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \tag{2.5}$$

can be transformed into a separable equation via variable change, where $a_1, a_2, b_1, b_2, c_1, c_2$ are constants.

- When $c_1 = c_2 = 0$, the equation becomes:

  $$\frac{dy}{dx} = f\left(\frac{a_1 + b_1\frac{y}{x}}{a_2 + b_2\frac{y}{x}}\right) = g\left(\frac{y}{x}\right).$$

  Let

  $$u = \frac{y}{x}, \text{ namely } y = ux.$$

  Differentiating both sides with respect to x, we get:

  $$\frac{dy}{dx} = x\frac{du}{dx} + u.$$

  Substituting the results into original equation and simplifying, we obtain:

  $$\frac{du}{dx} = \frac{g(u) - u}{x},$$

  which is a separable equation. It can be solved easily. Then, substituting $u = \frac{y}{x}$ back, the solution is derived.

- When c_1, c_2 are not entirely zero, the right-hand side of (2.5) consists of linear polynomials of x and y. Therefore:

$$\begin{cases} a_1x + b_1y + c_1 = 0, \\ a_2x + b_2y + c_2 = 0, \end{cases}$$

represents two intersecting straight lines on the Oxy plane. For the coefficient determinant of the system:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

two cases are analyzed:

1. If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, indicating that the two lines intersect at a unique point (α, β) on the Oxy plane. Let:

$$\begin{cases} X = x - \alpha, \\ Y = y - \beta, \end{cases}$$

then (2.3) becomes:

$$\begin{cases} a_1X + b_1Y = 0, \\ a_2X + b_2Y = 0. \end{cases}$$

Substituting into 2.5, it simplifies to:

$$\frac{dY}{dX} = f\left(\frac{a_1 + b_1\frac{Y}{X}}{a_2 + b_2\frac{Y}{X}}\right) = g\left(\frac{Y}{X}\right).$$

This is a homogeneous differential equation. Solving it by substitution and reverting back to the original variables yields the solution to equation 2.5.

2. When $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. To ensure this system holds, there are three possible scenarios:

(a). If $a_1 = b_1 = 0$, 2.5 becomes:

$$\frac{dy}{dx} = f\left(\frac{c_1}{a_2x + b_2y + c_2}\right),$$

and when $a_2 = b_2 = 0$, it becomes:

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{c_2}\right).$$

In this case, let

$$u = \frac{a_1x + b_1y + c_1}{c_2}.$$

Then it can be transformed into a separable equation.

(b). If $b_1 = b_2 = 0$, 2.5 transforms into:

$$\frac{dy}{dx} = f\left(\frac{a_1x + c_1}{a_2x + c_2}\right),$$

and

$$\frac{dy}{dx} = f\left(\frac{b_1y + c_1}{b_2y + c_2}\right),$$

when $a_1 = a_2 = 0$.

(c). If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$, let $u = a_2x + b_2y$. In this case:

$$\frac{du}{dx} = a_2 + b_2\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$f\left(\frac{k(a_2x + b_2y) + c_1}{(a_2x + b_2y) + c_2}\right) = f\left(\frac{ku + c_1}{u + c_2}\right) = g(u)$$

which simplifies to:

$$\frac{du}{dx} = a_2 + b_2g(u).$$

Example 2.1 A function $f(x, y)$ is called a quasihomogeneous function of degree d with generalized weights if

$$f(t^\alpha sx, t^\beta sy) = t^{ds}f(x, y),$$

where $t > 0$, α and β are positive constants with $\alpha + \beta = 1$, and $s \in \mathbb{R}$. Here, α and β are called the weights of x and y, respectively. Consider the differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0,$$

where $M(x, y)$ and $N(x, y)$ are quasihomogeneous functions of degree d_0 and d_1 with weights α and β for x and y, respectively. Proposition: When $d_0 = d_1 + \beta - \alpha$ the equation can be solved by elementary integration method.

2.4 Linear Equations

Definition 2.4 (First-Order Linear Equations)

A **first-order linear equation** is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x), \tag{2.6}$$

where $p(x)$ and $q(x)$ are continuous functions on the interval (a, b). In Equation (2.6), when $q(x) \equiv 0$, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0, \tag{2.7}$$

which is called a **first-order homogeneous linear equation** corresponding to Equation (2.6). Otherwise, it is called a first-order non-homogeneous linear equation. ♣

Note *It should be noted that the definition of a homogeneous equation here differs from that in the previous section.*

Firstly, we solve the first-order homogeneous linear equation. Equation 2.7 is separable, thus its general solution is given by:

$$y = ce^{-\int p(x)\,\mathrm{d}x},$$

where c is an arbitrary constant.

Since 2.7 is a special case of 2.6, the general solution of 2.6 can be expressed as:

$$y = c(x)e^{-\int p(x)\,\mathrm{d}x},$$

substituting it into 2.6 yields:

$$y = e^{-\int p(x)\,\mathrm{d}x}\left(c + \int q(x)e^{\int p(x)\,\mathrm{d}x}\,\mathrm{d}x\right).$$

This method of solving first-order linear equations is known as the **method of variation of constants**.

Definition 2.5 (Bernoulli's Equation)

A first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n, \quad n \neq 0, 1,$$

where n is a real number and $p(x)$ and $q(x)$ are continuous functions on the interval (a, b), is called a **Bernoulli's equation**. ♣

Bernoulli's equation can be transformed into a first-order linear equation by the substitution:

$$z = y^{1-n}.$$

Differentiating both sides with respect to x gives:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Substituting $\frac{\mathrm{d}y}{\mathrm{d}x}$ from Bernoulli's equation into the above expression yields:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)\left(-p(x)z + q(x)\right).$$

This is a first-order linear equation in z, which can be solved using the method for first-order linear equations.

2.5 Integrating Factors

Definition 2.6 (Integrating Factors)

An **integrating factor** for a first-order differential equation of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{2.8}$$

is a differentiable function $\mu(x,y)$ such that when multiplied by the equation:

$$\mu(x,y)M(x,y)\,\mathrm{d}x + \mu(x,y)N(x,y)\,\mathrm{d}y = 0,$$

it becomes an exact equation. Id est, there exists a function $\Phi(x,y)$ such that

$$\mu(x,y)M(x,y)\,\mathrm{d}x + \mu(x,y)N(x,y)\,\mathrm{d}y = \mathrm{d}U(x,y).$$

If such functions $\mu(x,y)$ and $U(x,y)$ exist, and $U(x,y)$ is smooth, then

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}\left(= \frac{\partial^2 U}{\partial x \partial y}\right). \tag{2.9}$$

In this case, $\mu(x,y)$ is called an integrating factor for equation (2.8). ♣

According to Equation (2.9), finding an integrating factor $\mu(x,y)$ for equation (2.8) is equivalent to solving the partial differential equation:

$$\frac{\partial \mu}{\partial x}N - \frac{\partial \mu}{\partial y}M = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu. \tag{2.10}$$

Theorem 2.3

1. For the partial differential equation 2.10 to have a solution $\mu(x)$ that depends only on x, the necessary and sufficient condition is:
   The function G defined below must depend only on x:

$$G = -\frac{1}{N(x,y)}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

In this case, we have:

$$\mu(x) = e^{\int_{x_0}^{x} G(t)\, dt}.$$

2. For the partial differential equation 2.10 to have a solution $\mu(y)$ that depends only on y, the necessary and sufficient condition is:
   The function H defined below must depend only on y:
   $$H = \frac{1}{M(x,y)}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$
   In this case, we have:
   $$\mu(y) = e^{\int_{y_0}^{y} H(s)\, ds}.$$
3. For equation 2.8 to have an integrating factor of the form $\mu = \mu(\phi(x,y))$, the necessary condition is:
   $$\frac{1}{\frac{\partial \phi}{\partial x}N - \frac{\partial \phi}{\partial y}M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(\phi(x,y)),$$
   where f is a certain univariate function.

Theorem 2.4

Let the functions $P(x,y)$, $Q(x,y)$, $\mu_1(x,y)$, and $\mu_2(x,y)$ be continuously differentiable. Suppose $\mu_1(x,y)$ and $\mu_2(x,y)$ are integrating factors for equation (2.8), and the ratio $\frac{\mu_1(x,y)}{\mu_2(x,y)}$ is not a constant. Then:

$$\frac{\mu_1(x,y)}{\mu_2(x,y)} = c$$

is a general solution to the equation, where c is an arbitrary constant.

2.6 Implicit Equations

This section discusses the problem of solving the first-order implicit differential equations,

$$F(x, y, y') = 0 \tag{2.11}$$

where F is a continuously differentiable function. A so-called implicit differential equation is one in which y' does not have an explicit solution, that is, the equation cannot be written in the form $y' = f(x,y)$.

¶ Differentiation Method

Suppose that Equation (2.11) can be solved for y, that is,

$$y = f(x,p), \quad p = \frac{dy}{dx}, \tag{2.12}$$

where $f(x,p)$ is a continuously differentiable function.

Differentiating both sides of $y = f(x,p)$ with respect to x, we obtain

$$p = \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx},$$

that is,

$$\frac{\partial f}{\partial p}\frac{dp}{dx} = p - \frac{\partial f}{\partial x}.$$

This is a first-order differential equation in the variables x, p, $\frac{dp}{dx}$. If a solution $p = p(x)$ can be found, then Equation (2.12) yields a solution

$$y = f(x, p(x)).$$

¶ **Parametric Method**

In general, Equation (2.11) represents a surface in the (x, y, p)-space. Therefore, the solution can be obtained using a parametric representation of the surface. Suppose the parametric form of the surface described by Equation (2.11) is

$$x = x(u, v), \quad y = y(u, v), \quad p = p(u, v) = y'.$$

Note that

$$dy = p\ \mathrm{d}x,$$

thus we obtain

$$y'_u \mathrm{d}u + y'_v \mathrm{d}v = p(u, v)(x'_u \mathrm{d}u + x'_v \mathrm{d}v).$$

This is an explicit differential equation in the variables u and v. Suppose it admits a solution

$$v = v(u, c),$$

where c is a constant, then Equation (2.11) has a solution

$$x = x(u, v(u, c)), \quad y = y(u, v(u, c)).$$

Chapter 3 Existence and Uniqueness Theorem

3.1 Picard-Lindelöf Theorem

Theorem 3.1 (Bellman-Gronwall Inequality)

Let $f(x), g(x)$ be continuous functions on the interval $[a,b]$, $g(x) \geqslant 0$, and c be a non-negative constant. If

$$f(x) \leqslant c + \int_a^x f(t)g(t)\,dt,$$

then

$$f(x) \leqslant c\exp\left(\int_a^x g(t)\,dt\right).$$

♡

For a Cauchy problem:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \\ y(x_0) = y_0, \end{cases} \tag{3.1}$$

give the existence and uniqueness theorem.

¶ **Picard-Lindelöf Theorem**

Theorem 3.2 (Picard-Lindelöf Theorem)

In the Cauchy problem (3.1), let D be a closed rectangle in the xy-plane:

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$$

. If the function $f(x,y)$ satisfies the following two conditions:

1. $f(x,y)$ is continuous in D.
2. $f(x,y)$ satisfies the Lipschitz condition with respect to y in D, i.e., there exists a constant $L > 0$ such that for any $(x,y_1), (x,y_2) \in D$,

$$|f(x,y_1) - f(x,y_2)| \leqslant L|y_1 - y_2|.$$

Then there exists a unique solution $y = \varphi(x)$ ($\varphi(x_0) = y_0$) to the Cauchy problem (3.1) in the interval $[x_0 - h, x_0 + h]$, where

$$h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(x,y)\in D} |f(x,y)|.$$

♡

Proposition 3.1

♠

¶ **Peano Theorem and Osgood Theorem**

In regard to the solutions for the Cauchy problem (3.1), we have the following two theorems, which are weaker than the Picard-Lindelöf theorem:

Definition 3.1 (Osgood Condition)

♣

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