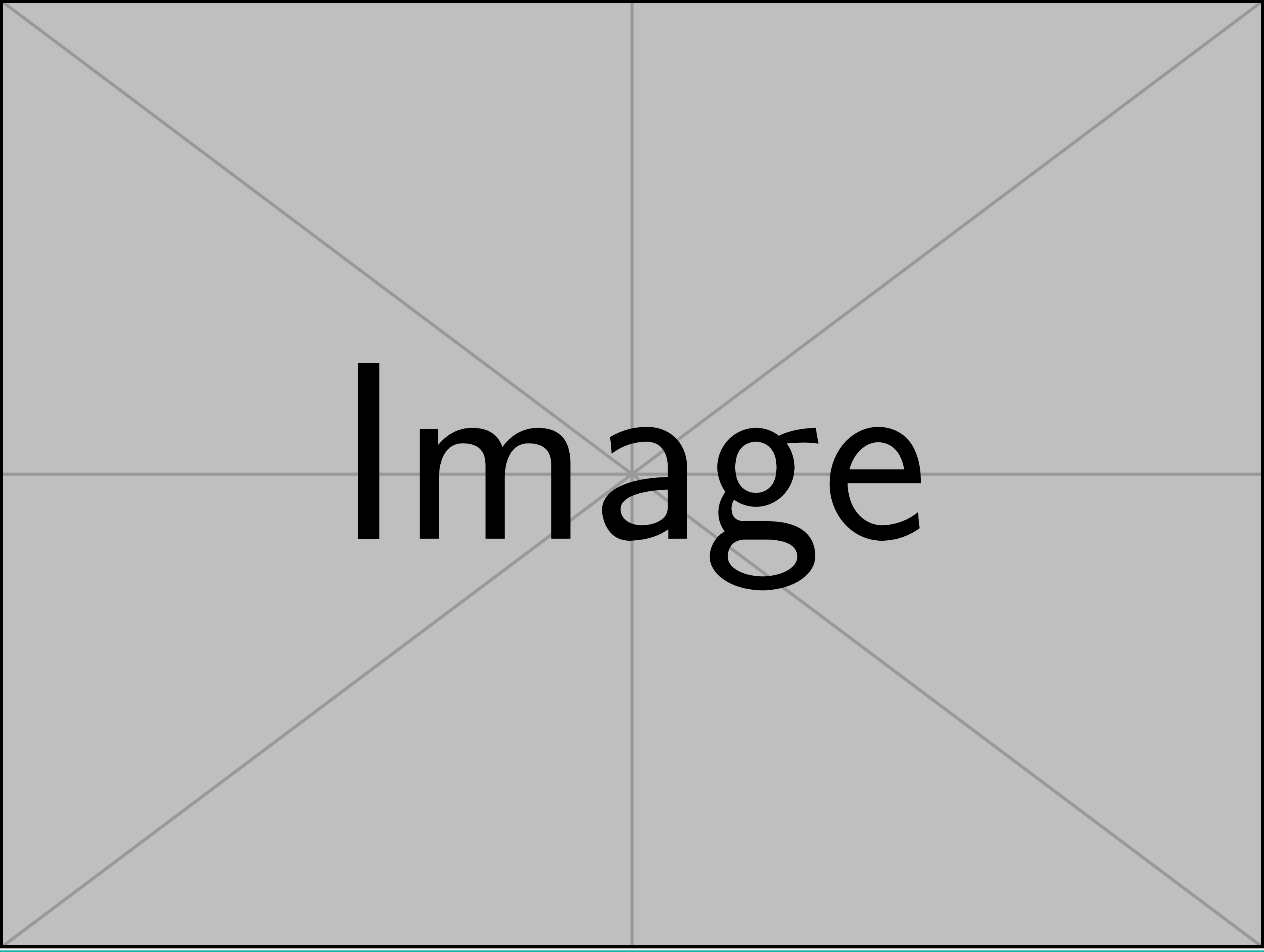

Équation Différentielle Ordinaire

Author: MonoCat

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Contents

Preface

This is the preface of the book…

Alert Throught this text, we focus exclusively on real-valued differential equations, where all quantities are assumed to be real unless stated otherwise.

Chapter 1 Introduction

1.1 Classification of Differential Equations

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a **differential equation**. Differential equations can be classified according to the following criteria:

¶ `Number of Independent Variables`

An **ordinary differential equation(ODE)** is defined as an equation of the following form:

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \dots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n}\right) = 0, \quad n \in \mathbb{N}, \tag{1.1}$$

or, using the prime notation for derivatives,

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0, \quad n \in \mathbb{N}.$$

If there are two or more independent variables, the equation is called a **partial differential equation(PDE)**.

¶ `Order`

The order of a differential equation is the order of the highest derivative present in the equation.

- A first-order equation has the form $F(x, y, y') = 0$.
- A second-order equation has the form $F(x, y, y', y'') = 0$.
- Higher-order equations involve derivatives of order three or more.

Note *Crucially, the order tells you how many initial conditions are needed to find a unique solution.*

¶ `Linearity`

An n-th order differential equation is linear if it can be written in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where the coefficients $a_i(x)$ and the term $g(x)$ depend only on the independent variable x. Otherwise, it is nonlinear.

Note *Specially, for the aforementioned equation, if $g(x) = 0$, it is called **homogeneous**, and **non-homogeneous** otherwise.*

1.2 Solution to a Ordinary Differential Equation

¶ `Particular and General Solutions`

Let J be an interval in $\mathbb{R}$. A function $y = \phi(x)$ defined on the interval J is called a solution to equation (1.1) if it satisfies:

$$F(x, \phi(x), \phi'(x), \phi''(x), \dots, \phi^{(n)}(x)) = 0 \quad x \in J.$$

The interval J is then called the interval of existence of the solution $y = \phi(x)$.

Generally speaking, the solution to equation (1.1) contains one or more arbitrary constants, the determination of which depends on other conditions that the solution must satisfy. If a solution to a differential equation does not contain any arbitrary constants, it is called a **particular solution** of the differential equation.

Suppose $y = \phi(x; c_1, c_2, \cdots, c_n)$ is a solution to equation (1.1), where $c_1, c_2, \dots, c_n$ are arbitrary constants. If $c_1, c_2, \dots, c_n$ are mutually independent, then $y = \phi(x; c_1, c_2, \dots, c_n)$ is called the **general solution** to equation (1.1). Here, "mutually independent" means that the Jacobian determinant is non-zero:

$$\det \frac{\partial(\phi, \phi', \ldots, \phi^{(n-1)})}{\partial(c_1, c_2, \ldots, c_n)} \neq 0, \quad x \in J.$$

When all the arbitrary constants in the general solution are determined, one obtains a particular solution to the differential equation.

Initial Conditions, Explicit and Implicit Solutions

Let $y = \phi(x)$ be a solution to equation (1.1) that also satisfies

$$\phi(x_0) = y_0, \quad \phi'(x_0) = y_0', \ldots, \quad \phi^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{1.2}$$

The conditions (1.2) are called the initial conditions for equation (1.1), and $y = \phi(x)$ is called the solution to equation (1.1) satisfying the initial conditions (1.2).

A function $y = \phi(x)$ that turns the differential equation (1.1) into an identity is called an **(explicit) solution** to the equation. If a solution $y = \phi(x)$ to the differential equation (1.1) is determined by the relation $\Phi(x, y) = 0$, then $\Phi(x, y) = 0$ is called an **implicit solution** to the differential equation (1.1). An implicit solution is also called an "integral".

Integral Curve and Direction Field

Consider the first-order differential equation:

$$\frac{dy}{dx} = f(x, y), \tag{1.3}$$

where f is continuous in a planar region G. Suppose

$$y = \phi(x), \quad x \in J$$

is a solution to this equation, where $J \subset \mathbb{R}$ is an interval. Then the set of points in the plane

$$\Gamma = (x, y) | y = \phi(x), x \in J$$

is a differentiable curve in the plane. This curve is called a solution curve or an **integral curve**.

Let $(x_0, y_0) \in \Gamma$. The slope of the tangent line to the curve Γ at this point is

$$\phi'(x_0) = f(x_0, y_0).$$

Therefore, the equation of the tangent line is

$$y - y_0 = f(x_0, y_0)(x - x_0).$$

This implies that even without knowing the explicit expression for ϕ, we can obtain the slope and equation of the tangent line to the solution curve at a given point from equation (1.3).

Remark Note that in a small neighborhood of a point on a differentiable curve, the tangent line can be seen as a first-order approximation of the curve. Utilizing this viewpoint, one can obtain an approximate solution to the differential equation. In fact, this is the fundamental idea behind Euler's method.

At each point P in the region G, we can draw a short line segment $l(P)$ with slope $f(P)$. We call $l(P)$ the line element of equation (1.3) at point P. The region G together with the entire collection of these line elements is called the lineal **linear element field** or **direction field** for equation (1.3).

Theorem 1.1

A necessary and sufficient condition for a continuously differentiable curve $\Gamma = \{(x, y) | y = \psi(x), x \in J\}$ in the plane to be an integral curve of equation (1.3) is that for every point (x, y) on the curve Γ, its tangent line at that point coincides with the line element determined by equation (1.3) at that point.

Proof The necessity follows from the preceding discussion. We now prove the sufficiency. For any point

$(x, y) = (x, \psi(x))$ on the curve Γ, the slope of the tangent line to Γ at this point is $\psi'(x)$. By the condition of the theorem, we have $\psi'(x) = f(x, y)$. Since (x, y) is an arbitrary point on the curve, it follows that $y = \psi(x)$ is a solution to equation (1.3). ■

Chapter 2 First Order Equations

2.1 Exact Equations

Definition 2.1 (Exact Equations)

An equation of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{2.1}$$

is called the symmetric form of a first-order differential equation.

If there exists a continuously differentiable function $u(x,y)$ such that

$$\mathrm{d}u(x,y) = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y,$$

then equation (2.1) is said to be an **exact equation** or a **total differential equation**.

It follows that, when equation (2.1) is exact, it can be rewritten as

$$\mathrm{d}u(x,y) = 0,$$

which implies

$$u(x,y) = c, \tag{2.2}$$

where c is an arbitrary constant. Equation (2.2) is called the **general integral** of equation (2.1). ♣

Remark It should be noted that, strictly speaking, equation (2.1) is not a differential equation. However, expressing a first-order differential equation in the form of (2.1) is extremely convenient for analysis. This formulation does not necessarily require y to be expressed as a function of x. For the sake of simplicity in description, we often refer to the symmetric form (2.1) as a differential equation, too.

Theorem 2.1

Let the functions $M(x,y)$ and $N(x,y)$ be continuous in a simply connected domain $D \subset \mathbb{R}^2$, and suppose their first-order partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are also continuous. Then a necessary and sufficient condition for equation (2.1) to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

in the domain D. When this condition holds, for any $(x_0,y_0),(x,y) \in D$, a general integral of equation (2.1) is given by

$$\int_\gamma M(x,y)dx + N(x,y)dy = c,$$

where γ is any curve composed of finitely many smooth segments within D connecting (x_0,y_0) and (x,y), and c is an arbitrary constant. ♡

Proof

■

2.2 Separable Equations

Definition 2.2 (Separable Equations)

If the functions $M(x, y)$ and $N(x, y)$ in Equation (2.1) can both be written as the product of a function of x and a function of y, that is,

$$M(x, y) = M_1(x)M_2(y), \quad N(x, y) = N_1(x)N_2(y),$$

then equation (2.1) is called a separable equation.

When equation (2.1) is a separable equation, it can be written as

$$M_1(x)M_2(y)\,\mathrm{d}x + N_1(x)N_2(y)\,\mathrm{d}y = 0. \tag{2.3}$$

♣

2.3 Homogeneous Equations

Definition 2.3

A first-order differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

is called a **homogeneous equation** if both M and N are homogeneous functions [a] of the same degree n.

[a] A function $f(x, y)$ is called a homogeneous function of degree n if it satisfies the condition $f(tx, ty) = t^n f(x, y)$ for all $t > 0$.

♣

Example 2.1 A function $f(x, y)$ is called a quasihomogeneous function of degree d with generalized weights if

$$f(t^\alpha sx, t^\beta sy) = t^{ds} f(x, y),$$

where $t > 0$, α and β are positive constants with $\alpha + \beta = 1$, and $s \in \mathbb{R}$. Here, α and β are called the weights of x and y, respectively. Consider the differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0,$$

where $M(x, y)$ and $N(x, y)$ are quasihomogeneous functions of degree d_0 and d_1 with weights α and β for x and y, respectively. Proposition: When $d_0 = d_1 + \beta - \alpha$ the equation can be solved by elementary integration method.

2.4 Linear Equations

Definition 2.4 (First-Order Linear Equations)

A **first-order linear equation** is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x), \tag{2.4}$$

where $p(x)$ and $q(x)$ are continuous functions on the interval (a, b). In Equation (2.4), when $q(x) \equiv 0$, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0,$$

which is called a **first-order homogeneous linear equation** corresponding to Equation (2.4). Otherwise, it is called a first-order non-homogeneous linear equation.

♣

Note *It should be noted that the definition of a homogeneous equation here differs from that in the previous section.*

Definition 2.5 (Bernoulli's Equation)

A first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n, \quad n \neq 0, 1,$$

where n is a real number and $p(x)$ and $q(x)$ are continuous functions on the interval (a, b), is called a **Bernoulli's equation.**

♣

2.5 Integrating Factors

Definition 2.6 (Integrating Factors)

An **integrating factor** for a first-order differential equation of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{2.5}$$

is a differentiable function $\mu(x,y)$ such that when multiplied by the equation:

$$\mu(x,y)M(x,y)\,\mathrm{d}x + \mu(x,y)N(x,y)\,\mathrm{d}y = 0,$$

it becomes an exact equation. Id est, there exists a function $\Phi(x,y)$ such that

$$\mu(x,y)M(x,y)\,\mathrm{d}x + \mu(x,y)N(x,y)\,\mathrm{d}y = \mathrm{d}\Phi(x,y).$$

If such functions $\mu(x,y)$ and $\Phi(x,y)$ exist, and $\Phi(x,y)$ is smooth, then

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \left(= \frac{\partial^2 \Phi}{\partial x \partial y} \right).$$

In this case, $\mu(x,y)$ is called an integrating factor for equation (2.5).

♣

2.6 Implicit Equations

This section discusses the problem of solving the first-order implicit differential equations,

$$F(x, y, y') = 0 \tag{2.6}$$

where F is a continuously differentiable function. A so-called implicit differential equation is one in which y' does not have an explicit solution, that is, the equation cannot be written in the form $y' = f(x,y)$.

¶ **Differentiation Method**

Suppose that Equation (2.6) can be solved for y, that is,

$$y = f(x,p), \quad p = \frac{dy}{dx}, \tag{2.7}$$

where $f(x,p)$ is a continuously differentiable function.

Differentiating both sides of $y = f(x,p)$ with respect to x, we obtain

$$p = \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx},$$

that is,

$$\frac{\partial f}{\partial p}\frac{dp}{dx} = p - \frac{\partial f}{\partial x}.$$

This is a first-order differential equation in the variables x, p, $\frac{dp}{dx}$. If a solution $p = p(x)$ can be found, then Equation (2.7) yields a solution

$$y = f(x, p(x)).$$

¶ **Parametric Method**

In general, Equation (2.6) represents a surface in the (x, y, p)-space. Therefore, the solution can be obtained using a parametric representation of the surface. Suppose the parametric form of the surface described by Equation (2.6) is

$$x = x(u, v), \quad y = y(u, v), \quad p = p(u, v) = y'.$$

Note that

$$dy = p \, \mathrm{d}x,$$

thus we obtain

$$y'_u \mathrm{d}u + y'_v \mathrm{d}v = p(u, v)(x'_u \mathrm{d}u + x'_v \mathrm{d}v).$$

This is an explicit differential equation in the variables u and v. Suppose it admits a solution

$$v = v(u, c),$$

where c is a constant, then Equation (2.6) has a solution

$$x = x(u, v(u, c)), \quad y = y(u, v(u, c)).$$

Chapter 3 Existence and Uniqueness Theorem

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