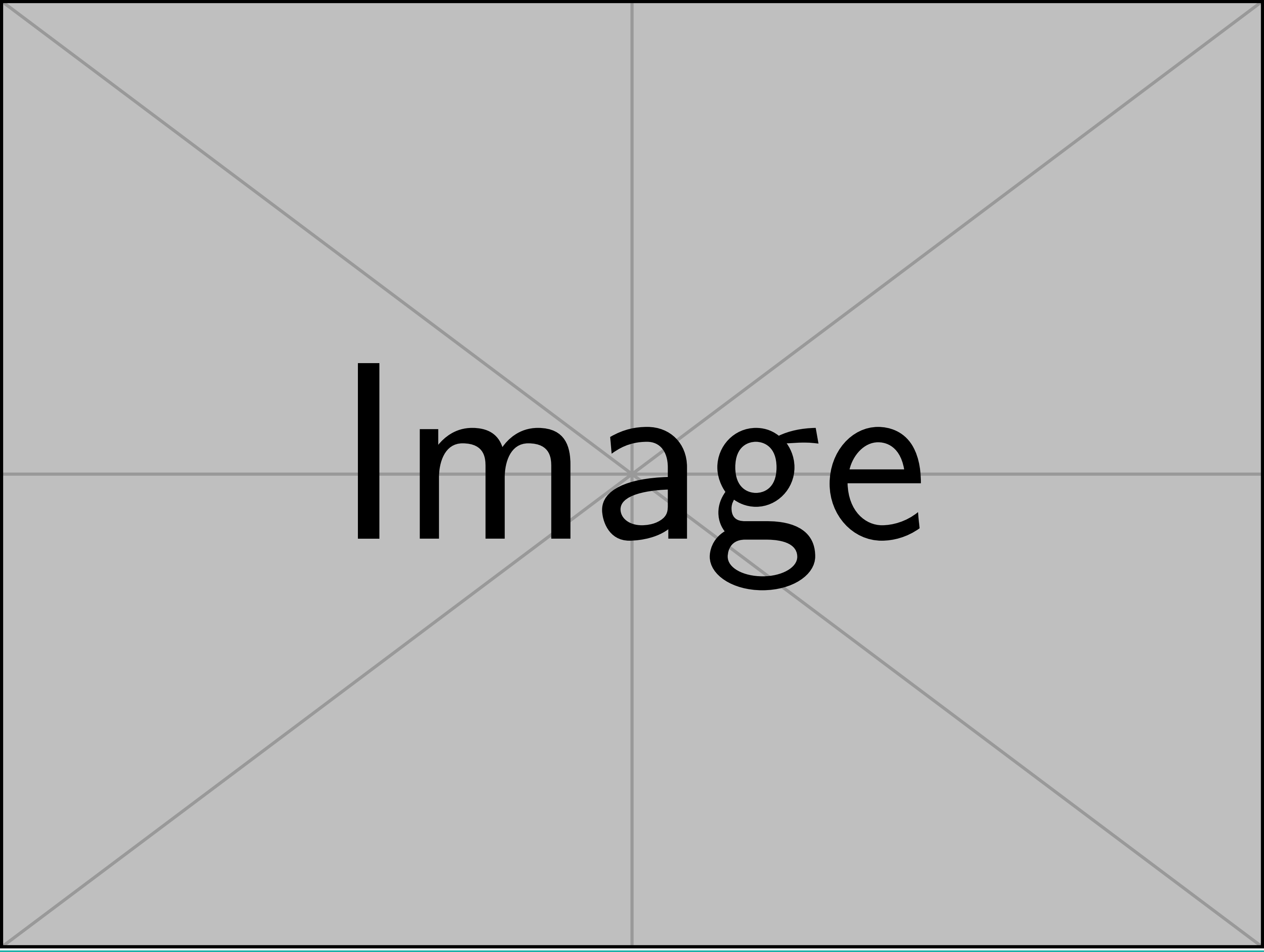

Équation Différentielle Ordinaire

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Preface

Chapter 1 Introduction

1.1 Classification of Differential Equations

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a **differential equation**. Differential equations can be classified according to the following criteria:

¶ `Number of Independent Variables`

An **ordinary differential equation(ODE)** is defined as an equation of the following form:

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \dots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n}\right) = 0, \quad n \in \mathbb{N}, \tag{1.1}$$

or, using the prime notation for derivatives,

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0, \quad n \in \mathbb{N}.$$

If there are two or more independent variables, the equation is called a **partial differential equation(PDE)**.

¶ `Order`

The order of a differential equation is the order of the highest derivative present in the equation.

- A first-order equation has the form $F(x, y, y') = 0$.
- A second-order equation has the form $F(x, y, y', y'') = 0$.
- Higher-order equations involve derivatives of order three or more.

Note *Crucially, the order tells you how many initial conditions are needed to find a unique solution.*

¶ `Linearity`

An n-th order differential equation is linear if it can be written in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where the coefficients $a_i(x)$ and the term $g(x)$ depend only on the independent variable x. Otherwise, it is nonlinear.

Note *Specially, for the aforementioned equation, if $g(x) = 0$, it is called **homogeneous**, and **non-homogeneous** otherwise.*

1.2 Solution to a Ordinary Differential Equation

¶ `Particular and General Solutions`

Let J be an interval in $\mathbb{R}$. A function $y = \phi(x)$ defined on the interval J is called a solution to equation (1.1) if it satisfies:

$$F(x, \phi(x), \phi'(x), \phi''(x), \dots, \phi^{(n)}(x)) = 0 \quad x \in J.$$

The interval J is then called the interval of existence of the solution $y = \phi(x)$.

Generally speaking, the solution to equation (1.1) contains one or more arbitrary constants, the determination of which depends on other conditions that the solution must satisfy. If a solution to a differential equation does not contain any arbitrary constants, it is called a **particular solution** of the differential equation.

Suppose $y = \phi(x; c_1, c_2, \cdots, c_n)$ is a solution to equation (1.1), where $c_1, c_2, \dots, c_n$ are arbitrary constants. If $c_1, c_2, \dots, c_n$ are mutually independent, then $y = \phi(x; c_1, c_2, \dots, c_n)$ is called the **general solution** to equation (1.1). Here, "mutually independent" means that the Jacobian determinant is non-zero:

$$\det \frac{\partial(\phi, \phi', \ldots, \phi^{(n-1)})}{\partial(c_1, c_2, \ldots, c_n)} \neq 0, \quad x \in J.$$

When all the arbitrary constants in the general solution are determined, one obtains a particular solution to the differential equation.

¶ **Initial Conditions, Explicit and Implicit Solutions**

Let $y = \phi(x)$ be a solution to equation (1.1) that also satisfies

$$\phi(x_0) = y_0, \quad \phi'(x_0) = y_0', \ldots, \quad \phi^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{1.2}$$

The conditions (1.2) are called the **initial conditions** for equation (1.1), and $y = \phi(x)$ is called the solution to equation (1.1) satisfying the initial conditions (1.2). Such initial value problems are often referred to as **Cauchy problems**.

A function $y = \phi(x)$ that turns the differential equation (1.1) into an identity is called an **(explicit) solution** to the equation. If a solution $y = \phi(x)$ to the differential equation (1.1) is determined by the relation $\Phi(x, y) = 0$, then $\Phi(x, y) = 0$ is called an **implicit solution** to the differential equation (1.1). An implicit solution is also called an "integral".

¶ **Integral Curve and Direction Field**

Consider the first-order differential equation:

$$\frac{dy}{dx} = f(x, y), \tag{1.3}$$

where f is continuous in a planar region G. Suppose

$$y = \phi(x), \quad x \in J$$

is a solution to this equation, where $J \subset \mathbb{R}$ is an interval. Then the set of points in the plane

$$\Gamma = (x, y)|y = \phi(x), x \in J$$

is a differentiable curve in the plane. This curve is called a solution curve or an **integral curve**.

Let $(x_0, y_0) \in \Gamma$. The slope of the tangent line to the curve Γ at this point is

$$\phi'(x_0) = f(x_0, y_0).$$

Therefore, the equation of the tangent line is

$$y - y_0 = f(x_0, y_0)(x - x_0).$$

This implies that even without knowing the explicit expression for ϕ, we can obtain the slope and equation of the tangent line to the solution curve at a given point from equation (1.3).

Remark Note that in a small neighborhood of a point on a differentiable curve, the tangent line can be seen as a first-order approximation of the curve. Utilizing this viewpoint, one can obtain an approximate solution to the differential equation. In fact, this is the fundamental idea behind Euler's method.

At each point P in the region G, we can draw a short line segment $l(P)$ with slope $f(P)$. We call $l(P)$ the line element of equation (1.3) at point P. The region G together with the entire collection of these line elements is called the lineal **linear element field** or **direction field** for equation (1.3).

Theorem 1.1

A necessary and sufficient condition for a continuously differentiable curve $\Gamma = \{(x, y)|y = \psi(x), x \in J\}$ in the plane to be an integral curve of equation (1.3) is that for every point (x, y) on the curve Γ, its tangent line at that point coincides with the line element determined by equation (1.3) at that point.

Proof The necessity follows from the preceding discussion. We now prove the sufficiency. For any point $(x, y) = (x, \psi(x))$ on the curve Γ, the slope of the tangent line to Γ at this point is $\psi'(x)$. By the condition of the theorem, we have $\psi'(x) = f(x, y)$. Since (x, y) is an arbitrary point on the curve, it follows that $y = \psi(x)$ is a solution to equation (1.3). ■

Chapter 2 First Order Equations

2.1 Exact Equations

Definition 2.1 (Exact Equations)

An equation of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{2.1}$$

is called the symmetric form of a first-order differential equation.

If there exists a continuously differentiable function $u(x,y)$ such that

$$\mathrm{d}U(x,y) = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y,$$

then equation (2.1) is said to be an **exact equation** or a **total differential equation**.

It follows that, when equation (2.1) is exact, it can be rewritten as

$$\mathrm{d}(U(x,y)) = 0,$$

which implies

$$U(x,y) = c, \tag{2.2}$$

where c is an arbitrary constant. Equation (2.2) is called the **general integral** of equation (2.1).

Remark It should be noted that, strictly speaking, equation (2.1) is not a differential equation. However, expressing a first-order differential equation in the form of (2.1) is extremely convenient for analysis. This formulation does not necessarily require y to be expressed as a function of x. For the sake of simplicity in description, we often refer to the symmetric form (2.1) as a differential equation, too.

Theorem 2.1

Let the functions $M(x,y)$ and $N(x,y)$ be continuous in a simply connected domain $D \subset \mathbb{R}^2$, and suppose their first-order partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are also continuous. Then a necessary and sufficient condition for equation (2.1) to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

in the domain D. When this condition holds, for any $(x_0,y_0),(x,y) \in D$, a general integral of equation (2.1) is given by

$$\int_\gamma M(x,y)dx + N(x,y)dy = c,$$

where γ is any curve composed of finitely many smooth segments within D connecting (x_0,y_0) and (x,y), and c is an arbitrary constant.

Proof

■

The aforementioned proof also serves as a method for determining the bivariate function $U(x,y)$ that satisfies specific conditions. In addition to this approach, there exist two simpler methods for solving $U(x,y)$.

Utilizing Curve Integrals to Solve $U(x,y)$

Term Combination Method Utilizing the properties of bivariate differential functions, we combine the terms of the differential equation into a full differential form. This method requires familiarity with some simple bivariate differential functions, such as:

$$\begin{aligned} y\mathrm{d}x + x\mathrm{d}y &= \mathrm{d}(xy), \\ \frac{y\mathrm{d}x - x\mathrm{d}y}{y^2} &= \mathrm{d}\left(\frac{x}{y}\right), \\ \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2} &= \mathrm{d}\left(\frac{y}{x}\right), \\ \frac{1}{x}\mathrm{d}x + \frac{1}{y}\mathrm{d}y = \frac{y\mathrm{d}x + x\mathrm{d}y}{xy} &= \mathrm{d}(\ln|xy|), \\ \frac{1}{x}\mathrm{d}x - \frac{1}{y}\mathrm{d}y = \frac{y\mathrm{d}x - x\mathrm{d}y}{xy} &= \mathrm{d}(\ln|\frac{x}{y}|), \\ \frac{y\mathrm{d}x - x\mathrm{d}y}{x^2 - y^2} &= \frac{1}{2}\mathrm{d}\left(\ln\left|\frac{x-y}{x+y}\right|\right), \\ \frac{y\mathrm{d}x + x\mathrm{d}y}{x^2 + y^2} &= \mathrm{d}\left(\arctan\frac{y}{x}\right), \\ \frac{y\mathrm{d}x - x\mathrm{d}y}{x^2 + y^2} &= \mathrm{d}\left(\operatorname{arccot}\frac{y}{x}\right). \end{aligned}$$

The theory above can also be rewritten in differential form:

Let:

$$\omega^1 = M(x,y)\,dx + N(x,y)\,dy.$$

The differential form ω^1 is said to be **closed** if $\mathrm{d}\omega^1 = 0$. It is called **exact** if there exists a function $U(x,y)$ such that $\omega^1 = \mathrm{d}U(x,y)$. By the Poincaré theorem, it can be concluded that on $\mathbb{R}^2$, a first-order differential form is exact if and only if it is closed. Note that:

$$\mathrm{d}\omega^1 = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y.$$

Clearly, $\mathrm{d}\omega^1 = 0$ holds if and only if:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Under this condition, the expression for the function $U(x,y)$ is:

$$U(x,y) = \int \omega^1.$$

2.2 Separable Equations

Definition 2.2 (Separable Equations)

If the functions $M(x,y)$ and $N(x,y)$ in Equation (2.1) can both be written as the product of a function of x and a function of y, that is,

$$M(x,y) = M_1(x)M_2(y), \quad N(x,y) = N_1(x)N_2(y),$$

then equation (2.1) is called a separable equation.

When equation (2.1) is a separable equation, it can be written as

$$M_1(x)M_2(y)\,\mathrm{d}x + N_1(x)N_2(y)\,\mathrm{d}y = 0, \tag{2.3}$$

or more conveniently as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \left(= -\frac{M_1(x)}{N_1(x)} \cdot \frac{N_2(y)}{M_2(y)} \right). \tag{2.4}$$

Theorem 2.2 (Solutions to Separable Equations)

All the solutions to the separable equation (2.3) are given by:

$$\int_{x_0}^{x} \frac{M_1(t)}{N_1(t)} \mathrm{d}t + \int_{y_0}^{y} \frac{N_2(s)}{M_2(s)} \mathrm{d}s = c,$$

and

$$y \equiv b_i, \quad i = 1, 2, \ldots, m, x \equiv a_j, \quad j = 1, 2, \ldots, n,$$

where $M_2(b_i) = 0 \quad (i = 1, 2, \ldots, m)$ and $N_1(a_j) = 0 \quad (j = 1, 2, \ldots, n)$, c is arbitrary constant.

2.3 Homogeneous Equations

Definition 2.3

A first-order differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

is called a **homogeneous equation** if both M and N are homogeneous functions [a] of the same degree n. In other words, for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

$f(x, y)$ can be rewritten as $g\left(\frac{y}{x}\right)$.

[a] A function $f(x, y)$ is called a homogeneous function of degree n if it satisfies the condition $f(tx, ty) = t^n f(x, y)$ for all $t > 0$.

The equation

$$\frac{dy}{dx} = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right) \tag{2.5}$$

can be transformed into a separable equation via variable change, where $a_1, a_2, b_1, b_2, c_1, c_2$ are constants.

- When $c_1 = c_2 = 0$, the equation becomes:

  $$\frac{dy}{dx} = f\left(\frac{a_1 + b_1 \frac{y}{x}}{a_2 + b_2 \frac{y}{x}}\right) = g\left(\frac{y}{x}\right).$$

  Let

  $$u = \frac{y}{x}, \text{ namely } y = ux.$$

  Differentiating both sides with respect to x, we get:

  $$\frac{dy}{dx} = x\frac{du}{dx} + u.$$

  Substituting the results into original equation and simplifying, we obtain:

  $$\frac{du}{dx} = \frac{g(u) - u}{x},$$

  which is a separable equation. It can be solved easily. Then, substituting $u = \frac{y}{x}$ back, the solution is derived.

- When c_1, c_2 are not entirely zero, the right-hand side of (2.5) consists of linear polynomials of x and y. Therefore:

$$\begin{cases} a_1x + b_1y + c_1 = 0, \\ a_2x + b_2y + c_2 = 0, \end{cases}$$

represents two intersecting straight lines on the Oxy plane. For the coefficient determinant of the system:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

two cases are analyzed:

1. If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, indicating that the two lines intersect at a unique point (α, β) on the Oxy plane. Let:

$$\begin{cases} X = x - \alpha, \\ Y = y - \beta, \end{cases}$$

then (2.3) becomes:

$$\begin{cases} a_1X + b_1Y = 0, \\ a_2X + b_2Y = 0. \end{cases}$$

Substituting into 2.5, it simplifies to:

$$\frac{dY}{dX} = f\left(\frac{a_1 + b_1\frac{Y}{X}}{a_2 + b_2\frac{Y}{X}}\right) = g\left(\frac{Y}{X}\right).$$

This is a homogeneous differential equation. Solving it by substitution and reverting back to the original variables yields the solution to equation 2.5.

2. When $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. To ensure this system holds, there are three possible scenarios:

(a). If $a_1 = b_1 = 0$, 2.5 becomes:

$$\frac{dy}{dx} = f\left(\frac{c_1}{a_2x + b_2y + c_2}\right),$$

and when $a_2 = b_2 = 0$, it becomes:

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{c_2}\right).$$

In this case, let

$$u = \frac{a_1x + b_1y + c_1}{c_2}.$$

Then it can be transformed into a separable equation.

(b). If $b_1 = b_2 = 0$, 2.5 transforms into:

$$\frac{dy}{dx} = f\left(\frac{a_1x + c_1}{a_2x + c_2}\right),$$

and

$$\frac{dy}{dx} = f\left(\frac{b_1y + c_1}{b_2y + c_2}\right),$$

when $a_1 = a_2 = 0$.

(c). If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$, let $u = a_2x + b_2y$. In this case:

$$\frac{du}{dx} = a_2 + b_2\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$f\left(\frac{k(a_2x + b_2y) + c_1}{(a_2x + b_2y) + c_2}\right) = f\left(\frac{ku + c_1}{u + c_2}\right) = g(u)$$

which simplifies to:

$$\frac{du}{dx} = a_2 + b_2g(u).$$

Example 2.1 A function $f(x, y)$ is called a quasihomogeneous function of degree d with generalized weights if

$$f(t^\alpha sx, t^\beta sy) = t^{ds}f(x, y),$$

where $t > 0$, α and β are positive constants with $\alpha + \beta = 1$, and $s \in \mathbb{R}$. Here, α and β are called the weights of x and y, respectively. Consider the differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0,$$

where $M(x, y)$ and $N(x, y)$ are quasihomogeneous functions of degree d_0 and d_1 with weights α and β for x and y, respectively. Proposition: When $d_0 = d_1 + \beta - \alpha$ the equation can be solved by elementary integration method.

2.4 Linear Equations

Definition 2.4 (First-Order Linear Equations)

A **first-order linear equation** is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x), \tag{2.6}$$

where $p(x)$ and $q(x)$ are continuous functions on the interval (a, b). In Equation (2.6), when $q(x) \equiv 0$, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0, \tag{2.7}$$

which is called a **first-order homogeneous linear equation** corresponding to Equation (2.6). Otherwise, it is called a first-order non-homogeneous linear equation. ♣

Note *It should be noted that the definition of a homogeneous equation here differs from that in the previous section.*

Firstly, we solve the first-order homogeneous linear equation. Equation 2.7 is separable, thus its general solution is given by:

$$y = ce^{-\int p(x)\,\mathrm{d}x},$$

where c is an arbitrary constant.

Since 2.7 is a special case of 2.6, the general solution of 2.6 can be expressed as:

$$y = c(x)e^{-\int p(x)\,\mathrm{d}x},$$

substituting it into 2.6 yields:

$$y = e^{-\int p(x)\,\mathrm{d}x}\left(c + \int q(x)e^{\int p(x)\,\mathrm{d}x}\,\mathrm{d}x\right).$$

This method of solving first-order linear equations is known as the **method of variation of constants**.

Definition 2.5 (Bernoulli's Equation)

A first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n, \quad n \neq 0, 1,$$

where n is a real number and $p(x)$ and $q(x)$ are continuous functions on the interval (a, b), is called a **Bernoulli's equation**.

♣

Bernoulli's equation can be transformed into a first-order linear equation by the substitution:

$$z = y^{1-n}.$$

Differentiating both sides with respect to x gives:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Substituting $\frac{\mathrm{d}y}{\mathrm{d}x}$ from Bernoulli's equation into the above expression yields:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)\left(-p(x)z + q(x)\right).$$

This is a first-order linear equation in z, which can be solved using the method for first-order linear equations.

2.5 Integrating Factors

Definition 2.6 (Integrating Factors)

An **integrating factor** for a first-order differential equation of the form

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{2.8}$$

is a differentiable function $\mu(x, y)$ such that when multiplied by the equation:

$$\mu(x, y)M(x, y)\,\mathrm{d}x + \mu(x, y)N(x, y)\,\mathrm{d}y = 0,$$

it becomes an exact equation. Id est, there exists a function $\Phi(x, y)$ such that

$$\mu(x, y)M(x, y)\,\mathrm{d}x + \mu(x, y)N(x, y)\,\mathrm{d}y = \mathrm{d}U(x, y).$$

If such functions $\mu(x, y)$ and $U(x, y)$ exist, and $U(x, y)$ is smooth, then

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}\left(= \frac{\partial^2 U}{\partial x \partial y}\right). \tag{2.9}$$

In this case, $\mu(x, y)$ is called an integrating factor for equation (2.8).

♣

According to Equation (2.9), finding an integrating factor $\mu(x, y)$ for equation (2.8) is equivalent to solving the partial differential equation:

$$\frac{\partial \mu}{\partial x}N - \frac{\partial \mu}{\partial y}M = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu. \tag{2.10}$$

Theorem 2.3

1. For the partial differential equation 2.10 to have a solution $\mu(x)$ that depends only on x, the necessary and sufficient condition is:

   The function G defined below must depend only on x:

$$G = -\frac{1}{N(x, y)}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

In this case, we have:

$$\mu(x) = e^{\int_{x_0}^{x} G(t)\,dt}.$$

2. For the partial differential equation 2.10 to have a solution $\mu(y)$ that depends only on y, the necessary and sufficient condition is:
   The function H defined below must depend only on y:
   $$H = \frac{1}{M(x,y)}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$
   In this case, we have:
   $$\mu(y) = e^{\int_{y_0}^{y} H(s)\,ds}.$$

3. For equation 2.8 to have an integrating factor of the form $\mu = \mu(\phi(x,y))$, the necessary condition is:
   $$\frac{1}{\frac{\partial \phi}{\partial x}N - \frac{\partial \phi}{\partial y}M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(\phi(x,y)),$$
   where f is a certain univariate function.

Theorem 2.4

Let the functions $P(x,y)$, $Q(x,y)$, $\mu_1(x,y)$, and $\mu_2(x,y)$ be continuously differentiable. Suppose $\mu_1(x,y)$ and $\mu_2(x,y)$ are integrating factors for equation (2.8), and the ratio $\frac{\mu_1(x,y)}{\mu_2(x,y)}$ is not a constant. Then:

$$\frac{\mu_1(x,y)}{\mu_2(x,y)} = c$$

is a general solution to the equation, where c is an arbitrary constant.

2.6 Implicit Equations

This section discusses the problem of solving the first-order implicit differential equations,

$$F(x,y,y') = 0 \tag{2.11}$$

where F is a continuously differentiable function. A so-called implicit differential equation is one in which y' does not have an explicit solution, that is, the equation cannot be written in the form $y' = f(x,y)$.

¶ **Differentiation Method**

Suppose that Equation (2.11) can be solved for y, that is,

$$y = f(x,p), \quad p = \frac{dy}{dx}, \tag{2.12}$$

where $f(x,p)$ is a continuously differentiable function.

Differentiating both sides of $y = f(x,p)$ with respect to x, we obtain

$$p = \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx},$$

that is,

$$\frac{\partial f}{\partial p}\frac{dp}{dx} = p - \frac{\partial f}{\partial x}.$$

This is a first-order differential equation in the variables x, p, $\frac{dp}{dx}$. If a solution $p = p(x)$ can be found, then Equation (2.12) yields a solution

$$y = f(x, p(x)).$$

¶ **Parametric Method**

In general, Equation (2.11) represents a surface in the (x, y, p)-space. Therefore, the solution can be obtained using a parametric representation of the surface. Suppose the parametric form of the surface described by Equation (2.11) is

$$x = x(u, v), \quad y = y(u, v), \quad p = p(u, v) = y'.$$

Note that

$$dy = p \, \mathrm{d}x,$$

thus we obtain

$$y'_u \mathrm{d}u + y'_v \mathrm{d}v = p(u, v)(x'_u \mathrm{d}u + x'_v \mathrm{d}v).$$

This is an explicit differential equation in the variables u and v. Suppose it admits a solution

$$v = v(u, c),$$

where c is a constant, then Equation (2.11) has a solution

$$x = x(u, v(u, c)), \quad y = y(u, v(u, c)).$$

Chapter 3 Existence and Uniqueness Theorem

3.1 Picard-Lindelöf Theorem

Theorem 3.1 (Bellman-Gronwall Inequality)

Let $f(x), g(x)$ be continuous functions on the interval $[a, b]$, $g(x) \geqslant 0$, and c be a non-negative constant. If

$$f(x) \leqslant c + \int_a^x f(t)g(t)\,dt,$$

then

$$f(x) \leqslant c \exp\left(\int_a^x g(t)\,dt\right).$$

♡

For a Cauchy problem:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \\ y(x_0) = y_0, \end{cases} \tag{3.1}$$

give the existence and uniqueness theorem.

¶ **Picard-Lindelöf Theorem**

Theorem 3.2 (Picard-Lindelöf Theorem)

In the Cauchy problem (3.1), let D be a closed rectangle in the xy-plane:

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b].$$

If the function $f(x, y)$ satisfies the following two conditions:

1. $f(x, y)$ is continuous in D.
2. $f(x, y)$ satisfies the Lipschitz condition with respect to y in D, i.e., there exists a constant $L > 0$ such that for any $(x, y_1), (x, y_2) \in D$,

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|.$$

Then there exists a unique solution $y = \varphi(x)$ $(\varphi(x_0) = y_0)$ to the Cauchy problem (3.1) in the interval $[x_0 - h, x_0 + h]$, where

$$h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(x,y)\in D} |f(x, y)|.$$

♡

Proposition 3.1

♠

¶ **Peano Theorem and Osgood Theorem**

In regard to the solutions for the Cauchy problem (3.1), we have the following two theorems, which are weaker than the Picard-Lindelöf theorem:

Definition 3.1 (Osgood Condition)

Let $f(x, y)$ be a continuous function in the region D. If for any $(x, y_1), (x, y_2) \in D$,

$$|f(x, y_1) - f(x, y_2)| \leqslant F(|y_1 - y_2|),$$

where $F(t) > 0$ $(t > 0)$ is a continuous function, and

$$\int_0^{\varepsilon} \frac{1}{F(t)}\, dt = +\infty, \quad \forall \varepsilon > 0,$$

then $f(x, y)$ is said to satisfy the **Osgood condition** with respect to y in D.

Remark If $f(x, y)$ satisfies Lipschitz condition, then it also satisfies the Osgood condition. In fact, in this case, we can take $F(t) = Lt$.

Theorem 3.3 (Peano Theorem)

In the Cauchy problem (3.1), let D be a closed rectangle in the xy-plane:

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$$

. If the function $f(x, y)$ is continuous in D, then there exists at least one solution $y = \varphi(x)$ $(\varphi(x_0) = y_0)$ to the Cauchy problem (3.1) in the interval $[x_0 - h, x_0 + h]$, where

$$h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(x,y)\in D} |f(x, y)|.$$

Theorem 3.4 (Osgood Theorem)

In the Cauchy problem (3.1), let D be a closed rectangle in the xy-plane:

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$$

. If the function $f(x, y)$ satisfies the Osgood condition with respect to y in D, then there exists a unique solution for any $(x_0, y_0) \in D$ to the Cauchy problem (3.1) in the interval $[x_0 - h, x_0 + h]$, where

$$h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(x,y)\in D} |f(x, y)|.$$

3.2 Continuation of the Solution

¶ Uncontinuable Solutions

Definition 3.2 (Uncontinuable Solutions)

Let $y = \varphi(x)$ be a solution to the Cauchy problem (3.1) in the interval $I_1 \subset \mathbb{R}$. If there exists an another solution $y = \varphi_2(x)$ to the Cauchy problem (3.1) in any interval $I_2 \supsetneq I_1$ such that

$$\varphi_2(x) \equiv \varphi(x), \quad x \in I_1,$$

then $y = \varphi_1(x)$ is called **continuable**, and $y = \varphi_2(x)$ is called a **continuation** of $y = \varphi_1(x)$. If there does not exist such a solution $y = \varphi_2(x)$, then $y = \varphi_1(x)$ is called **uncontinuable**, or **saturated**.

Theorem 3.5

In the Cauchy problem (3.1), let D be a bounded closed rectangle in the xy-plane. If the function $f(x, y)$ is continuous in D, and satisfies the local Lipschitz condition with respect to y in D, then any solution $y = \varphi(x)$ passing through $(x_0, y_0) \in D$ to the Cauchy problem (3.1) can be continued until it arbitrarily approaches the boundary of D.

¶ Comparison Theorem

3.3 Singular Solutions and Envelopes

3.4 Dependency of Solutions on Initial Data

Chapter 4 System of First-Order Linear Equations

4.1 System of First-Order Linear Equations

¶ Common Forms

System of first-order equations with n variables is of the form:

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}x} = f_1(x, y_1, y_2, \ldots, y_n), \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} = f_2(x, y_1, y_2, \ldots, y_n), \\ \quad\vdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}x} = f_n(x, y_1, y_2, \ldots, y_n). \end{cases} \tag{4.1}$$

If the right-hand side of each equation in system (4.1) does not include explicitly x, then the system is called **autonomous**.

The solution to system (4.1) is an n-tuple of functions

$$y_1 = \varphi_1(x), y_2 = \varphi_2(x), \ldots, y_n = \varphi_n(x),$$

which satisfy all equations in system (4.1) simultaneously.

Solution containing arbitrary constants $C_1, C_2, \ldots, C_n$

$$\begin{cases} y_1 = \varphi_1(x, C_1, C_2, \ldots, C_n), \\ y_2 = \varphi_2(x, C_1, C_2, \ldots, C_n), \\ \quad\vdots \\ y_n = \varphi_n(x, C_1, C_2, \ldots, C_n) \end{cases}$$

is called the **general solution** of system (4.1). If general solution satisfies

$$\begin{cases} \Phi_1(x, y_1, \cdots, y_n, C_1, \cdots, C_n) = 0, \\ \Phi_2(x, y_1, \cdots, y_n, C_1, \cdots, C_n) = 0, \\ \quad\vdots \\ \Phi_n(x, y_1, \cdots, y_n, C_1, \cdots, C_n) = 0, \end{cases}$$

then it is called the **general integral** of system (4.1).

For convenience, we rewrite system (4.1) in matrix form:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{F}(x, \mathbf{Y}), \tag{4.2}$$

and autonomous system as:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{F}(\mathbf{Y}), \tag{4.3}$$

and Cauchy problem for system (4.2) as:

$$\begin{cases} \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{F}(x, \mathbf{Y}), \\ \mathbf{Y}(x_0) = \mathbf{Y}_0, \end{cases} \tag{4.4}$$

where

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{F}(x, \mathbf{Y}) = \begin{bmatrix} f_1(x, y_1, y_2, \dots, y_n) \\ f_2(x, y_1, y_2, \dots, y_n) \\ \vdots \\ f_n(x, y_1, y_2, \dots, y_n) \end{bmatrix}, \quad \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \begin{bmatrix} \frac{\mathrm{d}y_1}{\mathrm{d}x} \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} \\ \vdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}x} \end{bmatrix}.$$

With these notations, from the perspective of form, the system of first-order linear equations is similar to first-order equations.

In system (4.1), if $f_i(x, y_1, y_2, \dots, y_n)$ is a linear function of $y_1, y_2, \dots, y_n$, i.e., it can be rewritten as:

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}x} = a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n + f_1(x), \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} = a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots + a_{2n}(x)y_n + f_2(x), \\ \quad \vdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}x} = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n + f_n(x). \end{cases}$$

It is called a **system of first-order linear equations**. $a_{ij}(x)$ and $f_i(x)$ are always assumed to be continuous on some interval $I \subset \mathbb{R}$, where $i, j = 1, 2, \dots, n$.

For convenience, we rewrite the system of first-order linear equations in matrix form:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{A}(x)\mathbf{Y} + \mathbf{F}(x), \tag{4.5}$$

where

$$\mathbf{A}(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}, \quad \mathbf{F}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}.$$

On the interval I, if $\mathbf{F}(x) \equiv \mathbf{0}$, that is,

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{A}(x)\mathbf{Y} \tag{4.6}$$

then system (4.5) is called a **homogeneous system**, otherwise, it is called a **non-homogeneous system**.

¶ General Theory

Theorem 4.1 (Existence and Uniqueness Theorem for System of First-Order Linear Equations)

In the Cauchy problem (4.4), let D be a closed region in the $\mathbb{R}^{n+1}$:

$$D = [x_0 - a, x_0 + a] \times [\mathbf{Y}_0 - b, \mathbf{Y}_0 + b].$$

If the function $\mathbf{F}(x, \mathbf{Y})$ satisfies the following two conditions:

1. $\mathbf{F}(x, \mathbf{Y})$ is continuous in D.
2. $\mathbf{F}(x, \mathbf{Y})$ satisfies the Lipschitz condition with respect to $\mathbf{Y}$ in D, i.e., there exists a constant $L > 0$ such that for any $(x, \mathbf{Y}_1), (x, \mathbf{Y}_2) \in D$,

$$\|\mathbf{F}(x, \mathbf{Y}_1) - \mathbf{F}(x, \mathbf{Y}_2)\| \leqslant L\|\mathbf{Y}_1 - \mathbf{Y}_2\|.$$

Then there exists a unique solution $\mathbf{Y} = \mathbf{\Phi}(x)$ ($\mathbf{\Phi}(x_0) = \mathbf{Y}_0$) to the Cauchy problem (4.4) in the interval $[x_0 - h, x_0 + h]$, where

$$h = \min\{a, \frac{b}{M}\}, \quad M = \max_{(x, \mathbf{Y}) \in D} \|\mathbf{F}(x, \mathbf{Y})\|.$$

♡

4.2 General Theory of Homogeneous Linear Systems

Similar to linear homogeneous systems of algebraic equations, the linear combination of solutions to homogeneous linear systems of differential equations is still a solution to the system.

Proposition 4.1

If $\mathbf{Y}_1(x)$ and $\mathbf{Y}_2(x)$ are two solutions to the homogeneous linear system (4.6), then any linear combination of them

$$\mathbf{Y}(x) = C_1\mathbf{Y}_1(x) + C_2\mathbf{Y}_2(x),$$

where C_1 and C_2 are arbitrary constants, is also a solution to the system.
Three or more solutions also have this property.

With this proposition, it is easy to verify that the set of all solutions to the homogeneous linear system (4.6) forms a linear space. And similarly, linear independence of solutions can be defined. Then we can introduce the concept of fundamental solution matrix.

Definition 4.1 (Fundamental Solution Matrix)

Let $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ be n linearly independent solutions to the homogeneous linear system (4.6). Then the matrix

$$\mathbf{\Phi}(x) = \begin{pmatrix} \mathbf{Y}_1(x) & \mathbf{Y}_2(x) & \cdots & \mathbf{Y}_n(x) \end{pmatrix} = \begin{pmatrix} y_{11}(x) & y_{12}(x) & \cdots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \cdots & y_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(x) & y_{n2}(x) & \cdots & y_{nn}(x) \end{pmatrix}$$

is called a **fundamental solution matrix** of the system.
Simultaneously, such a set of solutions is called a **fundamental solution system**.

¶ **Criteria for Linear Dependence**

Given n vector functions with n components each:

$$\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x), \tag{4.7}$$

criteria for their linear independence on the definition interval I is provided by the following theorem.

Theorem 4.2 (Wronskian Determinant Theorem)

For vector functions (4.7), let

$$W(x) = \det \begin{pmatrix} y_{11}(x) & y_{12}(x) & \cdots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \cdots & y_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(x) & y_{n2}(x) & \cdots & y_{nn}(x) \end{pmatrix}$$

be their Wronskian determinant. Then, if (4.7) are linearly dependent on I, then $W(x) \equiv 0$ for all $x \in I$;

Furthermore, if (4.7) are solutions to the homogeneous linear system (4.6), then the following conclusion holds:

Theorem 4.3

If (4.7) are linearly independent solutions to the homogeneous linear system (4.6), then $W(x) \not\equiv 0$ for all $x \in I$.

Combine the above two theorems, we have the following corollary:

Corollary 4.1 (Criterion for Linear Independence)

For vector functions (4.7), if their Wronskian determinant $W(x_0) \neq 0$ for some $x_0 \in I$, then they are linearly independent on I.

As for the relation between the solutions and the coefficient, we have the following theorem.

Theorem 4.4 (Liouville's Formula)

Let $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ be n solutions to the homogeneous linear system (4.6), and $W(x)$ be their Wronskian determinant. Then

$$W(x) = W(x_0) \exp\left(\int_{x_0}^{x} \operatorname{tr}(\mathbf{A}(t))\, dt \right),$$

where $\operatorname{tr}(\mathbf{A}(t))$ is the trace of matrix $\mathbf{A}(t)$.

¶ **Solution Space**

With the above conclusions, we can give the existence of fundamental solution systems.

Theorem 4.5

The fundamental solution system to the homogeneous linear system (4.6) does exist.

Proof Due to the existence and uniqueness theorem for system of first-order linear equations (Theorem 4.1), for initial conditions

$$\mathbf{Y}_1(x_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{Y}_2(x_0) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \mathbf{Y}_n(x_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad x_0 \in I, \tag{4.8}$$

there exist n solutions $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ to the homogeneous linear system (4.6). Their Wronskian determinant at $x = x_0$ is

$$W(x_0) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0.$$

Therefore, by the criterion for linear independence (Corollary 4.1), $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ are linearly independent on I, i.e., they form a fundamental solution system to the homogeneous linear system (4.6). ■

Fundamental solution systems satisfying (4.8) are called **standard fundamental systems**, and their fundamental solution matrices are called **standard fundamental solution matrices**. Obviously, standard fundamental solution matrices is identity matrix at $x = x_0$.

Then, general solution can be also derived.

Theorem 4.6 (General Solution to Homogeneous Linear Systems)

If $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ is a fundamental solution system to the homogeneous linear system (4.6), then the **general solution** to the system is given by:

$$\mathbf{Y}(x) = C_1\mathbf{Y}_1(x) + C_2\mathbf{Y}_2(x) + \cdots + C_n\mathbf{Y}_n(x) = \mathbf{\Phi}(x)\mathbf{C},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants, and $\mathbf{\Phi}(x)$ is the fundamental matrix solution.

Proof

■

Therefore, the number of linearly independent solutions to the homogeneous linear system (4.6) can be not exceed n, and the solution space of the system is an n-dimensional linear space.

4.3 General Theory of Non-Homogeneous Linear Systems

For the non-homogeneous linear system (4.5), similar to linear systems of algebraic equations, we have the following conclusion:

- The difference between any two solutions to the non-homogeneous linear system (4.5) is a solution to the corresponding homogeneous linear system (4.6).
- If $\tilde{\mathbf{Y}}(x)$ is a particular solution to the non-homogeneous linear system (4.5), then
$$\mathbf{Y}(x) = \mathbf{Y}_0(x) + \tilde{\mathbf{Y}}(x),$$
is still a solution to the system, where $\mathbf{Y}_0(x)$ is the general solution to the corresponding homogeneous linear system (4.6).

Then we can give the general solution to the non-homogeneous linear system (4.5).

Theorem 4.7 (General Solution to Non-Homogeneous Linear Systems)

If $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ is a fundamental solution system to the corresponding homogeneous linear system (4.6), then the general solution to the non-homogeneous linear system (4.5) is given by:

$$\mathbf{Y}(x) = C_1\mathbf{Y}_1(x) + C_2\mathbf{Y}_2(x) + \cdots + C_n\mathbf{Y}_n(x) + \tilde{\mathbf{Y}}(x),$$

where $\tilde{\mathbf{Y}}(x)$ is a particular solution to the non-homogeneous linear system (4.5).

For non-homogeneous linear systems, method of variation of constants can also be used to find particular solutions. According to Theorem 4.6, the general solution to the corresponding homogeneous linear system is given by:

$$\mathbf{Y}(x) = \mathbf{\Phi}(x)\mathbf{C},$$

where $\mathbf{\Phi}(x)$ is the fundamental matrix solution, and $\mathbf{C} = \begin{pmatrix} C_1 & C_2 & \cdots & C_n \end{pmatrix}^{\mathrm{T}}$ is a constant vector. Now find a particular solution to the non-homogeneous linear system in the form:

$$\mathbf{Y}(x) = \mathbf{\Phi}(x)\mathbf{C}(x),$$

where $\mathbf{C}(x) = \begin{pmatrix} C_1(x) & C_2(x) & \cdots & C_n(x) \end{pmatrix}^{\mathrm{T}}$ is a vector function to be determined. Substituting it into the non-homogeneous linear system (4.5), we have:

$$\mathbf{\Phi}(x)\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}x} = \mathbf{F}(x). \tag{4.9}$$

Since $\mathbf{\Phi}(x)$ is invertible, we obtain:

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}x} = \mathbf{\Phi}^{-1}(x)\mathbf{F}(x). \tag{4.10}$$

Integrating both sides of Equation (4.10), we have:

$$\mathbf{C}(x) = \int_{x_0}^{x} \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,\mathrm{d}t, \tag{4.11}$$

where x_0 is an arbitrary constant. Then substituting Equation (4.11) into $\mathbf{Y}(x) = \mathbf{\Phi}(x)\mathbf{C}(x)$, we obtain a particular solution to the non-homogeneous linear system (4.5):

$$\tilde{\mathbf{Y}}(x) = \mathbf{\Phi}(x)\int_{x_0}^{x} \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,\mathrm{d}t. \tag{4.12}$$

Remark If $\mathbf{\Phi}(x)^{-1}$ is difficult to compute, we can use (4.9) directly to find $\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}x}$.

4.4 Solution to Constant Coefficient Linear Systems

For autonomous linear systems with constant coefficients:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{A}\mathbf{Y}, \tag{4.13}$$

we have the following conclusion:

Theorem 4.8

Matrix exponential function $\Phi(x) = e^{\mathbf{A}x}$ is a fundamental solution matrix to the homogeneous linear system (4.13).

For according non-homogeneous linear systems with constant coefficients:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{A}\mathbf{Y} + \mathbf{F}(x), \tag{4.14}$$

we can also use method of variation of constants to find particular solutions.

Theorem 4.9

The general solution to the non-homogeneous linear system (4.14) is given by:

$$\mathbf{Y}(x) = e^{\mathbf{A}x}\mathbf{C} + \int_{x_0}^{x} e^{\mathbf{A}(x-s)}\mathbf{F}(s)\,\mathrm{d}s,$$

where $\mathbf{C}$ is a constant vector. The solution satisfying the initial condition $\mathbf{Y}(x_0) = \mathbf{Y}_0$ is given by:

$$\mathbf{Y}(x) = e^{\mathbf{A}(x-x_0)}\mathbf{Y}_0 + \int_{x_0}^{x} e^{\mathbf{A}(x-s)}\mathbf{F}(s)\,\mathrm{d}s.$$

The problem we confront is: Can $e^{\mathbf{A}x}$ be expressed in a finite form of elementary functions? If so, how can it be expressed?

In fact, if $\mathbf{A}$ is an n-order Jordan block, i.e.,

$$\begin{aligned}\mathbf{A} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}_{n\times n} &= \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\ &=: \mathrm{diag}(\lambda, \lambda, \ldots, \lambda) + \mathbf{Z}_n,\end{aligned}$$

then we have:

$$\begin{aligned} e^{\mathbf{A}x} &= e^{\operatorname{diag}(\lambda,\lambda,\ldots,\lambda)x+\mathbf{Z}_n x} = e^{\operatorname{diag}(\lambda,\lambda,\ldots,\lambda)x} \cdot e^{\mathbf{Z}_n x} \\ &= e^{\lambda x} \cdot \left(\mathbf{E} + \frac{\mathbf{Z}_n x}{1!} + \frac{(\mathbf{Z}_n x)^2}{2!} + \cdots + \frac{(\mathbf{Z}_n x)^{n-1}}{(n-1)!} \right) \\ &= e^{\lambda x} \cdot \begin{pmatrix} 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^{n-1}}{(n-1)!} \\ 0 & 1 & x & \cdots & \frac{x^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{x^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n} . \end{aligned} \tag{4.15}$$

And Jordan canonical form can be deemed as a block diagonal matrix composed of Jordan blocks. Therefore, if we can compute the Jordan canonical form of matrix $\mathbf{A}$, then we can express $e^{\mathbf{A}x}$ in a finite form of elementary functions.

According to the theory of Jordan canonical form, for any n-order square matrix $\mathbf{A} \in M_n(\mathbb{C})$, there exists an invertible matrix $\mathbf{P} \in M_n(\mathbb{C})$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J},$$

where $\mathbf{J}$ is the Jordan canonical form of $\mathbf{A}$,

$$\mathbf{J} = \operatorname{diag}(\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_k),$$

and $\mathbf{J}_i$ is a Jordan block corresponding to eigenvalue λ_i of $\mathbf{A}$. Then we have:

$$e^{\mathbf{A}x} = e^{\mathbf{P}\mathbf{J}\mathbf{P}^{-1}x} = \mathbf{P}e^{\mathbf{J}x}\mathbf{P}^{-1} = \mathbf{P}\operatorname{diag}\left(e^{\mathbf{J}_1 x}, e^{\mathbf{J}_2 x}, \ldots, e^{\mathbf{J}_k x}\right)\mathbf{P}^{-1}. \tag{4.16}$$

However, computing the Jordan canonical form and transition matrix is not always easy. Note that $e^{\mathbf{A}x}$ is a fundamental solution matrix to the homogeneous linear system (4.13), since $\mathbf{P}$ is invertible, $e^{\mathbf{A}x}\mathbf{P}$ is also a fundamental solution matrix to (4.13). Then according to (4.16), $\mathbf{P}e^{\mathbf{J}x}$ is also a fundamental solution matrix to (4.13).

By (4.15), we can utilize method of undetermined coefficients to find n linearly independent solution matrix to (4.13). In the following, we classify the discussion based on whether $\mathbf{A}$ has repeated eigenvalues.

¶ Distinct Eigenvalues

Theorem 4.10

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n distinct eigenvalues of matrix $\mathbf{A}$, then the fundamental solution matrix to the homogeneous linear system (4.13) is given by:

$$\Phi(x) = \begin{pmatrix} e^{\lambda_1 x}\xi_1 & e^{\lambda_2 x}\xi_2 & \cdots & e^{\lambda_n x}\xi_n \end{pmatrix},$$

where ξ_i is the eigenvector corresponding to eigenvalue λ_i of matrix $\mathbf{A}$ $(i = 1, 2, \ldots, n)$.

♡

¶ Repeated Eigenvalues

Assume that $\lambda_1, \lambda_2, \cdots, \lambda_s$ are all distinct eigenvalues of matrix $\mathbf{A}$, with algebraic multiplicities $n_1, n_2, \ldots, n_s$ respectively, where $n_1 + n_2 + \cdots + n_s = n$. Note that the fundamental solution matrix $e^{\mathbf{A}x}\mathbf{P} = \mathbf{P}e^{\mathbf{J}x}$, hence in the expression of $\mathbf{P}e^{\mathbf{J}x}$, all column vectors corresponding to the eigenvalue λ_j have the form:

$$\mathbf{Y} = e^{\lambda_j x}\left[\xi_0 + \frac{x}{1!}\xi_1 + \frac{x^2}{2!}\xi_2 + \cdots + \frac{x^{n_j-1}}{(n_j-1)!}\xi_{n_j-1}\right], \tag{4.17}$$

where $\xi_0, \xi_1, \ldots, \xi_{n_j-1}$ are constant vectors.

Lemma 4.1

Let λ_j be an eigenvalue of matrix $\mathbf{A}$ with algebraic multiplicity n_j, then (4.13) has non-zero solutions of the form (4.17) if and only if ξ_0 is a non-zero solution to

$$(\mathbf{A} - \lambda_j \mathbf{E})^{n_j} \xi_0 = 0, \tag{4.18}$$

and $\xi_1, \xi_2, \ldots, \xi_{n_j-1}$ satisfy the following chain of equations:

$$\begin{cases} \xi_1 = (\mathbf{A} - \lambda_j \mathbf{E})\xi_0, \\ \xi_2 = (\mathbf{A} - \lambda_j \mathbf{E})^2 \xi_1, \\ \vdots \\ \xi_{n_j-1} = (\mathbf{A} - \lambda_j \mathbf{E})^{n_j-1} \xi_0. \end{cases} \tag{4.19}$$

♡

Lemma 4.2

Under the same conditions as above, denote the linear space of all constant vectors of n degrees as V, then

1. The subspace of V

$$V_j = \{\xi \in V \mid (\mathbf{A} - \lambda_j \mathbf{E})^{n_j} \xi = 0\}, \quad j = 1, 2, \ldots, s$$

is invariant under $\mathbf{A}$.

2. There exists a direct sum decomposition of V:

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s.$$

♡

Theorem 4.11

Under the same conditions as above, the fundamental solution matrix to the homogeneous linear system (4.13) is given by:

$$\Phi(x) = \left(e^{\lambda_1 x} \mathbf{P}_1^{(1)}(x) \quad e^{\lambda_2 x} \mathbf{P}_2^{(2)}(x) \quad \cdots \quad e^{\lambda_s x} \mathbf{P}_s^{(s)}(x) \right),$$

where for each $j = 1, 2, \ldots, s$,

$$\mathbf{P}_j^{(j)}(x) = \xi_{j0}^{(i)} + \frac{x}{1!}\xi_{j1}^{(i)} + \frac{x^2}{2!}\xi_{j2}^{(i)} + \cdots + \frac{x^{n_j-1}}{(n_j-1)!}\xi_{j,n_j-1}^{(i)},$$

which is the j-th vector polynomial corresponding to eigenvalue λ_i $(i = 1, 2, \ldots, n_j; j = 1, 2, \ldots, n_i)$. $\xi_{10}^{(i)}, \cdots, \xi_{n_i 0}^{(i)}$ is n_i linearly independent solutions to (4.18), and the other $\xi_{jl}^{(i)}$ $(j = 1, 2, \ldots, n_i; l = 1, 2, \ldots, n_j - 1)$ is obtained by replacing corresponding $\xi_{j0}^{(i)}$ with ξ_0 in (4.19).

♡

4.5 Periodic Coefficient Linear Differential Equation Systems

Chapter 5 System of Higher-Order Linear Equations

Chapter 6 Boundary Value Problems

6.1 Sturm-Liouville Problems

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