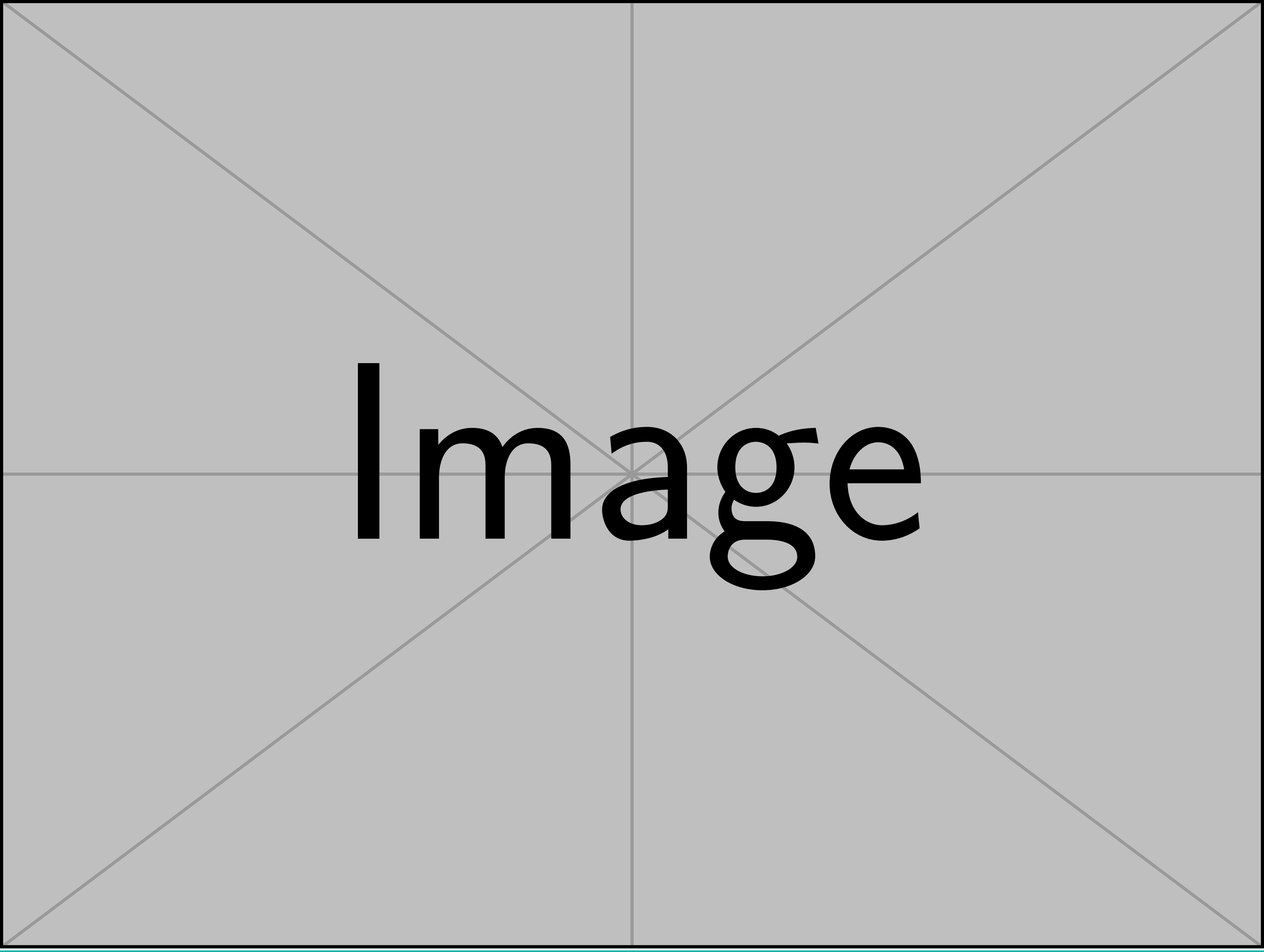

Combinatoire

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Preface

This is the preface of the book…

Part I

Basic Counting

Chapter 1 Basic Counting Principles

1.1 Addition and Multiplication Principles

1.2 Bijection Principle

1.3 Permutations and Combinations

Definition 1.1 (Permutation and Combination)

Let n be a non-negative integer, and k be an integer such that $0 \leq k \leq n$. The number of ways to choose k elements from a set of n distinct elements and arrange them in a specific order is called the number of permutations of n elements taken k at a time, denoted as $P(n,k)$ (or ${}_nP_k$, P_n^k or A_n^k). It is given by the formula:

$$P(n,k) = \frac{n!}{(n-k)!}.$$

The number of ways to choose k elements from a set of n distinct elements without regard to the order of selection is called the number of combinations of n elements taken k at a time, denoted as $C(n,k)$ (or ${}_nC_k$, C_n^k or $\binom{n}{k}$). It is given by the formula:

$$C(n,k) = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

♣

Property The following properties hold for permutations and combinations:

1. $A_n^0 = 1$ and $A_n^n = n!$.
2. $C_n^0 = 1$ and $C_n^n = 1$.
3. $C_n^k = C_n^{n-k}$.
4. $A_n^k = k! \cdot C_n^k$.
5. $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$ (**Pascal's triangle/YangHui's triangle**)[1].
   In Pascal's triangle, each element is equal to the sum of the two elements directly above it, as shown in Figure 1.1.

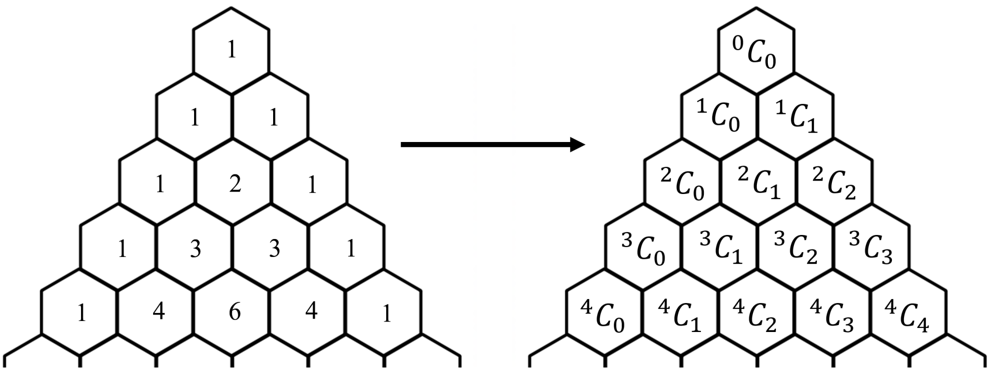

Figure 1.1: Pascal's triangle (YangHui's triangle).

[1]This property can also be understood that to choose k elements from $n+1$, you can first take one element A:
(a). The number of ways that include A is C_n^{k-1};
(b). The number of ways that does not include A is C_n^k.

6. $(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$ (**Binomial theorem**).
   Therefore, we can see the relationship between Pascal's triangle and the Binomial theorem, as shown in Figure 1.2. Here, C_n^k is the element in the n-th row and k-th column of Pascal's triangle.

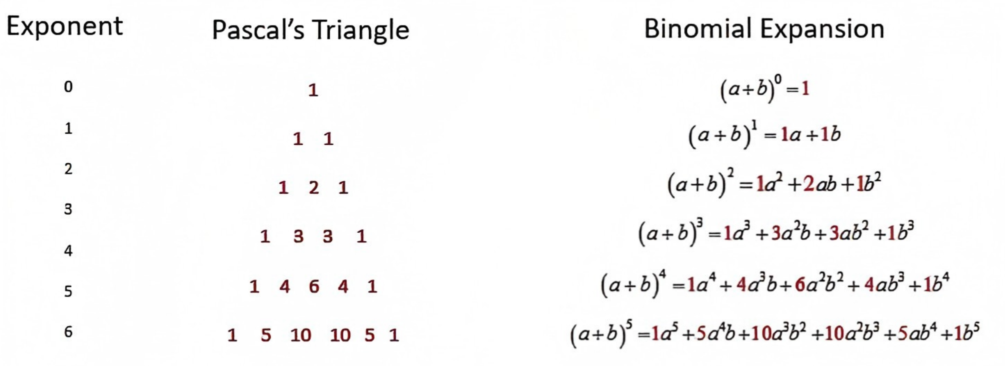

Figure 1.2: Pascal's triangle and Binomial theorem.

Chapter 2 Binomial Coefficients

Part II

Advanced Counting

Chapter 3 Recurrence Relations and Generating Functions

3.1 Recurrence Relations

Recurrence relations are equations that define sequences recursively, expressing each term as a function of its preceding terms, with the form of:

$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-k}), \quad n \geq k,$$

where k is the order of the recurrence relation, and f is a function that combines the previous k terms.

Recurrence relations can be classified into several types:

Homogeneous vs. Non-homogeneous A recurrence relation is homogeneous if it can be expressed solely in terms of previous terms of the sequence. If it includes additional functions or constants, it is non-homogeneous.

Linear vs. Non-linear A recurrence relation is linear if each term is a linear combination of previous terms:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

If it involves products or other non-linear operations, it is non-linear.

Constant Coefficients vs. Variable Coefficients A recurrence relation has constant coefficients if the coefficients in the linear combination are constants. If they vary with n, it has variable coefficients.

¶ `Methods for Solving Recurrence Relations`

¶ `Common Recurrence Relations`

Example 3.1 Define the Fibonacci sequence $\{F_n\}$ by the recurrence relation:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

which is a linear homogeneous recurrence relation with constant coefficients.

Example 3.2 The tower of Hanoi problem (Figure 3.1) is a classic example of a problem that can be solved using a recurrence relation. Move n plates from A to C, using B as an auxiliary, with the condition that only one plate can be moved at a time and a larger plate cannot be placed on a smaller one.

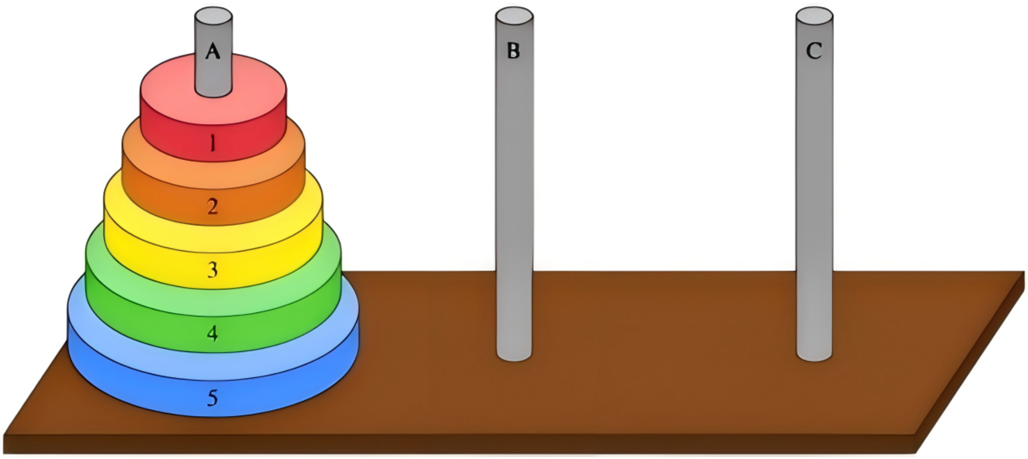

Figure 3.1: Tower of Hanoi problem

It can be described by the recurrence relation:

$$T(n) = 2T(n-1) + 1, \quad T(1) = 1,$$

where $T(n)$ represents the minimum number of moves required to transfer n disks from one peg to another.

3.2 Generating Functions

Definition 3.1

The **ordinary generating function** (OGF) of a sequence $\{a_n\}$ is defined as the formal power series:

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n,$$

where x is an indeterminate.

The **exponential generating function** (EGF) of a sequence $\{a_n\}$ is defined as:

$$E(a_n; x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

The **Dirichlet generating function** (DGF) of a sequence $\{a_n\}$ is defined as:

$$D(a_n; s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where s is a complex variable.

♣

¶ `Solving Recurrence Relations Using Generating Functions`

¶ `Integer Partitions`

Chapter 4 Inclusion-Exclusion Principle

4.1 Inclusion-Exclusion Principle

Theorem 4.1 (Inclusion-Exclusion Principle)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then the number of elements in the union of these sets is given by:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Denote $S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$ for $k = 1, 2, \ldots, n$. Then the formula can be rewritten as:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} S_k.$$

♡

Remark *Mnemonic: "Add odd, subtract even".*

Specially, when $n = 2$, we have:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

When $n = 3$, we have:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

The inclusion-exclusion principle also has a more commonly used form, **the complement form (property counting method)**:

Let U be the universal set that contains all the elements under consideration, and $\overline{A_i} = U \setminus A_i$ be the complement of A_i in U. Then we have:

$$\left|\bigcup_{i=1}^{n} A_i\right| = |U| - \left|\bigcap_{i=1}^{n} \overline{A_i}\right| = |U| - \sum_{k=0}^{n} (-1)^k S_k.$$

4.2 Möbius Inversion

The inclusion-exclusion principle holds essentially because the inclusion relations among sets form a partially ordered set. Möbius inversion is a highly generalized form of the inclusion-exclusion principle in number theory functions and partially ordered sets.

Definition 4.1 (Arithmetic Function)

An **arithmetic function** is a function defined on the set of positive integers and taking values in the complex numbers:

$$f : \mathbb{N}^* \to \mathbb{C}.$$

Examples include the divisor function $d(n)$, the Euler totient function $\phi(n)$, and the Möbius function $\mu(n)$. ♣

Definition 4.2 (Möbius Function)

The Möbius function $\mu(n)$ is defined on the positive integers as follows:

$$\mu(n) = \begin{cases} 1 & n = 1, \\ (-1)^k & n \text{ is a product of } k \text{ distinct primes} (n = p_1 p_2 \cdots p_k), \\ 0 & n \text{ has a squared prime factor} (4, 9, 16, \ldots). \end{cases}$$

♣

Theorem 4.2 (Möbius Inversion)

Let f and g be two arithmetic functions defined on the positive integers. If for every positive integer n, we have:

$$g(n) = \sum_{d|n} f(d),$$

then we can express $f(n)$ in terms of $g(n)$ using the Möbius function $\mu(d)$:

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right).$$

♡

4.3 Generalizations of Inclusion-Exclusion

Chapter 5 Special Counting Sequences

5.1 Catalan Numbers

Definition 5.1 (Catalan Numbers)

The n-th Catalan number C_n is given by:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \binom{2n}{n} - \binom{2n}{n+1}.$$

The first ten Catalan numbers are:

$$C_0 = 1, \quad C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14,$$
$$C_5 = 42, \quad C_6 = 132, \quad C_7 = 429, \quad C_8 = 1430, \quad C_9 = 4862.$$

♣

Property The Catalan numbers satisfy multiple recurrence relations:

1. $$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad (n \geq 1), \quad C_0 = 1.$$

   This recurrence relation reflects the self-similarity of Catalan numbers.

2. $$C_n = \frac{2(2n-1)}{n+1} C_{n-1} \quad (n \geq 1), \quad C_0 = 1.$$

   This recurrence relation can be derived from the closed-form expression of Catalan numbers.

3. Let $G(x) = \sum_{n=0}^{\infty} C_n x^n$ be the generating function of Catalan numbers. Then $G(x)$ satisfies the functional equation:

   $$G(x) = 1 + xG(x)^2,$$

   id est,

   $$G(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

   This functional equation can be used to derive the closed-form expression of Catalan numbers using the Lagrange inversion formula.

Catalan numbers is the answer to many combinatorial problems,

Ballot problem There is an $n \times n$ grid graph, with the bottom-left corner at $(0,0)$ and the top-right corner at (n,n). Starting from the bottom-left corner, and *moving only right or up one unit at each step*, the total number of paths to reach the top-right corner without going above the diagonal $y = x$ (but allowing touching it) is denoted as C_n.

Dyck path counting problem A Dyck path of semilength n is a lattice path from $(0,0)$ to $(2n,0)$ that never dips below the x-axis and consists of steps $(1,1)$ (up step) and $(1,-1)$ (down step). The number of Dyck paths of semilength n is C_n.

Counting non-intersecting chords in a circle There are $2n$ points on a circle. The number of ways to pair these points with n chords such that no two chords intersect is the Catalan number C_n.

Triangulation counting problem The number of ways to divide a convex $(n+2)$-sided region into triangular regions without intersecting diagonals is C_n.

Binary Tree Counting Problem The number of structurally different binary trees with n nodes is C_n. Equivalently, the number of structurally different full binary trees with n non-leaf nodes is C_n.

Counting problem of parenthesis sequences The number of valid parenthesis sequences consisting of n pairs of parentheses is C_n.

Stack popping sequence counting problem The push sequence of a stack (of infinite size) is $1, 2, \ldots, n$, and the number of valid popping sequences is C_n.

Sequence Counting Problem The number of sequences $a_1, a_2, \ldots, a_{2n}$ consisting of n $+1$'s and n -1's such that the partial sums satisfy $a_1 + a_2 + \ldots + a_k \geq 0$ ($k = 1, 2, 3, \ldots, 2n$) is C_n.

5.2 Stirling Numbers

5.3 Bell Numbers

5.4 Schröder Numbers

Part III

Existence and Extremal

Chapter 6 Pigeonhole Principle

Chapter 7 Extremal Principle

Chapter 8 Ramsey Theory

Part IV

Structure and Algebra

Chapter 9 Design Theory

Chapter 10 Pólya Counting

Bibliography

[1] Richard A. Brualdi. *Introductory Combinatorics 5th Edition*. 机械工业出版社, 2012.