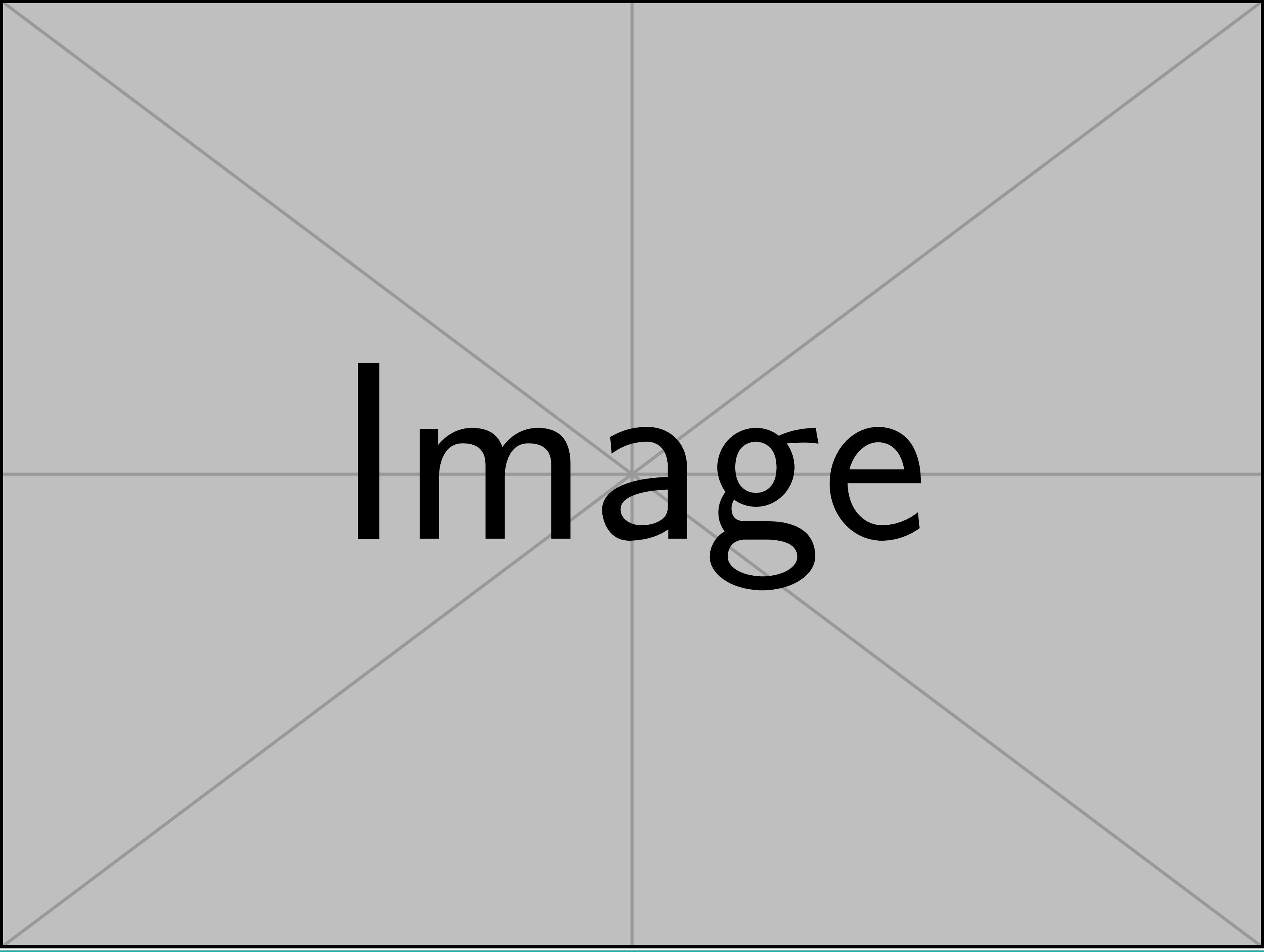

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This is the preface of the book…

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1.4 Special Determinants

Definition 1.1 (Vandermonde Determinant)

The Vandermonde determinant is defined as

$$V_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

where $x_1, x_2, \ldots, x_n$ are distinct variables. ♣

The value of the Vandermonde determinant is given by

$$V_n = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Definition 1.2 (Arrow Determinant)

The Arrow determinant (↖) is defined as

$$A_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & 0 & \cdots & a_{nn} \end{vmatrix}.$$

The value of the Arrow determinant is given by

$$A_n = \left(a_{11} - \sum_{k=2}^{n} \frac{a_{1k}a_{k1}}{a_{kk}} \right) \prod_{k=2}^{n} a_{kk}.$$

♣

From the first column sequentially, subtract $\frac{a_{21}}{a_{22}}$ times the second column, $\cdots$, $\frac{a_{n1}}{a_{nn}}$ times the n-th column, so that the first column becomes:

$$\begin{bmatrix} a_{11} - \sum\limits_{k=2}^{n} \frac{a_{1k}a_{k1}}{a_{kk}} & 0 & 0 & \vdots & 0 \end{bmatrix}^{\mathrm{T}}.$$

Then expand along the first column.

Definition 1.3 (Two-Triangular Determinant)

If the determinant satisfies

$$a_{ij} = \begin{cases} a, & i < j, \\ x_i, & i = j, \\ b, & i > j, \end{cases}$$

, then D_n is called a two-triangular determinant. ♣

The value of the two-triangular determinant is given by

$$\begin{vmatrix} x_1 & a & a & \dots & a \\ b & x_2 & a & \dots & a \\ b & b & x_3 & \dots & a \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \dots & x_n \end{vmatrix} = \begin{cases} \left[x_1 + a \sum\limits_{k=2}^{n} \frac{x_1 - a}{x_k - a}\right] \cdot \prod\limits_{k=2}^{n} (x_k - a), & a = b \\ (x_n - b)D_{n-1} + \prod\limits_{k=1}^{n-1} (x_k - a), & a \neq b \end{cases}$$

Chapter 2 System of Linear Equations

Definition 2.1 (System of Linear Equations)

A system of linear equations is a collection of one or more linear equations involving the same set of variables. For example, a system of m linear equations in n variables can be written as:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $x_1, x_2, \ldots, x_n$ are the variables, a_{ij} are the coefficients, and b_i are the constants.

A solution to the system is an ordered set of values for the variables that satisfies all equations simultaneously. If two systems have the same solution set, they are called equivalent systems, whose relationship are equivalence. ♣

2.1 Elimination Method

2.2 Linear Space

Definition 2.2 (Linear Space over Field F)

Let V be a non-empty set, and F be a field. If the following two operations (binary mapping) are defined on V:

Vector addition For any $\alpha, \beta \in V$, there exists $\gamma \in V$ such that $\gamma = \alpha + \beta$.

Scalar multiplication For any $\alpha \in V$ and $c \in F$, there exists $\gamma \in V$ such that $\gamma = c \cdot \alpha$.

and the following axioms are satisfied (for any $\alpha, \beta, \gamma \in V$ and $k, l \in F$):

A1. Commutativity of Addition $\alpha + \beta = \beta + \alpha$.

A2. Associativity of Addition $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

A3. Existence of Additive Identity There exists an element $0 \in V$ such that $\alpha + 0 = 0 + \alpha = \alpha$.

A4. Existence of Additive Inverse There exists an element $-\alpha \in V$ such that $\alpha + (-\alpha) = (-\alpha) + \alpha = 0$.

M1. Compatibility of Scalar Multiplication with Field Multiplication $k(l\alpha) = (kl)\alpha$.

M2. Identity Element of Scalar Multiplication $1 \cdot \alpha = \alpha$, where 1 is the multiplicative identity in F.

D1. Distributivity of Scalar Multiplication with respect to Vector Addition $k(\alpha + \beta) = k\alpha + k\beta$.

D2. Distributivity of Scalar Multiplication with respect to Field Addition $(k + l)\alpha = k\alpha + l\alpha$.

Then V is called a linear space (or vector space) over the field F. ♣

2.3 Matrix Representation of Systems of Linear Equations

Chapter 3 Matrices

3.1 Basic Operations

¶ Addition

¶ Scalar Multiplication

¶ Transpose

¶ Matrix Multiplication

Theorem 3.1 (Cauchy-Binet Formula)

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times m}$:

1. If $m > n$, then $|AB| = 0$;
2. If $m \leq n$, then $|AB|$ is equal to the sum of products of all m-step minors of A and the corresponding m-step minors of B, that is:

$$|AB| = \sum_{1\leq v_1<v_2<\cdots<v_m\leq n} \left| A\begin{pmatrix} 1,2,\ldots,m \\ v_1,v_2,\ldots,v_m \end{pmatrix} \right| \cdot \left| B\begin{pmatrix} v_1,v_2,\ldots,v_m \\ 1,2,\ldots,m \end{pmatrix} \right|.$$

♡

3.2 Matrix Equivalence

3.3 Special Matrices

3.4 Inverse Matrix

¶ Inverse Matrix and Its Operations

¶ Equivalent Propositions and Method of Inversion

¶ Generalized Inverse

3.5 Block Matrix

Theorem 3.2 (Determinant Reduction Formula)

Let $A_{m\times m}, B_{m\times n}, C_{n\times m}, D_{n\times n}$ be matrices. Then:

1. If A is invertible, then:
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\cdot|D - CA^{-1}B|.$$
2. If D is invertible, then:
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D|\cdot|A - BD^{-1}C|.$$
3. If both A and D are invertible, then:
$$|D|\cdot|A - BD^{-1}C| = |A|\cdot|D - CA^{-1}B|.$$

♡

Remark The mnemonic is: For $\begin{vmatrix} A & B \\ C & D \end{vmatrix}$, for example, if A is invertible, one factor is $|A|$, and the other factor is D (the diagonal element of A) minus the product of the other three terms arranged clockwise, where the middle one is the inverse matrix.

3.6 Operations of Rank

Proposition 3.1

The matrices A and B in the following operations do not need to be square matrices; they only need to be compatible for multiplication or addition.

1. Addition

$$\operatorname{rank}(A+B) \leqslant \operatorname{rank}(A) + \operatorname{rank}(B).$$

2. Multiplication

$$\operatorname{rank}(AB) \leqslant \operatorname{rank}(A), \quad \operatorname{rank}(AB) \leqslant \operatorname{rank}(B).$$

2.1. Sylvester's Inequality

$$\operatorname{rank}(AB) \geqslant \operatorname{rank}(A) + \operatorname{rank}(B) - n \quad (A_{s\times n}, B_{n\times m}).$$

Specially, if $AB = O$, then:

$$\operatorname{rank}(A) + \operatorname{rank}(B) \leqslant n.$$

2.2. Frobenius Inequality

$$\operatorname{rank}(ABC) \geqslant \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B).$$

3. Transpose

$$\operatorname{rank}(AA^T) = \operatorname{rank}(A^TA) = \operatorname{rank}(A) = \operatorname{rank}(A^T).$$

4. Inverse

$$\operatorname{rank}(A) = \operatorname{rank}(A^{-1}) = n.$$

3.7 Low-Rank Update

Due to all the row and column vectors of a rank-1 matrix are linearly dependent, it can be expressed as the outer product of two non-zero vectors; in other words, a rank-1 matrix can be expressed as $\alpha\beta^T$, where α and β are non-zero column vectors.

Based on the decomposition $A = \alpha\beta^T$, the matrix of rank-1 has simplified calculation rules:

Property

Exponentiation *For any positive integer $k \geq 1$,*

$$A^k = (\beta^T\alpha)^{k-1} \cdot A,$$

where $\beta^T\alpha$ is a constant (the inner product of vectors).

Rank Transmission *If B is any matrix, then:*

$$\operatorname{rank}(AB) \leq 1 \quad \textit{and} \quad \operatorname{rank}(BA) \leq 1,$$

(rank 1 matrices multiplied by arbitrary matrices result in ranks not exceeding 1).

Theorem 3.3 (Sherman-Morrison Formula)

If $A \in \mathbb{R}^{n\times n}$ is an invertible matrix, and $\alpha, \beta \in \mathbb{R}^n$ are column vectors, then $A + \alpha\beta^T$ is invertible if and only if $1 + \beta^T A^{-1}\alpha \neq 0$. In this case, the inverse of $A + \alpha\beta^T$ is given by:

$$\left(A + \alpha\beta^T\right)^{-1} = A^{-1} - \frac{A^{-1}\alpha\beta^T A^{-1}}{1 + \beta^T A^{-1}\alpha},$$

where $\alpha\beta^{\mathrm{T}}$ is the outer product of α and β.

Note *Combining the properties of determinants, we can derive the determinant version of the Sherman-Morrison formula:*

$$\left|A + \alpha\beta^T\right| = |A| \cdot \left(1 + \beta^T A^{-1}\alpha\right),$$

which is known as the ***matrix determinant lemma****.*

The theorem can also be stated in terms of the adjugate matrix of A:

$$\det(A + uv^T) = \det(A) + v^T \operatorname{adj}(A)u,$$

in which case it applies whether or not the matrix A is invertible.

Chapter 4 Linear Spaces

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