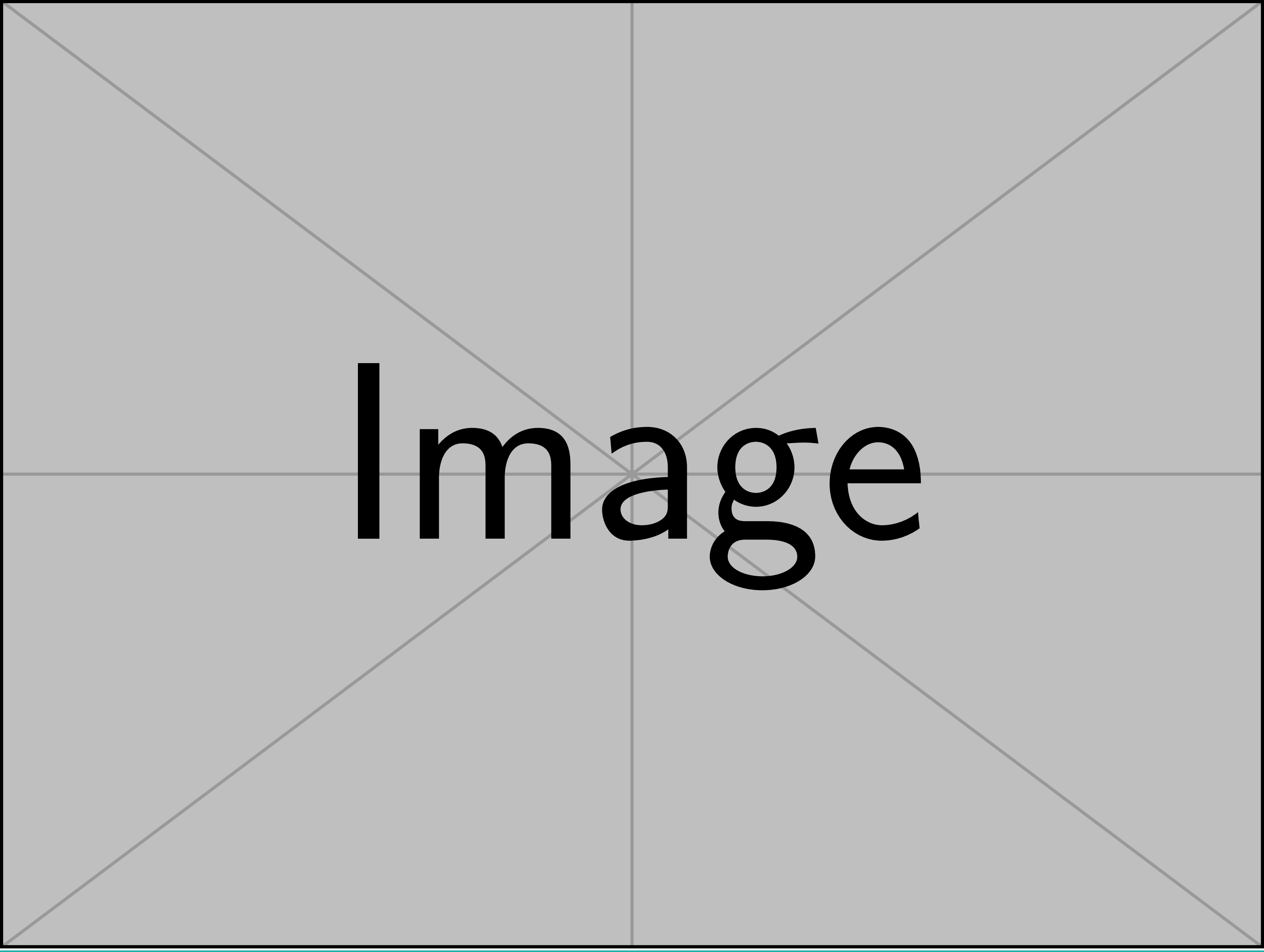

Analyse Mathématique

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8.1 Directional Derivatives and Total Differential

Definition 8.1 (Directional Derivative)

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^1$, $\boldsymbol{e}$ is a unit vector in $\mathbb{R}^n$, $\boldsymbol{x}^0 \in U$. Define

$$u(t) = f(\boldsymbol{x}^0 + t\boldsymbol{e}).$$

If the derivative of u at $t = 0$

$$u'(0) = \lim_{t \to 0} \frac{u(t) - u(0)}{t} = \lim_{t \to 0} \frac{f(\boldsymbol{x}^0 + t\boldsymbol{e}) - f(\boldsymbol{x}^0)}{t}$$

exists and is finite, it is called the **directional derivative** of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{e}$, denoted by $\frac{\partial f}{\partial \boldsymbol{e}}(\boldsymbol{x}_0)$. It is the rate of change of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{e}$.

Consider the following set of unit coordinate vectors: $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$. For a function f, the directional derivative of f at the point $\boldsymbol{x}_0$ in the direction of $\boldsymbol{e}_i$ is called the ith first-order **partial derivative** of f at $\boldsymbol{x}^0$, denoted by

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}^0) \quad \text{or} \quad \mathrm{D}_i f(\boldsymbol{x}^0).$$

$\mathrm{D}_i = \frac{\partial}{\partial x_i}$ is called the ith partial differential operator ($i = 1, 2, \cdots, n$).

Note *Let* $\mathbf{e}$ *be a direction, then* $\|-\mathbf{e}\| = \|\mathbf{e}\| = 1$*, which implies that* $-\mathbf{e}$ *is also a direction. At this point, we have:*

$$\frac{\partial f}{\partial(-\mathbf{e})}(\boldsymbol{x}^0) = -\frac{\partial f}{\partial \mathbf{e}}(\boldsymbol{x}^0).$$

If the first-order partial derivative of f, $\frac{\partial f}{\partial x_i}$, itself possesses partial derivatives, then the second-order partial derivative of f is defined, and is denoted as follows:

$$f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right), \quad f_{x_i x_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right), \quad i, j = 1, 2, \ldots, n.$$

Similarly, higher-order partial derivatives of order $3, 4, \cdots m, \cdots$ can be defined.

Definition 8.2 (Jacobian Matrix (Gradient))

Let

$$\boldsymbol{J} f(\boldsymbol{x}) = (\mathrm{D}_1 f(\boldsymbol{x}), \mathrm{D}_2 f(\boldsymbol{x}), \ldots, \mathrm{D}_n f(\boldsymbol{x})),$$

which is called the **Jacobian matrix** of the function f at the point $\boldsymbol{x}$, (a $1 \times n$ matrix) whose counterpart is the first-order derivative of a single-variable function.

Henceforth, we represent the point $\boldsymbol{x}$ in $\mathbb{R}^n$ and its increments $\boldsymbol{h}$ as column vectors. In this way, the differential of the function can be expressed using matrix multiplication as follows:

$$\mathrm{d}f(\boldsymbol{x}^0)(\boldsymbol{\Delta x}) = \boldsymbol{J} f(\boldsymbol{x}^0)\boldsymbol{\Delta x}.$$

The Jacobian matrix of the function f is also frequently denoted as $\mathbf{grad}\, f$ (or ∇f), that is,

$$\mathbf{grad}\, f(\boldsymbol{x}) = \boldsymbol{J} f(\boldsymbol{x}),$$

which is called the **gradient** of the scalar function f.

Definition 8.3 (Total Differential)

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^1$, $\boldsymbol{x}^0 \in U$, $\boldsymbol{\Delta x} = (\Delta x_1, \Delta x_2, \cdots, \Delta x_n) \in \mathbb{R}^n$. If

$$f(\boldsymbol{x}^0 + \boldsymbol{\Delta x}) - f(\boldsymbol{x}^0) = \sum_{i=1}^{n} A_i \Delta x_i + o(\|\boldsymbol{\Delta x}\|) \qquad (\|\boldsymbol{\Delta x}\| \to 0),$$

where $A_1, A_2, \dots, A_n$ are constants independent of $\boldsymbol{\Delta x}$, then the function f is said to be **differentiable** at the point $\boldsymbol{x}^0$, and the linear main part $\sum_{i=1}^{n} A_i \Delta x_i$ is called the **total differential** of f at $\boldsymbol{x}^0$, denoted as

$$df(\boldsymbol{x}^0)(\boldsymbol{\Delta x}) = \sum_{i=1}^{n} A_i \Delta x_i.$$

If f is differentiable at every point in the open set U, then f is called a differentiable function on U. ♣

Theorem 8.1 (Conditions of Differentiability)

Necessary Condition If an n-variable function f is differentiable at the point $\boldsymbol{x}_0$, then f is continuous at $\boldsymbol{x}^0$ and possesses first-order partial derivatives $\frac{\partial f}{\partial x_i}(\boldsymbol{x}^0)$ at $\boldsymbol{x}^0$ for $i = 1, 2, \dots, n$, and

$$\boldsymbol{A} = (A_1, A_2, \dots, A_n) = \boldsymbol{J} f(\boldsymbol{x}^0) = \left(\mathrm{D}_1 f(\boldsymbol{x}^0), \mathrm{D}_2 f(\boldsymbol{x}^0), \dots, \mathrm{D}_n f(\boldsymbol{x}^0)\right).$$

[a] However, the converse is not true.

Sufficient Condition Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^1$ be an n-variable function. If $\boldsymbol{J} f = (\mathrm{D}_1 f, \mathrm{D}_2 f, \dots, \mathrm{D}_n f)$ is continuous at $\boldsymbol{x}^0$ (i.e., $\frac{\partial f}{\partial x_i}$ is continuous at $\boldsymbol{x}^0$ for $i = 1, 2, \dots, n$), then f is differentiable at $\boldsymbol{x}^0$. However, the converse is not necessarily true.

[a] It is referred to as the total differential formula, and the more common form is

$$\mathrm{d}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\,\mathrm{d}x + \frac{\partial f}{\partial y}(x_0, y_0)\,\mathrm{d}y.$$

♡

Note *(At some point)*

1. *Differentiable*
   - $\Longrightarrow$ *Continuous*
   - $\Longrightarrow$ *Partial derivatives exist:* $\mathrm{D}_{\vec{u}} = \nabla f \cdot \vec{u}$
2. *Directional Derivative*
   - *All directional derivatives exist* $\not\Longrightarrow$ *differentiable or continuous.*
   - *All directional derivatives exist and are equal* $\not\Longrightarrow$ *differentiable.*
3. *Partial Derivative*
   - *The continuity and existence of directional/partial derivatives are mutually exclusive.*

Chapter 9 Multiple Integrals

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