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## Analyse Mathématique

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## Preface

For an interval  $I$ , a open interval  $(a, b)$  and a closed interval  $[a, b]$ , we denote  $C(I)$ ,  $C(a, b)$  and  $C[a, b]$  as the set of continuous univariate functions on  $I$ ,  $(a, b)$  and  $[a, b]$  respectively. Similarly, the following notations are used:

Notation	Meaning
$D(I)$	Set of derivative (differential) functions on $I$
$D(a, b)$	Set of derivative (differential) functions on $(a, b)$
$D[a, b]$	Set of derivative (differential) functions on $[a, b]$
$D^k(I)$	Set of $k$ -th order derivative (differential) functions on $I$

Let  $U \subset \mathbb{R}^n$  be an open set, and  $\mathbf{f} : U \rightarrow \mathbb{R}^m$  be a  $C^k$  mapping:

- $k = 0$ ,  $\mathbf{f}$  is a continuous mapping;
- $0 < k < +\infty$ ,  $f_i$  has continuous partial derivatives up to order  $k$ ,  $i = 1, 2, \dots, m$ ;
- $k = +\infty$ ,  $f_i$  has continuous partial derivatives of all orders,  $i = 1, 2, \dots, m$ ;
- $k = \omega$ ,  $f_i$  is really analytic, i.e., in the neighborhood of any point  $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in U$ ,  $f_i$  can be expanded into a convergent ( $n$ -dimensional) power series,  $i = 1, 2, \dots, m$ .

Let  $C^k(U, \mathbb{R}^m)$  denote the set of  $C^k$  mappings from  $U$  to  $\mathbb{R}^m$ .

# Chapter 1 Preliminaries

## 1.1 Trigonometric Formulas

### Product-to-Sum Formulas:

$$\begin{aligned}\sin \alpha \cos \beta &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\ \cos \alpha \sin \beta &= \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)] \\ \cos \alpha \cos \beta &= \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] \\ \sin \alpha \sin \beta &= -\frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]\end{aligned}$$

### Sum and Difference Formulas:

$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta\end{aligned}$$

### Sum-to-Product Formulas:

$$\begin{aligned}\sin \alpha + \sin \beta &= 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) \\ \sin \alpha - \sin \beta &= 2 \sin \left( \frac{\alpha - \beta}{2} \right) \cos \left( \frac{\alpha + \beta}{2} \right) \\ \cos \alpha + \cos \beta &= 2 \cos \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) \\ \cos \alpha - \cos \beta &= -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right)\end{aligned}$$

### Double Angle Formulas:

$$\begin{aligned}\sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha \\ \tan 2\alpha &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha}\end{aligned}$$

### Half Angle Formulas:

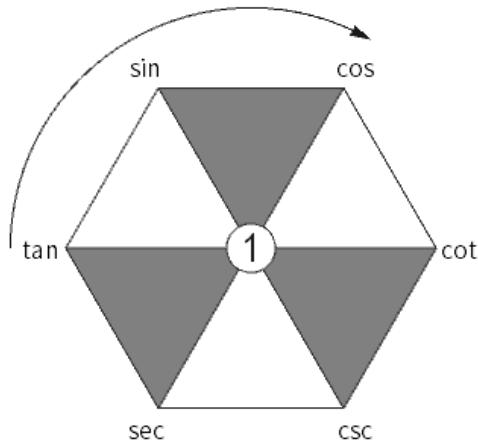
$$\begin{aligned}\sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}} \\ \cos \frac{\alpha}{2} &= \pm \sqrt{\frac{1 + \cos \alpha}{2}} \\ \tan \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}\end{aligned}$$

### Power-Reducing Formulas:

$$\begin{aligned}\sin^2 \alpha &= \frac{1 - \cos 2\alpha}{2} \\ \cos^2 \alpha &= \frac{1 + \cos 2\alpha}{2}\end{aligned}$$

**Angle Decomposition Formulas:**

$$\begin{aligned}\sin^2 \alpha - \sin^2 \beta &= \sin(\alpha + \beta) \sin(\alpha - \beta) \\ \cos^2 \alpha - \sin^2 \beta &= \cos(\alpha + \beta) \cos(\alpha - \beta)\end{aligned}$$

**Remark**

- On the gray triangle, the sum of the squares of the two numbers above is equal to the square of the number below, for instance,  $\tan^2 x + 1 = \sec^2 x$
- The three trigonometric functions in the clockwise direction have the following properties:  $\tan x = \frac{\sin x}{\cos x}$ , etc.

## 1.2 Common Inequalities

Some common inequalities:

$$\frac{x}{1+x} < \ln(1+x) < x, \quad x > 0;$$

## 1.3 Factorial Power

### Definition 1.1

Rising factorials and falling factorials can be expressed in multiple notations.

The Pochhammer symbol, introduced by Leo August Pochhammer, is one of the commonly used notations, represented as  $x^{(n)}$  or  $(x)_n$ .

Ronald Graham, Donald Ervin Knuth, and Oren Patashnik introduced the symbols  $x^{\bar{n}}$  and  $x^n$  in their book *Concrete Mathematics*.

### Definitions:

- Rising factorial:**

$$x^{\bar{n}} = x(x+1)(x+2)\dots(x+n-1) = \frac{(x+n-1)!}{(x-1)!}.$$

- **Falling factorial:**

$$x^{\underline{n}} = x(x-1)(x-2)\dots(x-n+1) = \frac{x!}{(x-n)!}.$$

**Relationships:**

- Relationship between rising and falling factorials:

$$x^{\bar{n}} = (x+n-1)^{\underline{n}}.$$

- Relationship with factorial:

$$1^{\bar{n}} = n^{\underline{n}} = n!.$$



## 1.4 Combination

**Definition 1.2 (Combination)**

The number of ways to choose  $k$  elements from a set of  $n$  distinct elements, denoted as  $C_n^k$  or  $\binom{n}{k}$ , is given by:

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$



**Property**

$$C_n^k = \frac{A_n^k}{k!} = \frac{n!}{(n-k)!k!}$$

$$C_n^k = C_n^{n-k}$$

$$C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$$

**Remark** The third property can be understood that to choose  $k$  elements from  $n+1$ , you can first take one element  $A$ :

1. The number of ways that include  $A$  is  $C_n^{k-1}$ ;
2. The number of ways that do not include  $A$  is  $C_{n-1}^k$ .

# Chapter 2 Limits of Sequences and Continuity of Real Number System

## 2.1 Convergent Sequences

- ¶ Convergent Sequences
- ¶ Properties of Convergent Sequences
- ¶ Cauchy Proposition and Fitting Method

*Proposition 2.1 (Cauchy Proposition)*

Let  $\lim_{n \rightarrow \infty} x_n = l$ , then:

$$\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = l.$$



### Note

1. In the proposition,  $l$  can be  $+\infty$  or  $-\infty$ .
2. Let  $\lim_{n \rightarrow \infty} x_n = l$ , then:

$$\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{x_1 x_2 \dots x_n} = \lim_{n \rightarrow \infty} \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}} = l.$$

It can be proved directly by Stolz theorem 2.1. On top of that, it can also be proved by the **fitting method**.

### Proof



**Remark** To prove  $\lim_{n \rightarrow \infty} x_n = A$ , the key is to show that  $|x_n - A|$  can be arbitrarily small. For this purpose, it is generally recommended to simplify the expression of  $x_n$  as much as possible. However, in some cases,  $A$  can also be transformed into a form similar to  $x_n$ . This method is called the fitting method. The core idea behind the method of fitting is to appropriately divide into units of 1 for analysis.

## 2.2 Indeterminate Form

- ¶ Infinitely Large Quantities and Infinitesimal Quantities
- ¶ Indeterminate Forms

*Theorem 2.1 (Stolz-Cesàro theorem)*

**Type  $\frac{0}{0}$**  Let  $\{a_n\}, \{b_n\}$  be two infinitesimal sequences, where  $\{a_n\}$  is also a strictly monotonic decreasing sequence. If

$$\lim_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l \text{ (finite or } \pm \infty\text{),}$$

then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l.$$

**Type  $\frac{\infty}{\infty}$**  Let  $\{a_n\}$  be a strictly monotonic increasing sequence of divergent large quantities. If

$$\lim_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l \text{ (finite or } \pm \infty\text{),}$$

then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l.$$



### Note

1. The inverse proposition of Stolz's Theorem does not hold.
2. If  $a_1$  is an undefined infinite quantity  $\infty$ , Stolz Theorem does not hold.

### Theorem 2.2 (Silverman-Toeplitz Theorem)

Let

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{bmatrix},$$

where the infinite triangular matrix satisfies:

1.  $\forall j, \lim_{n \rightarrow \infty} a_{nj} = 0$ . (Every column sequence converges to 0.)
2.  $\sup_{i \in \mathbb{N}} \sum_{j=1}^i |a_{ij}| < \infty$ . (The absolute row sums are bounded.)

And  $\lim_{n \rightarrow \infty} x_n = l$ . We denote  $y_n$  as the weighted sum sequence:  $y_n = \sum_{j=1}^n a_{nj}x_j$ . Then the following results hold:

1. If  $l = 0$ , then  $\lim_{n \rightarrow \infty} y_n = 0$ .
2. If  $l \neq 0$  and  $\lim_{n \rightarrow \infty} \sum_{j=1}^n a_{nj} = 1$ , then  $\lim_{n \rightarrow \infty} y_n = l$ .



## 2.3 Subsequences

### ¶ Subsequences

### ¶ Upper Limits and Lower Limits

## 2.4 Completeness of The Real Numbers

### ¶ Dedkind Completeness

### ¶ Least Upper Bound Property

### ¶ Monotone Convergence Theorem

### ¶ Bolzano-Weierstrass Theorem

### ¶ Nested Interval Theorem

### ¶ Cauchy Completeness

### Definition 2.1 (Cauchy Sequence)

A sequence  $\{x_n\}$  is called a **Cauchy sequence** if for any  $\varepsilon > 0$ , there exists a positive integer  $N$  such that when  $m, n > N$ ,

$$|x_n - x_m| < \varepsilon.$$



**Theorem 2.3 (Cauchy Convergence Criterion for Sequences)**

A sequence  $\{x_n\}$  converges if and only if it is a Cauchy sequence.



### ¶ Heine-Borel Theorem

## 2.5 Iterative Sequences

Formally,  $x_0$  is a **fixed point** of the function  $f$  if  $f(x_0) = x_0$ .

**Theorem 2.4 (Banach Fixed-Point Theorem (Contraction Mapping Theorem))**

There exists a contraction mapping (in 3.2)  $f$  on an interval  $I$ , which admits a unique fixed point  $x^* \in I$ . Furthermore,  $x^*$  can be found as follows: start with an arbitrary point  $x_0 \in I$  and define the iterative sequence  $x_{n+1} = f(x_n)$  for  $n = 0, 1, 2, \dots$ . Then  $\lim_{n \rightarrow \infty} x_n = x^*$ .



**Remark** The following inequalities are equivalent and describe the speed of convergence:

$$\begin{aligned}|x_n - x^*| &\leq \frac{L^n}{1-L} |x_1 - x_0|, \\ |x_{n+1} - x^*| &\leq \frac{L}{1-L} |x_{n+1} - x_n|, \\ |x_{n+1} - x^*| &\leq L |x_n - x^*|.\end{aligned}$$

Any such value of  $L < 1$  is the Lipschitz constant for  $f$ , and the smallest one is sometimes called **the best Lipschitz constant of  $L$** .

# Chapter 3 Limits and Continuity of Functions

## 3.1 Limits of Functions

### ¶ Definition of Limit

### ¶ Limits of Functions and Sequences

#### Theorem 3.1 (Heine Theorem)

Let  $f$  be a function defined on a deleted neighborhood  $\mathring{U}(x_0)$  of  $x_0$ . The following two statements are equivalent:

1.  $\lim_{x \rightarrow x_0} f(x) = A$ .
2. For any sequence  $\{x_n\} \subset \mathring{U}(x_0)$  with  $\lim_{n \rightarrow \infty} x_n = x_0$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = A$  for the sequence  $\{f(x_n)\}$ .



## 3.2 Continuous Functions

## 3.3 Infinitesimal and Infinite Quantities

## 3.4 Continuous Functions on Closed Intervals

### ¶ Concerning Theorems

#### Theorem 3.2 (The Bolzano-Cauchy Intermediate-Value Theorem)



#### Theorem 3.3 (Zero Point Existence Theorem)



### ¶ Uniform Continuity and Lipschitz Continuity

#### Definition 3.1 (Uniform Continuity)



#### Theorem 3.4 (Uniform Continuity Theorem)



#### Theorem 3.5 (Cantor's Theorem)



#### Definition 3.2 (Lipschitz Continuity)

If there exists a constant  $L > 0$  such that for any  $x_1, x_2 \in I$ ,

$$|f(x_1) - f(x_2)| \leq L |x_1 - x_2|,$$

then  $f$  is called **Lipschitz continuous** on  $I$ .

Specially, if  $L < 1$ , then  $f$  is called a **contraction mapping** on  $I$ .



### ¶ Remark

- If  $f$  is Lipschitz continuous on  $I$ , then  $f$  is uniformly continuous on  $I$ . ( $\forall \varepsilon > 0$ , just let  $\delta = \frac{\varepsilon}{L}$ )
- If  $f$  is uniformly continuous on  $I$ , then  $f$  is continuous on  $I$ .
- The converse of the above two statements does not hold.

## 3.5 Period Three Implies Chaos

## 3.6 Functional Equations

# Chapter 4 Differential

## 4.1 Differential and Derivative

### ¶ Basic Differential Rules and Formulas

	<b>Derivative Rules</b>	<b>Differential Rules</b>
Linear Combination	$(c_1f + c_2g)' = c_1f' + c_2g'$	$d(c_1f + c_2g) = c_1df + c_2dg$
Product Rule	$(fg)' = f'g + fg'$	$d(fg) = gdf + fdg$
Quotient Rule	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$	$d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$
Inverse Function	$[f^{-1}(y)]' = \frac{1}{f'(x)}$	$dx = \frac{dy}{f'(x)} = [f^{-1}(y)]' dy$
Chain Rule	$[f(g(x))]' = f'(u)g'(x)$	$d[f(g(x))] = f'(u)g'(x)dx$

<b>Derivative</b>	<b>Differential</b>
$(C)' = 0$	$d(C) = 0 \cdot dx = 0$
$(x^\alpha)' = \alpha x^{\alpha-1}$	$d(x^\alpha) = \alpha x^{\alpha-1} dx$
$(\sin x)' = \cos x$	$d(\sin x) = \cos x dx$
$(\cos x)' = -\sin x$	$d(\cos x) = -\sin x dx$
$(\tan x)' = \sec^2 x$	$d(\tan x) = \sec^2 x dx$
$(\cot x)' = -\csc^2 x$	$d(\cot x) = -\csc^2 x dx$
$(\sec x)' = \tan x \sec x$	$d(\sec x) = \tan x \sec x dx$
$(\csc x)' = -\cot x \csc x$	$d(\csc x) = -\cot x \csc x dx$
$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$d(\arcsin x) = \frac{1}{\sqrt{1-x^2}} dx$
$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$d(\arccos x) = -\frac{1}{\sqrt{1-x^2}} dx$
$(\arctan x)' = \frac{1}{1+x^2}$	$d(\arctan x) = \frac{1}{1+x^2} dx$
$(\text{arccot} x)' = -\frac{1}{1+x^2}$	$d(\text{arccot} x) = -\frac{1}{1+x^2} dx$
$(a^x)' = \ln a \cdot a^x, (e^x)' = e^x$	$d(a^x) = \ln a \cdot a^x dx, d(e^x) = e^x dx$
$(\log_a x)' = \frac{1}{x \ln a}, (\ln x)' = \frac{1}{x}$	$d(\log_a x) = \frac{1}{x \ln a} dx, d(\ln x) = \frac{1}{x} dx$
$(\text{sh} x)' = \text{ch} x$	$d(\text{sh} x) = \text{ch} x dx$
$(\text{ch} x)' = \text{sh} x$	$d(\text{ch} x) = \text{sh} x dx$
$(\text{th} x)' = \text{sech}^2 x$	$d(\text{th} x) = \text{sech}^2 x dx$
$(\text{cth} x)' = -\text{csch}^2 x$	$d(\text{cth} x) = -\text{csch}^2 x dx$
$(\text{arcsh} x)' = \frac{1}{\sqrt{1+x^2}}$	$d(\text{arcsh} x) = \frac{1}{\sqrt{1+x^2}} dx$
$(\text{arcch} x)' = \frac{1}{\sqrt{x^2-1}}$	$d(\text{arcch} x) = \frac{1}{\sqrt{x^2-1}} dx$
$(\text{arcth} x)' = (\text{arccth} x)' = \frac{1}{1-x^2}$	$d(\text{arcth} x) = d(\text{arccth} x) = \frac{1}{1-x^2} dx$
$\ln(x + \sqrt{x^2 + a^2})' = \frac{1}{\sqrt{x^2+a^2}}$	$d[\ln(x + \sqrt{x^2 + a^2})] = \frac{dx}{\sqrt{x^2+a^2}}$

## 4.2 Higher-Order Derivatives

Some useful formulas of higher-order derivatives:

$$\begin{aligned}(a^x)^{(n)} &= (\ln a)^n a^x, \\ (\sin \alpha x)^{(n)} &= \alpha^n \sin\left(\alpha x + \frac{n\pi}{2}\right), \\ (\cos \alpha x)^{(n)} &= \alpha^n \cos\left(\alpha x + \frac{n\pi}{2}\right), \\ (\ln x)^{(n)} &= \frac{(-1)^{n-1}(n-1)!}{x^n}, \\ (x^\alpha)^{(n)} &= \alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}.\end{aligned}$$

In order to obtain the higher-order derivative of two or more functions' linear combination and product, we need to use the following theorems.

### Theorem 4.1 (Linear Operation of Higher-Order Derivatives)

If  $f, g \in D^{(n)}(I)$ , then for any constants  $c_1, c_2 \in \mathbb{R}$ ,

$$(c_1 f + c_2 g)^{(n)} = c_1 f^{(n)} + c_2 g^{(n)}.$$



### Theorem 4.2 (Leibniz's Formula)

If  $f, g \in D^{(n)}(I)$ , then

$$(fg)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}.$$



### ⚠ Caution Note the distinction:

- $dx^2$  represents the square of the differential of the independent variable, i.e.,  $(dx)^2$ ;
- $d^2x$  represents the second differential of the independent variable,  $d(dx)$ ;
- $d(x^2)$  represents the differential of  $x^2$ , which is  $2x dx$ .

## 4.3 Differential Mean Value Theorems

### Definition 4.1 (Argmax and Argmin)

Let  $f(x)$  is defined on  $(a, b)$ ,  $x_0 \in (a, b)$ . If there exists  $U(x_0, \delta) \subset (a, b)$  such that  $f(x) \leq f(x_0)$  on it, then  $x_0$  is called a arguments of the maxima point of  $f$ , and  $f(x_0)$  is referred to as the corresponding arguments of the maxima (abbreviated arg max or argmax).

The definition of the argmin is analogous.



### Lemma 4.1 (Fermat's Lemma)

If  $f$  is differentiable at  $x_0$  which is a local extremum, then  $f'(x_0) = 0$ .



### Theorem 4.3 (Rolle's Theorem)

If  $f \in C[a, b]$ ,  $f \in D(a, b)$  and  $f(a) = f(b)$ , then there exists  $\xi \in (a, b)$  such that  $f'(\xi) = 0$ .

Enhanced Version: If  $f \in D(a, b)$  (finite or infinite interval), and  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow b^-} f(x)$ , then

there exists  $\xi \in (a, b)$  such that  $f''(\xi) = 0$ .



#### Theorem 4.4 (Lagrange's Mean Value Theorem)

If  $f \in C[a, b]$ ,  $f \in D(a, b)$ , then there exists  $\xi \in (a, b)$  such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$



#### Theorem 4.5 (Cauchy's Mean Value Theorem)

If  $f, g \in C[a, b]$ ,  $f, g \in D(a, b)$  and  $g'(x) \neq 0$  for all  $x \in (a, b)$ , then there exists  $\xi \in (a, b)$  such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$



**Note** The following types of problems commonly appear in proofs related to intermediate values in differential calculus:

1. Prove the existence of a point  $\xi$  such that  $F(\xi, f(\xi), f'(\xi)) = 0$ . Problems of this type generally involve constructing auxiliary functions and applying Rolle's theorem. The commonly used auxiliary functions include:

$$\begin{aligned} \xi f'(\xi) + f(\xi) &= 0, & xf(x), \\ \xi f'(\xi) + nf(\xi) &= 0, & x^n f(x), \\ \xi f'(\xi) - f(\xi) &= 0, & e^x f(x), \\ f'(\xi) + \lambda f(\xi) &= 0, & e^{-x} f(x), \\ f'(\xi) + f(\xi) &= 0, & x^n f(x), \\ f'(\xi) - f(\xi) &= 0, & xf(x). \end{aligned}$$

2. Prove the existence of two points  $\xi, \eta$  (i.e., two intermediate values) such that  $F(\xi, f(\xi), f'(\xi), \eta, f(\eta), f'(\eta)) = 0$ . These problems can be divided into the following categories:

$\xi \neq \eta$  Problems of this type usually occur in the same interval  $[a, b]$  and employ theorems of double differentiation intermediate values such as the Lagrange mean value theorem or Cauchy's mean value theorem. The specific choice of auxiliary functions often includes terms like  $\xi$  and other variables determined after decomposition.

$\xi = \eta$  Such problems cannot occur within the same interval  $[a, b]$ . They use double differentiation mean value theorems by splitting  $[a, b]$  into two intervals  $[a, c]$  and  $[c, b]$ , applying the Lagrange mean value theorem separately to each interval. Here, the selection of  $\xi$  and  $\eta$  is key.

3. As a rule, when conditions in a theorem involve additional constraints about higher-order derivatives, it is necessary to use Taylor's intermediate value theorem.

## 4.4 Theorems about Derivatives

#### Theorem 4.6 (Darboux's Intermediate Value Theorem for Derivatives)

If  $f(x) \in D[a, b]$ , and  $f'_+(a) \cdot f'_{-}(b) < 0$ , then there at least exists  $\xi \in (a, b)$  such that  $f'(\xi) = 0$ .



#### Theorem 4.7 (Theorem on the Limit of Derivatives)

If  $f(x) \in C(U(x_0))$ ,  $D(\mathring{U}(x_0))$ , and  $\lim_{x \rightarrow x_0} f'(x) = A$ , then  $f$  is differentiable at  $x_0$  and  $f'(x_0) = A$ .



**Remark** In fact,  $\lim_{x \rightarrow x_0} f'(x) = A$  has already been shown to imply that  $f \in D(\mathring{U}(x_0))$ .

The mnemonic for this theorem is: Continuous function + limit of derivative  $\Rightarrow$  derivative at the point.

## 4.5 Taylor Theorem

¶ L'Hôpital's Rule

¶ Taylor Formula

¶ Maclaurin Formula

### Lemma 4.2

If  $f(x)$  has  $n+2$  derivatives in some neighborhood of  $x_0$ , then the derivative of its  $n+1$ th degree Taylor polynomial is exactly the  $n$ th degree Taylor polynomial of  $f'(x)$ . 

Taylor formula at  $x_0 = 0$  is called the **Maclaurin formula**. Some common Maclaurin formulas are as follows:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n),$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + o(x^n),$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + o(x^{2n}),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1}),$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + o(x^{2n}),$$

$$\arcsin x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \cdots + \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1} + o(x^{2n+2}).$$

Specially,

$$(1+x)^\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{k} x^k + o(x^n),$$

- if  $\alpha = n \in \mathbb{N}^+$ , that is Newton's binomial formula  $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$ ;
- if  $\alpha = \frac{1}{2}$ , then  $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$ ;
- if  $\alpha = -1$ , then  $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ ;
- if  $\alpha = -\frac{1}{2}$ , then  $(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots$ .

¶ Euler and Bernoulli Numbers

### Definition 4.2 (Euler Numbers)

The Euler numbers  $E_n$  are defined by the Taylor series expansion of the secant function:

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

The odd-indexed Euler numbers are all zero. The even-indexed ones have alternating signs. Some values are:

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385.$$



#### *Definition 4.3 (Bernoulli Numbers)*

The Bernoulli numbers  $B_n$  are defined by the Taylor series expansion of the function  $\frac{x}{e^x - 1}$ :

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Some values are:

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}.$$

Notably, all odd-indexed Bernoulli numbers (except  $B_1 = -\frac{1}{2}$ ) are zero.



**Remark** Euler and Bernoulli numbers are widely used in number theory, combinatorics, and numerical analysis. For example, in the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}, \quad k \in \mathbb{N}^+,$$

when  $k = 1$ , it gives the famous Basel problem result:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

With the help of Bernoulli numbers, we have

$$\tan x = \sum_{n=0}^{\infty} \frac{B_{2n}}{2n} \frac{x^{2n}}{(2n)!} = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \dots$$

## 4.6 Properties of Functions

### Monotonicity and Convexity

#### *Definition 4.4 (Convex Function)*

A function  $f$  is called **convex** on an interval  $I$  if for any  $x_1, x_2 \in I$  and  $t \in [0, 1]$ , the following inequality holds:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

If the inequality is strict for  $x_1 \neq x_2$  and  $t \in (0, 1)$ , then  $f$  is called **strictly convex** on  $I$ .

Conversely, if the inequality is reversed, then  $f$  is called **concave** or **concave down** on  $I$ .



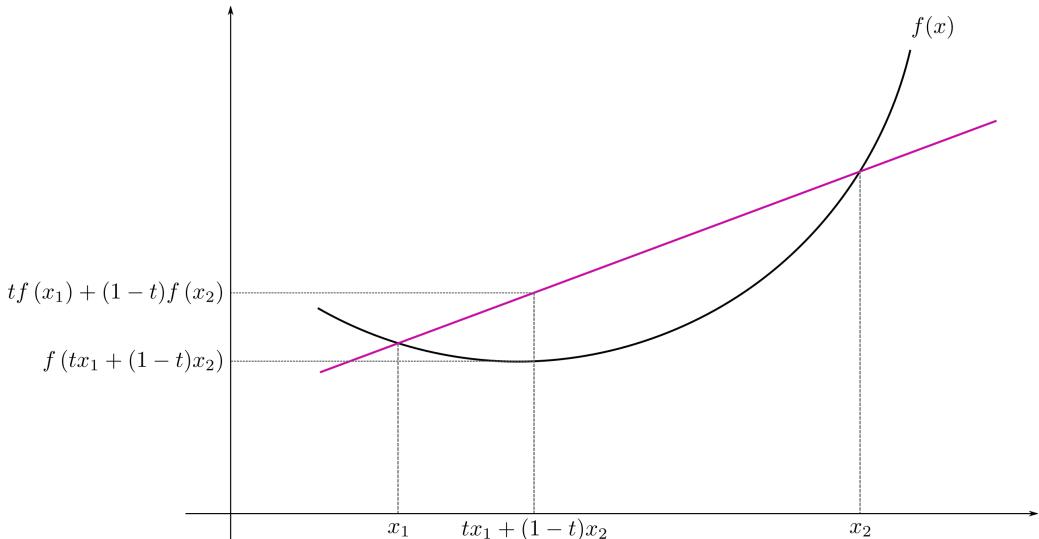
A related concept is that of **inflection points**: a point on the graph of a function at which the concavity changes.

#### *Theorem 4.8*

Mark above definition as Definition 1, give the following statements:

2. (Jensen Definition) A function  $f$  is called convex on an interval  $I$  if for any  $x_1, x_2 \in I$ :

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$



3. A function  $f$  is called convex on an interval  $I$  if for any  $x_1, x_2, \dots, x_n \in I$ :

$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}.$$

4. A function  $f$  is called convex on an interval  $I$  if the tangent line at any point lies below the graph of the function.

Then,

- Definitions 2 and 3 are equivalent.
- When  $f$  is continuous, Definition 1, 2, 3 is equivalent.
- When  $f$  is differentiable, all four definitions are equivalent.



#### Theorem 4.9 (Jensen Inequality)

If  $f$  is convex on an interval  $I$ , then for any  $x_1, x_2, \dots, x_n \in I$  and any  $t_1, t_2, \dots, t_n > 0$  such that  $t_1 + t_2 + \dots + t_n = 1$ , the following inequality holds:

$$f(t_1x_1 + t_2x_2 + \dots + t_nx_n) \leq t_1f(x_1) + t_2f(x_2) + \dots + t_nf(x_n).$$

Specially, when  $t_1 = t_2 = \dots = t_n = \frac{1}{n}$ , it reduces to Definition 3.



Next, we present derivative-based criteria for monotonicity and convexity:

#### Theorem 4.10

1. If  $f \in D(I)$ , then  $f$  is increasing (decreasing) on  $I$  if and only if  $f'(x) \geq 0$  ( $f'(x) \leq 0$ ) for all  $x \in I$ .
2. If  $f \in D^{(2)}(I)$ , then  $f$  is convex (concave) on  $I$  if and only if  $f''(x) \geq 0$  ( $f''(x) \leq 0$ ) for all  $x \in I$ .



**Note** If  $f'(x) > 0$  ( $f''(x) > 0$ ) for all  $x \in I$ , then  $f$  is strictly increasing (convex) on  $I$ . Even though the condition weakens to holding except at finitely many points, the conclusion of strict monotonicity (convexity) still holds. For example,  $f(x) = x^3$  is strictly increasing on  $\mathbb{R}$  despite  $f'(0) = 0$ .

#### Argmax and Argmin

**Definition 4.5 (Stationary Point)**

Stationary points are points where the first derivative of a function is zero or non-existent.



Stationary points can be classified into three types:

**Argmax and argmin points** Points where the function attains its maximum or minimum values.

**Inflection points** Points where the function changes concavity.

**Trivial points** Points that are neither local maxima nor local minima.

**¶ Asymptote**

## 4.7 Applications

# Chapter 5 Indefinite Integral

## 5.1 Two Common Integration Methods

### ¶ Integration Methods

#### *Definition 5.1 (Integration by Parts)*

Let  $u(x)$  and  $v(x)$  be two differentiable functions, and at least one of them has an antiderivative. Then the integration by parts formula states that:

$$\int u \, dv = uv - \int v \, du.$$



#### *Definition 5.2 (Substitution Method)*



Some common substitutions are as follows:

**Trigonometric Substitution** When restoring variables, auxiliary right triangles is often utilized.

**Sine**  $\sqrt{a^2 - x^2}$ :  $x = a \sin t$  or  $x = a \cos t$

**Tangent**  $\sqrt{a^2 + x^2}$ :  $x = a \tan t$  or  $x = a \sinh t$

**Secant**  $\sqrt{x^2 - a^2}$ :  $x = a \sec t$  or  $x = a \cosh t$

**Irreational Substitution** • If the integrand contains  $\sqrt[n]{x}$ , one can use the substitution  $t = \sqrt[n]{x}$  to simplify the expression.

• If the integrand contains  $\sqrt[n]{\frac{\alpha x + \beta}{\gamma x + \delta}}$ , one can use the substitution  $t = \sqrt[n]{\frac{\alpha x + \beta}{\gamma x + \delta}}$  to simplify the expression.

**Reciprocal Substitution** If the degree of the numerator is lower than that of the denominator according to  $x$  one can use the substitution  $x = \frac{1}{t}$  to reduce the degree.

### ¶ Basic Integration Formulas

Integral	Result
$\int a \, dx$	$ax + C$ (a is constant)
$\int x^n \, dx$	$\frac{x^{n+1}}{n+1} + C$ ( $n \neq -1$ )
$\int \frac{1}{x} \, dx$	$\ln  x  + C$
$\int e^x \, dx$	$e^x + C$
$\int a^x \, dx$	$\frac{a^x}{\ln a} + C$ ( $a > 0, a \neq 1$ )
$\int \sin x \, dx$	$-\cos x + C$
$\int \cos x \, dx$	$\sin x + C$
$\int \tan x \, dx$	$-\ln  \cos x  + C$
$\int \cot x \, dx$	$\ln  \sin x  + C$
$\int \sec x \, dx$	$\ln  \sec x + \tan x  + C$
$\int \csc x \, dx$	$\ln  \csc x - \cot x  + C$
$\int \sec x \tan x \, dx$	$\sec x + C$
$\int \csc x \cot x \, dx$	$-\csc x + C$
$\int \sec^2 x \, dx$	$\tan x + C$
$\int \csc^2 x \, dx$	$-\cot x + C$
$\int \frac{1}{\sqrt{a^2-x^2}} \, dx$	$\arcsin\left(\frac{x}{a}\right) + C$
$\int \frac{-1}{\sqrt{a^2-x^2}} \, dx$	$\arccos\left(\frac{x}{a}\right) + C$
$\int \frac{1}{a^2+x^2} \, dx$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
$\int \frac{-1}{a^2+x^2} \, dx$	$\frac{1}{a} \operatorname{arccot}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{x^2+a^2}} \, dx$	$\ln x+\sqrt{x^2+a^2}  + C$
$\int \frac{1}{\sqrt{x^2-a^2}} \, dx$	$\ln x+\sqrt{x^2-a^2}  + C$ ( $x > a$ or $x < -a$ )
$\int \sinh x \, dx$	$\cosh x + C$
$\int \cosh x \, dx$	$\sinh x + C$

# Chapter 6 Definite Integral

## 6.1 Riemann Integral

### ¶ Riemann Integral

#### Definition 6.1 (Riemann Integral)

Let  $f(x)$  be a bounded function defined on  $[a, b]$ . Take any set of division points  $\{x_i\}_{i=0}^n$  on  $[a, b]$  to form a partition  $P : a = x_0 < x_1 < \dots < x_n = b$ , and choose arbitrary points  $\xi_i \in [x_{i-1}, x_i]$ . Denote the length of the sub-interval  $[x_{i-1}, x_i]$  as  $\Delta x_i = x_i - x_{i-1}$ , and let  $\lambda = \max_{1 \leq i \leq n} (\Delta x_i)$ . If the limit

$$\lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i$$

exists as  $\lambda \rightarrow 0$ , and the limit is independent of the partition  $P$  and the choice of  $\xi_i$ , then  $f(x)$  is said to be **Riemann integrable** on  $[a, b]$ .

The summation

$$S_n = \sum_{i=1}^n f(\xi_i) \Delta x_i$$

is called the Riemann sum, and its limit  $I$  is called the definite integral of  $f(x)$  on  $[a, b]$ , denoted as:

$$I = \int_a^b f(x) dx,$$

where  $a$  and  $b$  are called the lower and upper limits of the definite integral, respectively.

Alternatively, it can also be expressed as:

$$\exists I, \forall \varepsilon > 0, \exists \delta > 0, \text{s.t. } \forall P(\lambda = \max_{1 \leq i \leq n} (\Delta x_i) < \delta), \forall \{\xi_i\} : \left| \sum_{i=1}^n f(\xi_i) \Delta x_i - I \right| < \varepsilon.$$

Then  $f(x)$  is said to be Riemann integrable on  $[a, b]$ , and  $I$  is the definite integral of  $f(x)$  on  $[a, b]$ .



★ **Remark** Partition → Intermediate points → Summation → Take the limit.

### ¶ Darboux Sum

#### Definition 6.2 (Darboux Sum)

Let the supremum and infimum of  $f(x)$  on  $[a, b]$  be  $M$  and  $m$ , respectively. Clearly,  $m \leq f(x) \leq M$ . Let the supremum and infimum of  $f(x)$  on  $[x_{i-1}, x_i]$  be  $M_i$  and  $m_i$  ( $i = 1, 2, \dots, n$ ), respectively, i.e.,

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}, \quad m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

After fixing the partition  $P$ , define the sums:

$$\bar{S}(P) = \sum_{i=1}^n M_i \Delta x_i, \quad \underline{S}(P) = \sum_{i=1}^n m_i \Delta x_i,$$

which are called the Darboux upper sum and Darboux lower sum corresponding to the partition  $P$ , respectively.



### ⊖ Property

1.  $\underline{S}(P) \leq \sum_{i=1}^n f(\xi_i) \Delta x_i \leq \bar{S}(P)$ .
2. *If a new partition is formed by adding division points to the original partition, the upper sum does not increase, and the lower sum does not decrease.*

3. Let  $\bar{S}$  denote the set of Darboux upper sums and  $S$  denote the set of Darboux lower sums. For any  $\bar{S}(P_1) \in \bar{S}, S(P_2) \in S$ , it always holds that:

$$m(b-a) \leq S(P_2) \leq \bar{S}(P_1) \leq M(b-a).$$

4. Let  $L = \inf\{\bar{S}(P) \mid \bar{S}(P) \in \bar{S}\}, l = \sup\{S(P) \mid S(P) \in S\}$ , which are called the upper integral and lower integral, respectively. It always holds that:  $l \leq L$ .

5. **Darboux's Theorem:** For any  $f(x) \in B[a, b]$ , it always holds that:

$$\lim_{\lambda \rightarrow 0} \bar{S}(P) = L, \quad \lim_{\lambda \rightarrow 0} S(P) = l.$$

### ¶ Riemann-Stieltjes Integral

#### Definition 6.3 (Riemann-Stieltjes Integral)

Let  $\alpha$  be a bounded, monotonically increasing function on  $[a, b]$ . For every partition  $P$  of  $[a, b]$ , let  $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$  (clearly  $\Delta\alpha_i \geq 0$ ). For a bounded real function  $f(x)$  on  $[a, b]$ , define the Stieltjes upper sum and lower sum as:

$$\bar{S}(P, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i, \quad S(P, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i,$$

and define the upper and lower integrals as:

$$L = \inf\{\bar{S}(P, \alpha) \mid \bar{S}(P, \alpha) \in \bar{S}\}, \quad l = \sup\{S(P, \alpha) \mid S(P, \alpha) \in S\},$$

where  $\bar{S}, S$  are the sets of Stieltjes upper and lower sums respectively.

If  $L = l$ , then:

$$\int_a^b f(x) d\alpha(x) = L = l,$$

and  $f(x)$  is said to be **Riemann-Stieltjes integrable** on  $[a, b]$  with respect to  $\alpha$ , or simply Stieltjes integrable. ♣

When  $\alpha(x) = x$ , this reduces to the Riemann integral. However, in general,  $\alpha(x)$  does not even need to be continuous.

The properties of Darboux sums also apply to Stieltjes sums.

## 6.2 Integrability Criteria

### ¶ Common Integrability Criteria

#### Theorem 6.1 (Integrability Criterion)

A bounded function  $f(x)$  is Riemann integrable on  $[a, b]$  if and only if:

- The upper and lower integrals are equal, i.e.,

$$\forall P(\lambda = \max_{1 \leq i \leq n} (\Delta x_i) < \delta) : \lim_{\lambda \rightarrow 0} \bar{S}(P) = L = l = \lim_{\lambda \rightarrow 0} S(P).$$

- Let  $\omega_i = M_i - m_i$  be the oscillation of  $f(x)$  on  $[x_{i-1}, x_i]$ . Then: The limit of the sum of oscillations is zero, i.e.,

$$\forall P(\lambda = \max_{1 \leq i \leq n} (\Delta x_i) < \delta) : \lim_{\lambda \rightarrow 0} \sum_{i=1}^n \omega_i \Delta x_i = 0.$$

**Corollary 1** Continuous functions on closed intervals are necessarily integrable.

**Corollary 2** Monotonic functions on closed intervals are necessarily integrable.

- For all  $\varepsilon > 0$ , there exists a partition  $P$  such that:

$$\sum_{i=1}^n \omega_i \Delta x_i < \varepsilon.$$

**Corollary 1** The total length of intervals where oscillation  $\omega$  cannot be arbitrarily small can be made arbitrarily small, i.e.,

$$\forall \varepsilon, \eta > 0, \exists P, \text{s.t. } \sum_{\omega \geq \eta} \Delta x_i < \varepsilon.$$

**Corollary 2** Bounded functions with only finitely many discontinuities on closed intervals are necessarily integrable.



### Proof



## ¶ Lesbesgue's Theorem

### Definition 6.4 (Null Set)

A set  $E \subset \mathbb{R}$  is called a **null set** (or measure zero set) if for any  $\varepsilon > 0$ , there exists a countable collection of open intervals  $\{I_n | n \in \mathbb{N}^*\}$  such that:

$$E \subset \bigcup_{i=1}^{\infty} I_n \quad \text{and} \quad \sum_{i=1}^{\infty} |I_n| < \varepsilon.$$



If some property holds for all  $x \in A$  except for a null set  $E \subset A$ , we say that the property holds **almost everywhere** on  $A$ .

### Lemma 6.1

- Let  $\omega$  be the oscillation of bounded function  $f(x)$  on  $[a, b]$ , then:

$$\omega = \sup\{f(y_1) - f(y_0) | y_0, y_1 \in [a, b]\}.$$

- $f(x)$  is continuous at point  $x_0$  if and only if the oscillation of  $f(x)$  at  $x_0$  is zero, i.e.,  $\omega_f(x_0) = 0$ .
- Let  $D(f)$  be the set of discontinuities of bounded function  $f(x)$  on  $[a, b]$ . For  $\delta > 0$ , denote  $D_\delta = \{x \in [a, b] | \omega_f(x) \geq \delta\}$ . Then

$$D(f) = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}.$$

- If there exists a series of open intervals  $(\alpha_j, \beta_j)$  ( $j = 1, 2, \dots$ ) such that  $D(f) \subset \bigcup_{j=1}^{\infty} (\alpha_j, \beta_j)$ , and let  $K = [a, b] \setminus \bigcup_{j=1}^{\infty} (\alpha_j, \beta_j)$ . Then:

$$\forall \varepsilon > 0, \exists \delta > 0, \text{s.t. } \forall x \in K, y \in [a, b] (|x - y| < \delta) : |f(x) - f(y)| < \varepsilon.$$



### Theorem 6.2 (Lesbesgue's Theorem)

Let  $f(x) \in B[a, b]$ , then  $f(x)$  is Riemann integrable on  $[a, b]$  if and only if  $f(x)$  is continuous almost everywhere on  $[a, b]$ .



## 6.3 Properties of Definite Integrals

## ¶ Properties of Riemann Integrals

### Property

**Linearity** Let  $f(x), g(x) \in R[a, b]$ , and  $k_1, k_2$  are constants. Then the function  $k_1 f(x) + k_2 g(x) \in R[a, b]$ , and:

$$\int_a^b [k_1 f(x) + k_2 g(x)] dx = k_1 \int_a^b f(x) dx + k_2 \int_a^b g(x) dx.$$

**Multiplicative Integrability** Let  $f(x), g(x) \in R[a, b]$ , and  $k_1, k_2$ . Then  $f(x) \cdot g(x) \in R[a, b]$ . In general,

$$\int_a^b f(x)g(x) dx \neq \left( \int_a^b f(x) dx \right) \cdot \left( \int_a^b g(x) dx \right).$$

**Monotonicity** Let  $f(x), g(x) \in R[a, b]$ , and  $f(x) \geq g(x)$  (or  $f(x) > g(x)$ ) on  $[a, b]$ . Then:

$$\int_a^b f(x) dx \geq \int_a^b g(x) dx \quad \left( \int_a^b f(x) dx > \int_a^b g(x) dx \right).$$

**Corollary 1** If  $f(x) \in C[a, b]$ ,  $f(x) \geq 0$ ,  $f(x) \not\equiv 0$ , then:

$$\int_a^b f(x) dx > 0.$$

**Corollary 2** If  $f(x) \in R[a, b]$ ,  $f(x) > 0$ , then:

$$\int_a^b f(x) dx > 0.$$

**Absolute Value Integrability** Let  $f(x) \in R[a, b]$ . Then  $|f(x)| \in R[a, b]$ , and:

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

The inverse statement of this property is not true.

**Additivity Over Intervals** Let  $f(x) \in R[a, b]$ . For any point  $c \in [a, b]$ ,  $f(x)$  is integrable on  $[a, b]$  and  $[c, d]$ . Conversely, if  $f \in R[a, c] \cup [c, b]$ , then  $f(x)$  is integrable on  $[a, b]$ , and:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

### Theorem 6.3 (Integral Mean Value Theorem)

**First Integral Mean Value Theorem** Let  $f(x), g(x) \in R[a, b]$ , and  $g(x)$  does not change sign on  $[a, b]$ .

Then there exists  $\eta \in [m, M]$  such that:

$$\int_a^b f(x)g(x) dx = \eta \int_a^b g(x) dx,$$

where  $m, M$  represent the infimum and supremum of  $f(x)$  on  $[a, b]$ , respectively.

In particular, if  $f(x) \in C[a, b]$ , then there exists  $\xi \in [a, b]$  such that:

$$\int_a^b f(x)g(x) dx = f(\xi) \int_a^b g(x) dx.$$

A special case arises when  $f(x) \in C[a, b]$  and  $g(x) \equiv 1$ , then:

$$\int_a^b f(x) dx = f(\xi) \int_a^b 1 dx.$$

**Corollary** If  $f(x) \in C[a, b]$ , then there exists  $\xi \in (a, b)$  such that:

$$\int_a^b f(x) dx = f(\xi) \int_a^b 1 dx.$$

**Second Integral Mean Value Theorem (Bonnet Formula)** Let  $f(x) \in R[a, b]$ ,

- If  $g(x)$  is decreasing on  $[a, b]$  and  $g(x) \geq 0$  ( $x \in [a, b]$ ):

$$\exists \xi \in [a, b] : \int_a^b f(x)g(x) dx = g(a) \int_a^\xi f(x) dx.$$

- If  $g(x)$  is increasing on  $[a, b]$  and  $g(x) \geq 0$  ( $x \in [a, b]$ ):

$$\exists \eta \in [a, b] : \int_a^b f(x)g(x)dx = g(b) \int_\eta^b f(x)dx.$$

The general form is: Let  $f(x) \in R[a, b]$ , and  $g(x)$  be a monotonic function. Then:

$$\exists \xi \in [a, b], \int_a^b f(x)g(x)dx = g(a) \int_a^\xi f(x)dx + g(b) \int_\xi^b f(x)dx.$$



**Note** For the first integral mean value theorem,

- If  $f(x) \in C[a, b]$  is replaced with  $f(x) \in R[a, b]$ , the conclusion does not hold.
- If  $f(x) \in R[a, b]$  and  $\int f(x)dx$  exists, the conclusion holds.

#### ¶ Integrability of Composite Functions

**Outer Continuity, Inner Integrability** Let  $f(x) \in R[a, b]$ ,  $A \leq f(x) \leq B$ , and  $g(u) \in C[A, B]$ . Then the composite function  $g(f(x)) \in R[a, b]$ .

**Outer Integrability, Inner Continuity** In this case, the composite function may not be integrable.

**Both Inner and Outer Integrability** In this case, the composite function may not be integrable. In fact, even if both the inner and outer functions are not integrable, the composite function may still be integrable.

## 6.4 Fundamental Theorem of Calculus

#### ¶ Newton-Leibniz Formula

**Definition 6.5 (Variable Limit Integrals)**

Let  $f(x) \in R[a, b]$ . Define:

$$F(x) = \int_a^x f(t) dt \quad \text{and} \quad F(x) = \int_x^b f(t) dt,$$

which are referred to as the variable upper limit integral and variable lower limit integral, respectively.



#### ¶ Property

**Continuity of Antiderivative**  $F(x) \in C[a, b]$  (The variable upper limit integral satisfies the Lipschitz condition and is uniformly continuous on the closed interval).

**Fundamental Theorem of Calculus** Let  $x_0 \in [a, b]$  be a point where  $f(x)$  is continuous. Then:

$$F'(x_0) = f(x_0).$$

**Existence of Antiderivatives** If  $f(x) \in C[a, b]$ , then  $F(x) \in D[a, b]$  and  $F'(x) = f(x)$ .

**Rule of Derivation** If  $F(x) = \int_{u(x)}^{v(x)} f(t) dt$ , then:

$$F'(x) = f(v(x))v'(x) - f(u(x))u'(x).$$

In fact, the formula is the simplified version of the **Leibniz's law**.

**Remark** Differentiation can reduce the smoothness of functions (the original function may be differentiable, while the derivative may have second-type discontinuities), whereas integration can improve smoothness.

**Theorem 6.4 (Newton-Leibniz Formula)**

Let  $f(x) \in C[a, b]$ , and  $F(x)$  be an antiderivative of  $f(x)$  on  $[a, b]$ . Then:

$$\int_a^b f(x) dx = F(b) - F(a).$$

**Generalized Newton-Leibniz Formula** Let  $f(x) \in R[a, b]$ ,  $F(x) \in C[a, b]$ , and  $F'(x) = f(x)$  holds except for finitely many points. Then:

$$\int_a^b f(x) dx = F(b) - F(a).$$



## Common Questions concerning Integrals

## 6.5 Calculation of Definite Integrals

**Example 6.1** Prove the ignition formula (Wallis formula) with recursion method:

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & n \text{ is even;} \\ \frac{(n-1)!!}{n!!}, & n \text{ is odd.} \end{cases}$$

## 6.6 Integral Inequalities

**Theorem 6.5 (Integral Inequalities)**

**Hadamard Inequality** Let  $f(x)$  be a convex function on  $(a, b)$ . Then for any pair  $x_1, x_2 \in (a, b)$  with  $x_1 < x_2$ , we have:

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(t) dt \leq \frac{f(x_1) + f(x_2)}{2}.$$

**Schwarz Inequality** Let  $f(x), g(x) \in R[a, b]$ . Then:

$$\left( \int_a^b f(x)g(x) dx \right)^2 \leq \int_a^b f^2(x) dx \int_a^b g^2(x) dx.$$

**Hölder Inequality** Let  $f(x), g(x) \in R[a, b]$ , and  $p, q$  are conjugate numbers (i.e.,  $p > 0, q > 0, \frac{1}{p} + \frac{1}{q} = 1$ ).

Then:

$$\int_a^b |f(x)g(x)| dx \leq \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_a^b |g(x)|^q dx \right)^{\frac{1}{q}}.$$

**Young Inequality** Let  $y = f(x) \in C[0, +\infty)$ , strictly increasing, and  $f(0) = 0$ . Denote its inverse function as  $x = f^{-1}(y)$ . Then:

$$\int_0^a f(x) dx + \int_0^b f^{-1}(y) dy \geq ab \quad (a > 0, b > 0).$$

**Minkowski Inequality** Let  $f(x), g(x) \in R[a, b]$ . Then:

$$\left\{ \int_a^b [f(x) + g(x)]^2 dx \right\}^{\frac{1}{2}} \leq \left[ \int_a^b f^2(x) dx \right]^{\frac{1}{2}} + \left[ \int_a^b g^2(x) dx \right]^{\frac{1}{2}}.$$

**Чебышёв Inequality** Let  $f(x), g(x)$  be similarly ordered functions, i.e.,  $\forall x_1, x_2 : (f(x_1) - f(x_2))(g(x_1) - g(x_2)) \geq 0$ .

$g(x_2)) \geq 0$ . Then:

$$\int_a^b f(x) dx \int_a^b g(x) dx \leq (b-a) \int_a^b f(x)g(x) dx.$$

**Discrete Form** Let sequences  $\{a_n\}, \{b_n\}$  be similarly ordered, i.e.,  $\forall i, j : (a_i - a_j)(b_i - b_j) \geq 0$ . Then:

$$\left( \sum_{i=1}^n a_i \right) \left( \sum_{i=1}^n b_i \right) \leq n \sum_{i=1}^n a_i b_i.$$

If the sequences are oppositely ordered, the inequality reverses.



**Example 6.2** Let  $f(t)$  be convex on  $[0, 1]$ , prove that:

$$\int_0^1 t(1-t)f(t) dt \leq \frac{1}{3} \int_0^1 (t^3 + (1-t)^3) f(t) dt.$$

Proof Since  $f(t)$  is convex on  $[0, 1]$ , for any  $t \in (0, 1)$ , we have:

$$t = (1-t)(tx) + t(1-x+tx),$$

then

$$f(t) \leq (1-t)f(tx) + tf(1-x+tx).$$

Integrating both sides from 0 to 1 with respect to  $x$ , we get:

$$f(t) \leq (1-t) \int_0^1 f(tx) dx + t \int_0^1 f(1-x+tx) dx = \frac{1-t}{t} \int_0^t f(x) dx + \frac{t}{1-t} \int_t^1 f(x) dx.$$

Multiplying both sides by  $t(1-t)$  and integrating from 0 to 1 with respect to  $t$ , we have:

$$\int_0^1 t(1-t)f(t) dt \leq \int_0^1 \left[ (1-t)^2 \int_0^t f(x) dx \right] dt + \int_0^1 t^2 \left[ \int_t^1 f(x) dx \right] dt.$$

Change the order of integration in the right side:

$$\int_0^1 \left[ (1-t)^2 \int_0^t f(x) dx \right] dt + \int_0^1 t^2 \left[ \int_t^1 f(x) dx \right] dt = \frac{1}{3} \int_0^1 (t^3 + (1-t)^3) f(t) dt.$$

Thus, the desired inequality is proven. ■

## 6.7 Applications of Definite Integrals

### Arc Length

*Definition 6.6 (Arc Length)*

Let  $C = \widehat{AB}$  be a curve on the  $\mathbb{R}^2$  plane<sup>a</sup>, take any partition  $A = P_0, P_1, \dots, P_n = B$ , which divides the curve  $C$  into  $n$  segments, denoted as  $T$ . Then connect every two adjacent points  $P_{i-1}$  and  $P_i$  with a straight line segment, obtaining  $n$  chords  $\overline{P_{i-1}P_i}$  ( $i = 1, 2, \dots, n$ ), which in turn form an inscribed polygonal line  $C$ .

Let

$$\|T\| = \max_{1 \leq i \leq n} \|P_{i-1}P_i\|, \quad s_T = \sum_{i=1}^n \|P_{i-1}P_i\|.$$

If the limit

$$\lim_{\|T\| \rightarrow 0} s_T = s,$$

namely,

$$\forall \varepsilon > 0, \exists \delta > 0, \text{s.t. } \forall T (\|T\| < \delta) : |s_T - s| < \varepsilon,$$

and the limit is independent of the choice of partition  $T$ , then  $C$  is said to be rectifiable, and the limit  $s$  is called the arc length of the curve  $C$ .

<sup>a</sup>Or in  $\mathbb{R}^3$  space, even in a higher-dimensional Euclidean space.



#### Theorem 6.6 (Sufficient Condition for Rectifiability of Curves)

Let the curve  $C$  in  $\mathbb{R}^2$  be given by the parametric equations

$$(x, y) = (x(t), y(t)), \quad t \in [\alpha, \beta],$$

and let it be a  $C^1$  smooth regular curve<sup>a</sup> Then  $C$  is rectifiable, and its arc length is

$$s = \int_{\alpha}^{\beta} \sqrt{x'^2(t) + y'^2(t)} dt.$$

<sup>a</sup>I.e.,  $x(t)$  and  $y(t)$  are continuously differentiable, and  $x'^2(t) + y'^2(t) \neq 0$ ; a curve  $C$  satisfying this condition is called a regular point. Also see Definition 12.5



## ¶ Curvature

#### Definition 6.7 (Curvature)



## ¶ Polar Coordinate System

Category	Explicit Cartesian Equation	Parametric Cartesian Equation	Polar Equation
Equation	$y = f(x), x \in [a, b]$	$\begin{cases} x = x(t), t \in [T_1, T_2], \\ y = y(t), \end{cases}$	$r = r(\theta), \theta \in [\alpha, \beta]$
Area of Plane Shape	$\int_a^b f(x) dx$	$\int_{T_1}^{T_2}  y(t)x'(t)  dt$	$\frac{1}{2} \int_{\alpha}^{\beta} r^2(\theta) d\theta$
Infinitesimal Arc Length	$dl = \sqrt{1 + [f'(x)]^2} dx$	$dl = \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$	$dl = \sqrt{r^2(\theta) + r'^2(\theta)} d\theta$
Curve Length	$\int_a^b \sqrt{1 + [f'(x)]^2} dx$	$\int_{T_1}^{T_2} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$	$\int_{\alpha}^{\beta} \sqrt{r^2(\theta) + r'^2(\theta)} d\theta$
Volume of Solid of Revolution	$\pi \int_a^b [f(x)]^2 dx$	$\pi \int_{T_1}^{T_2} y^2(t)x'(t) dt$	$\frac{2}{3}\pi \int_{\alpha}^{\beta} r^3(\theta) \sin \theta d\theta$
Surface Area of Solid of Revolution	$2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx$	$2\pi \int_{T_1}^{T_2} y(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$	$2\pi \int_{\alpha}^{\beta} r(\theta) \sin \theta \sqrt{r^2(\theta) + r'^2(\theta)} d\theta$

# Chapter 7 Improper Integral

## 7.1 Infinite and Defective Integrals

## 7.2 Convergence Tests for Improper Integrals

### Definition 7.1 (Absolute and Conditional Convergence)

Let  $f(x) \in R[a, A] \subset [a, +\infty)$ , and suppose  $\int_a^{+\infty} |f(x)| dx$  converges. Then  $\int_a^{+\infty} f(x) dx$  is said to be **absolutely convergent** (or  $f(x)$  is **absolutely integrable** on  $[a, +\infty)$ ).

If  $\int_a^{+\infty} f(x) dx$  converges but is not absolutely convergent, then  $\int_a^{+\infty} f(x) dx$  is said to be **conditionally convergent**.



### ¶ Infinite Integrals

### Theorem 7.1 (Cauchy Convergence Criterion for Infinite Integrals)

The necessary and sufficient condition for the convergence of the infinite integral  $\int_a^{+\infty} f(x) dx$  is:

$$\forall \varepsilon > 0, \exists A_0 > \max\{a, 0\}, \forall A', A'' > A_0 : \left| \int_a^{A'} f(x) dx - \int_a^{A''} f(x) dx \right| = \left| \int_{A'}^{A''} f(x) dx \right| < \varepsilon.$$



From this, we can conclude that if  $\int_a^{+\infty} f(x) dx$  is absolutely convergent, then it must be convergent.

### Theorem 7.2 (Comparison Test for Infinite Integrals)

**Comparison Test** Let  $f(x), g(x)$  be functions defined on  $[a, +\infty)$ , and suppose  $f(x) \leq K g(x)$  (where  $K$  is a positive constant). Then:

- If  $\int_a^{+\infty} g(x) dx$  converges, then  $\int_a^{+\infty} f(x) dx$  also converges.
- If  $\int_a^{+\infty} f(x) dx$  diverges, then  $\int_a^{+\infty} g(x) dx$  also diverges.

**Limit Form** Let  $f(x), g(x) > 0$  be functions defined on  $[a, +\infty)$ , and suppose:

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = l.$$

Then:

- If  $0 < l < +\infty$ , and  $\int_a^{+\infty} g(x) dx$  converges, then  $\int_a^{+\infty} f(x) dx$  also converges.
- If  $0 < l < +\infty$ , and  $\int_a^{+\infty} g(x) dx$  diverges, then  $\int_a^{+\infty} f(x) dx$  also diverges.
- If  $l = +\infty$ ,  $\int_a^{+\infty} g(x) dx$  and  $\int_a^{+\infty} f(x) dx$  both converge or both diverge.

**Comparison with p-Integrals** Let  $f(x) > 0$  be defined on  $[a, +\infty)$ , and suppose:

$$\frac{f(x)}{x^p} \leq \frac{K}{x^p}, \quad \text{where } p > 0.$$

Then:

- If  $p > 1$ , then  $\int_a^{+\infty} f(x) dx$  converges.
- If  $p \leq 1$ , then  $\int_a^{+\infty} f(x) dx$  diverges.

**Limit Form** Let  $f(x) > 0$  be defined on  $[a, +\infty)$ , and suppose:

$$\lim_{x \rightarrow +\infty} x^p f(x) = l, \quad \text{where } l > 0.$$

Then:

- i) If  $p > 1$ , then  $\int_a^{+\infty} f(x) dx$  converges.
- ii) If  $p \leq 1$ , then  $\int_a^{+\infty} f(x) dx$  diverges.



### Theorem 7.3 (Abel-Dirichlet Test)

The infinite integral  $\int_a^{+\infty} f(x)g(x) dx$  converges if either of the following two conditions is satisfied:

**Abel**  $\int_a^{+\infty} f(x) dx$  converges, and  $g(x)$  is monotonic and bounded on  $[a, +\infty)$ .

**Dirichlet**  $F(A) = \int_a^A f(x) dx$  is bounded on  $[a, +\infty)$ ,  $g(x)$  is monotonic on  $[a, +\infty)$ , in the meanwhile  $\lim_{x \rightarrow +\infty} g(x) = 0$ .



## ¶ Defective Integrals

### ¶ Examples

**Example 7.1** Discuss the convergence of the following improper integrals:

1.

$$\int_0^{+\infty} \frac{\sin x}{x^p} dx$$

2.

$$\int_0^{+\infty} \frac{\sin x}{x^p + \sin x} dx$$

3.

$$\int_0^1 \frac{1}{x^p \ln x} dx$$

4.

$$\int_0^{+\infty} \frac{1}{x^p} \sin \frac{1}{x} dx$$

## 7.3 Special Integrals

### ¶ Definite Integrals

#### Dirichlet Integral

$$\int_0^\pi \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} dx = \pi, \quad n \in \mathbb{N},$$

where integrand  $D_n(x)$  is called the Dirichlet kernel.

#### Fejér Integral

$$\int_0^\pi \left( \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \right)^2 dx = n\pi, \quad n \in \mathbb{N},$$

### ¶ Improper Integrals

#### Euler Integral

$$\int_0^{\frac{\pi}{2}} \ln \sin x dx = -\frac{\pi}{2} \ln 2.$$

**Froullani Integral**

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} dx = [f(0) - f(+\infty)] \ln \frac{b}{a}, \quad a, b > 0,$$

where  $f(x)$  is continuous on  $(0, +\infty)$ , and both limits  $f(0)$  and  $f(+\infty)$  exist.

**Dirichlet Integral**

$$\int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

**Euler-Poisson Integral**

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

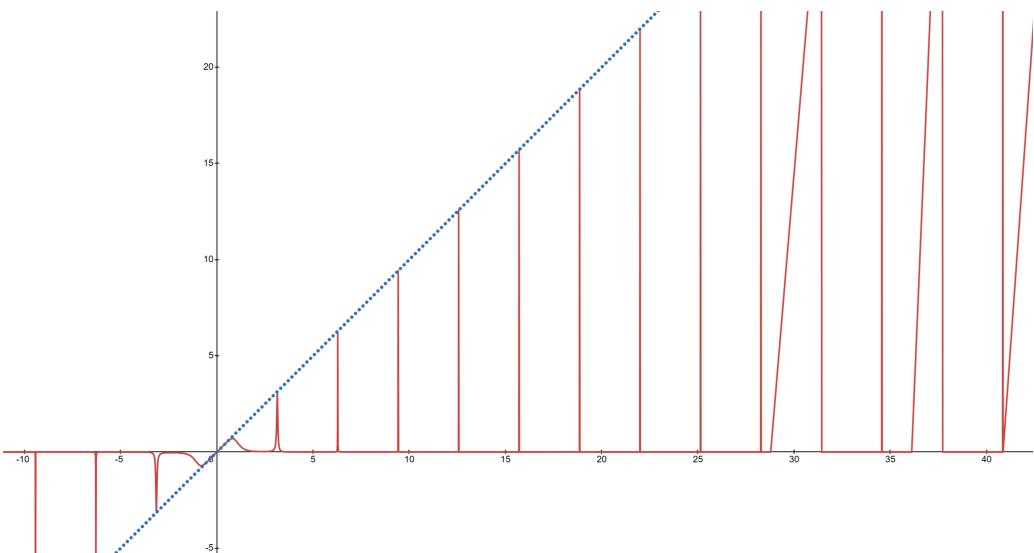
**Poisson Integral**

$$\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos x + r^2} dx, \quad (0 < r < 1)$$

**Special Integral**

$$\int_0^{+\infty} \frac{1}{1 + x^a \sin^b x} dx \quad (a > b, b > 0 \text{ and even})$$

When  $a = 6, b = 2$ , the figure is shown as Fig 7.1.



**Figure 7.1:** Graph of  $y = \frac{1}{1+x^6 \sin^2 x}$

## 7.4 Common Questions

### ¶ Square Integrable

*Definition 7.2 (Square Integrable Function)*

If  $f(x) \in R[a, +\infty)$  and  $\int_a^{+\infty} [f(x)]^2 dx$  converges, then  $f(x)$  is called a **square integrable function** on  $[a, +\infty)$ . For defective integrals, the definition is similar.



### 🔗 Property

### ¶ Properties of the Integrand of the Convergent Infinite Integral at Infinity

For the infinite integral

$$\int_0^{+\infty} \frac{1}{1+x^6 \sin^2 x} dx,$$

whose integrand is shown in Fig 7.1, we can deduce that even if the integral converges,  $f(+\infty)$  is not necessarily equal to 0. Moreover, it is possible that  $\overline{\lim}_{x \rightarrow +\infty} f(x) = +\infty$ .

# Chapter 8 Numerical Series

## 8.1 Convergence of Numerical Series

## 8.2 Positive Term Series and Its Convergence Tests

### Definition 8.1 (Positive Term Series)

If all terms of the series  $\sum_{n=1}^{\infty} x_n$  are non-negative real numbers, i.e.,  $x_n \geq 0$  ( $x_n > 0$ ),  $n = 1, 2, \dots$ , then this series is called a **positive term series** (or strictly positive term series).



**Note** The positive term series converges if and only if the partial sums of the sequence are bounded. If the partial sums are unbounded, the series must diverge to  $+\infty$ .

### Comparison Test

#### Theorem 8.1 (Comparison Test)

Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be positive term series. If  $\exists N \in \mathbb{N}$ , s.t.  $\forall n > N : a_n \leq b_n$ , then:

1. If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  also converges.
2. If  $\sum_{n=1}^{\infty} a_n$  diverges, then  $\sum_{n=1}^{\infty} b_n$  also diverges.

**Limit Form** Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be positive term series, and suppose  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists. Then:

1. If  $0 < l < +\infty$ ,  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  have the same convergence or divergence behavior.
2. If  $l = 0$ ,  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  also converges.
3. If  $l = +\infty$ ,  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  also diverges.



#### Theorem 8.2

**Cauchy Test** Let  $\sum_{n=1}^{\infty} a_n$  be a positive term series.

1. If  $\exists q \in [0, 1)$ , s.t.  $\sqrt[n]{a_n} \leq q < 1$  ( $n \geq N, N \in \mathbb{N}$ ), then the series converges.
2. If  $\sqrt[n]{a_n} \geq 1$  for infinitely many  $n$ , then the series diverges.

**Limit Form** Let  $\sum_{n=1}^{\infty} a_n$  be a positive term series, and suppose  $\overline{\lim}_{n \rightarrow +\infty} \sqrt[n]{a_n} = r$ . Then:

1. If  $0 \leq r < 1$ , the series  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $r > 1$ , the series  $\sum_{n=1}^{\infty} a_n$  diverges.
3. If  $r = 1$ , the test fails.

**D'Alembert Test** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series.

1. If  $\exists q \in [0, 1)$ , s.t.  $\frac{a_{n+1}}{a_n} \leq q < 1$  ( $n \geq N, N \in \mathbb{N}$ ), then the series converges.
2. If  $\frac{a_{n+1}}{a_n} \geq 1$  ( $n \geq N, N \in \mathbb{N}$ ), then the series diverges.

**Limit Form** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series. Then:

1. If  $\overline{\lim}_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = r \in (0, 1)$ , the series converges.
2. If  $\underline{\lim}_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = r' > 1$ , the series diverges.
3. If  $r = 1$  or  $r' = 1$ , the test fails.

**Raabe Test** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series.

1. If  $\exists r > 1, \exists N_0 \in \mathbb{N}$  s.t.  $\forall n > N_0 : n \left( \frac{a_n}{a_{n+1}} - 1 \right) \geq r$ , then the series converges.
2. If  $\exists N_0 \in \mathbb{N}$ , s.t.  $\forall n > N_0 : n \left( \frac{a_n}{a_{n+1}} - 1 \right) \leq 1$ , then the series diverges.

**Limit Form** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series. Then:

1. If  $\underline{\lim}_{n \rightarrow +\infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) = l > 1$ , the series converges.
2. If  $\overline{\lim}_{n \rightarrow +\infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) = l' < 1$ , the series diverges.
3. If  $l = 1$  or  $l' = 1$ , the test fails.

**Bertrand Test** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series.

1. If  $\underline{\lim}_{n \rightarrow +\infty} \ln n \left[ n \left( \frac{a_n}{a_{n+1}} - 1 \right) \right] = l > 1$ , the series converges.
2. If  $\overline{\lim}_{n \rightarrow +\infty} \ln n \left[ n \left( \frac{a_n}{a_{n+1}} - 1 \right) \right] = l' < 1$ , the series diverges.
3. If  $l = 1$  or  $l' = 1$ , the test fails.

**Gauß Test** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series, and suppose:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\delta}{n \ln n} + o\left(\frac{1}{n \ln n}\right), \quad (n \rightarrow +\infty).$$

Then:

1. If  $\delta > 1$ , the series converges.
2. If  $\delta < 1$ , the series diverges.
3. If  $\delta = 1$ , the criterion fails.

**Generalized Form** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series, and suppose:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\delta_n}{n \ln n} + o\left(\frac{1}{n \ln n}\right), \quad (n \rightarrow +\infty).$$

If  $\lim_{n \rightarrow \infty} \delta_n = \delta \in \mathbb{R}$ , then:

1. If  $\delta > 1$ , the series converges.
2. If  $\delta < 1$ , the series diverges.
3. If  $\delta = 1$ , the criterion fails.



**Note** The Bertrand test can be refined by considering series such as:

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p}, \quad \sum_{n=9}^{\infty} \frac{1}{n \ln n \ln \ln n (\ln \ln n)^p}, \dots$$

These refinements are collectively known as the Bertrand test.

**Remark** All the aforementioned criteria are derived from the Comparison Criterion.

- By comparing positive term series with the geometric series (or equal ratio series), the Cauchy Criterion and d'Alembert Criterion are derived.
- By comparing positive term series with the slower-converging series  $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$  ( $\alpha > 1$ ), the Raabe Criterion is derived.
- By comparing positive term series with the even slower-converging series  $\sum_{n=1}^{\infty} \frac{1}{n \ln^{\alpha} n}$  ( $\alpha > 1$ ), the Gauß Criterion is derived.

**General Observation** The slower the convergence of the series used for comparison, the more precise the derived criterion.

## Integral Test

**Theorem 8.3 (Cauchy Integral Test)**

Let  $f(x)$  be defined on  $[a, +\infty)$ , where  $f(x) \geq 0$ , and  $f(x)$  is Riemann integrable on any finite interval  $[a, A]$ . Consider a monotonic increasing sequence  $\{a_n\}$  such that  $a = a_1 < a_2 < \dots < a_n < \dots$ , and let:

$$u_n = \int_{a_n}^{a_{n+1}} f(x) dx.$$

Then the improper integral  $\int_a^{+\infty} f(x) dx$  and the positive term series  $\sum_{n=1}^{\infty} u_n$  converge or diverge to  $+\infty$  simultaneously. Moreover:

$$\int_a^{+\infty} f(x) dx = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} f(x) dx.$$

**¶ Other Tests****Theorem 8.4 (Cauchy Condensation Test)**

Let  $\{a_n\}$  be a monotonically decreasing sequence of positive numbers. Then the positive term series  $\sum_{n=1}^{\infty} a_n$  converges if and only if the condensed series:

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + \dots + 2^n a_{2^n} + \dots$$

converges.



## 8.3 General Term Series and Its Convergence Tests

**¶ Cauchy Convergence Criterion for Series****Theorem 8.5 (Cauchy Convergence Criterion for Series)**

The necessary and sufficient condition for the convergence of the series  $\sum_{n=1}^{\infty} x_n$  is:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N : |x_{n+1} + x_{n+2} + \dots + x_m| = \left| \sum_{k=n+1}^m x_k \right| < \varepsilon.$$

**¶ Alternative Series****Definition 8.2 (Alternative Series)**

A series of the form:

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (-1)^{n-1} u_n \quad (u_n > 0),$$

is called an **alternative series**.

Moreover, if  $u_n$  is a monotonically decreasing sequence and  $\lim_{n \rightarrow \infty} u_n = 0$ , then the series is called a **Leibniz series**.



**Theorem 8.6 (Leibniz Test)**

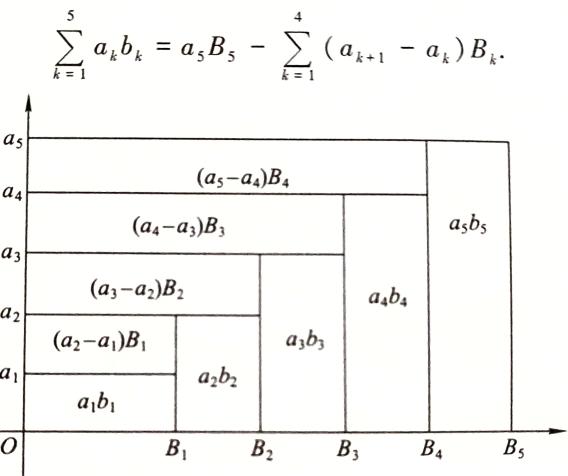
Leibniz series converges.

**¶ Abel-Dirichlet Test****Theorem 8.7 (Abel Transform (Discrete Integration by Parts/Summation by Parts))**

Let  $\{a_n\}, \{b_n\}$  be two sequences, then for any  $n \in \mathbb{N}^+$ ,

$$\sum_{k=1}^n a_k b_k = a_n B_n + \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k,$$

where  $B_n = \sum_{k=1}^n b_k$ .

**Lemma 8.1 (Abel Lemma (Discrete Second Integral Mean Value Theorem))**

Let  $\{a_n\}, \{b_n\}$  be two sequences, if  $\{a_n\}$  is a monotonic sequence and  $\{B_k\} = \sum_{k=1}^n b_k$  is a bounded sequence with bound  $M$ , then for any  $p \in \mathbb{N}^+$ ,

$$\left| \sum_{k=1}^p a_k b_k \right| \leq M (|a_1| + 2|a_p|).$$

**Theorem 8.8 (Abel-Dirichlet Test)**

The series  $\sum_{n=1}^{\infty} a_n b_n$  converges if one of the following two conditions is satisfied:

**Abel**  $\{a_n\}$  is a bounded monotonic sequence and  $\sum_{n=1}^{\infty} b_n$  converges.

**Dirichlet**  $\{a_n\}$  is a monotonic sequence,  $\lim_{n \rightarrow \infty} a_n = 0$ , and the partial sums  $B_n = \sum_{k=1}^n b_k$  are bounded.



## 8.4 Absolute and Conditional Convergence of Series

**Definition 8.3 (Absolute and Conditional Convergence of Series)**

If the series  $\sum_{n=1}^{\infty} |x_n|$  converges, then the series  $\sum_{n=1}^{\infty} x_n$  is said to be **absolutely convergent**.

If the series  $\sum_{n=1}^{\infty} x_n$  converges but is not absolutely convergent, then the series  $\sum_{n=1}^{\infty} x_n$  is said to be **conditionally convergent**.



## 8.5 Comparison of Convergence Speed of Series

The series  $\sum_{n=1}^{\infty} a_n$  is said to converge faster than the series  $\sum_{n=1}^{\infty} b_n$  if:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0.$$

**Theorem 8.9 (Du Bois-Reymond Theorem)**

For a given convergent positive term series  $\sum_{n=1}^{\infty} a_n$ , there always exists a convergent strictly positive term series  $\sum_{n=1}^{\infty} b_n$  such that:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0.$$



**Theorem 8.10 (Abel Theorem)**

For a given divergent positive term series  $\sum_{n=1}^{\infty} a_n$ , there always exists a divergent positive term series  $\sum_{n=1}^{\infty} b_n$  such that:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0.$$



**Remark** The above two theorems imply that the slowest converging positive term series does not exist.

## 8.6 Infinite Products

**¶ Infinite Products**

**¶ Two Formulas**

**Theorem 8.11 (Wallis Formula)**

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \left[ \frac{(2n)!!}{(2n-1)!!} \right]^2 = \frac{\pi}{2}.$$

Equivalently ( $n \rightarrow +\infty$ ),

$$\begin{aligned} \frac{(2n)!!}{(2n-1)!!} &\sim \sqrt{\pi n}, \\ \frac{(n!)^2 2^{2n}}{(2n)!} &\sim \sqrt{\pi n}. \end{aligned}$$



**Theorem 8.12 (Stirling Formula)**

$$n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + \frac{1}{12n} - \frac{1}{288n^2} + \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots + \frac{B_{2n}}{2k(2k-1)n^k} + \cdots \right),$$

where  $B_{2k}$  are Bernoulli numbers of order  $2k$ . Simplified form:

$$n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \quad (n \rightarrow +\infty),$$

or

$$n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\theta_n}, \quad \frac{1}{12n+1} < \theta_n < \frac{1}{12n}.$$



## 8.7 Special Series

### Geometric Series

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q},$$

it converges when  $|q| < 1$ , diverges otherwise.

### Telescoping Series

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n \rightarrow \infty} a_{n+1},$$

it converges when  $\lim_{n \rightarrow \infty} a_n$  exists, diverges otherwise.

### $p$ -Series/Hyperharmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

it converges when  $p > 1$ , diverges otherwise.

### $q$ -Series

$$\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^q},$$

it converges when  $q > 1$ , diverges otherwise.

### Generalized $q$ -Series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n) \cdots (\ln^{(k-1)} n) (\ln^{(k)} n)^q},$$

where  $\ln^{(k)} n$  denotes the  $k$ -th iterated logarithm, it converges when  $q > 1$ , diverges otherwise.

# Chapter 9 Series of Functions

## 9.1 Pointwise and Uniform Convergence

### ¶ Pointwise Convergence

#### Definition 9.1 (Function Term Series)

Let  $u_n(x)$  ( $n = 1, 2, 3, \dots$ ) be a sequence of functions with a common domain  $E$ . The sum of these infinitely many functions  $u_1(x) + u_2(x) + \dots + u_n(x) + \dots$  is called a **function term series**, denoted as:

$$\sum_{n=1}^{\infty} u_n(x).$$

For any fixed point  $x_0 \in E$ , if the numerical series  $\sum_{n=1}^{\infty} u_n(x_0)$  converges, then the function term series  $\sum_{n=1}^{\infty} u_n(x)$  is said to converge at  $x_0$ , or equivalently,  $x_0$  is called a **convergence point** of  $\sum_{n=1}^{\infty} u_n(x)$ .

The set of all convergence points is called the **domain of convergence** of  $\sum_{n=1}^{\infty} u_n(x)$ .



#### Definition 9.2 (Pointwise Convergence)

Let the domain of convergence of the function term series  $\sum_{n=1}^{\infty} u_n(x)$  be  $D \subset E$ . Then  $\sum_{n=1}^{\infty} u_n(x)$  defines a function  $S(x)$  on the set  $D$ , where:

$$S(x) = \sum_{n=1}^{\infty} u_n(x), \quad x \in D.$$

The function  $S(x)$  is called the **sum function** of the series, and  $\sum_{n=1}^{\infty} u_n(x)$  is said to **converge pointwise** to  $S(x)$  on  $D$ .



Define the **partial sum function** of the series as:

$$S_n(x) = \sum_{k=1}^n u_k(x).$$

It is evident that the set of all  $x$  for which  $\{S_n(x)\}$  converges is precisely  $D$ . Therefore, on  $D$ , we have:

$$S(x) = \lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n u_k(x).$$

Conversely, if a sequence of functions  $\{S_n(x)\}$  ( $x \in E$ ) is given, we can define:

$$\begin{cases} u_1(x) = S_1(x), \\ u_{n+1}(x) = S_{n+1}(x) - S_n(x), \quad n = 1, 2, \dots \end{cases}$$

to obtain the corresponding function term series.

Thus, the convergence behavior of a function term series and the corresponding sequence of partial sum functions is essentially the same.

However, it is important to note that the pointwise convergence has certain limitations.

**Continuity** The sum of finitely many continuous functions satisfies additive continuity:

$$\lim_{x \rightarrow x_0} [u_1(x) + u_2(x) + \cdots + u_n(x)] = \lim_{x \rightarrow x_0} u_1(x) + \lim_{x \rightarrow x_0} u_2(x) + \cdots + \lim_{x \rightarrow x_0} u_n(x).$$

If this property can be extended to infinitely many functions, that is: If  $u_n(x)$  is continuous on  $D$ , the sum function  $S(x) = \sum_{n=1}^{\infty} u_n(x)$  is also continuous on  $D$ . Moreover:

$$\lim_{x \rightarrow x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \rightarrow x_0} u_n(x),$$

meaning that the limit operation and infinite summation can be interchanged (also known as the fact that function term series can be evaluated termwise).

For the sequence of partial sums  $\{S_n(x)\}$ , the corresponding conclusion is that the limit function  $S(x) = \lim_{n \rightarrow \infty} S_n(x)$  is also continuous on  $D$ , and:

$$\lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} S_n(x),$$

meaning that the two limit operations can be interchanged.

Unfortunately, in the case of pointwise convergence, this property does not hold.

**Derivability** The sum of finitely many differentiable functions satisfies additive differentiability:

$$\frac{d}{dx} [u_1(x) + u_2(x) + \cdots + u_n(x)] = \frac{d}{dx} u_1(x) + \frac{d}{dx} u_2(x) + \cdots + \frac{d}{dx} u_n(x).$$

If this property can be extended to infinitely many functions, that is: If  $u_n(x)$  is differentiable on  $D$ , the sum function  $S(x) = \sum_{n=1}^{\infty} u_n(x)$  is also differentiable on  $D$ . Moreover:

$$\frac{d}{dx} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x),$$

meaning that the differentiation operation and infinite summation can be interchanged (also known as the fact that function term series can be differentiated termwise).

For the sequence of partial sums  $\{S_n(x)\}$ , the corresponding conclusion is that the limit function  $S(x) = \lim_{n \rightarrow \infty} S_n(x)$  is also differentiable on  $D$ , and:

$$\frac{d}{dx} \lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \frac{d}{dx} S_n(x),$$

meaning that the two operations can be interchanged.

Unfortunately, in the case of pointwise convergence, this property does not hold.

**Integrability** The sum of finitely many integrable functions satisfies additive integrability:

$$\int_a^b [u_1(x) + u_2(x) + \cdots + u_n(x)] dx = \int_a^b u_1(x) dx + \int_a^b u_2(x) dx + \cdots + \int_a^b u_n(x) dx.$$

If this property can be extended to infinitely many functions, that is: If  $u_n(x)$  is integrable on  $[a, b] \subset D$ ,

the sum function  $S(x) = \sum_{n=1}^{\infty} u_n(x)$  is also integrable on  $[a, b] \subset D$ . Moreover:

$$\int_a^b \sum_{n=1}^{\infty} u_n(x) dx = \sum_{n=1}^{\infty} \int_a^b u_n(x) dx,$$

meaning that the integration operation and infinite summation can be interchanged (also known as the fact that function term series can be integrated termwise).

For the sequence of partial sums  $\{S_n(x)\}$ , the corresponding conclusion is that the limit function  $S(x) = \lim_{n \rightarrow \infty} S_n(x)$  is also integrable on  $[a, b] \subset D$ , and:

$$\int_a^b \lim_{n \rightarrow \infty} S_n(x) dx = \lim_{n \rightarrow \infty} \int_a^b S_n(x) dx,$$

meaning that the two operations can be interchanged.

Unfortunately, in the case of pointwise convergence, this property does not hold.

## ¶ Uniform Convergence

### *Definition 9.3 (Uniform Convergence)*

Let  $\{S_n(x)\}(x \in D)$  be a sequence of functions. If:

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}^+, \forall n > N(\varepsilon) : |S_n(x) - S(x)| < \varepsilon \quad (\forall x \in D),$$

then  $\{S_n\}$  is said to **converge uniformly** to  $S(x)$  on  $D$ , denoted as:

$$S_n(x) \xrightarrow{D} S(x).$$

If the partial sum sequence  $\{S_n(x)\}$  of the function term series  $\sum_{n=1}^{\infty} u_n(x)(x \in D)$  converges uniformly to  $S(x)$  on  $D$ , then  $\sum_{n=1}^{\infty} u_n(x)$  is said to converge uniformly to  $S(x)$  on  $D$ .



Obviously, if the partial sum sequence  $\{S_n(x)\}$  of  $\sum_{n=1}^{\infty} u_n(x)$  satisfies:

$$S_n(x) \xrightarrow{D} S(x),$$

then:

$$u_n(x) \xrightarrow{D} 0.$$

### *Theorem 9.1 (Cauchy Criterion for Uniform Convergence)*

The necessary and sufficient condition for the sequence of functions  $\{S_n(x)\}$  to converge uniformly on  $D$  is:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall m > n > N : |S_m(x) - S_n(x)| < \varepsilon \quad (\forall x \in D).$$

Correspondingly, the necessary and sufficient condition for the function term series  $\sum_{n=1}^{\infty} u_n(x)$  to converge uniformly on  $D$  is:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall m > n > N : \left| \sum_{i=n+1}^m u_i(x) \right| < \varepsilon \quad (\forall x \in D).$$



**Theorem 9.2 (Necessary and Sufficient Conditions for Uniform Convergence)**

Let  $\{S_n(x)\}$  converge pointwise to  $S(x)$  on  $D$ . The necessary and sufficient conditions for  $S_n(x) \xrightarrow{D} S(x)$  are:

1.

$$\lim_{n \rightarrow \infty} d(S_n, S) = \lim_{n \rightarrow \infty} \sup_{x \in D} |S_n(x) - S(x)| = 0.$$

2. For any sequence  $\{x_n\}$  where  $x_n \in D$ , the following holds:

$$\lim_{n \rightarrow \infty} (S_n(x_n) - S(x_n)) = 0.$$



With the concept of uniform convergence, the flaws of pointwise convergence can be remedied, and the following properties can be established:

**Property**

**Continuity** Let  $f_n(x) \xrightarrow{I \subset \mathbb{R}} f(x)$ . If  $f_n(x)$  is continuous at  $x_0 \in I$  for  $n = 1, 2, 3, \dots$ , then  $f(x)$  is also continuous at  $x_0$ .

In particular, if  $f_n(x) \in C(I)$ , then  $f(x) \in C(I)$ .

**Termwise Limit** If  $\sum_{n=1}^{\infty} u_n(x) \xrightarrow{I \subset \mathbb{R}} S(x)$  and  $u_n(x) \in C(I)$ , then the sum function  $S(x) \in C(I)$ .

**Integrability** Let  $f_n(x) \xrightarrow{[a,b]} f(x)$ . If  $f_n(x) \in R[a, b]$ , then  $f(x) \in R[a, b]$ , and:

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = \int_a^b f(x) dx.$$

**Termwise Integration:** If  $\sum_{n=1}^{\infty} u_n(x) \xrightarrow{[a,b]} S(x)$  and  $u_n(x) \in R[a, b]$ , then  $S(x) \in R[a, b]$ .

**Differentiability** Let  $f'_n(x) \xrightarrow{[a,b]} \sigma(x)$ . If there exists  $x_0 \in [a, b]$  such that:  $\lim_{n \rightarrow \infty} f_n(x_0) = a$ , then there exists a function  $f(x)$  such that

$$f_n(x) \xrightarrow{[a,b]} f(x) \text{ and } f'(x) = \sigma(x).$$

**Termwise Differentiation** If  $\sum_{n=1}^{\infty} u'_n(x) \xrightarrow{[a,b]} \sigma(x)$  and there exists  $x_0 \in [a, b]$  such that:  $\sum_{n=1}^{\infty} u_n(x_0) \rightarrow a$ , then there exists a function  $S(x)$  such that

$$\sum_{n=1}^{\infty} u_n(x) \xrightarrow{[a,b]} S(x) \text{ and } S'(x) = \sigma(x).$$

**Corollary** Obviously, if we add the condition  $f'_n(x) \in C[a, b]$ , the conclusion still holds, and the proof becomes simpler.

**Note** Since continuity and differentiability are both local properties, it suffices to have **internally closed uniform convergence** of  $(a, b)$  to ensure that  $f(x)$  is continuous/differentiable.

**Quasi-Uniform Convergence****Definition 9.4 (Quasi-Uniform Convergence)**

The sequence of functions  $\{S_n(x)\}$  is said to **converge quasi-uniformly** on the interval  $[a, b]$  if it converges

pointwise to  $S(x)$  on  $[a, b]$ , and the following condition is satisfied:

$$\forall \varepsilon > 0, \forall N \in \mathbb{N}^*, \exists N_0 > N, \text{ s.t. } \forall x \in [a, b], \exists n_x \in [N, N_0] (n_x \in \mathbb{N}^*) : |S_{n_x}(x) - S(x)| < \varepsilon.$$



## 9.2 Uniform Convergence Tests

### ¶ Weierstrass Test (M-Test)

*Theorem 9.3 (Weierstrass Test (M-Test))*

If there exists a convergent positive term series  $\sum_{n=1}^{\infty} a_n$  such that:

$$|u_n(x)| \leq a_n, \quad \forall x \in E, n = 1, 2, 3, \dots$$

then the function term series  $\sum_{n=1}^{\infty} u_n(x)$  converges uniformly on  $E$ .

The positive term series  $\sum_{n=1}^{\infty} a_n$  is called a **majorant series** of  $\sum_{n=1}^{\infty} u_n(x)$ .

If replace the convergent positive term series  $\sum_{n=1}^{\infty} a_n$  with a uniform convergent series of functions  $\sum_{n=1}^{\infty} a_n(x)$ , the conclusion still holds.



### ¶ Abel-Dirichlet Test

*Theorem 9.4 (Abel-Dirichlet Test)*

If the series of functions  $\sum_{n=1}^{\infty} a_n(x)b_n(x)$  ( $x \in E$ ) satisfies at least one of the following two conditions, then it converges uniformly on  $E$ .

**Abel**  $\{a_n(x_0)\}$  ( $\forall x_0 \in E$ ) is monotonic and the series of functions  $\{a_n(x)\}$  is bounded uniformly on  $E$ .

Simultaneously, the series  $\sum_{n=1}^{\infty} b_n(x)$  converges uniformly on  $E$ .

**Dirichlet**  $\{a_n(x_0)\}$  ( $\forall x_0 \in E$ ) is a monotonic and  $a_n(x) \rightarrow 0$  uniformly convergent on  $E$  with limit 0.

Simultaneously, the partial sums  $B_n(x) = \sum_{k=1}^n b_k(x)$  are uniformly bounded on  $E$ .



### ¶ Dini Theorem

*Theorem 9.5 (Dini Theorem)*

Let the sequence of functions  $\{S_n(x)\}$  converges pointwise to  $S(x)$  on the closed interval  $[a, b]$ , if

1.  $S_n(x) \in C[a, b]$  ( $n = 1, 2, 3, \dots$ );
2.  $S(x) \in C[a, b]$ ;
3.  $\{S_n(x_0)\}$  ( $\forall x_0 \in [a, b]$ ) is monotonic;

then  $S_n(x) \xrightarrow{[a,b]} S(x)$ .



**Remark** Removing the condition of monotonicity, the Arzelà-Borel theorem (??) becomes the result of quasi-uniform convergence.

## 9.3 Special Cases

# Chapter 10 Power Series

## 10.1 Power Series and Its Convergence Radius

## 10.2 Expanding Functions into Power Series

## 10.3 Smooth Approximation of Functions

First, we use continuous functions to approximate Riemann integrable functions and smooth functions to approximate continuous functions, respectively.

### Theorem 10.1

Let  $f(x) \in R[a, b]$ . For any  $\varepsilon > 0$ , there exists a function  $g(x) \in C[a, b]$  such that:

$$\int_a^b |f(x) - g(x)| < \varepsilon.$$



### Theorem 10.2

Let  $f(x) \in C[a, b]$ . For any  $\varepsilon > 0$ , there exists a function  $g(x) \in C^\infty[a, b]$  such that:

$$|f(x) - g(x)| < \varepsilon, \quad \forall x \in [a, b].$$



Then, Weierstrass approximation theorem is stated as follows:

### Theorem 10.3 (Weierstrass First Approximation Theorem)

Let  $f(x) \in C[a, b]$ . For any  $\varepsilon > 0$ , there exists a polynomial  $P(x)$  such that:

$$|f(x) - P(x)| < \varepsilon, \quad \forall x \in [a, b].$$



### Theorem 10.4 (Weierstrass Second Approximation Theorem)

Let  $f(x)$  be a continuous periodic function with period  $2\pi$ . For any  $\varepsilon > 0$ , there exists a trigonometric polynomial sequence

$$\{T_n(x) = \frac{A_0}{2} + \sum_{k=1}^n A_k \cos(kx) + B_k \sin(kx)\}$$

such that:

$$T_n(x) \rightrightarrows f(x).$$



# Chapter 11 Limits and Continuity in Euclidean Spaces

## 11.1 Continuous Mappings

- ¶ Continuous Mappings on Compact Sets
- ¶ Continuous Mappings on Connected Sets

*Definition 11.1 (Connected Set)*

Let  $S$  be a set of points in  $\mathbb{R}^n$ . If a continuous mapping

$$\gamma : [0, 1] \rightarrow \mathbb{R}^n$$

satisfies that the range of  $\gamma([0, 1])$  lies entirely within  $S$ , we call  $\gamma$  a **path** in  $S$ , where  $\gamma(0)$  and  $\gamma(1)$  are referred to as the starting point and ending point of the path, respectively.

If for any two points  $x, y \in S$ , there exists a path in  $S$  with  $x$  as the starting point and  $y$  as the ending point,  $S$  is called path-connected, or equivalently,  $S$  is called a **connected set**.

A connected open set is called an **(open) region**. The closure of an (open) region is referred to as a closed region. 

**Remark** Intuitively, this means that any two points in  $S$  can be connected by a curve lying entirely within  $S$ . Clearly, a connected subset of  $\mathbb{R}$  is an interval, and a connected subset of  $\mathbb{R}^n$  is compact if and only if it is a closed interval.

# Chapter 12 Multi-variable Differential Calculus

## 12.1 Directional Derivatives and Total Differential

### ¶ Directional Derivative

*Definition 12.1 (Directional Derivative)*

Let  $U \subset \mathbb{R}^n$  be an open set,  $f : U \rightarrow \mathbb{R}^1$ ,  $\mathbf{e}$  is a unit vector in  $\mathbb{R}^n$ ,  $\mathbf{x}^0 \in U$ . Define

$$u(t) = f(\mathbf{x}^0 + t\mathbf{e}).$$

If the derivative of  $u$  at  $t = 0$

$$u'(0) = \lim_{t \rightarrow 0} \frac{u(t) - u(0)}{t} = \lim_{t \rightarrow 0} \frac{f(\mathbf{x}^0 + t\mathbf{e}) - f(\mathbf{x}^0)}{t}$$

exists and is finite, it is called the **directional derivative** of  $f$  at  $\mathbf{x}^0$  in the direction  $\mathbf{e}$ , denoted by  $\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0)$ . It is the rate of change of  $f$  at  $\mathbf{x}^0$  in the direction  $\mathbf{e}$ .



Consider the following set of unit coordinate vectors:  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ . Let  $\mathbf{e}_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$  denote the standard orthonormal basis in  $\mathbb{R}^n$ , where the 1 appears in the  $i$ -th position. That is,

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For a function  $f$ , the directional derivative of  $f$  at the point  $\mathbf{x}^0$  in the direction of  $\mathbf{e}_i$  is called the  $i$ th first-order **partial derivative** of  $f$  at  $\mathbf{x}^0$ , denoted by

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^0) \quad \text{or} \quad D_i f(\mathbf{x}^0) \quad \text{or} \quad f_{x_i}(\mathbf{x}^0) \quad (i = 1, 2, \dots, n).$$

$D_i = \frac{\partial}{\partial x_i}$  is called the  $i$ th partial differential operator ( $i = 1, 2, \dots, n$ ).

Let  $\mathbf{e}_i = \sum_{i=0}^n \mathbf{e}_i \cos \alpha$  be a unit vector, where  $\sum_{i=0}^n \cos^2 \alpha = 1$ . If  $\frac{\partial f}{\partial x_i}$  is continuous at  $\mathbf{x}^0$ , then the directional derivative of  $f$  at  $\mathbf{x}^0$  along the direction  $\mathbf{e}$  is given by:

$$\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}^0) \cos \alpha_i.$$

This is the formula for **expressing a directional derivative using partial derivatives**.

**Note** Let  $\mathbf{e}$  be a direction, then  $\|-\mathbf{e}\| = \|\mathbf{e}\| = 1$ , which implies that  $-\mathbf{e}$  is also a direction. At this point, we have:

$$\frac{\partial f}{\partial (-\mathbf{e})}(\mathbf{x}^0) = -\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0).$$

**Definition 12.2 (Jacobian Matrix (Gradient))**

Let

$$Jf(\mathbf{x}) = (D_1 f(\mathbf{x}), D_2 f(\mathbf{x}), \dots, D_n f(\mathbf{x})),$$

which is called the **Jacobian matrix** of the function  $f$  at the point  $\mathbf{x}$ , (a  $1 \times n$  matrix) whose counterpart is the first-order derivative of a single-variable function.

Henceforth, we represent the point  $\mathbf{x}$  in  $\mathbb{R}^n$  and its increments  $\Delta\mathbf{x}$  as column vectors. In this way, the differential of the function can be expressed using matrix multiplication as follows:

$$df(\mathbf{x}^0)(\Delta\mathbf{x}) = Jf(\mathbf{x}^0)\Delta\mathbf{x}.$$

The Jacobian matrix of the function  $f$  is also frequently denoted as  $\text{grad } f$  (or  $\nabla f$ ), that is,

$$\text{grad } f(\mathbf{x}) = Jf(\mathbf{x}),$$

which is called the **gradient** of the scalar function  $f$ .

**¶ Total Differential****Definition 12.3 (Total Differential)**

Let  $U \subset \mathbb{R}^n$  be an open set,  $f : U \rightarrow \mathbb{R}^1$ ,  $\mathbf{x}^0 \in U$ ,  $\Delta\mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n) \in \mathbb{R}^n$ . If

$$f(\mathbf{x}^0 + \Delta\mathbf{x}) - f(\mathbf{x}^0) = \sum_{i=1}^n A_i \Delta x_i + o(\|\Delta\mathbf{x}\|) \quad (\|\Delta\mathbf{x}\| \rightarrow 0),$$

where  $A_1, A_2, \dots, A_n$  are constants independent of  $\Delta\mathbf{x}$ , then the function  $f$  is said to be **differentiable** at the point  $\mathbf{x}^0$ , and the linear main part  $\sum_{i=1}^n A_i \Delta x_i$  is called the **total differential** of  $f$  at  $\mathbf{x}^0$ , denoted as

$$df(\mathbf{x}^0)(\Delta\mathbf{x}) = \sum_{i=1}^n A_i \Delta x_i.$$

If  $f$  is differentiable at every point in the open set  $U$ , then  $f$  is called a differentiable function on  $U$ .

**Theorem 12.1 (Conditions of Differentiability)**

**Necessary Condition** If an  $n$ -variable function  $f$  is differentiable at the point  $\mathbf{x}_0$ , then  $f$  is continuous at  $\mathbf{x}^0$  and possesses first-order partial derivatives  $\frac{\partial f}{\partial x_i}(\mathbf{x}^0)$  at  $\mathbf{x}^0$  for  $i = 1, 2, \dots, n$ , and

$$\mathbf{A} = (A_1, A_2, \dots, A_n) = Jf(\mathbf{x}^0) = (D_1 f(\mathbf{x}^0), D_2 f(\mathbf{x}^0), \dots, D_n f(\mathbf{x}^0)).$$

<sup>a</sup> However, the converse is not true.

**Sufficient Condition** Let  $U \subset \mathbb{R}^n$  be an open set, and let  $f : U \rightarrow \mathbb{R}^1$  be an  $n$ -variable function. If  $Jf = (D_1 f, D_2 f, \dots, D_n f)$  is continuous at  $\mathbf{x}^0$  (i.e.,  $\frac{\partial f}{\partial x_i}$  is continuous at  $\mathbf{x}^0$  for  $i = 1, 2, \dots, n$ ), then  $f$  is differentiable at  $\mathbf{x}^0$ . However, the converse is not necessarily true.

<sup>a</sup>It is referred to as the total differential formula, and the more common form is

$$df(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) dx + \frac{\partial f}{\partial y}(x_0, y_0) dy.$$



### 💡 Note

- The continuity of the derivative function at  $\mathbf{x}^0$  implies that the original function  $f$  is differentiable in some neighborhood of  $\mathbf{x}^0$ .
- In fact, this condition can be relaxed to require that one partial derivative exists at the point, while the remaining  $n - 1$  partial derivative functions are continuous at that point.

✍ **Proof** Taking a function of three variables as an example.

Assume the 3-ary function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  meets:

1. There exists  $f_z(x_0, y_0, z_0)$ .
2. The partial derivative functions  $f_x(x, y, z)$  and  $f_y(x, y, z)$  are continuous at  $(x_0, y_0, z_0)$ , i.e. there are partial derivatives in some neighborhood of  $(x_0, y_0, z_0)$ .

Consider the total increment of  $f$  at the point  $(x_0, y_0, z_0)$ :

$$\begin{aligned}\Delta f &= \underbrace{[f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0 + \Delta y, z_0 + \Delta z)]}_{I_1} \\ &\quad + \underbrace{[f(x_0, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0 + \Delta z)]}_{I_2} \\ &\quad + \underbrace{[f(x_0, y_0, z_0 + \Delta z) - f(x_0, y_0, z_0)]}_{I_3}.\end{aligned}$$

For  $I_1, I_2$ , by the Lagrange's Mean Value Theorem of unary functions, there exist  $\theta_1, \theta_2 \in (0, 1)$  such that

$$\begin{aligned}I_1 &= f_x(x_0 + \theta_1 \Delta x, y_0 + \Delta y, z_0 + \Delta z) \Delta x, \\ I_2 &= f_y(x_0, y_0 + \theta_2 \Delta y, z_0 + \Delta z) \Delta y.\end{aligned}$$

Then by the continuity of their partial derivatives at  $(x_0, y_0, z_0)$ , we have

$$\lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} I_1 = f_x(x_0, y_0, z_0) \Delta x, \quad \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} I_2 = f_y(x_0, y_0, z_0) \Delta y.$$

They can be expressed in terms of infinitesimals ( $\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ ):

$$\begin{aligned}I_1 &= f_x(x_0, y_0, z_0) \Delta x + \alpha_1 \Delta x, \quad \alpha_1 \rightarrow 0 (\rho \rightarrow 0), \\ I_2 &= f_y(x_0, y_0, z_0) \Delta y + \alpha_2 \Delta y, \quad \alpha_2 \rightarrow 0 (\rho \rightarrow 0).\end{aligned}$$

For  $I_3$ , by the definition of the partial derivative  $f_z(x, y, z)$  at  $(x_0, y_0, z_0)$ , we have

$$I_3 = f_z(x_0, y_0, z_0) \Delta z + \alpha_3 \Delta z, \quad \alpha_3 \rightarrow 0 (\rho \rightarrow 0).$$

Accordingly,

$$\begin{aligned}\Delta f &= I_1 + I_2 + I_3 \\ &= [f_x(x_0, y_0, z_0)\Delta x + \alpha_1\Delta x] + [f_y(x_0, y_0, z_0)\Delta y + \alpha_2\Delta y] + [f_z(x_0, y_0, z_0)\Delta z + \alpha_3\Delta z] \\ &= f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z + [\alpha_1\Delta x + \alpha_2\Delta y + \alpha_3\Delta z].\end{aligned}$$

Apparently,

$$\lim_{\rho \rightarrow 0} \frac{\alpha_1\Delta x + \alpha_2\Delta y + \alpha_3\Delta z}{\rho} = 0,$$

i.e.  $\alpha_1\Delta x + \alpha_2\Delta y + \alpha_3\Delta z = o(\rho)$ . Therefore,  $f(x, y, z)$  is differentiable at  $(x_0, y_0, z_0)$ , which completes the proof. ■

### 💡 Note (At some point)

#### 1. Differentiable

- $\implies$  Continuous
- $\implies$  Partial derivatives exist:  $D_{\vec{u}} = \nabla f \cdot \vec{u}$

#### 2. Directional Derivative

- All directional derivatives exist  $\iff$  differentiable or continuous.
- All directional derivatives exist and are equal  $\iff$  differentiable.

#### 3. Partial Derivative

- The continuity and existence of directional/partial derivatives are mutually exclusive.

## ¶ Higher-Order Partial Derivatives and Differential

If the first-order partial derivative of  $f$ ,  $\frac{\partial f}{\partial x_i}$ , itself possesses partial derivatives, then the second-order partial derivative of  $f$  is defined, and is denoted as follows(the first is also called the mixed partial derivative):

$$f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial x_i} \right), \quad f_{x_i x_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_i} \right), \quad i, j = 1, 2, \dots, n.$$

Similarly, higher-order partial derivatives of order  $3, 4, \dots, m, \dots$  can be defined.

The following theorem provides the conditions under which mixed partial derivatives are equal.

### Theorem 12.2 (Conditions for Equality of Mixed Partial Derivatives)

1. Let  $U \subset \mathbb{R}^2$  be an open set, and  $f : U \rightarrow \mathbb{R}$  be a function of two variables. If  $f_{xy}$  and  $f_{yx}$  are continuous at  $(x_0, y_0) \in U$ , then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

2. Let  $U \subset \mathbb{R}^n$  be an open set, and  $f : U \rightarrow \mathbb{R}$  be a function of  $n$  variables. If  $f$  has partial derivatives up to order  $k$  in  $D$ , and all of them are continuous at  $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in U$ , then

$$\frac{\partial^l f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}}(\mathbf{x}^0) = \frac{\partial^l f}{\partial x_{i_2} \partial x_{i_1} \cdots \partial x_{i_l}}(\mathbf{x}^0) = \cdots = \frac{\partial^l f}{\partial x_{i_l} \partial x_{i_{l-1}} \cdots \partial x_{i_1}}(\mathbf{x}^0),$$

that is, the order of taking partial derivatives  $l (\leq k)$  does not affect the result.<sup>a</sup>

<sup>a</sup>If the condition " $f_{xy}$  and  $f_{yx}$  are continuous at  $(x_0, y_0)$ ", is not satisfied, then the conclusion " $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$ " does not necessarily hold.



 **Proof** When  $k \neq 0, h \neq 0$ , define

$$\varphi(y) = f(x_0 + h, y) - f(x_0, y),$$

and

$$\psi(x) = f(x, y_0 + k) - f(x, y_0).$$

Applying the Lagrange Mean Value Theorem, we have

$$\begin{aligned} & [f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)] - [f(x_0 + h, y_0) - f(x_0, y_0)] \\ &= \varphi(y_0 + k) - \varphi(y_0) \\ &= \varphi'(y_0 + \theta_1 k)k \\ &= [f_y(x_0 + h, y_0 + \theta_1 k) - f_y(x_0, y_0 + \theta_1 k)]k \\ &= f_{yx}(x_0 + \theta_2 h, y_0 + \theta_1 k)hk, \quad 0 < \theta_1, \theta_2 < 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} & [f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)] - [f(x_0 + h, y_0) - f(x_0, y_0)] \\ &= [f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)] - [f(x_0, y_0 + k) - f(x_0, y_0)] \\ &= \psi(x_0 + h) - \psi(x_0) \\ &= \psi'(x_0 + \theta_3 h)h \\ &= [f_x(x_0 + \theta_3 h, y_0 + k) - f_x(x_0 + \theta_3 h, y_0)]h \\ &= f_{xy}(x_0 + \theta_3 h, y_0 + \theta_4 k)hk, \quad 0 < \theta_3, \theta_4 < 1. \end{aligned}$$

Therefore,

$$f_{xy}(x_0 + \theta_3 h, y_0 + \theta_4 k) = f_{yx}(x_0 + \theta_2 h, y_0 + \theta_1 k).$$

Since  $f_{xy}$  and  $f_{yx}$  are continuous at  $(x_0, y_0)$ , letting  $h \rightarrow 0, k \rightarrow 0$ , we obtain

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

By applying 12.1 and the principle of mathematical induction, one can immediately derive the following result.

■

Suppose  $z = f(x, y)$  has continuous partial derivatives in the domain  $U \subset \mathbb{R}^2$ . Then  $z$  is differentiable, and

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

If  $z$  also has continuous second-order partial derivatives, then  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  are also differentiable, and thus  $dz$  is differentiable. We call the differential of  $dz$  the second-order differential of  $z$ , denoted as

$$d^2z = d(dz).$$

In general, based on the  $k$ -th order differential  $d^k z$  of  $z$ , its  $(k+1)$ -th order differential (if it exists) is defined as

$$d^{k+1}z = d(d^k z), \quad k = 1, 2, \dots.$$

Due to the fact that for the independent variables  $x$  and  $y$ , we always have

$$d^2x = d(dx) = 0, \quad d^2y = d(dy) = 0,$$

the second-order differential of  $z = f(x, y)$  is given by

$$\begin{aligned} d^2z &= d(dz) = d\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) \\ &= d\left(\frac{\partial z}{\partial x}\right)dx + \frac{\partial z}{\partial x}d^2x + d\left(\frac{\partial z}{\partial y}\right)dy + \frac{\partial z}{\partial y}d^2y \\ &= \left(\frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial x \partial y}dy\right)dx + \left(\frac{\partial^2 z}{\partial y \partial x}dx + \frac{\partial^2 z}{\partial y^2}dy\right)dy \\ &= \frac{\partial^2 z}{\partial x^2}(dx)^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}(dy)^2, \end{aligned}$$

where  $(dx)^2$  and  $(dy)^2$  denote  $d^2x$  and  $d^2y$  respectively. If we treat  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$  as operators for partial differentiation and define

$$\left(\frac{\partial}{\partial x}\right)^2 = \frac{\partial^2}{\partial x^2}, \quad \left(\frac{\partial}{\partial y}\right)^2 = \frac{\partial^2}{\partial y^2}, \quad \left(\frac{\partial}{\partial x} \frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x \partial y},$$

then the formulas for the first and second differentials can be written as

$$dz = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)z,$$

$$d^2z = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^2 z.$$

Similarly, we define

$$\left(\frac{\partial}{\partial x}\right)^p \left(\frac{\partial}{\partial y}\right)^q = \frac{\partial^{p+q}}{\partial x^p \partial y^q} = \frac{\partial^q}{\partial y^q} \left(\frac{\partial}{\partial x}\right)^p, \quad (p, q = 1, 2, \dots)$$

It is easy to use mathematical induction to prove the formula for higher-order differentials:

$$d^k z = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^k z, \quad k = 1, 2, \dots$$

For an  $n$ -variable function  $u = f(x_1, x_2, \dots, x_n)$ , higher-order differentials can be similarly defined, and the following holds:

$$d^k u = \left(dx_1 \frac{\partial}{\partial x_1} + dx_2 \frac{\partial}{\partial x_2} + \dots + dx_n \frac{\partial}{\partial x_n}\right)^k u, \quad k = 1, 2, \dots$$

## 12.2 Differential of Vector-Valued Functions

Consider an  $n$ -dimensional vector-valued function defined on a domain  $U \subset \mathbb{R}^n$ :

$$\begin{aligned} f : U &\rightarrow \mathbb{R}^m, \\ \mathbf{x} \mapsto \mathbf{y} &= f(\mathbf{x}) \end{aligned}$$

Expressed in coordinate vector form:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_m(x_1, x_2, \dots, x_n) \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in U$$

- If each component function  $f_i(x_1, x_2, \dots, x_n)$  ( $i = 1, 2, \dots, m$ ) is partially differentiable at  $\mathbf{x}^0$ , then the vector-valued function  $\mathbf{f}$  is differentiable at  $\mathbf{x}^0$ , and we define the matrix

$$\left( \frac{\partial f}{\partial x_j}(\mathbf{x}^0) \right)_{m \times n} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}^0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}^0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}^0) \end{pmatrix}$$

This matrix is called the Jacobian matrix of  $\mathbf{f}$  at  $\mathbf{x}^0$ , denoted by  $f'(\mathbf{x}^0)$  (or  $Df(\mathbf{x}^0)$ ,  $J_f(\mathbf{x}^0)$ ).

For the special case  $m = 1$ , i.e.,  $n$ -variable scalar function  $z = f(x_1, x_2, \dots, x_n)$ , the derivative at  $\mathbf{x}^0$  is

$$f'(\mathbf{x}^0) = \left( \frac{\partial f}{\partial x_1}(\mathbf{x}^0), \frac{\partial f}{\partial x_2}(\mathbf{x}^0), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x}^0) \right)$$

If the vector-valued function  $\mathbf{f}$  is differentiable at every point in  $U$ , then  $\mathbf{f}$  is said to be differentiable on  $U$ , and the corresponding relationship is

$$\mathbf{x} \in U \mapsto f'(\mathbf{x}) = J_f(\mathbf{x})$$

where  $f'(\mathbf{x})$  (or  $Df(\mathbf{x})$ ,  $J_f(\mathbf{x})$ ) denotes the derivative of  $\mathbf{f}$  at  $\mathbf{x}$  in  $U$ .

- If every component function  $f_i(x_1, x_2, \dots, x_n)$  ( $i = 1, 2, \dots, m$ ) of  $\mathbf{f}$  has continuous partial derivatives at  $\mathbf{x}^0$ , then every element of the Jacobian matrix of  $\mathbf{f}$  is continuous at  $\mathbf{x}^0$ . In this case,  $\mathbf{f}$  is said to have a continuous derivative at  $\mathbf{x}^0$  as a vector-valued function.

If the derivative of a vector-valued function  $\mathbf{f}$  is continuous at every point in  $U$ , then  $\mathbf{f}$  is said to have a continuous derivative on  $U$ .

- If there exists an  $m \times n$  matrix  $A$  that depends only on  $\mathbf{x}^0$  (and not on  $\Delta\mathbf{x}$ ), such that in the neighborhood of  $\mathbf{x}^0$ ,

$$\Delta\mathbf{y} = f(\mathbf{x}^0 + \Delta\mathbf{x}) - f(\mathbf{x}^0) = A\Delta\mathbf{x} + o(\|\Delta\mathbf{x}\|)$$

(where  $\Delta\mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)^T$  is a column vector and  $\|\Delta\mathbf{x}\|$  denotes its norm), then  $f$  is said to be differentiable at  $\mathbf{x}^0$  as a vector-valued function, and  $A\Delta\mathbf{x}$  is called the differential of  $f$  at  $\mathbf{x}^0$ , denoted as  $d\mathbf{y}$ . If we denote  $\Delta\mathbf{x}$  by  $d\mathbf{x}$  ( $d\mathbf{x} = (dx_1, dx_2, \dots, dx_n)^T$ ), then

$$d\mathbf{y} = A d\mathbf{x}.$$

If the vector-valued function  $\mathbf{f}$  is differentiable at every point in  $U$ , then  $\mathbf{f}$  is said to be differentiable on  $U$ .

Combining the above three points, we obtain the following unified statement:

A vector-valued function  $\mathbf{f}$  is continuous, differentiable, and has derivatives if and only if each of its coor-

dinate component functions  $f_i(x_1, x_2, \dots, x_n)$  ( $i = 1, 2, \dots, m$ ) is continuous, differentiable, and has derivatives.

## 12.3 Derivatives of Composite Mappings (Chain Rule)

Let  $U \subset \mathbb{R}^l$  and  $V \subset \mathbb{R}^n$  be open sets, and let

$$\mathbf{g} : U \rightarrow V \quad \text{and} \quad \mathbf{f} : V \rightarrow \mathbb{R}^m$$

be mappings. If  $\mathbf{g}$  is derivative at  $\mathbf{u}^0 \in U$  and  $\mathbf{f}$  is differentiable at  $\mathbf{x}^0 = \mathbf{g}(\mathbf{u}^0)$ , then the composite mapping  $\mathbf{f} \circ \mathbf{g}$  is differentiable at  $\mathbf{u}^0$ , and:

$$J(\mathbf{f} \circ \mathbf{g})(\mathbf{u}^0) = J\mathbf{f}(\mathbf{x}^0)J\mathbf{g}(\mathbf{u}^0).$$

### Note

1. outer differentiable + inner derivative = total derivative
2. outer differentiable + inner differentiable = total differentiable
- 3.

Specially, define  $z = f(x, y)$ ,  $(x, y) \in D_f \subset \mathbb{R}^2$ ,  $\mathbf{g} : D_g \rightarrow \mathbb{R}^2$ ,  $(u, v) \mapsto (x(u, v), y(u, v))$ , and  $g(D_g) \subset D_f$ , then we have composite function

$$z = f \circ \mathbf{g} = f[x(u, v), y(u, v)], \quad (u, v) \in D_g.$$

$$\mathbb{R}^2 \xrightarrow{\mathbf{g}: \text{derivative}} \mathbb{R}^2 \xrightarrow{f: \text{differentiable}} \mathbb{R}$$

If  $\mathbf{g}$  is derivative at  $(u_0, v_0) \in D_g$ , and  $f$  is differentiable at  $(x_0, y_0) = \mathbf{g}(u_0, v_0)$ , then  $z = f \circ \mathbf{g}$  is differentiable at  $(u_0, v_0)$ , and at the point,

$$\begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

### Proof

■

## 12.4 Mean Value Theorem and Taylor's Formula

### Definition 12.4 (Convex Region)

Let  $D \subseteq \mathbb{R}^n$  be a region. If every line segment connecting any two points  $\mathbf{x}_0, \mathbf{x}_1 \in D$  (denoted by  $\overline{\mathbf{x}_0 \mathbf{x}_1}$ ) is entirely contained in  $D$ , i.e., for any  $\lambda \in [0, 1]$ , we have

$$\mathbf{x}_0 + \lambda(\mathbf{x}_1 - \mathbf{x}_0) \in D,$$

then  $D$  is called a convex region.

♣

**Theorem 12.3 (Lagrange's Mean Value Theorem)**

Let  $f$  be differentiable on a convex region  $D \subseteq \mathbb{R}^n$ . For any two points  $\mathbf{a}, \mathbf{b} \in D$ , there exists a point  $\xi \in \overline{\mathbf{ab}}$  such that:

$$f(\mathbf{b}) - f(\mathbf{a}) = Jf(\xi)(\mathbf{b} - \mathbf{a}).$$

**Proof****Theorem 12.4**

Let  $D$  be a region in  $\mathbb{R}^n$ . If for any  $\mathbf{x} \in D$ , we have

$$Jf(\mathbf{x}) = 0,$$

then  $f$  is constant on  $D$ .

**Proof****Theorem 12.5 (Taylor's Formula)**

**Lagrange's Remainder** Let  $D \subseteq \mathbb{R}^n$  be a convex region, and let  $f : D \rightarrow \mathbb{R}$  have  $m + 1$  continuous partial derivatives. For  $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$  and  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in D$ , there exists  $\xi \in \overline{\mathbf{x}^0 \mathbf{x}}$  such that:

$$f(\mathbf{x}) = f(\mathbf{x}^0) + \sum_{k=1}^m \frac{1}{k!} \left( \sum_{i=1}^n (x_i - x_i^0) \frac{\partial}{\partial x_i} \right)^k f(\mathbf{x}^0) + \frac{1}{(m+1)!} \left( \sum_{i=1}^n (x_i - x_i^0) \frac{\partial}{\partial x_i} \right)^{m+1} f(\xi).$$

**Peano's Remainder** Let  $D \subseteq \mathbb{R}^n$  be a convex region, and let  $f : D \rightarrow \mathbb{R}$  have  $m$  continuous partial derivatives. Then:

$$f(\mathbf{x}) = f(\mathbf{x}^0) + \sum_{k=1}^m \frac{1}{k!} \sum_{i_1, i_2, \dots, i_k=1}^n \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_k}}(\mathbf{x}^0) \prod_{j=1}^k (x_{i_j} - x_{i_j}^0) + R_m(\mathbf{x} - \mathbf{x}^0),$$

where  $R_m(\mathbf{x} - \mathbf{x}^0) = O(\|\mathbf{x} - \mathbf{x}^0\|^{m+1})$  or  $o(\|\mathbf{x} - \mathbf{x}^0\|^m)$ , as  $\|\mathbf{x} - \mathbf{x}^0\| \rightarrow 0$ .



In applications, particularly important is the expression of the first three terms in Taylor's formula, which is given as (let  $x_1 - x_1^0$  be denoted by  $\Delta x_1$ , and similarly for other variables;  $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$ ):

$$f(\mathbf{x}) = f(\mathbf{x}^0) + Jf(\mathbf{x}^0)(\Delta \mathbf{x}) + \frac{1}{2!}(\Delta \mathbf{x})Hf(\mathbf{x}^0)(\Delta \mathbf{x})^T + \dots,$$

where the matrix

$$Hf(\mathbf{x}^0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{\mathbf{x}^0}$$

is called the **Hessian matrix** of the function  $f$ . It is an  $n \times n$  symmetric matrix.

## 12.5 Implicit Function Theorem

*Theorem 12.6 (Implicit Function Theorem)*

Let  $U \subset \mathbb{R}^{n+1}$  be an open set, and  $F : U \rightarrow \mathbb{R}$  be an  $n + 1$ -variable function. If:

1.  $F \in C^k(U, \mathbb{R})$ , where  $1 \leq k \leq +\infty$ ;
2.  $F(\mathbf{x}^0, y^0) = 0$ , where  $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \mathbb{R}^n$ ,  $y^0 \in \mathbb{R}$ , and  $(\mathbf{x}^0, y^0) \in U$  (i.e., the equation  $F(\mathbf{x}, y) = 0$  has a solution  $(\mathbf{x}^0, y^0)$ );
3.  $F'_y(\mathbf{x}^0, y^0) \neq 0$ .

Then there exists an open interval  $I \times J$  containing  $(\mathbf{x}^0, y^0)$  ( $I$  being an open interval in  $\mathbb{R}^n$  containing  $\mathbf{x}^0$ , and  $J$  being an open interval in  $\mathbb{R}$  containing  $y^0$ ), as shown in Fig. 12.1, such that:

1.  $\forall x \in I$ , the equation  $F(\mathbf{x}, y) = 0$  has a unique solution  $y = f(x)$ , where  $f : I \rightarrow J$  is an  $n$ -variable function (called the **implicit function**  $f$ , hidden within the equation  $F(\mathbf{x}, f(\mathbf{x})) = 0$ , though not necessarily explicitly expressed);
2.  $y^0 = f(\mathbf{x}^0)$ ;
3.  $f \in C^k(I, \mathbb{R})$ ;
4. When  $x \in I$ ,  $\frac{\partial f}{\partial x_i} = \frac{\partial y}{\partial x_i} = -\frac{F_x(\mathbf{x}, y)}{F_y(\mathbf{x}, y)}$ ,  $i = 1, 2, \dots, n$ , where  $y = f(x)$ .

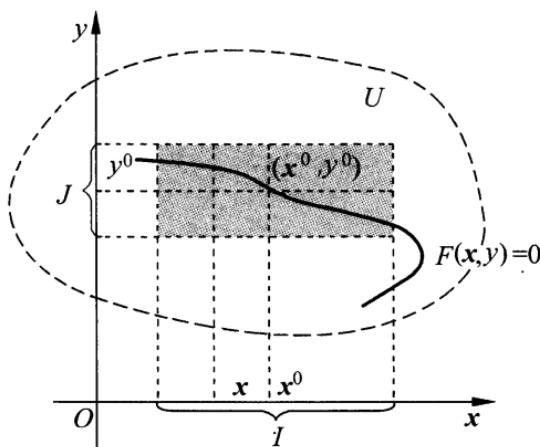


Figure 12.1: Implicit Function

**Proof** Only the single-variable implicit function theorem is proved; the multi-variable case can be derived using mathematical induction.

Without loss of generality, assume  $F_y(x^0, y^0) > 0$ .

First, prove the existence of the implicit function. From the continuity of  $F_y(x^0, y^0) > 0$  and  $F_y(x, y)$ , it is known that there exist closed rectangle:

$$D^* = \{(x, y) \mid |x - x_0| \leq \alpha, |y - y_0| \leq \beta\} \subset U,$$

where the following holds:

$$F_y(x, y) > 0.$$

Thus, for fixed  $x_0$ , the function  $F(x^0, y)$  is strictly monotonically increasing within  $[y^0 - \beta, y^0 + \beta]$ . Further-

more, since:

$$F(x^0, y^0) = 0,$$

it follows that:

$$F(x^0, y^0 - \beta) < 0, \quad F(x^0, y^0 + \beta) > 0.$$

Due to the continuity of  $F(x, y)$  within  $D^*$ , there exists  $\rho > 0$  such that along the line segment:

$$x = x^0 + \rho, \quad y = y^0 + \beta,$$

we have  $F(x, y) > 0$ , and along the line segment:

$$x = x^0 + \rho, \quad y = y^0 - \beta,$$

we have  $F(x, y) < 0$ . Therefore, for any point  $\bar{x} \in (x^0 - \rho, x^0 + \rho)$ , treat  $F(x, y)$  as a single-variable function of  $y$ . Within  $[y^0 - \beta, y^0 + \beta]$ , this function is continuous. From the previous discussion, we know:

$$F(\bar{x}, y^0 - \beta) < 0, \quad F(\bar{x}, y^0 + \beta) > 0.$$

According to the zero point existence theorem 3.3, there must exist a unique  $\bar{y} \in [y^0 - \beta, y^0 + \beta]$  such that  $F(\bar{x}, \bar{y}) = 0$ . Furthermore, because  $F_y(x, y) > 0$  within  $D^*$ , this  $\bar{y}$  is unique. Denote the corresponding relationship as  $\bar{y} = f(\bar{x})$ , then the function  $y = f(x)$  is defined within  $(x^0 - \rho, x^0 + \rho)$ , satisfying  $F(x, f(x)) = 0$ , and clearly:

$$y^0 = f(x^0).$$

Further proving the continuity of the implicit function  $y = f(x)$  on  $(x^0 - \rho, x^0 + \rho)$ : Let  $\bar{x} \in (x^0 - \rho, x^0 + \rho)$  be any point. For any given  $\varepsilon > 0$  ( $\varepsilon$  being sufficiently small), since  $F(\bar{x}, \bar{y}) = 0$  ( $\bar{y} = f(\bar{x})$ ), from the previous discussion we know:

$$F(\bar{x}, \bar{y} - \varepsilon) < 0, \quad F(\bar{x}, \bar{y} + \varepsilon) > 0.$$

Furthermore, due to the continuity of  $F(x, y)$  on  $D^*$ , there exists  $\delta > 0$  such that:

$$F(x, \bar{y} - \varepsilon) < 0, \quad F(x, \bar{y} + \varepsilon) > 0, \quad \text{when } x \in O(x^0, \delta).$$

By reasoning similar to the previous discussion, it can be obtained that when  $x \in O(x^0, \delta)$ , the corresponding implicit function value must satisfy  $f(x) \in (\bar{y} - \varepsilon, \bar{y} + \varepsilon)$ , i.e.,

$$|f(x) - f(x^0)| < \varepsilon.$$

This implies that  $y = f(x)$  is continuous on  $(x^0 - \rho, x^0 + \rho)$ .

Finally, prove the differentiability of  $y = f(x)$  on  $(x^0 - \rho, x^0 + \rho)$ : Let  $\bar{x} \in (x^0 - \rho, x^0 + \rho)$  be any point. Take  $\Delta x$  sufficiently small such that  $\bar{x} = x + \Delta x \in (x^0 - \rho, x^0 + \rho)$ . Denote  $\bar{y} = f(\bar{x})$  and  $\bar{y} + \Delta y = f(\bar{x} + \Delta x)$ . Clearly,

$$F(\bar{x}, \bar{y}) = 0 \quad \text{and} \quad F(\bar{x} + \Delta x, \bar{y} + \Delta y) = 0.$$

Using the multi-variable function's mean value theorem 12.3, we obtain:

$$\begin{aligned} 0 &= F(\bar{x}, \bar{y} + \Delta y) - F(\bar{x}, \bar{y}) \\ &= F_x(\bar{x} + \theta \Delta x, \bar{y} + \theta \Delta y) \Delta x + F_y(\bar{x} + \theta \Delta x, \bar{y} + \theta \Delta y) \Delta y, \end{aligned}$$

where  $0 < \theta < 1$ . Note that  $F_y \neq 0$  on  $D^*$ , hence:

$$\frac{\Delta y}{\Delta x} = -\frac{F_x(\bar{x} + \theta \Delta x, \bar{y} + \theta \Delta y)}{F_y(\bar{x} + \theta \Delta x, \bar{y} + \theta \Delta y)}.$$

Let  $\Delta x \rightarrow 0$ . Considering the continuity of  $F_x$  and  $F_y$ , we obtain:

$$\left. \frac{dy}{dx} \right|_{x=\bar{x}} = -\frac{F_x(\bar{x}, \bar{y})}{F_y(\bar{x}, \bar{y})}.$$

Thus:

$$f'(\bar{x}) = -\frac{F_x(\bar{x}, \bar{y})}{F_y(\bar{x}, \bar{y})}.$$

The proof is complete. ■

**Theorem 12.7 (Implicit Mapping Theorem)**

Let  $U \subset \mathbb{R}^{n+m}$  be an open set, and  $\mathbf{F} : U \rightarrow \mathbb{R}^m$  be a mapping. If:

1.  $\mathbf{F} \in C^k(U, \mathbb{R}^m)$ ,  $1 \leq k \leq \infty$ ;
2.  $\mathbf{F}(\mathbf{x}^0, \mathbf{y}^0) = \mathbf{0}$ , where  $\mathbf{x}^0 = (x_1, x_2, \dots, x_n)$ ,  $\mathbf{y}^0 = (y_1, y_2, \dots, y_m)$ ,  $(\mathbf{x}^0, \mathbf{y}^0) \in U$  (implying  $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$  has a solution at  $(\mathbf{x}^0, \mathbf{y}^0)$ );
3. The determinant

$$\det \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}_{(\mathbf{x}^0, \mathbf{y}^0)} = \det J_{\mathbf{y}} \mathbf{F}(\mathbf{x}^0, \mathbf{y}^0) \neq 0,$$

then there exists an open neighborhood  $I \times J \subset U \subset \mathbb{R}^{n+m}$  containing  $(\mathbf{x}^0, \mathbf{y}^0)$ , such that:

1. For all  $\mathbf{x} \in I$ , the system  $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$  has a unique solution  $\mathbf{y} = \mathbf{f}(\mathbf{x})$ , where  $\mathbf{f} : I \rightarrow J$  is a mapping (called  $\mathbf{f}$  the implicit function hidden in  $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$ );
2.  $\mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$ ;
3.  $\mathbf{f} \in C^k(I, \mathbb{R}^m)$ ;
4. For  $x \in I$ ,

$$J_{\mathbf{f}}(x) = -(J_{\mathbf{y}} \mathbf{F}(\mathbf{x}, \mathbf{f}(x)))^{-1} J_{\mathbf{x}} \mathbf{F}(\mathbf{x}, \mathbf{f}(x)) = - \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}_{(\mathbf{x}, \mathbf{f}(\mathbf{x}))}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix},$$

where  $\mathbf{y} = \mathbf{f}(\mathbf{x})$ . ♡

## 12.6 Applications of Multi-Variable Differential Calculus

### ¶ Surface and Tangent Space

#### Definition 12.5 (Parameterization of Surface)

Let  $\Delta$  be an open subset in  $\mathbb{R}^s$ , and  $\mathbf{x} : \Delta \rightarrow \mathbb{R}^n$  be a mapping, where  $\mathbf{u} = (u_1, u_2, \dots, u_s) \rightarrow \mathbf{x}(\mathbf{u}) = (x_1(u_1, u_2, \dots, u_s), x_2(u_1, u_2, \dots, u_s), \dots, x_n(u_1, u_2, \dots, u_s))$ . Then  $M = \mathbf{x}(\Delta) = \{\mathbf{x}(\mathbf{u}) \mid \mathbf{u} \in \Delta\}$  is called an  $s$ -dimensional surface, and  $\mathbf{x}(\mathbf{u})$  is referred to as the parameterization of  $M$ . When  $\mathbf{x}(\mathbf{u}) \in C^k$  ( $k \geq 0$ ),  $\mathbf{x}$  or  $M$  is called an  $s$ -dimensional  $C^k$  surface.

If  $\mathbf{x} \in C^k$  ( $k \geq 1$ ),  $\mathbf{x}$  or  $M$  is called an  $s$ -dimensional  $C^k$  smooth surface. When

$$\text{rank}(x'_1(\mathbf{u}^0), x'_2(\mathbf{u}^0), \dots, x'_s(\mathbf{u}^0)) = \text{rank} \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_s} \end{pmatrix}_{\mathbf{u}^0} = s,$$

we call  $\mathbf{u}^0$  or  $\mathbf{x}(\mathbf{u}^0)$  a **regular point** of the surface  $M$ . Otherwise, it is called a singular point. Every point that is a regular point of the surface is referred to as an  $s$ -dimensional  $C^k$  **regular surface**. At such points,  $\{x'_1, \dots, x'_s\}$  are linearly independent.



When  $s = 1$ ,  $t$  represents the parameter, a one-dimensional surface is commonly referred to as a curve. Considering a  $C^k$  ( $k \geq 1$ ) curve  $\mathbf{x}(t)$ , we have:

$$\mathbf{x}'(t) = (x'_1(t), x'_2(t), \dots, x'_n(t)).$$

If  $t$  is a regular point, then  $\text{rank}(\mathbf{x}'(t)) = \text{rank}(x'_1(t), x'_2(t), \dots, x'_n(t)) = 1$ ; this is equivalent to  $\mathbf{x}'(t) \neq 0$ , which means  $x'_1(t), x'_2(t), \dots, x'_n(t)$  are not all zero.

We refer to  $\mathbf{x}'(t)$  as the tangent vector of the curve  $\mathbf{x}(t)$  at point  $t$ . When  $t$  varies, a tangent vector field along the curve  $\mathbf{x}(t)$  is obtained. If  $\mathbf{x}(t)$  is a regular curve,  $\frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$  is the unit tangent vector field along the curve  $\mathbf{x}(t)$ . It should be emphasized that  $\mathbf{x}'(t)$  or  $\frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$  always points outward from point  $t$ .

#### Definition 12.6 (Tangent Vector)



### ¶ Unconditional Extremum

### ¶ Conditional Extremum

#### Definition 12.7 (Conditional Extremum)

Let  $f : D \rightarrow \mathbb{R}$  be a function with  $n + m$  variables defined on a open set  $D \subseteq \mathbb{R}^{n+m}$ , and let  $\Phi : D \rightarrow \mathbb{R}^m$  be a mapping,  $M = \{\mathbf{x} \in D \mid \Phi(\mathbf{x}) = 0\}$ . If there exists  $\mathbf{x}^0 \in M$  satisfying the constraints such that:

$$f(\mathbf{x}^0) \leq f(\mathbf{x}) \quad (\text{or } f(\mathbf{x}^0) \geq f(\mathbf{x})),$$

for all  $\mathbf{x} \in M$  that also satisfy the constraints, then  $f$  is said to have a conditional minimum (or maximum) at point  $\mathbf{x}^0$  under the given constraints.



**Theorem 12.8 (Lagrange Multiplier Method)**

Let  $f : D \rightarrow \mathbb{R}$  be a function with  $n + m$  variables defined on an open set  $D \subseteq \mathbb{R}^{n+m}$ , and let  $\Phi : D \rightarrow \mathbb{R}^m$  be a mapping,  $M = \{\mathbf{x} \in D \mid \Phi(\mathbf{x}) = 0\}$ . If:

1.  $f \in C^1(D, \mathbb{R})$ ,  $\Phi \in C^1(D, \mathbb{R}^m)$ ;
2.  $\text{rank}(J\Phi(\mathbf{x}^0)) = m$ ;
3.  $\mathbf{x}^0$  is a conditional extremum point of  $f$  on  $M$ ;

then there exist  $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ , such that:

$$\nabla f(\mathbf{x}^0) + \sum_{i=1}^m \lambda_i \nabla \Phi_i(\mathbf{x}^0) = 0.$$



# Chapter 13 Multiple Integrals

## 13.1 Multiple Integrals on Bounded Closed Regions

### ¶ Definition of Multiple Integral

Initially, we define the double integral on a closed interval.

*Definition 13.1 (Double Integral on a Closed Interval)*

Let  $I = [a, b] \times [c, d]$  be a closed interval in  $\mathbb{R}^2$ , (i.e., each boundary is parallel to the coordinate axes). Partition  $[a, b]$ :

$$T_x : a = x_0 < x_1 < \cdots < x_n = b.$$

Partition  $[c, d]$ :

$$T_y : c = y_0 < y_1 < \cdots < y_m = d.$$

Two sets of parallel lines  $x = x_i$  ( $i = 0, 1, \dots, n$ ) and  $y = y_j$  ( $j = 0, 1, \dots, m$ ) divide  $I$  into  $n \times m$  subrectangles:

$$[x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad i = 1, \dots, n, j = 1, \dots, m.$$

The union of these  $k$  subrectangles forms a partition  $T = T_x \times T_y = \{I_1, I_2, \dots, I_k\}$ . For each  $\xi^i \in I_i$  ( $i = 1, 2, \dots, k$ ), define the **Riemann sum** (also called a sum of integrals) as:

$$\sum_{i=1}^k f(\xi^i) v(I_i),$$

where  $v(I_i)$  is the area of the rectangle  $I_i$ , i.e., the product of its length and width. Denote:

$$\lambda = \max(\text{diam}(I_1), \text{diam}(I_2), \dots, \text{diam}(I_k)),$$

where  $\text{diam}(I)$  is the diagonal length of the rectangle  $I$ , and  $\lambda$  is called the modulus or width of the partition  $T$ . The points  $\xi = (\xi^1, \xi^2, \dots, \xi^k) \in I_1 \times I_2 \times \cdots \times I_k$  are called sampling points for the Riemann sum.

If there exists  $J \in \mathbb{R}$ , such that  $\forall \varepsilon > 0$ , there exists  $\delta > 0$ , such that when  $\lambda < \delta$ , for all  $\xi \in I_1 \times I_2 \times \cdots \times I_k$ , we have:

$$\left| \sum_{i=1}^k f(\xi^i) v(I_i) - J \right| < \varepsilon,$$

then  $f$  is said to be Riemann integrable on  $I$ , and:

$$J = \lim_{\lambda \rightarrow 0} \sum_{i=1}^k f(\xi^i) v(I_i) =: \iint_I f(x, y) \, dx \, dy \quad \text{or} \quad \int_I f \, dv \quad \text{or} \quad \int_I f.$$

The function  $f$  is said to have a double integral on  $I$ , or simply  $f$  is integrable on  $I$ . Here  $f$  is called the integrand,  $I$  is called the integration region, and  $dv = dx \, dy$  is called the integration element.

The defined double integral possesses properties similar to those of single-variable integrals.

On the basis of the above definition, we can extend it to the case of a bounded set.

**Definition 13.2 (Double Integral on a Bounded Set)**

Let  $\Omega \subset \mathbb{R}^2$  be a bounded set, and  $f : \Omega \rightarrow \mathbb{R}$  a two-dimensional function. Define:

$$f_\Omega(\mathbf{x}) = f_\Omega(x, y) = \begin{cases} f(x, y), & \text{if } \mathbf{x} = (x, y) \in \Omega, \\ 0, & \text{if } \mathbf{x} = (x, y) \notin \Omega, \end{cases}$$

and call this the **zero extension** of  $f$ . For any closed interval  $I \supset \Omega$ , if  $f_\Omega$  is Riemann integrable on  $I$ , then  $f$  is said to be **Riemann integrable** on  $\Omega$  (abbreviated as integrable). The integral of  $f$  on  $\Omega$ , denoted as:

$$\iint_{\Omega} f(x, y) dx dy = \int_{\Omega} f dV = \int_{\Omega} f = \int_{\Omega} f_\Omega = \iint_I f_\Omega(x, y) dx dy,$$

represents the Riemann integral of  $f$  on  $\Omega$ .



In above definition, the integral  $\int_{\Omega} f$  is independent of the choice of the closed interval  $I$  containing  $\Omega$  (this confirms the consistency of the definition).

It is worth noting that all the definitions and properties of double integrals can be extended to triple integrals and higher-dimensional integrals without excessive inconvenience.

**¶ Necessary and Sufficient Conditions for Integrability****Definition 13.3 (Set with Zero Area and Set with Zero Measure (Null Set))**

Let  $A \subset \mathbb{R}^2$ . If for any  $\varepsilon > 0$ , there exist finitely many closed intervals  $I_1, I_2, \dots, I_k$  such that:

$$\bigcup_{i=1}^k I_i \supset A, \quad \text{and} \quad \sum_{i=1}^k v(I_i) < \varepsilon,$$

then  $A$  is called a **set with zero area**.

Let  $A \subset \mathbb{R}^2$ . If for any  $\varepsilon > 0$ , there exist at most countably many closed intervals  $I_1, I_2, \dots, I_k, \dots$  such that:

$$\bigcup_{i=1}^{\infty} I_i \supset A, \quad \text{and} \quad \sum_{i=1}^{\infty} v(I_i) < \varepsilon,$$

then  $A$  is called a **set with zero measure (null set)**.

**Definition 13.4 (Set with Finite Area)**

Let  $\Omega \subset \mathbb{R}^2$  be a bounded set. If the constant function 1 is integrable on  $\Omega$ , then  $\Omega$  is called a **set with finite area**, and the area of  $\Omega$  is defined as:

$$v(\Omega) = \int_{\Omega} 1 = \iint_{\Omega} dx dy = \int_I 1_{\Omega}.$$



Obviously,  $\Omega$  is a set with zero area if and only if  $\Omega$  has finite area, and  $v(\Omega) = \int_{\Omega} 1 = 0$ .

## 13.2 Properties and Calculation of Multiple Integrals

**¶ Reduction of Double Integral to Iterated Integral**

**Theorem 13.1 (Reduction of Double Integral to Iterated Integral on a Closed Interval)**

Let  $f$  be integrable on the closed interval  $I = [a, b] \times [c, d]$ . If  $\forall x \in [a, b]$ , the function  $f(x, \cdot)$  is integrable on  $[c, d]$ , then:

$$\iint_I f = \int_a^b \left( \int_c^d f(x, y) dy \right) dx =: \int_a^b dx \int_c^d f(x, y) dy.$$

Similarly, if  $\forall y \in [c, d]$ , the function  $f(\cdot, y)$  is integrable on  $[a, b]$ , then:

$$\iint_I f = \int_c^d \left( \int_a^b f(x, y) dx \right) dy =: \int_c^d dy \int_a^b f(x, y) dx.$$



On the basis of the above theorem, we can extend it to the case of a bounded region.

**Theorem 13.2 (Reduction of Double Integral to Iterated Integral on a Bounded Set)**

Let  $\Omega \subset \mathbb{R}^2$  be a set with infinite area, and  $f : \Omega \rightarrow \mathbb{R}$  be bounded and continuous (13.1). Denote the vertical projection of  $\Omega$  onto the  $x$ -axis as:

$$I = \{x \in \mathbb{R} \mid \exists y, \text{ s.t. } (x, y) \in \Omega\}.$$

If  $\forall x \in I$ , let  $\Omega_x = \{y \in \mathbb{R} \mid (x, y) \in \Omega\}$  be an interval (possibly reducing to a single point), then:

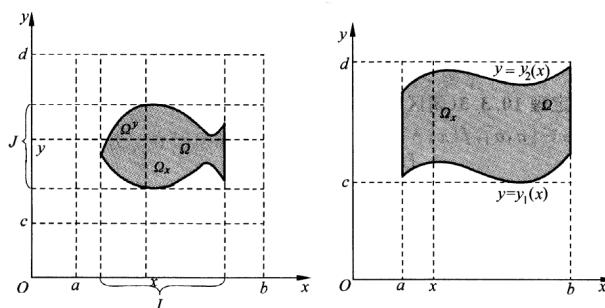
$$\int_{\Omega} f = \int_I dy \int_{\Omega_x} f(x, y) dx.$$

Similarly, denote the vertical projection of  $\Omega$  onto the  $y$ -axis as:

$$J = \{y \in \mathbb{R} \mid \exists x, \text{ s.t. } (x, y) \in \Omega\}.$$

If  $\forall y \in J$ , let  $\Omega_y = \{x \in \mathbb{R} \mid (x, y) \in \Omega\}$  be an interval (possibly reducing to a single point), then:

$$\int_{\Omega} f = \int_J dx \int_{\Omega_y} f(x, y) dy.$$



**Figure 13.1:** Double Integral on a Bounded Set

Specially, Let:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid y_1(x) \leq y \leq y_2(x), a \leq x \leq b\},$$

where the functions  $y_1$  and  $y_2$  are continuous on  $[a, b]$  (13.1) and the function  $f$  is integrable on  $\Omega$ . If  $\forall x \in [a, b]$ ,

the single-variable integral:

$$\int_{y_1(x)}^{y_2(x)} f(x, y) dy$$

exists, then:

$$\int_{\Omega} f = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy.$$

This area called the **type X region**, similarly, we can define the **type Y region**.

According to 13.1, we can derive the formula of multiplicative property for double integral.

**Theorem 13.3 (Formula of Multiplicative Property for Double Integral)**

Let  $f \in C([a, b])$ ,  $g \in C([c, d])$ . Then the function  $h(x, y) = f(x)g(y)$  is integrable on the closed interval  $I = [a, b] \times [c, d]$ , and:

$$\iint_I h(x, y) dx dy = \left( \int_a^b f(x) dx \right) \left( \int_c^d g(y) dy \right).$$



**Example 13.1** Let  $p(x) \in R[a, b]$ ,  $p(x) > 0$ ,  $x \in [a, b]$ , the monotonicity of  $f(x), g(x)$  is same, prove that

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx$$

**Proof** Let

$$I = \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx,$$

then

$$I = \int_a^b \int_a^b p(x)p(y)g(y)(f(x) - f(y))dx dy,$$

similarly,

$$I = \int_a^b \int_a^b p(x)p(y)g(x)(f(x) - f(y))dx dy.$$

Then

$$2I = \int_a^b \int_a^b p(x)p(y)(g(y) - g(x))(f(x) - f(y))dx dy \geq 0,$$

which implies

$$I \geq 0.$$

The proof is complete. ■

**¶ Calculation of Triple Integrals**

**Example 13.2** Calculating  $I = \iiint_{\Omega} z^2 dx dy dz$ , where  $\Omega$  is the cone defined by  $z^2 = \frac{h^2}{R^2}(x^2 + y^2)$  and  $z = h$  (13.2).

**Example 13.3** Calculating  $I = \iiint_{\Omega} xy dx dy dz$ , where  $\Omega$  is the region defined by  $0 \leq z \leq xy$ ,  $0 \leq y \leq 1 - x$ ,  $0 \leq x \leq 1$  (13.3).

With the help of examples above, we can derive **two methods for calculating triple integrals**.

**First 2 then 1 (Section Method)** Fix one variable (e.g.,  $z$ ), first perform a double integral over the other two

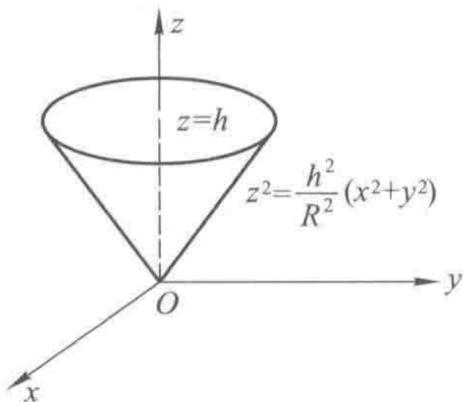


Figure 13.2: Cone Example.

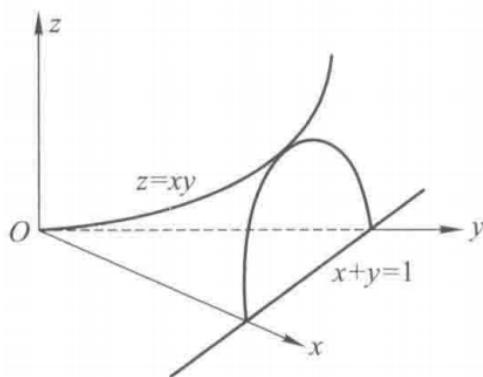


Figure 13.3: Project Method Example.

variables (e.g.,  $x, y$ ) on the "section region" corresponding to the fixed variable, and then perform a definite integral over the fixed variable ( $z$ ) within its range of values.

This method is convenient when the area of the section region is easy to calculate, or when the integrand is only related to the "later-integrated variable" (e.g., only related to  $z$ ).

In the example 13.2, the following steps are taken:

1. Determine the range of  $z$ :  $z \in [0, h]$ .
2. Determine the section region  $D_z$ : For a fixed  $z$ ,  $D_z$  is the region on the  $xy$ -plane satisfying  $\frac{h^2}{R^2}(x^2 + y^2) \leq z^2$ , which is a circle with radius  $\frac{R}{h}z$ .
3. Split the integral:

$$I = \int_0^h \left( \iint_{D_z} z^2 \, dx \, dy \right) dz.$$

Since  $z^2$  is independent of  $x$  and  $y$ , it can be factored out:  $I = \int_0^h z^2 \left( \iint_{D_z} \, dx \, dy \right) dz$ .

4. Calculate the double integral (area of the section):

$$\iint_{D_z} \, dx \, dy = \pi \left( \frac{R}{h} z \right)^2 = \pi \frac{R^2}{h^2} z^2.$$

5. Calculate the definite integral:

$$I = \int_0^h z^2 \cdot \pi \frac{R^2}{h^2} z^2 \, dz = \frac{\pi R^2 h^3}{5}.$$

**First 1 then 2 (Project Method)** Fix two variables (e.g.,  $x, y$ ), first perform a definite integral over the third variable (e.g.,  $z$ ) on the "vertical line segment" corresponding to the fixed variables, and then perform a double integral over the fixed two variables ( $x, y$ ) on their "projection region".

This method is convenient when the projection region of the integral region on a certain coordinate plane (e.g.,  $xy$ -plane) is easy to determine, and the upper and lower limits of a single variable (e.g.,  $z$ ) can be easily expressed by the other two variables.

In the example 13.3, the following steps are taken:

1. Determine the projection region  $D_{xy}$ :  $D_{xy}$  is the region on the  $xy$ -plane bounded by  $x + y \leq 1$ ,  $x \geq 0$ , and  $y \geq 0$ , which can be expressed as  $0 \leq x \leq 1$  and  $0 \leq y \leq 1 - x$ .
2. Determine the range of  $z$ :  $z \in [0, xy]$  (since  $z$  is bounded below by  $z = 0$  and above by  $z = xy$ ).
3. Split the integral:

$$I = \iint_{D_{xy}} \left( \int_0^{xy} xy dz \right) dx dy,$$

split the double integral on  $D_{xy}$  as:  $I = \int_0^1 dx \int_0^{1-x} dy \int_0^{xy} xy dz$ . (Since  $xy$  is independent of  $z$ , it can be factored out without affecting the integral:  $I = \int_0^1 dx \int_0^{1-x} xy dy \int_0^{xy} dz$ .)

4. Calculate the inner integral (with respect to  $z$ ):  $\int_0^{xy} xy dz = xy \cdot \int_0^{xy} dz = xy \cdot z|_0^{xy} = xy \cdot xy = x^2 y^2$ .
5. Calculate the middle integral (with respect to  $y$ ): Substitute the result of the inner integral,

$$\int_0^{1-x} x^2 y^2 dy = x^2 \cdot \frac{y^3}{3} \Big|_0^{1-x} = \frac{x^2 (1-x)^3}{3}.$$

6. Calculate the outer integral (with respect to  $x$ ): Substitute the result of the middle integral:

$$\begin{aligned} \int_0^1 \frac{x^2 (1-x)^3}{3} dx &= \frac{1}{3} \int_0^1 (x^2 - 3x^3 + 3x^4 - x^5) dx \\ &= \frac{1}{3} \left( \frac{x^3}{3} - \frac{3x^4}{4} + \frac{3x^5}{5} - \frac{x^6}{6} \Big|_0^1 \right) \\ &= \frac{1}{3} \left( \frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right) \\ &= \frac{1}{180}. \end{aligned}$$

Some tips for choosing between the two methods (take the above two examples as reference):

**First 2 then 1 (Section Method)**

Section area  $D_z$  is easy to calculate

Integrand is only related to  $z$

**First 1 then 2 (Project Method)**

Projection region  $D_{xy}$  is easy to determine

Upper and lower limits  $z$  can be easily expressed by the other two variables  $x, y$

## 13.3 Variable Substitution in Multiple Integrals

*Theorem 13.4 (Variable Substitution in Double Integral)*

Let  $\Omega \subset \mathbb{R}^2$  be an open set, and let the mapping:

$$\mathbf{F} : \Omega \rightarrow \mathbb{R}^2, \quad (u, v) \mapsto \mathbf{F}(u, v) = (x(u, v), y(u, v))$$

satisfy the following conditions:

1.  $\mathbf{F} \in C^1(\Omega, \mathbb{R}^2)$ ;
2.  $\frac{\partial(x, y)}{\partial(u, v)} = \det J\mathbf{F}(u, v) = \det J\mathbf{F}(\mathbf{p}) \neq 0, \quad \mathbf{p} = (u, v) \in \Omega$ ;
3.  $\mathbf{F}$  is injective.

If the set  $\Delta$  is a set with finite area and  $\overline{\Delta} \subset \Omega$ , and  $f$  is continuous on  $\mathbf{F}(\Omega)$ , then  $\mathbf{F}(\Delta)$  is also a set with finite area, and:

$$\iint_{\mathbf{F}(\Delta)} f = \iint_{\Delta} f \circ \mathbf{F} |\det J\mathbf{F}|,$$

i.e.,

$$\iint_{F(\Delta)} f(x, y) \, dx \, dy = \iint_{\Delta} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.$$



For triple and higher-dimensional integrals, the variable substitution theorem is similar to the above theorem.

Some common variable substitutions in multiple integrals are as follows:

### Polar Coordinates

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2}, & r \geq 0 \\ \theta = \arctan \left( \frac{y}{x} \right) & x \neq 0, \theta \in [0, 2\pi]. \end{cases}$$

and

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

### Cylindrical Coordinate System

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \\ z = z, \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2}, & r \geq 0 \\ \theta = \arctan \left( \frac{y}{x} \right) & x \neq 0, \theta \in [0, 2\pi], \\ z = z. \end{cases}$$

and

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = r.$$

### Spherical Coordinate System

$$\begin{cases} x = r \sin \varphi \cos \theta, \\ y = r \sin \varphi \sin \theta, \\ z = r \cos \varphi, \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2}, & r \geq 0 \\ \varphi = \arccos \left( \frac{z}{r} \right) & r \neq 0, \varphi \in [0, \pi], \\ \theta = \arctan \left( \frac{y}{x} \right) & x \neq 0, \theta \in [0, 2\pi]. \end{cases}$$

and

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = r^2 \sin \varphi.$$

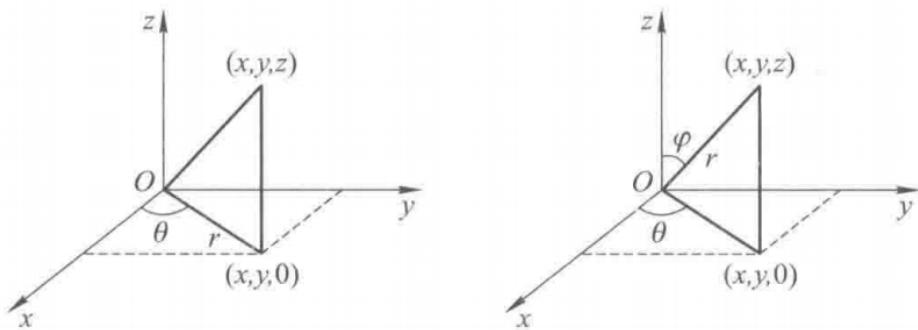


Figure 13.4: Cylindrical and Spherical Coordinate Systems

## 13.4 Improper Multiple Integrals

# Chapter 14 Line Integrals and Surface Integrals

## 14.1 Line Integrals and Surface Integrals of scalar fields

### ¶ Line Integral of Scalar Field

*Definition 14.1 (Line Integral of Scalar Field)*

Let  $L$  is a rectifiable continuous curve in  $\mathbb{R}^3$ , whose endpoints are  $A$  and  $B$ , and  $f(x, y, z)$  is bounded on  $L$ . Partition  $L$  into  $n$  segments by points  $A = P_0, P_1, \dots, P_n = B$ , and select a point  $\xi_i$  on each segment  $P_{i-1}P_i$  ( $i = 1, 2, \dots, n$ ). Remark that the length of segment  $P_{i-1}P_i$  is  $\Delta s_i$  ( $i = 1, 2, \dots, n$ ), and make the sum:

$$\sum_{i=1}^n f(\xi_i) \Delta s_i.$$

If when  $\lambda$  (the length of the longest segment) tends to 0, the above sum tends to a limit  $I$  independent of the partition and the choice of points  $\xi_i$ , then  $I$  is called the **line integral of the scalar field  $f$  along the curve  $L$** , denoted as:

$$\int_L f \, ds.$$

That is,

$$I = \int_L f(\xi) \, ds = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta s_i.$$



*Theorem 14.1*

Let  $L$  be a  $C^1$  smooth regular curve parameterized by  $\mathbf{x}(t) = (x(t), y(t), z(t))$ ,  $t \in [\alpha, \beta]$ , and  $f$  be continuous on  $L$ . Then:

$$\int_L f \, ds = \int_{\alpha}^{\beta} f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| \, dt. = \int_{\alpha}^{\beta} f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt.$$



### ¶ Surface Theory

### ¶ Surface Integrals of Scalar Fields

*Definition 14.2 (Surface Integral of Scalar Field)*

Let  $\Sigma$  be a piecewise smooth surface in  $\mathbb{R}^3$ , and  $f(x, y, z)$  be bounded on  $\Sigma$ . Partition  $\Sigma$  into  $n$  small pieces  $\Delta\Sigma_1, \Delta\Sigma_2, \dots, \Delta\Sigma_n$  with smooth curve webs, and select a point  $\xi_i$  on each piece  $\Delta\Sigma_i$  ( $i = 1, 2, \dots, n$ ). Remark that the area of piece  $\Delta\Sigma_i$  is  $\Delta S_i$  ( $i = 1, 2, \dots, n$ ), and make the sum:

$$\sum_{i=1}^n f(\xi_i) \Delta S_i.$$

If when  $\lambda$  (the area of the largest piece) tends to 0, the above sum tends to a limit  $I$  independent of the partition and the choice of points  $\xi_i$ , then  $I$  is called the **surface integral of the scalar field  $f$  over the surface  $\Sigma$** , denoted

as:

$$\iint_{\Sigma} f \, dS.$$

That is,

$$I = \iint_{\Sigma} f(\xi) \, dS = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta S_i.$$



### Theorem 14.2

Let  $\Sigma$  be a piecewise smooth closed surface parameterized by  $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ ,  $(u, v) \in D$ , and  $f$  be continuous on  $\Sigma$ .  $x, y, z$  have continuous first-order partial derivatives with respect to  $u$  and  $v$  on  $D$ , and according Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

is of full rank. Then:

$$\iint_{\Sigma} f \, dS = \iint_D f(\mathbf{r}(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, du \, dv = \iint_D f(x(u, v), y(u, v), z(u, v)) \sqrt{EG - F^2} \, du \, dv,$$

where

$$\begin{aligned} E &= \mathbf{r}_u \cdot \mathbf{r}_u = x_u^2 + y_u^2 + z_u^2, \\ F &= \mathbf{r}_u \cdot \mathbf{r}_v = x_u x_v + y_u y_v + z_u z_v, \\ G &= \mathbf{r}_v \cdot \mathbf{r}_v = x_v^2 + y_v^2 + z_v^2, \end{aligned}$$

and it is called the Gauß coefficients of the surface  $\Sigma$ .



## 14.2 Differential Form and Exterior Differentiation

Let  $dx_i, dx_j$  be differentials of independent variables  $x_i, x_j$ .

In  $\mathbb{R}^1$ :

0-form:  $f(x)$ ,

1-form:  $\omega = f(x)dx$ ,

k-form ( $k \geq 2$ ):  $\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1 i_2 \dots i_k}(x_1, x_2, \dots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k} = 0$ .

In  $\mathbb{R}^2$ :

0-form:  $f(x, y)$ ,

1-form:  $\omega = P(x, y)dx + Q(x, y)dy$ ,

2-form:  $\omega = f(x, y)dx \wedge dy$ ,

k-form ( $k \geq 3$ ):  $\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1 i_2 \dots i_k}(x_1, x_2, \dots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k} = 0$ .

In  $\mathbb{R}^3$ :

0-form:  $f(x, y, z)$ ,

1-form:  $\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$ ,

2-form:  $\omega = P(x, y, z)dy \wedge dz + Q(x, y, z)dz \wedge dx + R(x, y, z)dx \wedge dy$ ,

3-form:  $\omega = f(x, y, z)dx \wedge dy \wedge dz$ ,

$k$ -form ( $k \geq 4$ ):  $\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1 i_2 \dots i_k}(x_1, x_2, \dots, x_n)dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k} = 0$ .

Here,  $\wedge$  is called the **wedge product**, which satisfies:

1. Skew symmetric:  $dx_i \wedge dx_j = -dx_j \wedge dx_i$ ,
2. Associative:  $(dx_i \wedge dx_j) \wedge dx_k = dx_i \wedge (dx_j \wedge dx_k)$ ,
3. In a fixed dimension, the wedge product of  $k$ -forms becomes zero (as higher forms are not defined), for example, in 3-dimensional space, a 4-form is equal to 0.

**Differential form** is a skew symmetric tensor on vector field.

#### Definition 14.3 (Exterior Differentiation)

Let  $\omega$  be a  $k$ -form on  $\mathbb{R}^n$ ,

$$\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1 i_2 \dots i_k}(x_1, x_2, \dots, x_n)dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k},$$

where  $f_{i_1 i_2 \dots i_k}$  are functions with continuous first-order partial derivatives. The exterior differentiation of  $\omega$  is defined as:

$$d\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} df_{i_1 i_2 \dots i_k}(x_1, x_2, \dots, x_n) \wedge dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k},$$

where

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \dots + \frac{\partial f}{\partial x_n}dx_n.$$

Note that the exterior differentiation of a  $k$ -form is a  $k + 1$ -form.



#### Property

**Linearity**  $d(\alpha\omega + \beta\eta) = \alpha d\omega + \beta d\eta$ , where  $\alpha, \beta$  are constants.

**Leibniz Rule**  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ , where  $\omega$  is a  $k$ -form.

**Nilpotency**  $d(d\omega) = 0$ .

## 14.3 Orientation of Curves and Surfaces

## 14.4 Line Integrals and Surface Integrals of Vector Fields

### Line Integral of Vector Field

**Definition 14.4 (Line Integral of Vector Field)**

Let  $\vec{L}$  be a orientated smooth curve in  $\mathbb{R}^3$ , whose endpoints are  $A$  and  $B$ . Take unit tangent vector  $\tau = (\cos \alpha, \cos \beta, \cos \gamma)$  at each point of  $\vec{L}$ , making it consistent with the orientation of  $\vec{L}$ . Let  $\mathbf{f}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$  be a vector-valued function on  $\vec{L}$ , then

$$\int_{\vec{L}} \mathbf{f} \cdot \tau ds = \int_{\vec{L}} [P \cos \alpha + Q \cos \beta + R \cos \gamma] ds$$

is called the **line integral of the vector field  $\mathbf{f}$  along the oriented curve  $\vec{L}$**  (if the right-hand side exists). 

Consider a differential arc length element  $ds$  at a point  $(x, y, z)$  on the curve  $L$ . We form the vector  $ds = \tau ds$ , where  $\tau = (\cos \alpha, \cos \beta, \cos \gamma)$  represents the unit tangent vector of curve  $L$  at  $(x, y, z)$ , pointing along the direction of  $L$ . The projection of  $ds$  onto the  $x$ -axis is given by  $\cos \alpha ds$ . Therefore, we denote:

$$dx = \cos \alpha ds, \quad dy = \cos \beta ds, \quad dz = \cos \gamma ds.$$

Thus, the second type of line integral can be expressed as:

$$\int_L \mathbf{f} \cdot \tau ds = \int_L \mathbf{f} ds = \int_L P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz.$$

This line integral is also referred to as the integral of the 1-form:

$$\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz.$$

The second type of line integral of  $\omega$  along the curve  $L$  is denoted as:

$$\int_L \omega.$$

**Theorem 14.3**

Let  $\vec{L}$  be a  $C^1$  smooth regular oriented curve parameterized by  $\mathbf{x}(t) = (x(t), y(t), z(t))$ ,  $t \in [\alpha, \beta]$ , and  $\mathbf{f} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  be continuous on  $\vec{L}$ . Then:

$$\begin{aligned} \int_{\vec{L}} \mathbf{f} \cdot \tau ds &= \int_{\alpha}^{\beta} \mathbf{f}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt \\ &= \int_{\alpha}^{\beta} [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)] dt. \end{aligned}$$


**Surface Integral of Vector Field****Definition 14.5 (Surface Integral of Vector Field)**

Let  $\vec{\Sigma}$  be an orientated smooth surface in  $\mathbb{R}^3$ , and  $\mathbf{f}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$  be a

vector-valued function on  $\vec{\Sigma}$ . Each point appoints a unit normal vector  $\mathbf{n} = (\cos \alpha, \cos \beta, \cos \gamma)$ . Then

$$\iint_{\vec{\Sigma}} \mathbf{f} \cdot \mathbf{n} dS = \iint_{\vec{\Sigma}} [P \cos \alpha + Q \cos \beta + R \cos \gamma] dS$$

is called the **surface integral of the vector field  $\mathbf{f}$  over the oriented surface  $\vec{\Sigma}$**  (if the right-hand side exists). 

Consider a differential area element  $dS$  at a point  $(x, y, z)$  on the surface  $\Sigma$ . We form the vector  $d\mathbf{S} = \mathbf{n} dS$ , where  $\mathbf{n} = (\cos \alpha, \cos \beta, \cos \gamma)$  represents the unit normal vector of surface  $\Sigma$  at  $(x, y, z)$ , pointing along the orientation of  $\Sigma$ . The projection of  $dS$  onto the  $x$ -axis is given by  $\cos \alpha dS$ . Therefore, we denote:

$$dy \wedge dz = \cos \alpha dS, \quad dz \wedge dx = \cos \beta dS, \quad dx \wedge dy = \cos \gamma dS.$$

Thus, the surface integral can be expressed as:

$$\iint_{\vec{\Sigma}} \mathbf{f} \cdot \mathbf{n} dS = \iint_{\vec{\Sigma}} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy = \iint_{\Sigma} P dy dz + Q dz dx + R dx dy,$$

where  $dy dz$  is the simplified notation for  $dy \wedge dz$ , etc. This surface integral is also referred to as the integral of the 2-form:

$$\omega = P(x, y, z) dy \wedge dz + Q(x, y, z) dz \wedge dx + R(x, y, z) dx \wedge dy.$$

The second type of surface integral of  $\omega$  over the surface  $\Sigma$  is denoted as:

$$\iint_{\vec{\Sigma}} \omega.$$

#### Theorem 14.4

Let  $\vec{\Sigma}$  be a smooth oriented surface parameterized by  $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ ,  $(u, v) \in D$ , where  $D$  is a closed region with piecewise smooth boundary in  $uv$ -plane, and  $\mathbf{f} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  be continuous on  $\vec{\Sigma}$ .  $x, y, z$  have continuous first-order partial derivatives with respect to  $u$  and  $v$  on  $D$ , and according Jacobian matrix is of full rank. Then:

$$\begin{aligned} & \iint_{\vec{\Sigma}} \mathbf{f} \cdot \mathbf{n} dS \\ &= \iint_{\Sigma} [P \cos \alpha + Q \cos \beta + R \cos \gamma] dS \\ &= \iint_D \mathbf{f}(\mathbf{r}(u, v)) \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv \\ &= \pm \iint_D \left[ P(x(u, v), y(u, v), z(u, v)) \cdot \frac{\partial(y, z)}{\partial(u, v)} + Q(x(u, v), y(u, v), z(u, v)) \cdot \frac{\partial(z, x)}{\partial(u, v)} \right. \\ &\quad \left. + R(x(u, v), y(u, v), z(u, v)) \cdot \frac{\partial(x, y)}{\partial(u, v)} \right] dudv, \end{aligned}$$

where the sign  $\pm$  depends on whether the orientation of  $\vec{\Sigma}$  is consistent with the direction of  $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ . 

## 14.5 Stokes' Formula

¶ Newton-Leibniz Formula

¶ Green's Formula

Consider two kinds of special orientated closed regions in  $xy$ -plane as shown in Figure 14.1. As for the first region  $\vec{M}$ , it consists of four orientated curves:

- $\vec{C}_1$   $y = \varphi_1(x), x \in [a, b]$ ,
- $\vec{C}_2$   $x = b, y \in [\varphi_1(b), \varphi_2(b)]$ , can be reduced to a point,
- $\vec{C}_3$   $y = \varphi_2(x), x \in [a, b]$ ,
- $\vec{C}_4$   $x = a, y \in [\varphi_1(a), \varphi_2(a)]$ , can be reduced to a point.

The second region is similar.

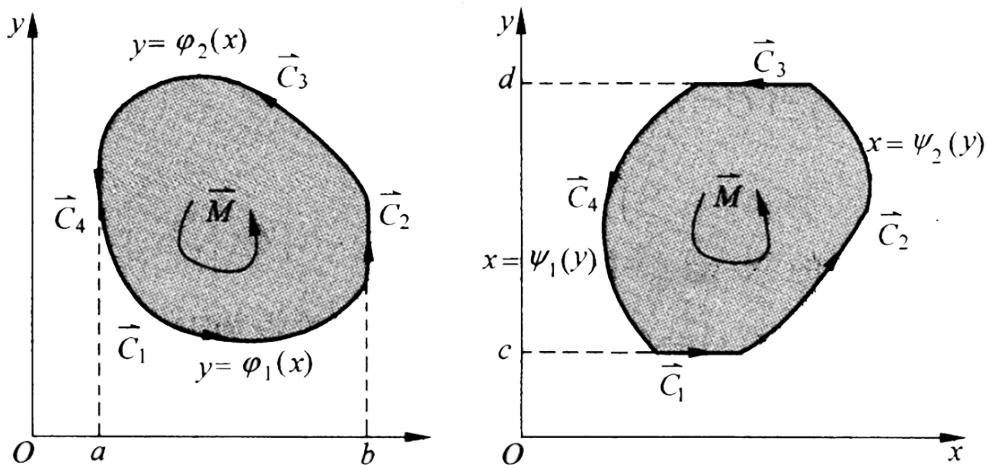


Figure 14.1: Two special orientated closed regions.

Denote  $\oint_{\partial M} P dx + Q dy$  as the line integral along the boundary of region  $\vec{M}$ , then we have the following lemma.

*Lemma 14.1*

1. Let  $\vec{\partial M}$  be the first region in Fig 14.1,  $P(x, y) \in C^1(M)$ , then:

$$\oint_{\vec{\partial M}} P dx = - \iint_{\vec{M}} \frac{\partial P}{\partial y} dx \wedge dy,$$

2. Let  $\vec{\partial M}$  be the second region in Fig 14.1,  $Q(x, y) \in C^1(M)$ , then:

$$\oint_{\vec{\partial M}} Q dy = \iint_{\vec{M}} \frac{\partial Q}{\partial x} dx \wedge dy.$$

*Theorem 14.5 (Green's Theorem)*

Let  $\vec{M}$  be an orientated closed region in  $\mathbb{R}^2$ , and  $\omega = Pdx + Qdy \in C^1(M)$ . If  $\vec{\partial M}$  can be split into finitely many first and second regions in Fig 14.1 simultaneously (non-overlapping, no shared interior points), then:

$$\oint_{\vec{\partial M}} P dx + Q dy = \iint_{\vec{M}} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy = \iint_M \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy,$$

<sup>a</sup> or equivalently,

$$\oint_{\partial M} \omega = \iint_M d\omega,$$

where  $\vec{\partial M}$  is the induced orientation of  $\vec{M}$ .

<sup>a</sup>Note that  $dx \wedge dy$  is directed area element, while  $dxdy$  is unsigned area element.



### Gauß's Formula

Consider three kinds of special orientated closed surfaces in  $\mathbb{R}^3$  as shown in Figure 14.2. As for the first surface  $\vec{M}$  ( $\vec{M}$  adopts a positive orientation (right-hand system), and  $\vec{\partial M}$  adopts the outward normal orientation), it consists of three orientated surfaces:

$$\Sigma_1 z = \varphi_1(x, y), (x, y) \in \Delta_1,$$

$$\Sigma_2 z = \varphi_2(x, y), (x, y) \in \Delta_1,$$

$\Sigma_3$  A cylindrical taking  $\partial\Delta_1$  as the directrix, with the generatrix paralleling to the  $Oz$ -axis. Of course, it can also be reduced as a closed curve.

The second and third surfaces are similar.

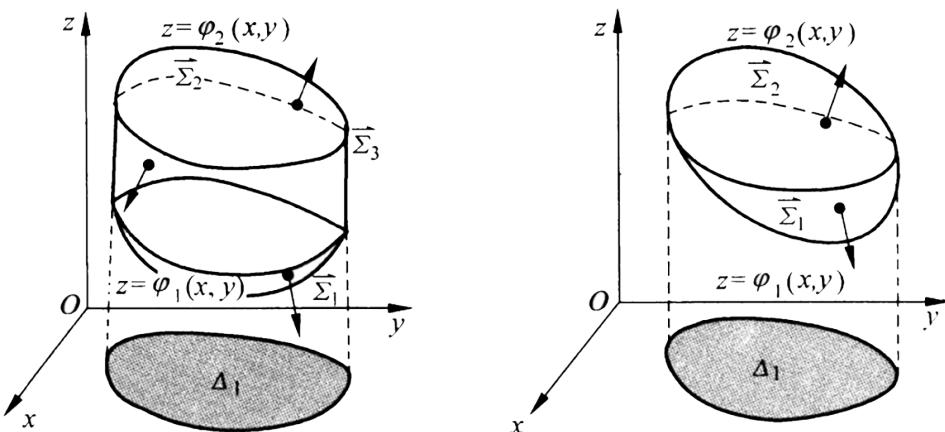


Figure 14.2: Three special orientated closed surfaces (only the first two are shown).

Denote  $\iint_{\partial M} \omega$  as the surface integral over the boundary of region  $M$ , then we have the following lemma.

#### Lemma 14.2

- Let  $\vec{\partial M}$  be the first surface in Fig 14.2,  $R(x, y, z) \in C^1(M)$ , then:

$$\iint_{\partial M} R dx \wedge dy = \iiint_{\vec{M}} \frac{\partial R}{\partial z} dx \wedge dy \wedge dz,$$

- Let  $\vec{\partial M}$  be the second surface in Fig 14.2,  $P(x, y, z) \in C^1(M)$ , then:

$$\iint_{\partial M} P dy \wedge dz = \iiint_{\vec{M}} \frac{\partial P}{\partial x} dx \wedge dy \wedge dz,$$

3. Let  $\vec{\partial M}$  be the third surface in Fig 14.2,  $Q(x, y, z) \in C^1(M)$ , then:

$$\oint_{\vec{\partial M}} Q \, dz \wedge dx = \iiint_{\vec{M}} \frac{\partial Q}{\partial y} \, dx \wedge dy \wedge dz.$$



**Theorem 14.6 (Gauß's Theorem)**

Let  $\vec{M}$  be an orientated closed region in  $\mathbb{R}^3$ , and  $\omega = P \, dy \wedge dz + Q \, dz \wedge dx + R \, dx \wedge dy \in C^1(M)$ . If  $\vec{\partial M}$  can be split into finitely many first, second and third regions in Fig 14.1 simultaneously (non-overlapping, no shared interior points), then: then:

$$\oint_{\vec{\partial M}} P \, dy \wedge dz + Q \, dz \wedge dx + R \, dx \wedge dy = \iiint_{\vec{M}} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \, dx \wedge dy \wedge dz,$$

or equivalently,

$$\oint_{\vec{\partial M}} \omega = \iiint_{\vec{M}} d\omega,$$

where  $\vec{\partial M}$  is the induced orientation of  $\vec{M}$ .



**Stokes' Formula**

**Theorem 14.7 (Stokes' Theorem)**

Let  $\vec{M}$  be an orientated smooth surface in  $\mathbb{R}^3$  with boundary  $\vec{\partial M}$ , and  $\omega = P \, dx + Q \, dy + R \, dz \in C^1(\Sigma)$ . Then:

$$\begin{aligned} & \oint_{\vec{\partial M}} P \, dx + Q \, dy + R \, dz \\ &= \iint_{\vec{M}} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \, dy \wedge dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \, dz \wedge dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \wedge dy \\ &= \iint_{\vec{M}} \begin{vmatrix} dy \wedge dz & dz \wedge dx & dx \wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \iint_{\vec{M}} \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \, dS, \end{aligned}$$

or equivalently,

$$\oint_{\vec{\partial M}} \omega = \iint_{\vec{M}} d\omega,$$

where  $\vec{\partial M}$  is the induced orientation of  $\vec{M}$ .



## 14.6 Closed and Exact Differential Forms

*Definition 14.6 (Closed and Exact Differential Forms)*

Let  $U \subset \mathbb{R}^n$  be an open set and  $\omega$  be a  $C^r$  ( $r \geq 1$ )  $k$ -form on  $U$ .

1. If  $d\omega = 0$ , then  $\omega$  is called a **closed form**.
2. If there exists a  $C^{r+1}$  ( $k - 1$ )-form  $\eta$  such that  $\omega = d\eta$ , then  $\omega$  is called an **exact differential form**.



## 14.7 Introduction to Field Theory

# Chapter 15 Integrals with Variable Parameters

## 15.1 Definite Integrals with Variable Parameters

**Definition 15.1 (Definite Integral with Variable Parameters)**

Let  $f(x, y)$  be defined on  $[a, b] \times [c, d]$ . For each fixed  $y \in [c, d]$ , if the definite integral

$$I(y) = \int_a^b f(x, y) dx$$

exists, then  $I(y)$  is called a **definite integral with variable parameter  $y$** .



## 15.2 Improper Integrals with Variable Parameters

There are two types of improper integrals with variable parameters: on infinite interval and with unbounded integrand. Here we only give the definition of improper integrals on infinite interval with variable parameters.

**Definition 15.2 (Improper Integral with Variable Parameters)**

Let  $f(x, y)$  be defined on  $[a, +\infty) \times [c, d]$ . For some fixed  $y_0 \in [c, d]$ , if the improper integral  $I(y_0) = \int_a^{+\infty} f(x, y_0) dx$  converges, then  $\int_a^{+\infty} f(x, y) dx$  is called convergent at  $y_0$ , and  $y_0$  is called its convergence point.

Let the set of all convergence points be  $E$ , then  $E$  is the domain of definition of the improper integral with variable parameters

$$I(y) = \int_a^{+\infty} f(x, y) dx,$$

also called the convergence domain of the improper integral  $\int_a^{+\infty} f(x, y) dx$ .



### ¶ Uniform Convergence and Its Tests

**Definition 15.3 (Uniform Convergence of Improper Integrals with Variable Parameters)**

Let  $f(x, y)$  be defined on  $[a, +\infty) \times [c, d]$ , where  $[c, d]$  is the convergence domain of the improper integral  $\int_a^{+\infty} f(x, y) dx$ . If for every  $\varepsilon > 0$ , there exists a number  $A_0 > a$  independent of  $y$ , such that for all  $A > A_0$  and for all  $y \in [c, d]$ ,

$$\left| \int_a^A f(x, y) dx - I(y) \right| = \left| \int_A^{+\infty} f(x, y) dx \right| < \varepsilon,$$

then the improper integral  $\int_a^{+\infty} f(x, y) dx$  is said to be **uniformly convergent** on  $[c, d]$ .



**Theorem 15.1 (Cauchy Criterion for Uniform Convergence of Improper Integrals with Variable Parameters)**

Let  $f(x, y)$  be defined on  $[a, +\infty) \times [c, d]$ , where  $[c, d]$  is the convergence domain of the improper integral  $\int_a^{+\infty} f(x, y) dx$ . The improper integral  $\int_a^{+\infty} f(x, y) dx$  is uniformly convergent on  $[c, d]$  if and only if for every  $\varepsilon > 0$ , there exists a number  $A_0 > a$  independent of  $y$ , such that for all  $A_1, A_2 > A_0$  and for all

$y \in [c, d]$ ,

$$\left| \int_{A_1}^{A_2} f(x, y) dx \right| < \varepsilon.$$



## ¶ Analysis Properties of Uniform Convergence

*Lemma 15.1*



*Theorem 15.2 (Uniform Convergence and Continuity)*

Let  $f(x, y)$  be continuous on  $[a, +\infty) \times [c, d]$ , and  $\int_a^{+\infty} f(x, y) dx$  is uniformly convergent on  $[c, d]$  with respect to  $y$ , then:

(i)

$$I(y) = \int_a^{+\infty} f(x, y) dx$$

is continuous on  $[c, d]$ , i.e.,

$$\lim_{y \rightarrow y_0} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \lim_{y \rightarrow y_0} f(x, y) dx, \quad y_0 \in [c, d],$$

that is, the limit and the integral can be interchanged.

(ii)

$$\int_c^d dy \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} dx \int_c^d f(x, y) dy,$$

that is, the order of integration can be interchanged.



When  $[c, d]$  is replaced by  $[c, +\infty)$ , the above theorem fails, but we have the following theorem.

*Theorem 15.3*

On the region  $D = [a, +\infty) \times [c, +\infty)$ ,

1. if  $f(x, y)$  satisfies:

- (a).  $f(x, y) \in C(D)$ ;
- (b).  $\int_a^{+\infty} f(x, y) dx$  internally closed uniformly converges with respect to  $y$ ;  $\int_c^{+\infty} f(x, y) dy$  internally closed uniformly converges with respect to  $x$ ;
- (c). One of the two integrals  $\int_a^{+\infty} dx \int_c^{+\infty} |f(x, y)| dy$  or  $\int_c^{+\infty} dy \int_a^{+\infty} |f(x, y)| dx$  converges;

then

$$\int_c^{+\infty} dy \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} dx \int_c^{+\infty} f(x, y) dy$$

2. if  $f(x, y)$  satisfies:

- (a).  $f(x, y) \in C(D)$  and  $f(x) \geq 0$  on  $D$ ;
- (b).  $\int_a^{+\infty} f(x, y) dx \in C[c, +\infty)$ ;  $\int_c^{+\infty} f(x, y) dy \in C[a, +\infty)$ ;
- (c). One of the two integrals  $\int_a^{+\infty} dx \int_c^{+\infty} f(x, y) dy$  or  $\int_c^{+\infty} dy \int_a^{+\infty} f(x, y) dx$  converges;

then

$$\int_c^{+\infty} dy \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} dx \int_c^{+\infty} f(x, y) dy$$



**Remark** One of the two integrals exists implies the other exists as well as the equality holds.

**Theorem 15.4 (Uniform Convergence and Differentiation)**

On the region  $D = [a, +\infty] \times [c, d]$ , if the following conditions are satisfied:

- (i)  $\frac{\partial}{\partial y} f(x, y) \in C(D)$ ;
- (ii)  $\int_a^{+\infty} \frac{\partial}{\partial y} f(x, y) dx$  converges uniformly with respect to  $y$  on  $[c, d]$ ;
- (iii) There exists a point  $y_0 \in [c, d]$ , such that  $\int_a^{+\infty} f(x, y_0) dx$  converges;
- (iv) For any  $[\alpha, \beta] \subset [a, +\infty)$ ,  $\int_\alpha^\beta f(x, y) dx$  exists.

Then  $I(y) = \int_a^{+\infty} f(x, y) dx$  is differentiable on  $[c, d]$ , and

$$\frac{d}{dy} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \frac{\partial}{\partial y} f(x, y) dx.$$



**Example 15.1** Compute Dirichlet's integral:

$$I = \int_0^{+\infty} \frac{\sin x}{x} dx.$$

*Solution*



## 15.3 Euler Integrals

### Beta Function

Beta function can be defined in the following equivalent forms:

1. For  $p > 0, q > 0$ :

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt.$$

2. Via substitution  $t = \frac{u}{1+u}$ :

$$B(p, q) = \int_0^{+\infty} \frac{u^{p-1}}{(1+u)^{p+q}} du.$$

3. Via substitution  $t = \sin^2 \theta$ :

$$B(p, q) = 2 \int_0^{\frac{\pi}{2}} \sin^{2p-1} \theta \cos^{2q-1} \theta d\theta.$$

Then we have:

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi, \quad B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\pi}{2}, \quad B(1, 1) = 1.$$

### Property

**Continuity**  $B(p, q) \in C(U)$ , where  $U = \{(p, q) | p > 0, q > 0\}$ .

**Symmetry**  $B(p, q) = B(q, p)$ .

**Recurrence Relation**  $B(p, q) = \frac{q-1}{p+q-1} B(p, q-1)$  for  $p > 0, q > 1$ .

### Gamma Function

Gamma function can be defined in the following equivalent forms:

1. For  $s > 0$ :

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx.$$

2. Via the limit:

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{n!}{s(s+1)(s+2)\cdots(s+n)}.$$

3. Via substitution  $x = t^2$ :

$$\Gamma(s) = \frac{1}{2} \int_0^{+\infty} t^{2s-1} e^{-t^2} dt.$$

Then we have:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \Gamma(1) = 1.$$

### Property

**Continuity**  $\Gamma(s) \in C(0, +\infty)$ .

**Recurrence Relation**  $\Gamma(s+1) = s\Gamma(s)$  for  $s > 0$ .

Gamma function can be extended to the whole complex plane except for non-positive integers, where it has simple poles.

## Relation between Beta and Gamma Functions

### Theorem 15.5

There holds the following relation between Beta and Gamma functions:

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p > 0, q > 0.$$

Next, we give three important formulas about Gamma function, which can be extended to the complex domain as well.

### Theorem 15.6 (Legendre's Duplication Formula)

For  $s > 0$ , there holds:

$$\Gamma(s)\Gamma(s + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s).$$

### Theorem 15.7 (Reflection Formula)

For  $0 < s < 1$ , there holds:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

### Theorem 15.8 (Stirling's Formula)

$$\Gamma(s+1) = \sqrt{2\pi s} \left(\frac{s}{e}\right)^s \exp\left(\frac{\theta}{12s}\right),$$

where  $0 < \theta < 1$ .

Specially, when  $s = n \in \mathbb{N}$ ,

$$\Gamma(n+1) = n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{\theta}{12n}\right),$$

where  $0 < \theta < 1$ .

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