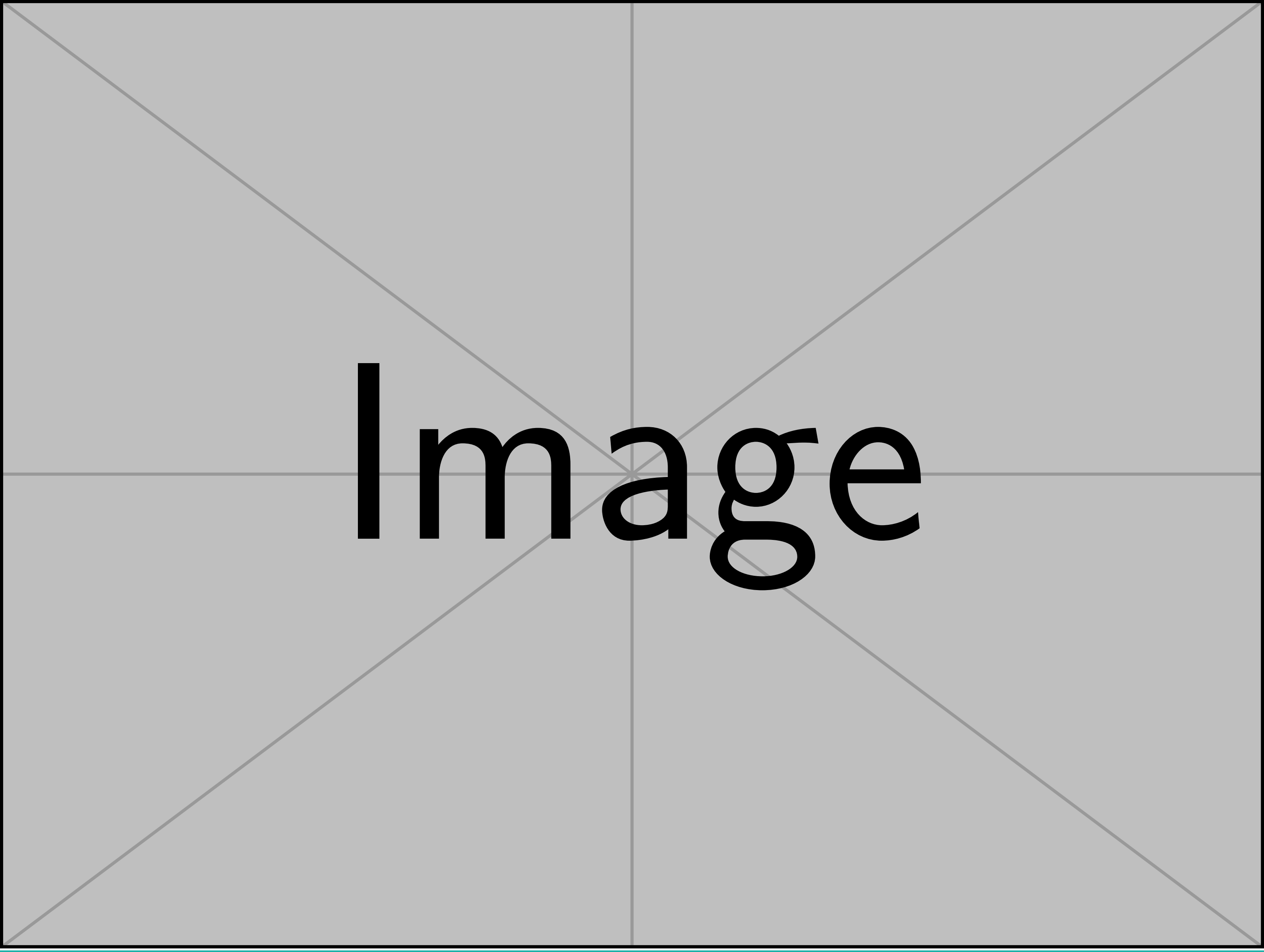

Analyse Harmonique

Author: CatMono

Date: November, 2025

Version: 0.1

Contents

Preface

This is the preface of the book…

Chapter 1 Classical Fourier Series

In this chapter, we will explore the Fourier series in such function space:

Set and Field The linear space we are working on is the set of all integrable (in the Riemann sense)[1] complex-valued periodic functions defined on $[-\pi, \pi]$[2], equipped with the usual addition and scalar multiplication of functions. We denote it as $\mathcal{R}[-\pi, \pi]$ that is a infinite-dimensional linear space. The field of scalars is the set of complex numbers $\mathbb{C}$.

Inner Product For any two functions $f(x), g(x)$ in this space, we define their inner product as:

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x,$$

where $\frac{1}{2\pi}$ is a normalization factor.

Norm The norm induced by this inner product is given by:

$$\|f\| = \sqrt{\langle f, f\rangle} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

In fact, we often assume that the functions are always piecewise continuous or piecewise smooth on $[-\pi, \pi]$, which is the most common case in engineering.

¶ **Function Defined on the Unit Circle**

For a periodic function $f(x) : \mathbb{R} \to \mathbb{C}$ with period 2π, we can explore it from the perspective of complex exponential functions on the unit circle in the complex plane. Let

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\},$$

which is one-dimensional torus, also known as the unit circle in the complex plane.

For any $\theta \in \mathbb{R}$, we can define:

$$f(\theta) = F(e^{i\theta}),$$

where $F : \mathbb{T} \to \mathbb{C}$ is a **function defined on the unit circle**. Thus, we can study the periodic function $f(x)$ by analyzing the function $F(z)$ on the unit circle $\mathbb{T}$. From the perspective of algebra, the set of all such functions $F(z)$ forms a function space over the unit circle, which is isomorphic to the space of periodic functions $f(x)$ with period 2π.

By introducing $\mathbb{T}$ that is a compact manifold without boundary in fact, we can not only eliminate the hassles of endpoints but also simplify many discussions. Furthermore, since $\mathbb{T}$ is a multiplicative group of complex numbers, we can better understand the essence of Fourier series: the duality theory on compact Abelian groups.

---

[1] For common integral, it should be Riemann integral; for defective integral, it should be absolute Riemann integral. For convenience, we just say Riemann integral in this context.

[2] It can be also defined on interval $[-T, T]$, but we choose $[-\pi, \pi]$ for simplicity.

1.1 Fourier Coefficients

Theorem 1.1

$$\mathcal{E} = \{e^{inx} : n \in \mathbb{Z}\}$$

or in real form:

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

is an orthonormal basis of the inner product space $\mathcal{R}[-\pi, \pi]$.

Definition 1.1

The Fourier coefficients $\hat{f}(n)$ of a function $f(x) \in \mathcal{R}[-\pi, \pi]$ is the projection of $f(x)$ onto the basis function e^{inx}:

$$\hat{f}(n) = \langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x, \quad n \in \mathbb{Z},$$

that is called Euler-Fourier formula.
Hence, the Fourier series of $f(x)$ is given by:

$$f(x) \sim \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{inx},$$

or in real form:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right],$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x, \quad n = 1, 2, \ldots \end{aligned}$$

and the symbol "$\sim$" indicates that the right-hand side is the Fourier series representation of $f(x)$.

It can be easily extended to any periodic function with period $2T$ by the substitution $x = \frac{\pi}{T} t$:

$$f(x) \sim \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{in\frac{\pi}{T}x},$$

or in real form:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[a_n \cos\left(n\frac{\pi}{T}x\right) + b_n \sin\left(n\frac{\pi}{T}x\right)\right].$$

When $f(x)$ is an even function, all sine terms vanish, and the Fourier series reduces to a cosine series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(nx).$$

When $f(x)$ is an odd function, all cosine terms vanish, and the Fourier series reduces to a sine series:

$$f(x) \sim \sum_{n=1}^{+\infty} b_n \sin(nx).$$

1.2 The Dirichlet Kernel

¶ **Dirichlet Kernel**

For partial sum of the first N terms of the Fourier series of $f(x)$:

$$S_N(f;x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(nx) + b_n \sin(nx)\right],$$

in order to study its convergence, we can transform it into integral form. By Euler-Fourier formula, we have:

$$\begin{aligned} S_N(f;x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \\ &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}t\right) e^{inx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(\sum_{n=-N}^{N} e^{in(x-t)}\right)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)\,\mathrm{d}t, \end{aligned}$$

where

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \sum_{n=1}^{N} 2\cos(nx) + 1 = \frac{\sin\left(\frac{2N+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)},$$

is called the **Dirichlet kernel**.

Dirichlet kernel possesses the following important properties:

Property

Evenness

$$D_N(-x) = D_N(x).$$

Normalization

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,\mathrm{d}x = 1.$$

However, $D_N(x)$ is like water waves, with both positive and negative values. This means that during convolution (weighted averaging), positive and negative offsets may lead to extremely unstable results. For example, for integral mean of the absolute value of the Dirichlet kernel, which is called the **Lebesgue constant**:

$$L_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|\,\mathrm{d}x \approx \frac{4}{\pi^2}\ln N, \quad (N \to +\infty).$$

It is precisely because the absolute integral of $D_N(x)$ tends to infinity that it is a "bad kernel function". It amplifies errors, causing the Fourier series of a continuous function to potentially diverge.

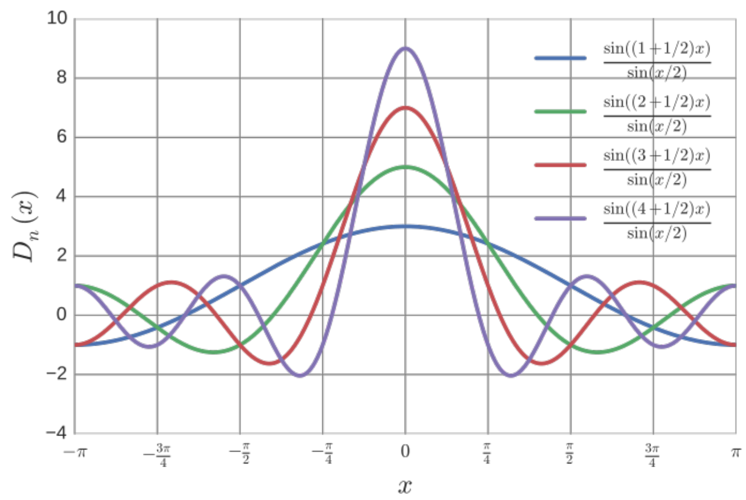

Figure 1.1: Dirichlet kernels for various values of N.

With the help of convolution theorem, we have:

$$\begin{aligned} S_N(f;x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)\,\mathrm{d}t \\ &\xlongequal{\text{Let } u=t-x} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)D_N(-u)\,\mathrm{d}u \\ &\xlongequal{D_N(-u)=D_N(u)} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)D_N(u)\,\mathrm{d}u \\ &\xlongequal{\text{Divide by 2}} \frac{1}{2\pi}\int_{0}^{\pi} \left[f(x+u)+f(x-u)\right]D_N(u)\,\mathrm{d}u. \end{aligned}$$

Then the convergence of $S_N(f;x)$ can be analyzed through the properties of the last integral that is called the **Dirichlet integral**.

Since the normalization property of Dirichlet kernel, we can analyze the difference between $S_N(f;x)$ and any a function $\sigma(x)$:

$$S_N(f;x)-\sigma(x) = \frac{1}{2\pi}\int_{0}^{\pi} \left[f(x+u)+f(x-u)-2\sigma(x)\right]D_N(u)\,\mathrm{d}u.$$

Denote $\varphi_\sigma(u,x) = f(x+u)+f(x-u)-2\sigma(x)$, then the convergence of $S_N(f;x)$ to $\sigma(x)$ is equivalent to:

$$\lim_{N\to+\infty}\int_0^{\pi}\varphi_\sigma(u,x)D_N(u)\,\mathrm{d}u = 0.$$

¶ Convolution

Definition 1.2 (Convolution)

For two functions $f(x), g(x)$ defined on $\mathbb{R}$, their convolution $f*g$ is defined as:

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)\,\mathrm{d}t.$$

Specially, if the functions are periodically defined on a finite interval $\mathbb{T}$ with period 2π, then the convolution is defined as:

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)\,\mathrm{d}t.$$

Here, $\frac{1}{2\pi}$ is a normalization factor. ♣

Remark From a physically intuitive perspective, convolution is a form of "weighted averaging" or "filtering". Here, $g(t)$ serves as the weight function (kernel), which samples and averages f within a "sliding window" around the point x.

Property

Commutativity $f * g = g * f$.

Associativity $f * (g * h) = (f * g) * h$.

Distributivity $f * (g + h) = f * g + f * h$.

Translation Invariance $(T_a f) * g = T_a(f * g)$, *where* $(T_a f)(x) = f(x - a)$.

With the definition of convolution, we can rewrite the partial sum of Fourier series as:

$$S_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x - t)\, \mathrm{d}t = (f * D_N)(x).$$

Actually, this is a special case of convolution theorem, and we have the following general conclusion:

Theorem 1.2 (Convolution Theorem)

Under suitable conditions the Fourier coefficients of a convolution of two functions (or signals) is the product of their Fourier coefficients,

$$\widehat{f * g}(n) = \hat{f}(n) \cdot \hat{g}(n).$$

In other words, the convolution in one domain corresponds to the product in another domain, for example, the convolution in the time domain corresponds to the product in the frequency domain.

Localization Theorem

First, we need the following important lemma:

Lemma 1.1 (Riemann-Lebesgue Lemma)

Let $f(x) \in R[a, b]$, $g(x)$ has a period T and $g(x) \in R[0, T]$, then:

$$\lim_{p \to +\infty} \int_a^b f(x) g(px)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x \cdot \frac{1}{T} \int_0^T g(t)\, \mathrm{d}t.$$

A special case is when $g(x) = \sin x$ or $g(x) = \cos x$, then:

$$\lim_{p \to +\infty} \int_a^b f(x) \sin(px)\, \mathrm{d}x = \int_a^b f(x) \cos(px)\, \mathrm{d}x = 0.$$

Proof

Special case. Prove for $g(x) = \sin x$, the case for $g(x) = \cos x$ is similar.

If $f(x) \in B[a, b]$, i.e., $f(x)$ is integrable in the common Riemann sense on $[a, b]$. Then there exists $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in [a, b]$. Denote $n = [\sqrt{p}]$, then when $p \to +\infty$, we have $n \to +\infty$. Divide the interval $[a, b]$ into n subintervals of equal length:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let ω_i be the oscillation of $f(x)$ on the i-th subinterval $[x_{i-1}, x_i]$.

By the integrability theory,

$$\lim_{n \to \infty} \sum_{i=1}^{n} \omega_i \Delta x_i = 0.$$

And we have:

$$\left|\int_{x_{i-1}}^{x_i} \sin(px)\,\mathrm{d}x\right| < \frac{2}{p}, \quad |\sin(px)| \leqslant 1.$$

Then we can estimate:

$$\begin{aligned}\left|\int_a^b f(x)\sin(px)\,\mathrm{d}x\right| &= \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\sin(px)\,\mathrm{d}x\right| \\ &\leqslant \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f(x)-f(x_i))\sin(px)\,\mathrm{d}x\right| + \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x_i)\sin(px)\,\mathrm{d}x\right| \\ &\leqslant \sum_{i=1}^n \omega_i \Delta x_i + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(x_i)|\,|\sin(px)|\,\mathrm{d}x \\ &\leqslant \sum_{i=1}^n \omega_i \Delta x_i + M\cdot n\cdot \frac{2}{p} \to 0, \quad (p\to+\infty).\end{aligned}$$

Thus, $\lim_{p\to\infty}\int_a^b f(x)\sin(px)\,\mathrm{d}x = 0$.

If $f(x)\notin B[a,b]$, i.e., $f(x)$ is absolutely integrable in the improper Riemann sense on $[a,b]$. Without loss of generality, assume that $f(x)$ is defective at point b. Then

$$\forall\varepsilon>0, \exists\delta>0, \forall\eta\in(0,\delta): \int_{b-\eta}^b |f(x)|\,\mathrm{d}x < \frac{\varepsilon}{2}.$$

Fix such η, then $f(x)\in R[a,b-\eta]$. According to the previous discussion, there exists $P>0$, such that when $p>P$:

$$\left|\int_a^{b-\eta} f(x)\sin(px)\,\mathrm{d}x\right| < \frac{\varepsilon}{2}.$$

Then we have:

$$\begin{aligned}\left|\int_a^b f(x)\sin(px)\,\mathrm{d}x\right| &\leqslant \left|\int_a^{b-\eta} f(x)\sin(px)\,\mathrm{d}x\right| + \left|\int_{b-\eta}^b f(x)\sin(px)\,\mathrm{d}x\right| \\ &< \frac{\varepsilon}{2} + \int_{b-\eta}^b |f(x)|\,\mathrm{d}x < \varepsilon.\end{aligned}$$

Thus, $\lim_{p\to\infty}\int_a^b f(x)\sin(px)\,\mathrm{d}x = 0$.

In summary, regardless of whether $f(x)$ is integrable in the common Riemann sense or absolutely integrable in the improper Riemann sense, we have proved the special case of Riemann-Lebesgue Lemma. ■

Then we can state Riemann's Localization Theorem:

Theorem 1.3 (Riemann's Localization Theorem)

The convergence or divergence of the Fourier series of a function $f(x)$ at a given point x depends only on the behavior of $f(x)$ in an arbitrarily small neighborhood of x.

Proof

■

Since the oscillation of $D_N(x)$ is so severe that it causes poor convergence, is there a way to "smooth it out"? In fact, we can use **Cesàro summation** and **Fejér kernel** to achieve this goal, which will be discussed in the next chapter.

1.3 Pointwise Convergence Tests

In this section, we will discuss several important convergence tests from coarse to fine for Fourier series.

Definition 1.3 (Hölder condition)

There exists a constant $L > 0$ and $\alpha \in (0, 1]$, such that for all sufficiently small δ:

$$|f(x \pm u) - f(x)| \leqslant Lu^{\alpha}, \quad 0 < u < \delta,$$

then f satisfies α-order **Hölder condition** at point x, denoted as $f \in \mathrm{Lip}_{\alpha}(x)$. When $\alpha = 1$, it is called **Lipschitz condition**.

♣

Theorem 1.4

Let $f(x) \in \mathcal{R}[-\pi, \pi]$, and satisfies one of the following conditions, then the Fourier series of $f(x)$ converges to $\frac{f(x+)+f(x-)}{2}$ at every point x:

Lipschitz's Test If $f \in \mathrm{Lip}_{\alpha}(x)$.

Since the condition is not easy to verify directly, we can use the following sufficient condition: the two quasi-uniliteral derivatives of f at point x exist, i.e.,

$$\lim_{h \to 0^+} \frac{f(x \pm h) - f(x\pm)}{h}$$

exist finitely.

Dini's Test There exists a $\delta > 0$, such that:

$$\int_0^{\delta} \frac{|f(x+u) + f(x-u) - 2S|}{u} \, \mathrm{d}u < +\infty,$$

where $S = \frac{f(x+)+f(x-)}{2}$.

Dirichlet-Jordan Test If $f(x)$ is of bounded variation in a neighborhood of point x, i.e., there exists a $\delta > 0$, such that $f \in BV(x - \delta, x + \delta)$.

♡

Example 1.1 Let $f(x)$ be a $2\pi-$periodic function defined as:

$$f(x) = \begin{cases} x, & x \in [-\pi, \pi), \\ -\pi, & x = \pi. \end{cases}$$

Find its Fourier series and $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

Solution By Euler-Fourier formula, we have:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x = 0,$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x = \frac{1}{\pi} \left[x \cdot \frac{\sin(nx)}{n} \bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n} \, \mathrm{d}x \right] = 0,$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x = \frac{1}{\pi} \left[-x \cdot \frac{\cos(nx)}{n} \bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(nx)}{n} \, \mathrm{d}x \right] = \frac{2(-1)^{n+1}}{n}.$$

Thus, the Fourier series of $f(x)$ is:

$$f(x) \sim \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{n} \sin(nx).$$

Since $f(x) \in BV[-\pi, \pi]$, by Dirichlet-Jordan test, its Fourier series converges to $f(x)$ at every continuous point x and to $\frac{f(x+)+f(x-)}{2}$ at every discontinuous point x.

Then

$$x = f(x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{n} \sin(nx), \quad x \in (-\pi, \pi).$$

Furthermore,

$$\begin{aligned}
\tilde{f}(x) &= \begin{cases} f(x), & x \neq 2k\pi + \pi, \\ \frac{f(\pi+)+f(\pi-)}{2}, & x = 2k\pi + \pi, \end{cases} \\
&= \begin{cases} f(x), & x \in (2k\pi - \pi, 2k\pi + \pi), \\ \frac{-\pi+\pi}{2} = 0, & x = 2k\pi + \pi, \end{cases} \\
&= \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{n} \sin(nx),
\end{aligned}$$

where $k \in \mathbb{Z}, x \in \mathbb{R}$.
Let $x = \frac{\pi}{2}$, then we have:

$$\frac{\pi}{2} = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{n} \sin\left(n \cdot \frac{\pi}{2}\right),$$

$$\frac{\pi}{4} = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n} \sin\left(n \cdot \frac{\pi}{2}\right) = \sum_{n=1}^{+\infty} (-1)^{2n+2} \frac{1}{2n+1} \sin\left((2n+1) \cdot \frac{\pi}{2}\right) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

□

¶ **Gibbs Phenomenon**

When a function $f(x)$ has jump discontinuities, its Fourier series converges to the midpoint of the jump at the discontinuity point. However, near the discontinuity, the Fourier series exhibits oscillations that overshoot and undershoot the function's actual values. This phenomenon is known as the **Gibbs phenomenon**.

For example, for a square wave function $f(x)$ with period 2π:

$$f(x) = \begin{cases} -1, & x \in [-\pi, 0) \\ 1, & x \in (0, \pi], \end{cases} \qquad f(x) \sim \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{\sin((2n+1)x)}{2n+1}.$$

Using Dirichlet-Jordan test, $f(x) \in BV$, then we can show that the Fourier series of $f(x)$ converges to:

$$\begin{cases} f(x), & x \text{ is continuous} \\ \frac{f(x+)+f(x-)}{2}, & x \text{ is discontinuous.} \end{cases}$$

Regarding the partial sum of the N-term Fourier series $S_N(f; x)$, it will exhibit significant overshoot near the discontinuity points (Fig. 1.2). Even if more sine terms are used, this approximation error will only converge to a limit of approximately 9% of the jump height, although the infinite Fourier series will eventually converge almost everywhere.

This phenomenon is an important consideration in signal processing, as it can cause high-frequency noise and ringing effects.

1.4 Properties of Fourier series

By Riemann-Lebesgue lemma (1.1), we have the following important conclusion of Fourier coefficients directly:

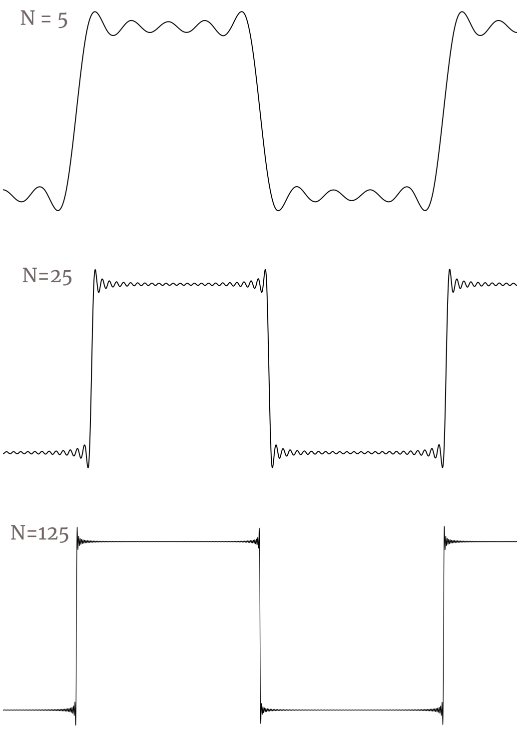

Figure 1.2: Gibbs phenomenon of square wave near a jump discontinuity.

Proposition 1.1

Let $f(x) \in \mathcal{R}[-\pi, \pi]$, then its Fourier coefficients satisfy:

$$\lim_{|n| \to +\infty} \hat{f}(n) = 0.$$

In real form, it is equivalent to:

$$\lim_{n \to +\infty} a_n = 0, \quad \lim_{n \to +\infty} b_n = 0.$$

And we give a theoretical property, used as a fallback option.

Theorem 1.5 (Uniqueness Theorem)

If $f(x), g(x) \in \mathcal{R}[-\pi, \pi]$ have the same Fourier coefficients, i.e., $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$, then $f(x) = g(x)$ almost everywhere on $[-\pi, \pi]$.

¶ Analytical Properties

Theorem 1.6 (Termwise Integration)

If $f(x) \in \mathcal{R}[-\pi, \pi]$ with Fourier series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right],$$

then its integral also has a Fourier series obtained by termwise integration:

$$\int_c^x f(t)\,\mathrm{d}t = \int_c^x \frac{a_0}{2}\,\mathrm{d}t + \sum_{n=1}^{+\infty} \int_c^x \left[a_n \cos(nt) + b_n \sin(nt)\right]\,\mathrm{d}t, \quad c, x \in [-\pi, \pi].$$

Note *Note that after termwise integration, the resulting Fourier series converges on $[-\pi, \pi]$ ($\sim$ to $=$). This is because integration smooths out the function, reducing oscillations and improving convergence behavior.*

Proof Here, we only prove the case that $f(x)$ has finitely many discontinuities of the first kind on $[-\pi, \pi]$.
■

Theorem 1.7 (Termwise Differentiation)

If $f(x) \in \mathcal{R}[-\pi, \pi]$ with Fourier series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right],$$

and if $f(x)$ is piecewise smooth on $[-\pi, \pi]$, then its derivative also has a Fourier series obtained by termwise differentiation:

$$f'(x) \sim \sum_{n=1}^{+\infty} \left[-na_n \sin(nx) + nb_n \cos(nx)\right].$$

¶ Smoothness and Decay Rate

The smoother and less angular a function is, its Fourier coefficients decay more rapidly (with fewer high-frequency components). The rougher a function is (with jumps), the more high-frequency components it has (and the slower the coefficients decay).

Function Smoothness	**Fourier Coefficient Decay Rate**
$f \in \mathcal{R}[-\pi, \pi]$	$\hat{f}(n) \to 0$ (slowest)
f has jump discontinuities	$\hat{f}(n) \sim O\left(\frac{1}{n}\right)$
f is continuous but not differentiable	$\hat{f}(n) \sim O\left(\frac{1}{n^2}\right)$
$f \in C^k$	$\hat{f}(n) \sim O\left(\frac{1}{n^{k+1}}\right)$
Analytic Function	Exponential Decay ($O(e^{-c\|n\|})$)

¶ Parseval's Identity

Theorem 1.8 (Square Approximation Property of Fourier Series)

Let $f(x) \in \mathcal{R}[-\pi, \pi]$, and W be an N-degree subspace of $\mathcal{R}[-\pi, \pi]$, then the best approximation of $f(x)$ in W is just given by its Fourier series partial sum $S_N(f; x)$:

$$S_N(f; x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(nx) + b_n \sin(nx)\right].$$

And the remainder $E_N(f; x) = f(x) - S_N(f; x)$ satisfies[a]:

$$\|E_N(f; x)\|^2 = \|f\|^2 - \sum_{n=-N}^{N} |\hat{f}(n)|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, \mathrm{d}x - \left[\frac{a_0^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2)\right].$$

[a] Refer to *Algèbre Linéaire - Section 10.5: Orthogonal Completion and Orthogonal Projection*.

Theorem 1.9 (Bessel's Inequality)

Let $f(x) \in \mathcal{R}[-\pi, \pi]$ with Fourier coefficients $\hat{f}(n)$, then for any integer $N \geqslant 0$:

$$\sum_{n=-N}^{N} |\hat{f}(n)|^2 \leqslant \|f\|^2,$$

or in real form:

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2) \leqslant \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, \mathrm{d}x.$$

Theorem 1.10 (Parseval's Identity)

Let $f(x) \in \mathcal{R}[-\pi, \pi]$ with Fourier coefficients $\hat{f}(n)$, then:

$$\sum_{n=-\infty}^{+\infty} |\hat{f}(n)|^2 = \|f\|^2,$$

or in real form:

$$\frac{a_0^2}{2} + \sum_{n=1}^{+\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, \mathrm{d}x.$$

The generalized form is:

$$\sum_{n=-\infty}^{+\infty} \hat{f}(n)\overline{\hat{g}(n)} = \langle f, g \rangle,$$

or in real form:

$$\frac{a_0 c_0}{2} + \sum_{n=1}^{+\infty} (a_n c_n + b_n d_n) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x,$$

where $g(x) \in \mathcal{R}[-\pi, \pi]$ with Fourier coefficients $\hat{g}(n)$,

$$g(x) \sim \frac{c_0}{2} + \sum_{n=1}^{+\infty} \left[c_n \cos(nx) + d_n \sin(nx)\right].$$

Proof

■

Theorem 1.11 (Wirtinger Inequality)

Let $f(x) \in C^1[-\pi, \pi]$, and satisfies:

$$\int_{-\pi}^{\pi} f(x) \, \mathrm{d}x = 0, \, f(-\pi) = f(\pi),$$

then:

$$\int_{-\pi}^{\pi} f^2(x) \, \mathrm{d}x \leqslant \int_{-\pi}^{\pi} f'^2(x) \, \mathrm{d}x,$$

with equality if and only if $f(x) = A \cos x + B \sin x$ for some constants $A, B \in \mathbb{R}$.

Theorem 1.12 (Poincaré Inequality)

Theorem 1.13 (Friedrichs Inequality)

Chapter 2 Cesàro Summation

In the previous chapter, we denote $\mathcal{R}[-\pi,\pi]$ as the set of all Riemann integrable or absolutely Riemann integrable functions on $[-\pi,\pi]$. Now, for convenience, we introduce the notation $\overline{\mathcal{R}}[-\pi,\pi]$ as the set of all Riemann integrable or square Riemann integrable functions on $[-\pi,\pi]$. Since for defective integral, the square integrability implies absolute integrability, we have

$$\mathcal{R}^2[-\pi,\pi] \subset \mathcal{R}[-\pi,\pi].$$

2.1 Cesàro Summation and Fejér Kernel

Until now, when we discuss the convergence of series $\sum_{n=1}^{\infty} a_n$, the convergence of partial sums $S_N = \sum_{n=1}^{N} a_n$ as $N \to \infty$ is considered de facto. The definition is put forward by Cauchy in 1821. However, there are other ways to define the convergence of series. Here we introduce Cesàro summation, which is a method to assign sums to some divergent series[1].

Definition 2.1 (Cesàro Summation)

A series $\sum_{n=1}^{\infty} a_n$ is said to be Cesàro summable to S if the arithmetical average σ_k of its partial sums converges to S, i.e.,

$$\lim_{k\to\infty} \sigma_k = S,$$

where

$$\sigma_k = \frac{S_1 + S_2 + \cdots + S_k}{k}.$$

♣

Due to

$$S_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) D_N(t)\,\mathrm{d}t,$$

then

$$\begin{aligned}
\sigma_N(f;x) &= \frac{1}{N}\sum_{k=0}^{N-1} S_k(f;x)\\
&= \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) D_k(t)\,\mathrm{d}t\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\frac{1}{N}\sum_{k=0}^{N-1} D_k(t)\right)\mathrm{d}t\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) F_N(t)\,\mathrm{d}t,
\end{aligned}$$

where

$$F_N(t) = \frac{1}{N}\sum_{k=0}^{N-1} D_k(t) = \frac{1}{N}\left(\frac{\sin\frac{Nt}{2}}{\sin\frac{t}{2}}\right)^2,$$

is called the **Fejér kernel**.

Fejér kernel has the following excellent properties:

---

[1] By Cauchy proposition (see *Analyse Mathématique - Section 2.1: Convergent Sequences*), if $\lim_{n\to\infty} x_n = l$, then $\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = l$. That is, the Cesàro sum of a convergent series is equal to its Cauchy sum.

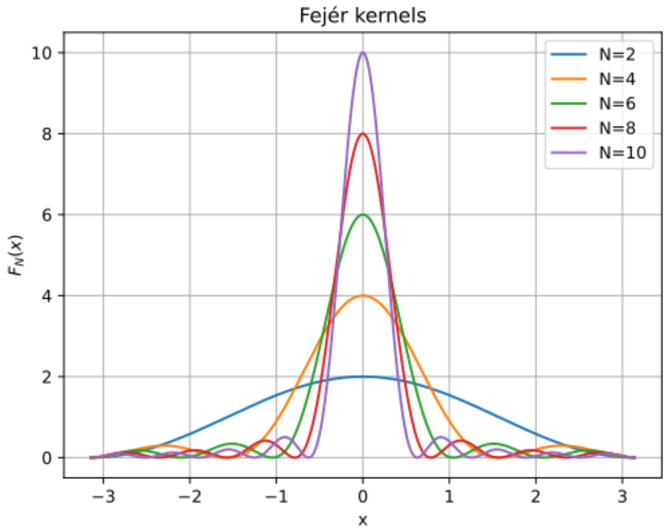

Figure 2.1: Fejér kernels $F_N(t)$ for $N = 2, 4, 6, 8, 10$.

Property

Positivity *For any integer N and real number t, $F_N(t) \geqslant 0$. This property significantly distinguishes Fejér kernel from Dirichlet kernel.*

Normalization *For any integer N,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(t)\,\mathrm{d}t = 1.$$

Concentration *For any $\delta \in (0, \pi)$,*

$$\lim_{N\to\infty}\int_{\delta}^{\pi} F_N(t)\,\mathrm{d}t = 0.$$

Bounded *Its L^1 norm is bounded:*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F_N(t)|\,\mathrm{d}t = 1.$$

Distinct from Dirichlet kernel, whose L^1 norm grows logarithmically with N:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|\,\mathrm{d}t \sim \frac{4}{\pi^2}\log N.$$

Property	**Dirichlet Kernel (D_N)**	**Fejér Kernel (K_N)**
Corresponding operation	Partial sum S_N (truncation)	Cesàro sum σ_N (averaging)
Formula feature	$\frac{\sin\left((N+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)}$ (1st-order)	$\frac{1}{N}\left(\frac{\sin\left(\frac{(N+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)^2$ (square)
Positivity	Violent oscillations (positive & negative)	Non-negative everywhere ($\geqslant 0$)
L^1 Norm	$\ln N \to \infty$ (divergent)	$= 1$ (bounded)
If $f \in C$	May diverge	Uniform convergence
Role	Projection operator	Approximation operator

With such a "good kernel", we can develop the following theorem:

Theorem 2.1 (Fejér Theorem)

If $f(x)$ is a continuous function defined on $\mathbb{T}$, then its Cesàro means $\sigma_N(f;x)$ converge uniformly to $f(x)$ on $\mathbb{T}$, i.e.,

$$\lim_{N\to\infty}\sup_{x\in\mathbb{T}}|\sigma_N(f;x)-f(x)|=0.$$

The generalized version is: if $f(x)\in\mathcal{R}[-\pi,\pi]$, and f has left and right limits at point $x_0\in[-\pi,\pi]$, then its Cesàro means $\sigma_N(f;x_0)$ converge to the average of the left and right limits, i.e.,

$$\lim_{N\to\infty}\sigma_N(f;x_0)=\frac{f(x_0^+)+f(x_0^-)}{2}.$$

Remark It means that as long as f is integrable or absolutely integrable, and has left and right limits at point x_0, then its Fourier series converges in Cesàro sense at x_0.

Compared with various convergence tests of Fourier series in previous chapter, Fejér theorem is brief and to the point.

With Fejér theorem, we can obtain that the trigonometric polynomials are dense in the continuous function space $C(\mathbb{T})$. and the Weierstrass second approximation theorem[2] can be proved easily.

2.2 Square mean Convergence

Definition 2.2 (Square mean Convergence)

A sequence of functions $f_n(x)$ is said to converge to $f(x)$ in the square mean (or L^2) sense on interval $[a,b]$, if

$$\lim_{n\to\infty}\|f_n-f\|^2=\lim_{n\to\infty}\int_a^b|f_n(x)-f(x)|^2\,\mathrm{d}x=0.$$

Theorem 2.2

If $f(x)\in\mathcal{R}^2[-\pi,\pi]$, then its Fourier series converges to $f(x)$ in the square mean sense on $[-\pi,\pi]$, i.e.,

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|f(x)-S_N(f;x)|^2\,\mathrm{d}x=0.$$

Example 2.1 Proof:

1. $$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6};$$
2. $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8};$$
3. $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12};$$

[2]The Weierstrass second approximation theorem states that any continuous function defined on a closed interval can be uniformly approximated by polynomials to any desired degree of accuracy. Refer to *Analyse Mathématique - Section 10.3: Smooth Appropriation of Functions* for details.

4.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720};$$

5.
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof

1. By 1.1,
$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in (-\pi, \pi).$$
Using Parseval's identity, we have
$$4\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \, \mathrm{d}x = \frac{2\pi^2}{3},$$
i.e.,
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. ■

2.3 Equidistribution

2.4 Poisson Kernel

Chapter 3 Modern Fourier Series

Chapter 4 Fourier Transform

4.1 Laplace Transform

Chapter 5 Sobolev Spaces

Bibliography

[1] Elias M. Stein, Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton University Press, 2016.

[2] 于品. *Fourier 分析*.

[3] Author2, Title2, Journal2, Year2. *This is another example of a reference.*