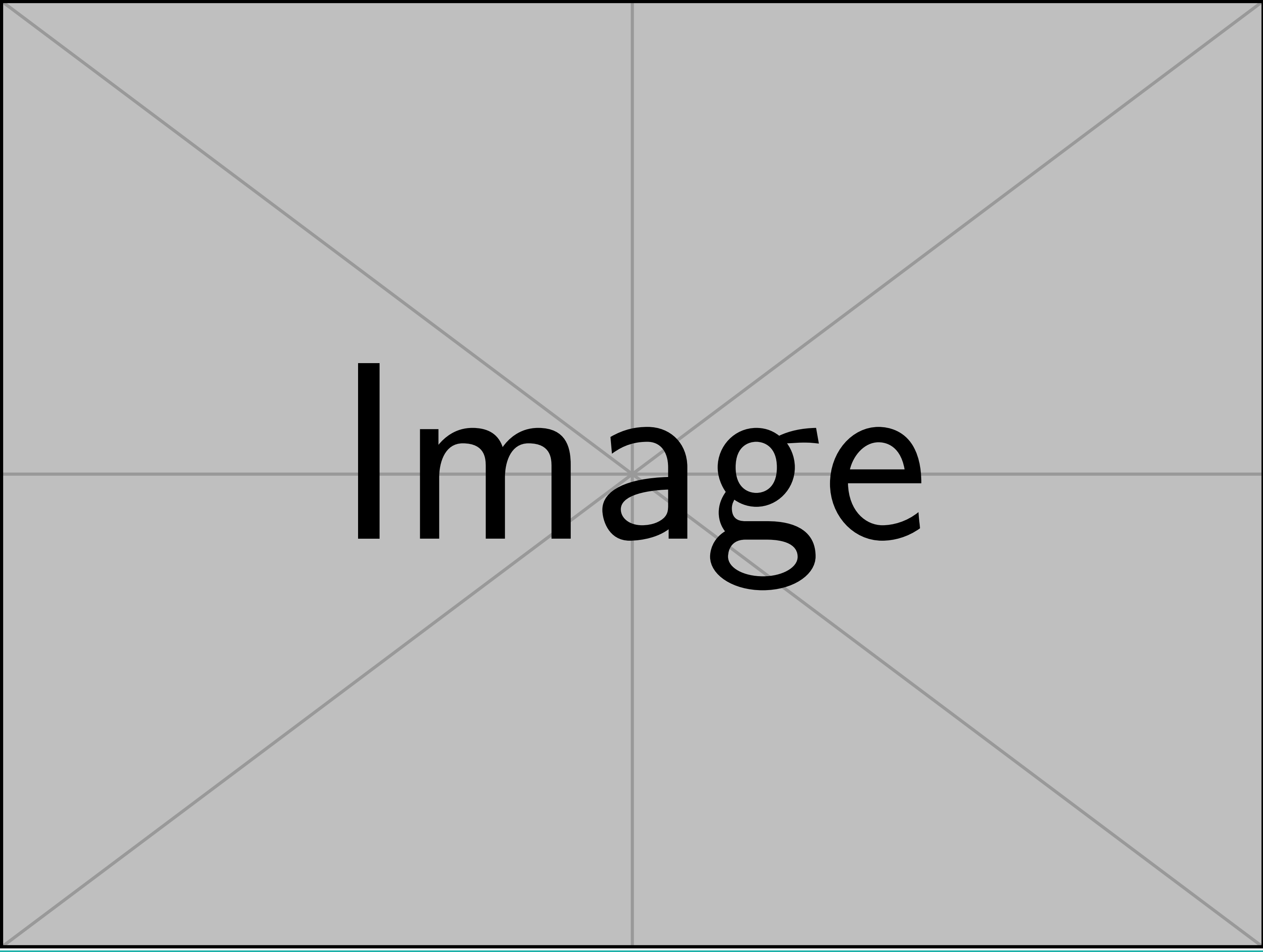

Analyse Mathématique

Author: CatMono

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8.1 Directional Derivatives and Total Differential

Definition 8.1 (Directional Derivative)

Let $U \subseteq \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^1$, $\boldsymbol{e}$ is a unit vector in $\mathbb{R}^n$, $x_0 \in U$. Define

$$u(t) = f(x_0 + t\boldsymbol{e}).$$

If the derivative of u at $t = 0$

$$u'(0) = \lim_{t\to 0} \frac{u(t) - u(0)}{t} = \lim_{t\to 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{e}) - f(\boldsymbol{x}_0)}{t}$$

exists and is finite, it is called the **directional derivative** of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{e}$, denoted by $\frac{\partial f}{\partial \boldsymbol{e}}(\boldsymbol{x}_0)$. It is the rate of change of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{e}$.

♣

Consider the following set of unit coordinate vectors:$\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$. For a function f, the directional derivative of f at the point $\boldsymbol{x}_0$ in the direction of $\boldsymbol{e}_i$ is called the ith first-order **partial derivative** of f at $\boldsymbol{x}_0$, denoted by

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0) \quad \text{or} \quad \mathrm{D}_i f(\boldsymbol{x}_0).$$

$\mathrm{D}_i = \frac{\partial}{\partial x_i}$ is called the ith **partial differential operator** $(i = 1, 2, \cdots, n)$.

If the first-order partial derivative of f, $\frac{\partial f}{\partial x_i}$, itself possesses partial derivatives, then the second-order partial derivative of f is defined, and is denoted as follows:

$$f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right), \quad f_{x_i x_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right), \quad i, j = 1, 2, \ldots, n.$$

Similarly, higher-order partial derivatives of order $3, 4, \cdots m, \cdots$ can be defined.

Definition 8.2 (Jacobian Matrix (Gradient))

Let

$$\boldsymbol{J}f(\boldsymbol{x}) = (\mathrm{D}_1 f(\boldsymbol{x}), \mathrm{D}_2 f(\boldsymbol{x}), \ldots, \mathrm{D}_n f(\boldsymbol{x})),$$

which is called the **Jacobian matrix** of the function f at the point x, a $1 \times n$ matrix which corresponds to the first-order derivative of a single-variable function.

Henceforth, we represent the point $\boldsymbol{x}$ in $\mathbb{R}^n$ and its increments $\boldsymbol{h}$ as column vectors. In this way, the differential of the function can be expressed using matrix multiplication as follows:

$$\mathrm{d}f(\boldsymbol{x}_0)(\boldsymbol{h}) = \boldsymbol{J}f(\boldsymbol{x}_0)\boldsymbol{h}.$$

The Jacobian matrix of the function f is also frequently denoted as $\mathbf{grad}\, f$ (or ∇f), that is,

$$\mathbf{grad}\, f(\boldsymbol{x}) = \boldsymbol{J}f(\boldsymbol{x}),$$

which is called the **gradient** of the scalar function f.

♣

Definition 8.3 (Total Differential)

Let $U \subseteq \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^1$, $\boldsymbol{x}_0 \in U$, $\boldsymbol{h} = (h_1, h_2, \cdots, h_n) \in \mathbb{R}^n$. If

$$f(\boldsymbol{x}_0 + \boldsymbol{h}) - f(\boldsymbol{x}_0) = \sum_{i=1}^{n} \lambda_i h_i + o(\|\boldsymbol{h}\|) \qquad (\|\boldsymbol{h}\| \to 0),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are constants independent of $\boldsymbol{h}$, then the function f is said to be **differentiable** at the point $\boldsymbol{x}_0$, and the linear main part $\sum_{i=1}^{n} \lambda_i h_i$ is called the **total differential** of f at $\boldsymbol{x}_0$, denoted as

$$df(x_0)(h) = \sum_{i=1}^{n} \lambda_i h_i.$$

If f is differentiable at every point in the open set U, then f is called a differentiable function on U. ♣

Theorem 8.1 (Conditions of Differentiability)

Necessary Condition If an n-variable function f is differentiable at the point $\boldsymbol{x}_0$, then f possesses first-order partial derivatives $\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0)$ at $\boldsymbol{x}_0$ for $i = 1, 2, \ldots, n$, and

$$\boldsymbol{A} = \boldsymbol{J}f(\boldsymbol{x}_0) = (\mathrm{D}_1 f(\boldsymbol{x}_0), \mathrm{D}_2 f(\boldsymbol{x}_0), \ldots, \mathrm{D}_n f(\boldsymbol{x}_0)).$$

However, the converse is not true.

Sufficient Condition Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^1$ be an n-variable function. If $\boldsymbol{J}f = (\mathrm{D}_1 f, \mathrm{D}_2 f, \ldots, \mathrm{D}_n f)$ is continuous at $\boldsymbol{x}_0$ (i.e., $\frac{\partial f}{\partial x_i}$ is continuous at $\boldsymbol{x}_0$ for $i = 1, 2, \ldots, n$), then f is differentiable at $\boldsymbol{x}_0$. However, the converse is not necessarily true. ♡

Chapter 9 Multiple Integrals

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