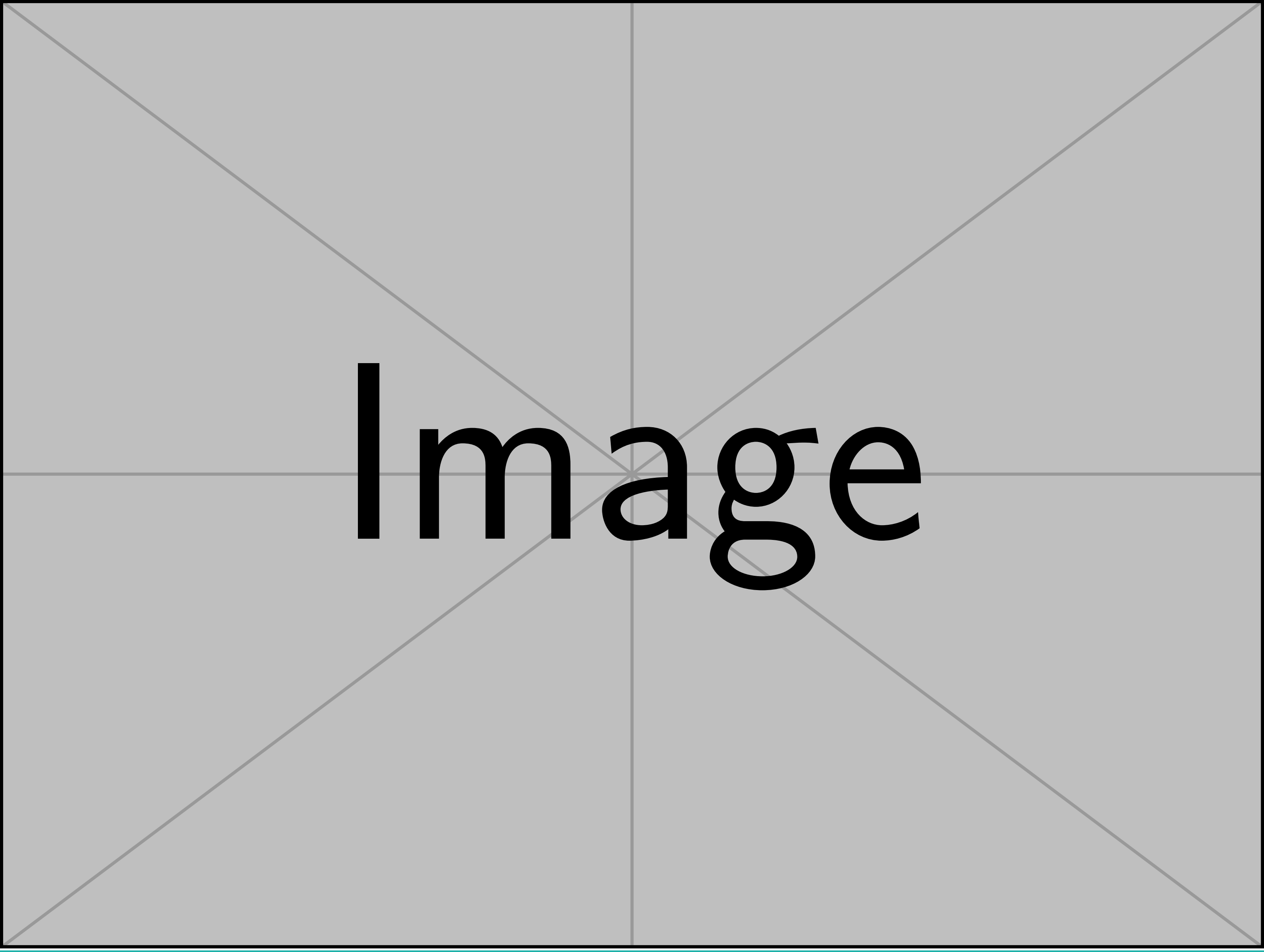

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Contents

Preface

This is the preface of the book…

Chapter 1 The Propositional Calculus

1.1 Connectives and Truth Tables

A proposition is a statement whose truth value can be determined (represented by 1 for true and 0 for false[1]), and a proposition is either true or false (within the framework of classical binary logic). We use lowercase letters such as $p, q, r \cdots$, to denote propositions.

There are five commonly used propositional connectives:

¶ **Negation**

If p is a proposition, then the **negation** of p is denoted by $\neg p$, which is true if and only if p is false. Its truth table is as follows:

p	$\neg p$
0	1
1	0

¶ **Conjunction**

The **conjunction** of two propositions p and q is denoted by $p \wedge q$, which is true if and only if both p and q are true. Its truth table is as follows:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

¶ **Disjunction**

The **disjunction** of two propositions p and q is denoted by $p \vee q$, which is true if and only if at least one of p or q is true. Its truth table is as follows:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

¶ **Implication**

Proposition p **implies** q, namely "if p, then q", is denoted by $p \to q$, which is false if and only if p is true and q is false. Its truth table is as follows:

[1] The set of logical truth values can be represented in various ways, such as $\{T, F\}$, $\{\top, \bot\}$, or {True, False}. $\{0, 1\}$ is applied in this book for simplicity.

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

¶ **Biconditional**

The **biconditional** of two propositions p and q is denoted by $p \leftrightarrow q$, which is true if and only if p and q have the same truth value. Its truth table is as follows:

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

1.2 Axiom System

1.2.1 Propositional Calculus

Propositional calculus (also called **Zero-Order Logic**) is a formal system $\mathcal{L}(A, \Omega, Z, I)$, whose formulas are constructed as follows:

A The infinite set consisting of propositional variables.

Ω The infinite set consisting of logical connectives[2].

Z The infinite set consisting of inference rules.

I The infinite set consisting of axioms (start point).

Parentheses, namely "(" and ")", are commonly used as auxiliaries to facilitate the construction of formulas.

Definition 1.1 (Well-formed Formula (WFF))

The language of $\mathcal{L}$, denoted by $L(A)$, is also called the set of well-formed formulas A **well-formed formula** (short for WFF or formula) is a finite sequence of symbols from A and Ω that is constructed recursively according to the following rules:

1. Any element of A is a formula.
2. If $p_1, p_2, \cdots p_j$ are formulas, $f \in \Omega_j$, then $(f p_1 p_2 \cdots p_j)$ is also a formula.
3. Every formula is generated by a finite number of applications of Rules 1 and 2.

♣

$L(A)$ can be layered as follows: TODO;

Although axiomatic proof has been used since the famous Ancient Greek textbook, Euclid's *Elements of Geometry*, in propositional logic it dates back to Gottlob Frege's 1879 *Begriffsschrift*. Frege's system used only implication and negation as connectives. It had six axioms ($p, q, r \in A$):

[2] It is divided into the following mutually disjoint subsets:

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m,$$

where Ω_j is the set of j-ary logical connectives(operators). In general, $\Omega_0 = \{0, 1\}, \Omega_1 = \{\neg\}, \Omega_2 = \{\wedge, \vee, \to, \leftrightarrow\}$.

$p \to (q \to p)$ (Law of Affirming the Consequent)
$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ (Law of Distribution of Implication)
$(p \to (q \to r)) \to (q \to (p \to r))$ (Law of Permutation)
$(p \to q) \to (\neg q \to \neg p)$ (Law of Transposition)
$\neg\neg p \to p$ (Law of Double Negation Elimination)
$p \to \neg\neg p$ (Law of Double Negation Introduction)

These were used by Frege together with modus ponens and a rule of substitution (which was used but never precisely stated) to yield a complete and consistent axiomatization of classical truth-functional propositional logic.

Jan Łukasiewicz showed that, in Frege's system, "the third axiom is superfluous since it can be derived from the preceding two axioms, and that the last three axioms can be replaced by the single sentence $CCNpNqqp$ or $\to\to\neg p\neg q \to qp$ with his Polish notation[3], namely $(\neg p \to \neg q) \to (q \to p)$. It is a simplified version of Frege's system, which is also known as the Łukasiewicz axiom system.

Definition 1.2 (Jan Łukasiewicz Axiom System)

Łukasiewicz axiom system is defined as $\mathcal{L} = (A, \Omega, Z, I)$, where:

- A contains sufficiently many propositional variables for discussion.
- $\Omega = \Omega_1 \cup \Omega_2$ is complete, where $\Omega_1 = \{\neg\}$ and $\Omega_2 = \{\to\}$.
- Z contains a single inference rule: Modus Ponens (MP), which states that if p and $p \to q$ are both formulas, then q is also a formula.
- I contains 3 axioms ($p, q, r \in A$):
  (L1) $p \to (q \to p)$ (Law of Affirming the Consequent)
  (L2) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ (Law of Distribution of Implication)
  (L3) $(\neg p \to \neg q) \to (q \to p)$ (Law of Contraposition)

♣

1.2.2 Propositional Algebra

1.2.3 Proof

Definition 1.3 (Proof)

Let $\Gamma \subseteq L(X), p \in L(X)$. When we say that "the formula p is provable from the set of formulas Γ", we mean that there exists a finite sequence of formulas $p_1, \ldots, p_n$ from $L(X)$, with the last term $p_n = p$, such that each p_k $(k = 1, \ldots, n)$ satisfies one of the following:

- p_k is an element of Γ.
- p_k is an axiom of the system.
- There exist $i, j < k$ such that $p_j = p_i \to p_k$ (MP).

Any finite sequence $p_1, \ldots, p_n$ possessing the above properties is called a "proof of p from Γ".

♣

If a formula p is provable from a set of formulas Γ, we write $\Gamma \vdash p$, or equivalently $\Gamma \vdash_{\mathcal{L}} p$. In this case, the formulas in Γ are called "assumptions," and p is called a syntactic consequence of Γ.

[3] Polish notation, also known as prefix notation, is a way of writing mathematical expressions in which operators precede their operands. It allows expressions to be written without parentheses to indicate order of operations and is commonly used in logic and computer science. For example, the expression $(5-6) \times 7$ in Polish notation is written as $\times - 567$.

If $\emptyset \vdash p$, then p is called a "theorem" (interior theorem) of $\mathcal{L}$, denoted $\vdash p$. A proof of p from $\emptyset$ in $\mathcal{L}$ is a sequence $p_1, \ldots, p_n$, abbreviated as a proof of p in $\mathcal{L}$.

In a proof, when $p_j = p_i \to p_k$ $(i, j < k)$, we say that p_k is obtained from p_i and $p_i \to p_k$ by using the rule of MP.

Definition 1.4 (Consistent Set of Formulas)

If, for any formula q, neither $\Gamma \vdash q$ nor $\Gamma \vdash \neg q$ hold simultaneously, then the set of formulas Γ is called a consistent set of formulas; otherwise, Γ is called an inconsistent set of formulas. ♣

Proposition 1.1

If Γ is a consistent set of formulas, then for any formula p in $L(A)$, we have $\Gamma \nvdash \neg p$. ♠

Proposition 1.2

1) $\vdash p \to p$ (Law of Identity)
2) $\vdash \neg q \to (q \to p)$ (Law of Denying the Antecedent)
3) $((x_1 \to (x_2 \to x_3)) \to (x_1 \to x_2)) \to ((x_1 \to (x_2 \to x_3)) \to (x_1 \to x_3))$ (12312→12313) ♠

Proof

■

1.2.4 Deduction Theorems

Theorem 1.1 (Deduction Theorem)

$$\Gamma \cup \{p\} \vdash q \iff \Gamma \vdash p \to q$$

♡

Proof

■

Corollary 1.1 (Hypothetical Syllogism (HS))

$$\{p \to q, q \to r\} \vdash p \to r$$

♡

Proposition 1.3

1) $\vdash (p \to q) \to (\neg q \to \neg p)$ (Law of Transition)
2) $\vdash p \to \neg\neg p$ (Law of Double Negation Introduction)
3) $\vdash ((p \to q) \to p) \to p$ (Peirce's Law)
4) $\vdash \neg(p \to q) \to (q \to p)$ (Overall Negation Reversal) ♠

1.2.5 Law of Contradiction and Absurdum

Theorem 1.2 (Law of Contradiction)

$$\left.\begin{array}{l}\Gamma \cup \{\neg p\} \vdash q \\ \Gamma \cup \{\neg p\} \vdash \neg q\end{array}\right\} \Longrightarrow \Gamma \vdash p.$$

Theorem 1.3 (Law of Absurdum)

$$\left.\begin{array}{l}\Gamma \cup \{p\} \vdash q \\ \Gamma \cup \{p\} \vdash \neg q\end{array}\right\} \Longrightarrow \Gamma \vdash \neg p.$$

Corollary 1.2 (Law of Double Negation Introduction and Elimination)

1. $\vdash p \to \neg\neg p$
2. $\vdash \neg\neg p \to p$

Proposition 1.4

1) $\vdash (p \to \neg q) \to (q \to \neg p)$ (Negation Reversal)
2) $\vdash (\neg p \to q) \to (\neg q \to p)$ (Negation Reversal)
3) $\vdash \neg(p \to q) \to \neg q$ (Overall Negation)
4) $\vdash \neg(p \to q) \to p$ (Overall Negation)

1.2.6 Other Connectives

In $\mathcal{L}$, we can also define three other binary operators:

$$p \vee q = \neg p \to q,$$
$$p \wedge q = \neg(p \to \neg q),$$
$$p \leftrightarrow q = (p \to q) \wedge (q \to p).$$

1.3 Semantics of Propositional Calculus

Chapter 2 First-Order Logic

Chapter 3 Second-Order Logic

Bibliography

[1] 作者, Title1, Journal1, Year1. *This is an example of a reference.*

[2] Author2, Title2, Journal2, Year2. *This is another example of a reference.*