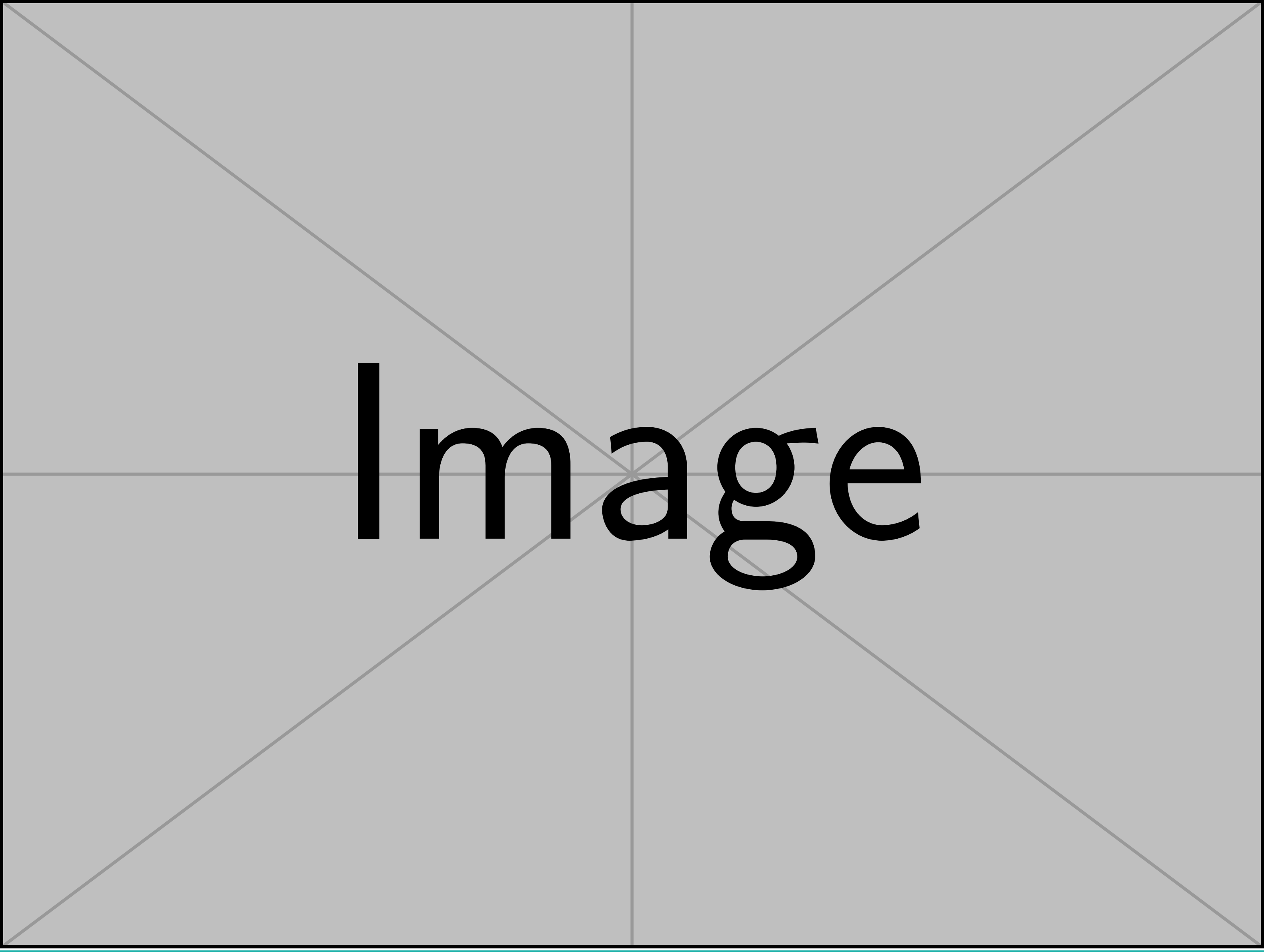

Théorie des Ensembles

Author: CatMono

Date: January, 2026

Version: 0.1

Contents

Preface

Some notations are used throughout this book:

- $\mathbb{N}$: Set of natural numbers (including 0).
- $\mathbb{N}^*/\mathbb{N}_+$: Set of natural numbers (excluding 0).
- $\mathbb{Z}$: Set of integers.
- $\mathbb{Q}$: Set of rational numbers.
- $\mathbb{R}$: Set of real numbers.
- $\mathcal{P}(A)$: Power set of set A.
- $|A|$: Cardinality of set A.

Chapter 1 Naive Set Theory

1.1 Sets and Their Operations

Definition 1.1 (Power Set)

Let X be a set. The **power set** of X, denoted by $\mathscr{P}(X)$, is defined as the set of all subsets of X:

$$\mathscr{P}(X) = \{A \mid A \subseteq X\}.$$

♣

1.2 Relations and Mappings

¶ Relations

Definition 1.2 (Cartesian Product)

Let X and Y be two sets. The **Cartesian product** (or direct product) of X and Y, denoted by $X \times Y$, is defined as the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$:

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

The Cartesian product can be extended to finitely many sets.
The Cartesian product of X and itself n times is denoted by X^n:

♣

Definition 1.3 (Relation)

Let X and Y be two sets. A **relation** R from X to Y is a subset of the Cartesian product $X \times Y$:

$$R \subseteq X \times Y.$$

If $(x, y) \in R$, we say that x is related to y by the relation R, denoted by xRy.
If $A \subseteq X$, then the subset of Y defined by

$$R(A) = \{y \in Y \mid \exists x \in A, (x, y) \in R\}$$

is called the **image** of A under the relation R. $R(X)$ is called the **range** of the relation R.

♣

There are several special types of relations:

Empty relation The empty set $\emptyset$ is a relation from X to Y.

Total relation The Cartesian product $X \times Y$ is a relation from X to Y.

Identity relation The relation $I_X = \{(x, x) \mid x \in X\}$ is called the **identity relation** on X.

When studying binary relations, we often focus on whether they have some special properties. For a binary relation R on a set X, we define the following special properties:

Reflexive $(\forall x \in X)\, xRx$.

Irreflexive $(\forall x \in X)\, \neg xRx$.

Symmetric $(\forall x, y \in X)\, (xRy \Leftrightarrow yRx)$.

Antisymmetric $(\forall x, y \in X)\, (xRy \wedge yRx) \implies x = y$.

Transitive $(\forall x, y, z \in X)\, (xRy \wedge yRz) \implies xRz$.

Connected (Total) $(\forall x, y \in X)\, x \neq y \implies (xRy \vee yRx)$.

Well-founded $(\exists x \in X \neq \emptyset)\, (\forall y \in X \setminus \{x\})\, \neg(yRx)$.

Transitive of incomparability $(\forall x, y, z \in X)\,(\neg(xRy \vee yRx) \wedge \neg(yRz \vee zRy)) \implies \neg(xRz \vee zRx)$.

Then we can define the equivalence relations based on these properties:

Definition 1.4 (Equivalence Relation)

A binary relation R on a set X is called an **equivalence relation** if it is reflexive, symmetric, and transitive. ♣

¶ Mappings

Definition 1.5 (Mapping (Function))

A **mapping** (or function) f from a set X to a set Y is a relation such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$. We denote this by $f : X \to Y$ and write $f(x) = y$.
The set X is called the **domain** of f, and the set Y is called the **codomain** of f.
The set $f(X) = \{f(x) \mid x \in X\}$ is called the **image** of f. ♣

There are several special types of mappings:

Identity mapping The mapping $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) = x$ for all $x \in X$ is called the **identity mapping** on X.

Constant mapping A mapping $f : X \to Y$ is called a **constant mapping** if there exists a fixed element $y_0 \in Y$ such that $f(x) = y_0$ for all $x \in X$.

Mappings can be classified based on their behavior:

Injective (One-to-One): A mapping $f : X \to Y$ is **injective** if for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Surjective (Onto): A mapping $f : X \to Y$ is **surjective** if for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.

Bijective: A mapping $f : X \to Y$ is **bijective** if it is both injective and surjective.

For $A \subseteq X$, let

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A, \end{cases}$$

be the **characteristic function** of set A.

Definition 1.6 (Inverse Mapping and Composition Mappings)

Let $f : X \to Y$ be a bijective mapping. The **inverse mapping** of f, denoted by $f^{-1} : Y \to X$, is defined by $f^{-1}(y) = x$ if and only if $f(x) = y$.
Let $f : X \to Y$ and $g : Y \to Z$ be two mappings. The **composition mapping** of f and g, denoted by $g \circ f : X \to Z$, is defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$. ♣

Definition 1.7 (Restriction and Extension)

Let $f : X \to Y$ be a mapping, and let $A \subseteq X$. The **restriction** of f to A, denoted by $f|_A$, is the mapping from A to Y defined by $f|_A(x) = f(x)$ for all $x \in A$.
Conversely, if $g : A \to Y$ is a mapping and $A \subseteq X$, an **extension** of g to X is a mapping $f : X \to Y$ such that $f|_A = g$. ♣

Chapter 2 Zermelo-Fraenkel Set Theory

2.1 Axioms of ZFC

Definition 2.1 (Zermelo-Fraenkel Set Theory with Choice (ZFC))

Zermelo-Fraenkel Set Theory with Choice (ZFC) is a formal system that provides a foundation for much of modern mathematics. It consists of a set of axioms that describe the properties and behavior of sets.

The axioms of ZFC are as follows:

Axiom of Extensionality Two sets are equal (are the same set) if they have the same elements.

$$\forall A \forall B\, (\forall x (x \in A \Leftrightarrow x \in B) \Rightarrow A = B)$$

Axiom of Regularity (Foundation) Every non-empty set A contains an element that is disjoint from A.

$$\forall A\, (A \neq \emptyset \Rightarrow \exists B (B \in A \wedge B \cap A = \emptyset))$$

Axiom Schema of Specification (Separation) For any set A and any property $P(x)$, there exists a subset B of A containing exactly those elements of A that satisfy the property $P(x)$.

$$\forall A\, \exists B\, \forall x\, (x \in B \Leftrightarrow (x \in A \wedge P(x)))$$

Axiom of Pairing For any two sets A and B, there exists a set C that contains exactly A and B as elements.

$$\forall A \forall B\, \exists C\, \forall x\, (x \in C \Leftrightarrow (x = A \vee x = B))$$

Axiom of Union For any set A, there exists a set B that contains exactly the elements of the elements of A.

$$\forall A\, \exists B\, \forall x\, (x \in B \Leftrightarrow \exists C (C \in A \wedge x \in C))$$

Axiom Schema of Replacement For any set A and any definable function F, there exists a set B that contains exactly the images of the elements of A under F.

$$\forall A\, \exists B\, \forall y\, (y \in B \Leftrightarrow \exists x (x \in A \wedge y = F(x)))$$

Axiom of Infinity There exists a set A that contains the empty set and is closed under the operation of taking the successor.

$$\exists A\, (\emptyset \in A \wedge \forall x (x \in A \Rightarrow x \cup \{x\} \in A))$$

Axiom of Power Set For any set A, there exists a set B that contains exactly the subsets of A.

$$\forall A\, \exists B\, \forall C\, (C \in B \Leftrightarrow C \subseteq A)$$

Axiom of Choice For any set A of non-empty sets, there exists a choice function f that selects exactly one element from each set in A.

$$\forall A\, (\forall B \in A\, B \neq \emptyset \Rightarrow \exists f : A \to \bigcup A\, \forall B \in A\, (f(B) \in B))$$

♣

2.2 Axiom of Choice

2.3 Von Neumann-Bernays-Gödel Set Theory

Chapter 3 Ordinals

3.1 Order

Definition 3.1 (Preordered Set)

A **preordered set** is a set P together with a binary relation $\preceq$ that is reflexive and transitive. ♣

Definition 3.2 (Partially Ordered Set (Poset))

A **partially ordered set** (or **poset**) is a set P together with a binary relation $\preceq$ that is reflexive, antisymmetric, and transitive. The relation $\preceq$ is called a **partial order** on P.

Sometimes, $\prec$ is called a **strict partial order** on P if it is irreflexive, antisymmetric, and transitive. ♣

Definition 3.3 (Totally Ordered Set (Chain))

A **totally ordered set** (or **chain**) is a poset P such that for every $a, b \in P$, either $a \preceq b$ or $b \preceq a$, that is, any two elements are comparable. ♣

Definition 3.4 (Well-Ordered Set)

A **well-ordered set** is a totally ordered set P such that every non-empty subset of P has a least element. ♣

Here is a table summarizing the different types of relations:

Table 3.1: Types of Relations

Binary Relation	Reflexive	Symmetric	Antisymmetric	Transitive	Connected	Well-founded
Equivalence	✓	✓		✓		
Preorder	✓			✓		
Partial Order	✓		✓	✓		
Total Order	✓		✓	✓	✓	
Well-Order	✓		✓	✓	✓	✓

3.2 Ordinal Numbers

Definition 3.5 (Transitive Set)

A set A is called **transitive** if every element of A is also a subset of A, i.e., $(\forall x \in A)\ (x \subseteq A)$. ♣

Definition 3.6 (Ordinal)

A set α is an ordinal number (an **ordinal**) if it is transitive and well-ordered by the membership relation $\in$.

All ordinals form a proper class denoted by Ord. ♣

Ordinals can be classified into three types:

Zero The empty set $\emptyset$ is the only ordinal that is neither a successor nor a limit.

Successor Ordinal An ordinal α is a **successor ordinal** if there exists an ordinal β such that $\alpha = \beta + 1 = \beta \cup \{\beta\}$.

Limit Ordinal An ordinal λ is a **limit ordinal** if it is nonzero and not a successor, i.e., $\lambda = \bigcup_{\beta<\lambda} \beta$.

Definition 3.7 (Natural Number)

Denote the least nonzero limit ordinal by ω (or $\mathbb{N}$). The ordinals less than ω are called **finite numbers**, or **natural numbers**. Specially,

$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \quad 3 = \{0, 1, 2\}, \quad \dots$$

A set X is finite if there is a one-to-one mapping of X onto some $n \in \mathbb{N}$. X is infinite if it is not finite. ♣

3.3 Induction and Recursion

Theorem 3.1 (Transfinite Induction)

Let C be a class of ordinals and assume that:

(i) $0 \in C$.

(ii) If $\alpha \in C$, then $\alpha + 1 \in C$.

(iii) If λ is an nonzero limit ordinal and $(\forall \beta < \lambda)\, \beta \in C$, then $\lambda \in C$.

Then $C = \mathrm{Ord}$. ♡

Theorem 3.2 (Transfinite Recursion)

Let F be a class function that assigns to each ordinal α an element $F(\alpha, g)$, where g is a function with domain α. Then there exists a unique class function G with domain Ord such that for every ordinal α,

$$G(\alpha) = F(\alpha, G \restriction \alpha),$$

where $G \restriction \alpha$ is the restriction of G to the domain α. ♡

3.4 Ordinal Arithmetic

Theorem 3.3 (Cantor's Normal Form)

Every ordinal $\alpha > 0$ can be uniquely expressed in the form

$$\alpha = \omega^{\beta_1} \cdot c_1 + \omega^{\beta_2} \cdot c_2 + \cdots + \omega^{\beta_n} \cdot c_n,$$

where n is a positive integer, $c_1, c_2, \dots, c_n$ are positive integers, and $\beta_1 > \beta_2 > \cdots > \beta_n$ are ordinals. ♡

Chapter 4 Cardinals

4.1 Cardinality

¶ Equinumerosity and Cardinality

Definition 4.1 (Equinumerosity and Cardinality)

Two sets A and B are said to be **equinumerous** (or have the same cardinality), denoted by $|A| = |B|$, if there exists a bijection $f : A \to B$.

The **cardinality** of a set A is the least ordinal κ such that $|A| = |\kappa|$.

♣

Definition 4.2 (Aleph Numbers)

The **aleph numbers** are a sequence of cardinal numbers defined as follows:

- $\aleph_0$ is the cardinality of the set of natural numbers $\mathbb{N}$.
- For any ordinal α, $\aleph_{\alpha+1}$ is the least cardinal number greater than $\aleph_\alpha$.
- For any limit ordinal λ, $\aleph_\lambda = \sup\{\aleph_\beta \mid \beta < \lambda\}$.

♣

Theorem 4.1 (Cantor-Bernstein-Schröder Theorem)

If there exist injections $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \to B$. In particular, $|A| = |B|$.

♡

¶ Countable and Uncountable Sets

Theorem 4.2

♡

¶ Continuum Hypothesis

4.2 Cardinal Arithmetic

4.3 The Canonical Well-Ordering of $\alpha \times \alpha$

4.4 Cofinality

Chapter 5 Real Numbers

5.1 Construction of Real Numbers and the Cardinality of the Continuum

5.2 Point Sets in Euclidean Space

In this section, we explore the point sets in Euclidean space. Furthermore, these concepts can be generalized to metric spaces and topological spaces.

Definition 5.1 (Diameter and Bounded Set)

Let A be a subset of the Euclidean space $\mathbb{R}^n$. The **diameter** of set A is defined as

$$\operatorname{diam}(A) = \sup\{d(x,y) \mid x, y \in A\},$$

where $d(x,y)$ denotes the Euclidean distance between points x and y.
A set A is called **bounded** if there exists a real number $M > 0$ such that

$$d(x,y) < M, \quad \forall x, y \in A.$$

Let $x_0 \in \mathbb{R}^n$, $\delta > 0$, the set

$$B(x_0, \delta) = \{x \in \mathbb{R}^n \mid d(x, x_0) < \delta\}$$

is called the **open ball** (or **neighborhood**) with center x_0 and radius δ[a]. Similarly, the closed ball can be defined as

$$\bar{B}(x_0, \delta) = \{x \in \mathbb{R}^n \mid d(x, x_0) \leq \delta\}.$$

Let a_i, b_i $(i = 1, 2, \ldots, n)$ be real numbers with $a_i < b_i$, the set

$$\prod_{i=1}^{n} [a_i, b_i] = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for all } i = 1, 2, \ldots, n\}$$

is called a **rectangle** (or **box**) in $\mathbb{R}^n$. If all the edge lengths are equal, i.e., $b_i - a_i = c$ for some constant $c > 0$ and for all i, then the rectangle is called a **cube** with side length c. Similarly, we can define the open rectangle (or open box) as

$$\prod_{i=1}^{n} (a_i, b_i) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_i < x_i < b_i \text{ for all } i = 1, 2, \ldots, n\}.$$

Rectangles are often denoted by $I, J, \ldots$ and their volumes by $|I|, |J|, \ldots$.

[a] It can be also denoted as $N(x_0, \delta)$ or $U(x_0, \delta)$. When δ does not need to be emphasized, it can also be abbreviated as $B(x_0)$. ♣

Definition 5.2 (Limit)

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that $\{x_k\}$ **converges** to x, or x is the **limit** of the sequence $\{x_k\}$, if for every $\varepsilon > 0$, there exists a natural number N such that

$$d(x_k, x) < \varepsilon, \quad \forall k > N.$$

In this case, we write

$$\lim_{k \to \infty} x_k = x.$$

♣

¶ Classification of Points

Definition 5.3 (Classification of Points)

Let E be a subset of the Euclidean space $\mathbb{R}^n$. Points in $\mathbb{R}^n$ can be classified based on their relationship to set E:

Interior Point A point $x \in E$ is called an **interior point** of set E if there exists $U(x)$ such that $U(x) \subseteq E$.

Exterior Point A point $x \in \mathbb{R}^n \setminus E$ is called an **exterior point** of set E if there exists $U(x)$ such that $U(x) \subseteq \mathbb{R}^n \setminus E$, or equivalently, $U(x) \cap E = \emptyset$.

Boundary Point A point $x \in \mathbb{R}^n$ is called a **boundary point** of set E if for every $U(x)$, the set $U(x)$ contains points in both E and $\mathbb{R}^n \setminus E$.

Accumulation Point (Limit Point) A point $x \in \mathbb{R}^n$ is called an **accumulation point** (or **limit point**) of set E if for every $U(x)$, the set $U(x)$ contains at least one point of E different from x[a].

Isolated Point A point $x \in E$ is called an **isolated point** of set E if x is not an accumulation point of E, i.e., there exists $U(x)$ such that $U(x) \cap E = \{x\}$.

[a] Obviously, only infinite sets can have accumulation points. In fact, here, containing at least one (distinct) point in the neighborhood is equivalent to containing infinitely many points. ♣

Any point $x \in \mathbb{R}^n$ can be uniquely classified into one of the following three categories:

$$\begin{cases} \text{Interior Point} & \text{if } \exists U(x) \subseteq E; \\ \text{Boundary Point} & \text{if } \forall U(x) \cap (\mathbb{R}^n \setminus E) \neq \emptyset; \\ \text{Exterior Point} & \text{if } \exists U(x) \subseteq \mathbb{R}^n \setminus E; \end{cases}$$

Or it can be uniquely classified into one of the following three categories:

$$\begin{cases} \text{Accumulation Point} & \text{if } \forall U(x) \cap (E \setminus \{x\}) \neq \emptyset; \\ \text{Isolated Point} & \text{if } \exists U(x) \cap E = \{x\}; \\ \text{Exterior Point} & \text{if } \exists U(x) \cap E = \emptyset; \end{cases}$$

Definition 5.4

Let E be a subset of the Euclidean space $\mathbb{R}^n$.

Derived Set The **derived set** of E, denoted by E', is the set of all accumulation points of E.

Interior The **interior** of set E, denoted by $\mathrm{int}(E)$, or $\mathring{E}$, is the set of all interior points of E.

Boundary The **boundary** of set E, denoted by ∂E, is the set of all boundary points of E, or equivalently, $\partial E = \bar{E} \setminus \mathring{E}$.

Closure The **closure** of set E, denoted by $\bar{E}$, is the union of E and its accumulation points, i.e., $\bar{E} = E \cup E'$. ♣

Property

- $\left(\mathring{E}\right)^c = \overline{E^c}$, $\quad \left(\overline{E}\right)^c = \mathring{E^c}$;
- *Let $A \subseteq B$, then $A' \subseteq B'$, $\mathring{A} \subseteq \mathring{B}$ and $\overline{A} \subseteq \overline{B}$;*
- $(A \cup B)' = A' \cup B'$.

Note *In a metric space, an alternative definition of accumulation point can be given: A point x is an accumulation point of set E if and only if it is the limit of some sequence of points in E.*

Remark By replacing the Euclidean distance with a general metric d, all the above definitions can be naturally extended to a general metric space (X, d).

By replacing the metric d with the family of open sets in a general topological structure, all the above definitions can be extended to a general topological space (X, τ).

¶ Open and Closed Sets

Definition 5.5 (Classification of Point Sets)

Let E be a subset of the Euclidean space $\mathbb{R}^n$. Point sets can be classified:

Closed Set A set E is called a **closed set** if it contains all its accumulation points.

Open Set A set E is called an **open set** if every point in E is an interior point of E.

Compact Set A set E is called a **compact set** if every open cover of E has a finite subcover, or equivalently, if E is closed and bounded (Heine-Borel Theorem).

Perfect Set A set E is called a **perfect set** if it is closed and has no isolated points, i.e., every point in E is an accumulation point of E, or equivalently, $E = E'$. ♣

Chapter 6 Special Classes of Sets

Chapter 7 Filters and Boolean Algebras

Bibliography

[1] Elias M. Stein, Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton University Press, 2016.

[2] Author2, Title2, Journal2, Year2. *This is another example of a reference.*