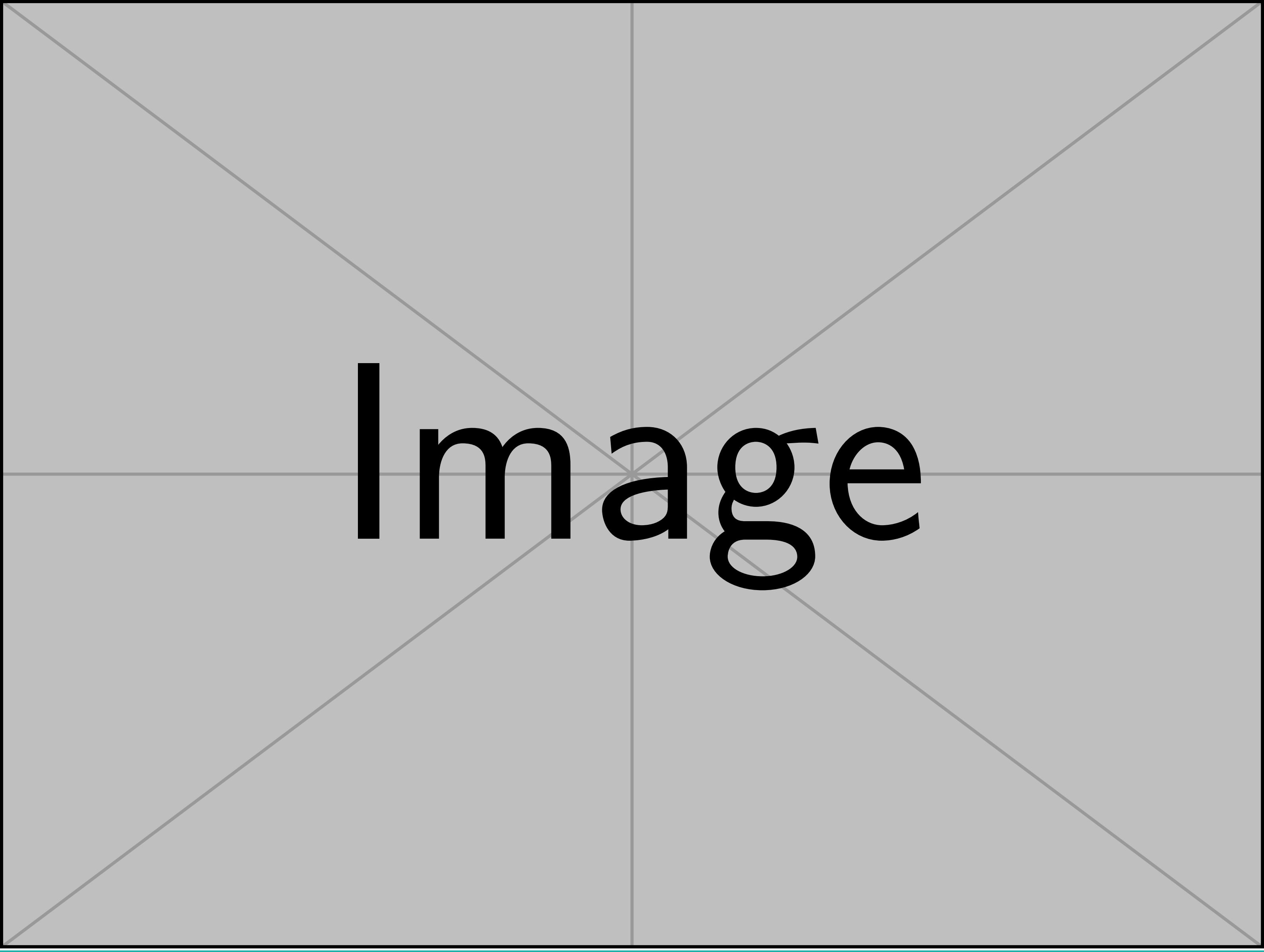

Équation Différentielle Ordinaire

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Preface

Chapter 1 Introduction

1.1 Classification of Differential Equations

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a **differential equation**. Differential equations can be classified according to the following criteria:

¶ `Number of Independent Variables`

An **ordinary differential equation(ODE)** is defined as an equation of the following form:

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \ldots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n}\right) = 0, \quad n \in \mathbb{N}, \tag{1.1}$$

or, using the prime notation for derivatives,

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0, \quad n \in \mathbb{N}.$$

If there are two or more independent variables, the equation is called a **partial differential equation(PDE)**.

¶ `Order`

The order of a differential equation is the order of the highest derivative present in the equation.

- A first-order equation has the form $F(x, y, y') = 0$.
- A second-order equation has the form $F(x, y, y', y'') = 0$.
- Higher-order equations involve derivatives of order three or more.

Note *Crucially, the order tells you how many initial conditions are needed to find a unique solution.*

¶ `Linearity`

An n-th order differential equation is linear if it can be written in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where the coefficients $a_i(x)$ and the term $g(x)$ depend only on the independent variable x. Otherwise, it is nonlinear.

Note *Specially, for the aforementioned equation, if $g(x) = 0$, it is called **homogeneous**, and **non-homogeneous** otherwise.*

1.2 Solution to a Ordinary Differential Equation

¶ `Particular and General Solutions`

Let J be an interval in $\mathbb{R}$. A function $y = \phi(x)$ defined on the interval J is called a solution to equation (1.1) if it satisfies:

$$F(x, \phi(x), \phi'(x), \phi''(x), \ldots, \phi^{(n)}(x)) = 0 \quad x \in J.$$

The interval J is then called the interval of existence of the solution $y = \phi(x)$.

Generally speaking, the solution to equation (1.1) contains one or more arbitrary constants, the determination of which depends on other conditions that the solution must satisfy. If a solution to a differential equation does not contain any arbitrary constants, it is called a **particular solution** of the differential equation.

Suppose $y = \phi(x; c_1, c_2, \cdots, c_n)$ is a solution to equation (1.1), where $c_1, c_2, \ldots, c_n$ are arbitrary constants. If $c_1, c_2, \ldots, c_n$ are mutually independent, then $y = \phi(x; c_1, c_2, \ldots, c_n)$ is called the **general solution** to equation (1.1). Here, "mutually independent" means that the Jacobian determinant is non-zero:

$$\det \frac{\partial(\phi, \phi', \dots, \phi^{(n-1)})}{\partial(c_1, c_2, \dots, c_n)} \neq 0, \quad x \in J.$$

When all the arbitrary constants in the general solution are determined, one obtains a particular solution to the differential equation.

¶ **Initial Conditions, Explicit and Implicit Solutions**

Let $y = \phi(x)$ be a solution to equation (1.1) that also satisfies

$$\phi(x_0) = y_0, \quad \phi'(x_0) = y_0', \dots, \quad \phi^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{1.2}$$

The conditions (1.2) are called the **initial conditions** for equation (1.1), and $y = \phi(x)$ is called the solution to equation (1.1) satisfying the initial conditions (1.2). Such initial value problems are often referred to as **Cauchy problems**.

A function $y = \phi(x)$ that turns the differential equation (1.1) into an identity is called an **(explicit) solution** to the equation. If a solution $y = \phi(x)$ to the differential equation (1.1) is determined by the relation $\Phi(x, y) = 0$, then $\Phi(x, y) = 0$ is called an **implicit solution** to the differential equation (1.1). An implicit solution is also called an "integral".

¶ **Integral Curve and Direction Field**

Consider the first-order differential equation:

$$\frac{dy}{dx} = f(x, y), \tag{1.3}$$

where f is continuous in a planar region G. Suppose

$$y = \phi(x), \quad x \in J$$

is a solution to this equation, where $J \subset \mathbb{R}$ is an interval. Then the set of points in the plane

$$\Gamma = (x, y) | y = \phi(x), x \in J$$

is a differentiable curve in the plane. This curve is called a solution curve or an **integral curve**.

Let $(x_0, y_0) \in \Gamma$. The slope of the tangent line to the curve Γ at this point is

$$\phi'(x_0) = f(x_0, y_0).$$

Therefore, the equation of the tangent line is

$$y - y_0 = f(x_0, y_0)(x - x_0).$$

This implies that even without knowing the explicit expression for ϕ, we can obtain the slope and equation of the tangent line to the solution curve at a given point from equation (1.3).

Remark Note that in a small neighborhood of a point on a differentiable curve, the tangent line can be seen as a first-order approximation of the curve. Utilizing this viewpoint, one can obtain an approximate solution to the differential equation. In fact, this is the fundamental idea behind Euler's method.

At each point P in the region G, we can draw a short line segment $l(P)$ with slope $f(P)$. We call $l(P)$ the line element of equation (1.3) at point P. The region G together with the entire collection of these line elements is called the lineal **linear element field** or **direction field** for equation (1.3).

Theorem 1.1

A necessary and sufficient condition for a continuously differentiable curve $\Gamma = \{(x, y) | y = \psi(x), x \in J\}$ in the plane to be an integral curve of equation (1.3) is that for every point (x, y) on the curve Γ, its tangent line at that point coincides with the line element determined by equation (1.3) at that point.

Proof The necessity follows from the preceding discussion. We now prove the sufficiency. For any point $(x, y) = (x, \psi(x))$ on the curve Γ, the slope of the tangent line to Γ at this point is $\psi'(x)$. By the condition of the theorem, we have $\psi'(x) = f(x, y)$. Since (x, y) is an arbitrary point on the curve, it follows that $y = \psi(x)$ is a solution to equation (1.3). ■

Chapter 2 First Order Equations

2.1 Exact Equations

Definition 2.1 (Exact Equations)

An equation of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{2.1}$$

is called the symmetric form of a first-order differential equation.

If there exists a continuously differentiable function $u(x,y)$ such that

$$\mathrm{d}U(x,y) = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y,$$

then equation (2.1) is said to be an **exact equation** or a **total differential equation**.

It follows that, when equation (2.1) is exact, it can be rewritten as

$$\mathrm{d}(U(x,y)) = 0,$$

which implies

$$U(x,y) = c, \tag{2.2}$$

where c is an arbitrary constant. Equation (2.2) is called the **general integral** of equation (2.1). ♣

Remark It should be noted that, strictly speaking, equation (2.1) is not a differential equation. However, expressing a first-order differential equation in the form of (2.1) is extremely convenient for analysis. This formulation does not necessarily require y to be expressed as a function of x. For the sake of simplicity in description, we often refer to the symmetric form (2.1) as a differential equation, too.

Theorem 2.1

Let the functions $M(x,y)$ and $N(x,y)$ be continuous in a simply connected domain $D \subset \mathbb{R}^2$, and suppose their first-order partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are also continuous. Then a necessary and sufficient condition for equation (2.1) to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

in the domain D. When this condition holds, for any $(x_0, y_0), (x,y) \in D$, a general integral of equation (2.1) is given by

$$\int_\gamma M(x,y)dx + N(x,y)dy = c,$$

where γ is any curve composed of finitely many smooth segments within D connecting (x_0,y_0) and (x,y), and c is an arbitrary constant. ♡

Proof

■

The aforementioned proof also serves as a method for determining the bivariate function $U(x,y)$ that satisfies specific conditions. In addition to this approach, there exist two simpler methods for solving $U(x,y)$.

Utilizing Curve Integrals to Solve $U(x,y)$

Term Combination Method Utilizing the properties of bivariate differential functions, we combine the terms of the differential equation into a full differential form. This method requires familiarity with some simple bivariate differential functions, such as:

$$
\begin{aligned}
y\mathrm{d}x + x\mathrm{d}y &= \mathrm{d}(xy), \\
\frac{y\mathrm{d}x - x\mathrm{d}y}{y^2} &= \mathrm{d}\left(\frac{x}{y}\right), \\
\frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2} &= \mathrm{d}\left(\frac{y}{x}\right), \\
\frac{1}{x}\mathrm{d}x + \frac{1}{y}\mathrm{d}y = \frac{y\mathrm{d}x + x\mathrm{d}y}{xy} &= \mathrm{d}(\ln|xy|), \\
\frac{1}{x}\mathrm{d}x - \frac{1}{y}\mathrm{d}y = \frac{y\mathrm{d}x - x\mathrm{d}y}{xy} &= \mathrm{d}(\ln|\frac{x}{y}|), \\
\frac{y\mathrm{d}x - x\mathrm{d}y}{x^2 - y^2} &= \frac{1}{2}\mathrm{d}\left(\ln\left|\frac{x-y}{x+y}\right|\right), \\
\frac{y\mathrm{d}x + x\mathrm{d}y}{x^2 + y^2} &= \mathrm{d}\left(\arctan\frac{y}{x}\right), \\
\frac{y\mathrm{d}x - x\mathrm{d}y}{x^2 + y^2} &= \mathrm{d}\left(\operatorname{arccot}\frac{y}{x}\right).
\end{aligned}
$$

The theory above can also be rewritten in differential form:

Let:

$$\omega^1 = M(x,y)\,dx + N(x,y)\,dy.$$

The differential form ω^1 is said to be **closed** if $\mathrm{d}\omega^1 = 0$. It is called **exact** if there exists a function $U(x,y)$ such that $\omega^1 = \mathrm{d}U(x,y)$. By the Poincaré theorem, it can be concluded that on $\mathbb{R}^2$, a first-order differential form is exact if and only if it is closed. Note that:

$$\mathrm{d}\omega^1 = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y.$$

Clearly, $\mathrm{d}\omega^1 = 0$ holds if and only if:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Under this condition, the expression for the function $U(x,y)$ is:

$$U(x,y) = \int \omega^1.$$

2.2 Separable Equations

Definition 2.2 (Separable Equations)

If the functions $M(x,y)$ and $N(x,y)$ in Equation (2.1) can both be written as the product of a function of x and a function of y, that is,

$$M(x,y) = M_1(x)M_2(y), \quad N(x,y) = N_1(x)N_2(y),$$

then equation (2.1) is called a separable equation.

When equation (2.1) is a separable equation, it can be written as

$$M_1(x)M_2(y)\,\mathrm{d}x + N_1(x)N_2(y)\,\mathrm{d}y = 0, \tag{2.3}$$

or more conveniently as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \left(= -\frac{M_1(x)}{N_1(x)} \cdot \frac{N_2(y)}{M_2(y)} \right). \tag{2.4}$$

♣

Theorem 2.2 (Solutions to Separable Equations)

All the solutions to the separable equation (2.3) are given by:

$$\int_{x_0}^{x} \frac{M_1(t)}{N_1(t)}\mathrm{d}t + \int_{y_0}^{y} \frac{N_2(s)}{M_2(s)}\mathrm{d}s = c,$$

and

$$y \equiv b_i, \quad i = 1, 2, \dots, m, x \equiv a_j, \quad j = 1, 2, \dots, n,$$

where $M_2(b_i) = 0 \quad (i = 1, 2, \dots, m)$ and $N_1(a_j) = 0 \quad (j = 1, 2, \dots, n)$, c is arbitrary constant.

♡

2.3 Homogeneous Equations

Definition 2.3 (Homogeneous Functions)

A function $f(x, y)$ is called a **homogeneous function** of degree n if it satisfies the condition:

$$f(tx, ty) = t^n f(x, y)$$

for all $t > 0$.

A function $f(x, y)$ is called a **quasihomogeneous function** of degree d with generalized weights if

$$f(t^\alpha sx, t^\beta sy) = t^{ds} f(x, y),$$

where $t > 0$, α and β are positive constants with $\alpha + \beta = 1$, and $s \in \mathbb{R}$. Here, α and β are called the weights of x and y, respectively.

♣

Definition 2.4

A first-order differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

is called a **homogeneous equation** if both M and N are homogeneous functions of the same degree n. In other words, for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

$f(x, y)$ can be rewritten as $g\left(\frac{y}{x}\right)$.

♣

The equation

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \tag{2.5}$$

can be transformed into a separable equation via variable change, where $a_1, a_2, b_1, b_2, c_1, c_2$ are constants.

- When $c_1 = c_2 = 0$, the equation becomes:

$$\frac{dy}{dx} = f\left(\frac{a_1 + b_1\frac{y}{x}}{a_2 + b_2\frac{y}{x}}\right) = g\left(\frac{y}{x}\right).$$

  Let

$$u = \frac{y}{x}, \text{ namely } y = ux.$$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = x\frac{du}{dx} + u.$$

Substituting the results into original equation and simplifying, we obtain:

$$\frac{du}{dx} = \frac{g(u) - u}{x},$$

which is a separable equation. It can be solved easily. Then, substituting $u = \frac{y}{x}$ back, the solution is derived.

- When c_1, c_2 are not entirely zero, the right-hand side of (2.5) consists of linear polynomials of x and y. Therefore:

$$\begin{cases} a_1x + b_1y + c_1 = 0, \\ a_2x + b_2y + c_2 = 0, \end{cases}$$

represents two intersecting straight lines on the Oxy plane. For the coefficient determinant of the system:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

two cases are analyzed:

1. If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, indicating that the two lines intersect at a unique point (α, β) on the Oxy plane. Let:

$$\begin{cases} X = x - \alpha, \\ Y = y - \beta, \end{cases}$$

then (2.3) becomes:

$$\begin{cases} a_1X + b_1Y = 0, \\ a_2X + b_2Y = 0. \end{cases}$$

Substituting into 2.5, it simplifies to:

$$\frac{dY}{dX} = f\left(\frac{a_1 + b_1\frac{Y}{X}}{a_2 + b_2\frac{Y}{X}}\right) = g\left(\frac{Y}{X}\right).$$

This is a homogeneous differential equation. Solving it by substitution and reverting back to the original variables yields the solution to equation 2.5.

2. When $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. To ensure this system holds, there are three possible scenarios:

(a). If $a_1 = b_1 = 0$, 2.5 becomes:

$$\frac{dy}{dx} = f\left(\frac{c_1}{a_2x + b_2y + c_2}\right),$$

and when $a_2 = b_2 = 0$, it becomes:

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{c_2}\right).$$

In this case, let

$$u = \frac{a_1x + b_1y + c_1}{c_2}.$$

Then it can be transformed into a separable equation.

(b). If $b_1 = b_2 = 0$, 2.5 transforms into:

$$\frac{dy}{dx} = f\left(\frac{a_1 x + c_1}{a_2 x + c_2}\right),$$

and

$$\frac{dy}{dx} = f\left(\frac{b_1 y + c_1}{b_2 y + c_2}\right),$$

when $a_1 = a_2 = 0$.

(c). If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$, let $u = a_2 x + b_2 y$. In this case:

$$\frac{du}{dx} = a_2 + b_2 \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$f\left(\frac{k(a_2 x + b_2 y) + c_1}{(a_2 x + b_2 y) + c_2}\right) = f\left(\frac{ku + c_1}{u + c_2}\right) = g(u)$$

which simplifies to:

$$\frac{du}{dx} = a_2 + b_2 g(u).$$

Example 2.1 Consider the differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0,$$

where $M(x, y)$ and $N(x, y)$ are quasihomogeneous functions of degree d_0 and d_1 with weights α and β for x and y, respectively. Proposition: When $d_0 = d_1 + \beta - \alpha$ the equation can be solved by elementary integration method.

2.4 Linear Equations

Definition 2.5 (First-Order Linear Equations)

A **first-order linear equation** is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x), \tag{2.6}$$

where $p(x)$ and $q(x)$ are continuous functions on the interval (a, b). In Equation (2.6), when $q(x) \equiv 0$, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0, \tag{2.7}$$

which is called a **first-order homogeneous linear equation** corresponding to Equation (2.6). Otherwise, it is called a first-order non-homogeneous linear equation. ♣

Note *It should be noted that the definition of a homogeneous equation here differs from that in the previous section.*

Firstly, we solve the first-order homogeneous linear equation. Equation 2.7 is separable, thus its general solution is given by:

$$y = ce^{-\int p(x)\,\mathrm{d}x},$$

where c is an arbitrary constant.

Since 2.7 is a special case of 2.6, the general solution of 2.6 can be expressed as:

$$y = c(x)e^{-\int p(x)\,\mathrm{d}x},$$

substituting it into 2.6 yields:

$$y = e^{-\int p(x)\,\mathrm{d}x}\left(c + \int q(x)e^{\int p(x)\,\mathrm{d}x}\,\mathrm{d}x\right).$$

This method of solving first-order linear equations is known as the **method of variation of constants**.

Definition 2.6 (Bernoulli's Equation)

A first-order differential equation of the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n, \quad n \neq 0, 1,$$
where n is a real number and $p(x)$ and $q(x)$ are continuous functions on the interval (a, b), is called a **Bernoulli's equation.** ♣

Bernoulli's equation can be transformed into a first-order linear equation by the substitution:
$$z = y^{1-n}.$$
Differentiating both sides with respect to x gives:
$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$
Substituting $\frac{\mathrm{d}y}{\mathrm{d}x}$ from Bernoulli's equation into the above expression yields:
$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)\left(-p(x)z + q(x)\right).$$
This is a first-order linear equation in z, which can be solved using the method for first-order linear equations.

2.5 Integrating Factors

Definition 2.7 (Integrating Factors)

An **integrating factor** for a first-order differential equation of the form
$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{2.8}$$
is a differentiable function $\mu(x, y)$ such that when multiplied by the equation:
$$\mu(x, y)M(x, y)\,\mathrm{d}x + \mu(x, y)N(x, y)\,\mathrm{d}y = 0,$$
it becomes an exact equation. Id est, there exists a function $\Phi(x, y)$ such that
$$\mu(x, y)M(x, y)\,\mathrm{d}x + \mu(x, y)N(x, y)\,\mathrm{d}y = \mathrm{d}U(x, y).$$
If such functions $\mu(x, y)$ and $U(x, y)$ exist, and $U(x, y)$ is smooth, then
$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}\left(= \frac{\partial^2 U}{\partial x \partial y}\right). \tag{2.9}$$
In this case, $\mu(x, y)$ is called an integrating factor for equation (2.8). ♣

According to Equation (2.9), finding an integrating factor $\mu(x, y)$ for equation (2.8) is equivalent to solving the partial differential equation:
$$\frac{\partial \mu}{\partial x}N - \frac{\partial \mu}{\partial y}M = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu. \tag{2.10}$$

Theorem 2.3

1. For the partial differential equation 2.10 to have a solution $\mu(x)$ that depends only on x, the necessary and sufficient condition is:

The function G defined below must depend only on x:

$$G = -\frac{1}{N(x,y)}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

In this case, we have:

$$\mu(x) = e^{\int_{x_0}^{x} G(t)\,dt}.$$

2. For the partial differential equation 2.10 to have a solution $\mu(y)$ that depends only on y, the necessary and sufficient condition is:
   The function H defined below must depend only on y:

$$H = \frac{1}{M(x,y)}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

   In this case, we have:

$$\mu(y) = e^{\int_{y_0}^{y} H(s)\,ds}.$$

3. For equation 2.8 to have an integrating factor of the form $\mu = \mu(\phi(x,y))$, the necessary condition is:

$$\frac{1}{\frac{\partial \phi}{\partial x}N - \frac{\partial \phi}{\partial y}M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(\phi(x,y)),$$

   where f is a certain univariate function.

Theorem 2.4

Let the functions $P(x,y)$, $Q(x,y)$, $\mu_1(x,y)$, and $\mu_2(x,y)$ be continuously differentiable. Suppose $\mu_1(x,y)$ and $\mu_2(x,y)$ are integrating factors for equation (2.8), and the ratio $\frac{\mu_1(x,y)}{\mu_2(x,y)}$ is not a constant. Then:

$$\frac{\mu_1(x,y)}{\mu_2(x,y)} = c$$

is a general solution to the equation, where c is an arbitrary constant.

2.6 Implicit Equations

This section discusses the problem of solving the first-order implicit differential equations,

$$F(x,y,y') = 0 \tag{2.11}$$

where F is a continuously differentiable function. A so-called implicit differential equation is one in which y' does not have an explicit solution, that is, the equation cannot be written in the form $y' = f(x,y)$.

¶ **Differentiation Method**

Suppose that Equation (2.11) can be solved for y, that is,

$$y = f(x,p), \quad p = \frac{dy}{dx}, \tag{2.12}$$

where $f(x,p)$ is a continuously differentiable function.

Differentiating both sides of $y = f(x,p)$ with respect to x, we obtain

$$p = \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx},$$

that is,

$$\frac{\partial f}{\partial p}\frac{dp}{dx} = p - \frac{\partial f}{\partial x}.$$

This is a first-order differential equation in the variables x, p, $\frac{dp}{dx}$. If a solution $p = p(x)$ can be found, then Equation (2.12) yields a solution

$$y = f(x, p(x)).$$

¶ **Parametric Method**

In general, Equation (2.11) represents a surface in the (x, y, p)-space. Therefore, the solution can be obtained using a parametric representation of the surface. Suppose the parametric form of the surface described by Equation (2.11) is

$$x = x(u, v), \quad y = y(u, v), \quad p = p(u, v) = y'.$$

Note that

$$dy = p \, \mathrm{d}x,$$

thus we obtain

$$y'_u \mathrm{d}u + y'_v \mathrm{d}v = p(u, v)(x'_u \mathrm{d}u + x'_v \mathrm{d}v).$$

This is an explicit differential equation in the variables u and v. Suppose it admits a solution

$$v = v(u, c),$$

where c is a constant, then Equation (2.11) has a solution

$$x = x(u, v(u, c)), \quad y = y(u, v(u, c)).$$

Chapter 3 Existence and Uniqueness Theorem

3.1 Picard-Lindelöf Theorem

Theorem 3.1 (Bellman-Gronwall Inequality)

Let $f(x), g(x)$ be continuous functions on the interval $[a, b]$, $g(x) \geqslant 0$, and c be a non-negative constant. If

$$f(x) \leqslant c + \int_a^x f(t)g(t)\,dt,$$

then

$$f(x) \leqslant c\exp\left(\int_a^x g(t)\,dt\right).$$

♡

For a Cauchy problem:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \\ y(x_0) = y_0, \end{cases} \tag{3.1}$$

give the existence and uniqueness theorem.

¶ **Picard-Lindelöf Theorem**

Theorem 3.2 (Picard-Lindelöf Theorem)

In the Cauchy problem (3.1), let D be a closed rectangle in the xy-plane:

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b].$$

If the function $f(x, y)$ satisfies the following two conditions:

1. $f(x, y)$ is continuous in D.
2. $f(x, y)$ satisfies the Lipschitz condition with respect to y in D, i.e., there exists a constant $L > 0$ such that for any $(x, y_1), (x, y_2) \in D$,

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|.$$

Then there exists a unique solution $y = \varphi(x)$ $(\varphi(x_0) = y_0)$ to the Cauchy problem (3.1) in the interval $[x_0 - h, x_0 + h]$, where

$$h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(x,y)\in D} |f(x, y)|.$$

♡

Proposition 3.1

♠

¶ **Peano Theorem and Osgood Theorem**

In regard to the solutions for the Cauchy problem (3.1), we have the following two theorems, which are weaker than the Picard-Lindelöf theorem:

Definition 3.1 (Osgood Condition)

Let $f(x, y)$ be a continuous function in the region D. If for any $(x, y_1), (x, y_2) \in D$,

$$|f(x, y_1) - f(x, y_2)| \leqslant F(|y_1 - y_2|),$$

where $F(t) > 0$ ($t > 0$) is a continuous function, and

$$\int_0^{\varepsilon} \frac{1}{F(t)}\, dt = +\infty, \quad \forall \varepsilon > 0,$$

then $f(x, y)$ is said to satisfy the **Osgood condition** with respect to y in D. ♣

Remark If $f(x, y)$ satisfies Lipschitz condition, then it also satisfies the Osgood condition. In fact, in this case, we can take $F(t) = Lt$.

Theorem 3.3 (Peano Theorem)

In the Cauchy problem (3.1), let D be a closed rectangle in the xy-plane:

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$$

. If the function $f(x, y)$ is continuous in D, then there exists at least one solution $y = \varphi(x)$ ($\varphi(x_0) = y_0$) to the Cauchy problem (3.1) in the interval $[x_0 - h, x_0 + h]$, where

$$h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(x,y)\in D} |f(x, y)|.$$

♡

Theorem 3.4 (Osgood Theorem)

In the Cauchy problem (3.1), let D be a closed rectangle in the xy-plane:

$$D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$$

. If the function $f(x, y)$ satisfies the Osgood condition with respect to y in D, then there exists a unique solution for any $(x_0, y_0) \in D$ to the Cauchy problem (3.1) in the interval $[x_0 - h, x_0 + h]$, where

$$h = \min\left\{a, \frac{b}{M}\right\}, M = \max_{(x,y)\in D} |f(x, y)|.$$

♡

3.2 Continuation of the Solution

¶ Uncontinuable Solutions

Definition 3.2 (Uncontinuable Solutions)

Let $y = \varphi(x)$ be a solution to the Cauchy problem (3.1) in the interval $I_1 \subset \mathbb{R}$. If there exists an another solution $y = \varphi_2(x)$ to the Cauchy problem (3.1) in any interval $I_2 \supsetneq I_1$ such that

$$\varphi_2(x) \equiv \varphi(x), \quad x \in I_1,$$

then $y = \varphi_1(x)$ is called **continuable**, and $y = \varphi_2(x)$ is called a **continuation** of $y = \varphi_1(x)$. If there does not exist such a solution $y = \varphi_2(x)$, then $y = \varphi_1(x)$ is called **uncontinuable**, or **saturated**. ♣

Theorem 3.5

In the Cauchy problem (3.1), let D be a bounded closed rectangle in the xy-plane. If the function $f(x, y)$ is continuous in D, and satisfies the local Lipschitz condition with respect to y in D, then any solution $y = \varphi(x)$ passing through $(x_0, y_0) \in D$ to the Cauchy problem (3.1) can be continued until it arbitrarily approaches the boundary of D. ♡

¶ Comparison Theorem

3.3 Singular Solutions and Envelopes

3.4 Dependency of Solutions on Initial Data

Chapter 4 System of First-Order Linear Equations

4.1 System of First-Order Linear Equations

¶ Common Forms

System of first-order equations with n variables is of the form:

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}x} = f_1(x, y_1, y_2, \ldots, y_n), \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} = f_2(x, y_1, y_2, \ldots, y_n), \\ \quad\vdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}x} = f_n(x, y_1, y_2, \ldots, y_n). \end{cases} \tag{4.1}$$

If the right-hand side of each equation in system (4.1) does not include explicitly x, then the system is called **autonomous**.

The solution to system (4.1) is an n-tuple of functions

$$y_1 = \varphi_1(x), y_2 = \varphi_2(x), \ldots, y_n = \varphi_n(x),$$

which satisfy all equations in system (4.1) simultaneously.

Solution containing arbitrary constants $C_1, C_2, \ldots, C_n$

$$\begin{cases} y_1 = \varphi_1(x, C_1, C_2, \ldots, C_n), \\ y_2 = \varphi_2(x, C_1, C_2, \ldots, C_n), \\ \quad\vdots \\ y_n = \varphi_n(x, C_1, C_2, \ldots, C_n) \end{cases}$$

is called the **general solution** of system (4.1). If general solution satisfies

$$\begin{cases} \Phi_1(x, y_1, \cdots, y_n, C_1, \cdots, C_n) = 0, \\ \Phi_2(x, y_1, \cdots, y_n, C_1, \cdots, C_n) = 0, \\ \quad\vdots \\ \Phi_n(x, y_1, \cdots, y_n, C_1, \cdots, C_n) = 0, \end{cases}$$

then it is called the **general integral** of system (4.1).

For convenience, we rewrite system (4.1) in matrix form:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{F}(x, \mathbf{Y}), \tag{4.2}$$

and autonomous system as:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{F}(\mathbf{Y}), \tag{4.3}$$

and Cauchy problem for system (4.2) as:

$$\begin{cases} \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{F}(x, \mathbf{Y}), \\ \mathbf{Y}(x_0) = \mathbf{Y}_0, \end{cases} \tag{4.4}$$

where

$$\mathbf{Y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{F}(x,\mathbf{Y})=\begin{bmatrix} f_1(x,y_1,y_2,\ldots,y_n) \\ f_2(x,y_1,y_2,\ldots,y_n) \\ \vdots \\ f_n(x,y_1,y_2,\ldots,y_n) \end{bmatrix}, \quad \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x}=\begin{bmatrix} \frac{\mathrm{d}y_1}{\mathrm{d}x} \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} \\ \vdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}x} \end{bmatrix}.$$

With these notations, from the perspective of form, the system of first-order linear equations is similar to first-order equations.

In system (4.1), if $f_i(x,y_1,y_2,\ldots,y_n)$ is a linear function of $y_1,y_2,\ldots,y_n$, i.e., it can be rewritten as:

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}x}=a_{11}(x)y_1+a_{12}(x)y_2+\cdots+a_{1n}(x)y_n+f_1(x), \\ \frac{\mathrm{d}y_2}{\mathrm{d}x}=a_{21}(x)y_1+a_{22}(x)y_2+\cdots+a_{2n}(x)y_n+f_2(x), \\ \quad\vdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}x}=a_{n1}(x)y_1+a_{n2}(x)y_2+\cdots+a_{nn}(x)y_n+f_n(x). \end{cases}$$

It is called a **system of first-order linear equations**. $a_{ij}(x)$ and $f_i(x)$ are always assumed to be continuous on some interval $I\subset\mathbb{R}$, where $i,j=1,2,\ldots,n$.

For convenience, we rewrite the system of first-order linear equations in matrix form:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x}=\mathbf{A}(x)\mathbf{Y}+\mathbf{F}(x), \tag{4.5}$$

where

$$\mathbf{A}(x)=\begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}, \quad \mathbf{F}(x)=\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}.$$

On the interval I, if $\mathbf{F}(x)\equiv\mathbf{0}$, that is,

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x}=\mathbf{A}(x)\mathbf{Y} \tag{4.6}$$

then system (4.5) is called a **homogeneous system**, otherwise, it is called a **non-homogeneous system**.

¶ `General Theory`

Theorem 4.1 (Existence and Uniqueness Theorem for System of First-Order Linear Equations)

In the Cauchy problem (4.4), let D be a closed region in the $\mathbb{R}^{n+1}$:

$$D=[x_0-a,x_0+a]\times[\mathbf{Y}_0-b,\mathbf{Y}_0+b].$$

If the function $\mathbf{F}(x,\mathbf{Y})$ satisfies the following two conditions:

1. $\mathbf{F}(x,\mathbf{Y})$ is continuous in D.
2. $\mathbf{F}(x,\mathbf{Y})$ satisfies the Lipschitz condition with respect to $\mathbf{Y}$ in D, i.e., there exists a constant $L>0$ such that for any $(x,\mathbf{Y}_1),(x,\mathbf{Y}_2)\in D$,

$$\|\mathbf{F}(x,\mathbf{Y}_1)-\mathbf{F}(x,\mathbf{Y}_2)\|\leqslant L\|\mathbf{Y}_1-\mathbf{Y}_2\|.$$

Then there exists a unique solution $\mathbf{Y}=\mathbf{\Phi}(x)$ $(\mathbf{\Phi}(x_0)=\mathbf{Y}_0)$ to the Cauchy problem (4.4) in the interval $[x_0-h,x_0+h]$, where

$$h=\min\{a,\frac{b}{M}\},\quad M=\max_{(x,\mathbf{Y})\in D}\|\mathbf{F}(x,\mathbf{Y})\|.$$

♡

4.2 General Theory of Homogeneous Linear Systems

Similar to linear homogeneous systems of algebraic equations, the linear combination of solutions to homogeneous linear systems of differential equations is still a solution to the system.

Proposition 4.1

If $\mathbf{Y}_1(x)$ and $\mathbf{Y}_2(x)$ are two solutions to the homogeneous linear system (4.6), then any linear combination of them

$$\mathbf{Y}(x) = C_1\mathbf{Y}_1(x) + C_2\mathbf{Y}_2(x),$$

where C_1 and C_2 are arbitrary constants, is also a solution to the system.
Three or more solutions also have this property.

With this proposition, it is easy to verify that the set of all solutions to the homogeneous linear system (4.6) forms a linear space. And similarly, linear independence of solutions can be defined. Then we can introduce the concept of fundamental solution matrix.

Definition 4.1 (Fundamental Solution Matrix)

Let $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ be n linearly independent solutions to the homogeneous linear system (4.6). Then the matrix

$$\mathbf{\Phi}(x) = \begin{pmatrix} \mathbf{Y}_1(x) & \mathbf{Y}_2(x) & \cdots & \mathbf{Y}_n(x) \end{pmatrix} = \begin{pmatrix} y_{11}(x) & y_{12}(x) & \cdots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \cdots & y_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(x) & y_{n2}(x) & \cdots & y_{nn}(x) \end{pmatrix}$$

is called a **fundamental solution matrix** of the system.
Simultaneously, such a set of solutions is called a **fundamental solution system**.

¶ Criteria for Linear Dependence

Given n vector functions with n components each:

$$\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x), \tag{4.7}$$

criteria for their linear independence on the definition interval I is provided by the following theorem.

Theorem 4.2 (Wronskian Determinant Theorem)

For vector functions (4.7), let

$$W(x) = \det \begin{pmatrix} y_{11}(x) & y_{12}(x) & \cdots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \cdots & y_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(x) & y_{n2}(x) & \cdots & y_{nn}(x) \end{pmatrix}$$

be their Wronskian determinant. Then, if (4.7) are linearly dependent on I, then $W(x) \equiv 0$ for all $x \in I$;

Furthermore, if (4.7) are solutions to the homogeneous linear system (4.6), then the following conclusion holds:

Theorem 4.3

If (4.7) are linearly independent solutions to the homogeneous linear system (4.6), then $W(x) \not\equiv 0$ for all $x \in I$.

Combine the above two theorems, we have the following corollary:

Corollary 4.1 (Criterion for Linear Independence)

For vector functions (4.7), if their Wronskian determinant $W(x_0) \neq 0$ for some $x_0 \in I$, then they are linearly independent on I.

As for the relation between the solutions and the coefficient, we have the following theorem.

Theorem 4.4 (Liouville's Formula)

Let $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ be n solutions to the homogeneous linear system (4.6), and $W(x)$ be their Wronskian determinant. Then

$$W(x) = W(x_0) \exp\left(\int_{x_0}^{x} \operatorname{tr}(\mathbf{A}(t))\, dt\right),$$

where $\operatorname{tr}(\mathbf{A}(t))$ is the trace of matrix $\mathbf{A}(t)$.

¶ Solution Space

With the above conclusions, we can give the existence of fundamental solution systems.

Theorem 4.5

The fundamental solution system to the homogeneous linear system (4.6) does exist.

Proof Due to the existence and uniqueness theorem for system of first-order linear equations (Theorem 4.1), for initial conditions

$$\mathbf{Y}_1(x_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{Y}_2(x_0) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \mathbf{Y}_n(x_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad x_0 \in I, \tag{4.8}$$

there exist n solutions $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ to the homogeneous linear system (4.6). Their Wronskian determinant at $x = x_0$ is

$$W(x_0) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0.$$

Therefore, by the criterion for linear independence (Corollary 4.1), $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ are linearly independent on I, i.e., they form a fundamental solution system to the homogeneous linear system (4.6). ■

Fundamental solution systems satisfying (4.8) are called **standard fundamental systems**, and their fundamental solution matrices are called **standard fundamental solution matrices**. Obviously, standard fundamental solution matrices is identity matrix at $x = x_0$.

Then, general solution can be also derived.

Theorem 4.6 (General Solution to Homogeneous Linear Systems)

If $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ is a fundamental solution system to the homogeneous linear system (4.6), then the **general solution** to the system is given by:

$$\mathbf{Y}(x) = C_1\mathbf{Y}_1(x) + C_2\mathbf{Y}_2(x) + \cdots + C_n\mathbf{Y}_n(x) = \mathbf{\Phi}(x)\mathbf{C},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants, and $\mathbf{\Phi}(x)$ is the fundamental matrix solution.

Proof

■

Therefore, the number of linearly independent solutions to the homogeneous linear system (4.6) can be not exceed n, and the solution space of the system is an n-dimensional linear space.

4.3 General Theory of Non-Homogeneous Linear Systems

For the non-homogeneous linear system (4.5), similar to linear systems of algebraic equations, we have the following conclusion:

- The difference between any two solutions to the non-homogeneous linear system (4.5) is a solution to the corresponding homogeneous linear system (4.6).
- If $\tilde{\mathbf{Y}}(x)$ is a particular solution to the non-homogeneous linear system (4.5), then

$$\mathbf{Y}(x) = \mathbf{Y}_0(x) + \tilde{\mathbf{Y}}(x),$$

  is still a solution to the system, where $\mathbf{Y}_0(x)$ is the general solution to the corresponding homogeneous linear system (4.6).

Then we can give the general solution to the non-homogeneous linear system (4.5).

Theorem 4.7 (General Solution to Non-Homogeneous Linear Systems)

If $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \ldots, \mathbf{Y}_n(x)$ is a fundamental solution system to the corresponding homogeneous linear system (4.6), then the general solution to the non-homogeneous linear system (4.5) is given by:

$$\mathbf{Y}(x) = C_1\mathbf{Y}_1(x) + C_2\mathbf{Y}_2(x) + \cdots + C_n\mathbf{Y}_n(x) + \tilde{\mathbf{Y}}(x),$$

where $\tilde{\mathbf{Y}}(x)$ is a particular solution to the non-homogeneous linear system (4.5).

For non-homogeneous linear systems, method of variation of constants can also be used to find particular solutions. According to Theorem 4.6, the general solution to the corresponding homogeneous linear system is given by:

$$\mathbf{Y}(x) = \mathbf{\Phi}(x)\mathbf{C},$$

where $\mathbf{\Phi}(x)$ is the fundamental matrix solution, and $\mathbf{C} = \begin{pmatrix} C_1 & C_2 & \cdots & C_n \end{pmatrix}^{\mathrm{T}}$ is a constant vector. Now find a particular solution to the non-homogeneous linear system in the form:

$$\mathbf{Y}(x) = \mathbf{\Phi}(x)\mathbf{C}(x),$$

where $\mathbf{C}(x) = \begin{pmatrix} C_1(x) & C_2(x) & \cdots & C_n(x) \end{pmatrix}^{\mathrm{T}}$ is a vector function to be determined. Substituting it into the non-homogeneous linear system (4.5), we have:

$$\mathbf{\Phi}(x)\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}x} = \mathbf{F}(x). \tag{4.9}$$

Since $\mathbf{\Phi}(x)$ is invertible, we obtain:

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}x} = \mathbf{\Phi}^{-1}(x)\mathbf{F}(x). \tag{4.10}$$

Integrating both sides of Equation (4.10), we have:

$$\mathbf{C}(x) = \int_{x_0}^{x} \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,\mathrm{d}t, \tag{4.11}$$

where x_0 is an arbitrary constant. Then substituting Equation (4.11) into $\mathbf{Y}(x) = \mathbf{\Phi}(x)\mathbf{C}(x)$, we obtain a particular solution to the non-homogeneous linear system (4.5):

$$\tilde{\mathbf{Y}}(x) = \mathbf{\Phi}(x) \int_{x_0}^{x} \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,\mathrm{d}t. \tag{4.12}$$

Remark If $\mathbf{\Phi}(x)^{-1}$ is difficult to compute, we can use (4.9) directly to find $\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}x}$.

4.4 Solution to Constant Coefficient Homogeneous Linear Systems

For autonomous linear systems with constant coefficients:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{A}\mathbf{Y}, \tag{4.13}$$

we have the following conclusion:

Theorem 4.8

Matrix exponential function $\Phi(x) = e^{\mathbf{A}x}$ is a fundamental solution matrix to the homogeneous linear system (4.13).

For according non-homogeneous linear systems with constant coefficients:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{A}\mathbf{Y} + \mathbf{F}(x), \tag{4.14}$$

we can also use method of variation of constants to find particular solutions.

Theorem 4.9

The general solution to the non-homogeneous linear system (4.14) is given by:

$$\mathbf{Y}(x) = e^{\mathbf{A}x}\mathbf{C} + \int_{x_0}^{x} e^{\mathbf{A}(x-s)}\mathbf{F}(s)\,\mathrm{d}s,$$

where $\mathbf{C}$ is a constant vector. The solution satisfying the initial condition $\mathbf{Y}(x_0) = \mathbf{Y}_0$ is given by:

$$\mathbf{Y}(x) = e^{\mathbf{A}(x-x_0)}\mathbf{Y}_0 + \int_{x_0}^{x} e^{\mathbf{A}(x-s)}\mathbf{F}(s)\,\mathrm{d}s.$$

The problem we confront is: Can $e^{\mathbf{A}x}$ be expressed in a finite form of elementary functions? If so, how can it be expressed?

In fact, if $\mathbf{A}$ is an n-order Jordan block, i.e.,

$$\mathbf{A} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}_{n\times n} = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$
$$=: \operatorname{diag}(\lambda, \lambda, \ldots, \lambda) + \mathbf{Z}_n,$$

then we have:

$$\begin{aligned} e^{\mathbf{A}x} &= e^{\operatorname{diag}(\lambda,\lambda,\ldots,\lambda)x+\mathbf{Z}_n x} = e^{\operatorname{diag}(\lambda,\lambda,\ldots,\lambda)x} \cdot e^{\mathbf{Z}_n x} \\ &= e^{\lambda x} \cdot \left(\mathbf{E} + \frac{\mathbf{Z}_n x}{1!} + \frac{(\mathbf{Z}_n x)^2}{2!} + \cdots + \frac{(\mathbf{Z}_n x)^{n-1}}{(n-1)!} \right) \\ &= e^{\lambda x} \cdot \begin{pmatrix} 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^{n-1}}{(n-1)!} \\ 0 & 1 & x & \cdots & \frac{x^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{x^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n} . \end{aligned} \tag{4.15}$$

And Jordan canonical form can be deemed as a block diagonal matrix composed of Jordan blocks. Therefore, if we can compute the Jordan canonical form of matrix $\mathbf{A}$, then we can express $e^{\mathbf{A}x}$ in a finite form of elementary functions.

According to the theory of Jordan canonical form, for any n-order square matrix $\mathbf{A} \in M_n(\mathbb{C})$, there exists an invertible matrix $\mathbf{P} \in M_n(\mathbb{C})$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J},$$

where $\mathbf{J}$ is the Jordan canonical form of $\mathbf{A}$,

$$\mathbf{J} = \operatorname{diag}(\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_k),$$

and $\mathbf{J}_i$ is a Jordan block corresponding to eigenvalue λ_i of $\mathbf{A}$. Then we have:

$$e^{\mathbf{A}x} = e^{\mathbf{P}\mathbf{J}\mathbf{P}^{-1}x} = \mathbf{P}e^{\mathbf{J}x}\mathbf{P}^{-1} = \mathbf{P} \operatorname{diag}\left(e^{\mathbf{J}_1 x}, e^{\mathbf{J}_2 x}, \ldots, e^{\mathbf{J}_k x}\right) \mathbf{P}^{-1}. \tag{4.16}$$

However, computing the Jordan canonical form and transition matrix is not always easy. Note that $e^{\mathbf{A}x}$ is a fundamental solution matrix to the homogeneous linear system (4.13), since $\mathbf{P}$ is invertible, $e^{\mathbf{A}x}\mathbf{P}$ is also a fundamental solution matrix to (4.13). Then according to (4.16), $\mathbf{P}e^{\mathbf{J}x}$ is also a fundamental solution matrix to (4.13).

By (4.15), we can utilize method of undetermined coefficients to find n linearly independent solution matrix to (4.13). In the following, we classify the discussion based on whether $\mathbf{A}$ has repeated eigenvalues.

¶ **Distinct Eigenvalues**

Theorem 4.10

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n distinct eigenvalues of matrix $\mathbf{A}$, then the fundamental solution matrix to the homogeneous linear system (4.13) is given by:

$$\Phi(x) = \begin{pmatrix} e^{\lambda_1 x}\boldsymbol{\xi}_1 & e^{\lambda_2 x}\boldsymbol{\xi}_2 & \cdots & e^{\lambda_n x}\boldsymbol{\xi}_n \end{pmatrix},$$

where $\boldsymbol{\xi}_i$ is the eigenvector corresponding to eigenvalue λ_i of matrix $\mathbf{A}$ $(i = 1, 2, \ldots, n)$.

If there is a complex eigenvalue $\lambda = \alpha + \beta i$ $(\beta \neq 0)$ of matrix $\mathbf{A}$, then its complex conjugate $\bar{\lambda} = \alpha - \beta i$ is also an eigenvalue of $\mathbf{A}$. And their corresponding eigenvectors are also complex conjugates of each other, i.e. $\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}$. At this time, the pair of conjugate eigenvalues and eigenvectors will produce a complex solution:

$$\begin{aligned} \mathbf{Y} = e^{\lambda x}\boldsymbol{\xi} &= e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right](\boldsymbol{\xi}_1 + i\boldsymbol{\xi}_2) \\ &= e^{\alpha x}\left[(\cos(\beta x)\boldsymbol{\xi}_1 - \sin(\beta x)\boldsymbol{\xi}_2) + i\left(\sin(\beta x)\boldsymbol{\xi}_1 + \cos(\beta x)\boldsymbol{\xi}_2\right)\right]. \end{aligned}$$

Since the equation (4.13) is linear, the real and imaginary parts of this complex solution are each real solutions to the original equation. Thus, we obtain two linearly independent real solutions:

$$\begin{aligned} \mathbf{Y}_1 &= e^{\alpha x}\left(\cos(\beta x)\boldsymbol{\xi}_1 - \sin(\beta x)\boldsymbol{\xi}_2\right), \\ \mathbf{Y}_2 &= e^{\alpha x}\left(\sin(\beta x)\boldsymbol{\xi}_1 + \cos(\beta x)\boldsymbol{\xi}_2\right). \end{aligned}$$

¶ **Repeated Eigenvalues**

Assume that $\lambda_1, \lambda_2, \cdots, \lambda_s$ are all distinct eigenvalues of matrix $\mathbf{A}$, with algebraic multiplicities $n_1, n_2, \ldots, n_s$ respectively, where $n_1 + n_2 + \cdots + n_s = n$. Note that the fundamental solution matrix $e^{\mathbf{A}x}\mathbf{P} = \mathbf{P}e^{\mathbf{J}x}$, hence in the expression of $\mathbf{P}e^{\mathbf{J}x}$, all column vectors corresponding to the eigenvalue λ_j have the form:

$$\mathbf{Y} = e^{\lambda_j x}\left[\boldsymbol{\xi}_0 + \frac{x}{1!}\boldsymbol{\xi}_1 + \frac{x^2}{2!}\boldsymbol{\xi}_2 + \cdots + \frac{x^{n_j-1}}{(n_j-1)!}\boldsymbol{\xi}_{n_j-1}\right], \tag{4.17}$$

where $\xi_0, \xi_1, \ldots, \xi_{n_j-1}$ are constant vectors.

Lemma 4.1

Let λ_j be an eigenvalue of matrix $\mathbf{A}$ with algebraic multiplicity n_j, then (4.13) has non-zero solutions of the form (4.17) if and only if $\boldsymbol{\xi}_0$ is a non-zero solution to

$$(\mathbf{A} - \lambda_j\mathbf{E})^{n_j}\boldsymbol{\xi} = 0, \tag{4.18}$$

and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_{n_j-1}$ satisfy the following chain of generalized eigenvector equations[a]:

$$\begin{cases} \boldsymbol{\xi}_1 = (\mathbf{A} - \lambda_j\mathbf{E})\boldsymbol{\xi}_0, \\ \boldsymbol{\xi}_2 = (\mathbf{A} - \lambda_j\mathbf{E})\boldsymbol{\xi}_1, \\ \vdots \\ \boldsymbol{\xi}_{n_j-1} = (\mathbf{A} - \lambda_j\mathbf{E})\boldsymbol{\xi}_{n_j-2}. \end{cases} \tag{4.19}$$

[a] In fact, $\boldsymbol{\xi}_{n_j-1}$ is the true eigenvector corresponding to eigenvalue λ_j, and $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n_j-2}$ are generalized eigenvectors of orders $n_j, n_j - 1, \ldots, 2$ respectively.

♡

Lemma 4.2

Under the same conditions as above, denote the linear space of all constant vectors of n degrees as V, then

1. The subspace of V

   $$V_j = \{\xi \in V \mid (\mathbf{A} - \lambda_j\mathbf{E})^{n_j}\xi = 0\}, \quad j = 1, 2, \ldots, s$$

   is invariant under $\mathbf{A}$.
2. There exists a direct sum decomposition of V:

   $$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s.$$

♡

Theorem 4.11

Under the same conditions as above, the fundamental solution matrix to the homogeneous linear system (4.13) is given by:

$$\Phi(x) = \left(e^{\lambda_1 x}\mathbf{P}_1^{(1)}(x) \quad e^{\lambda_2 x}\mathbf{P}_2^{(2)}(x) \quad \cdots \quad e^{\lambda_s x}\mathbf{P}_s^{(s)}(x) \right),$$

where for each $j = 1, 2, \ldots, s$,

$$\mathbf{P}_j^{(j)}(x) = \boldsymbol{\xi}_{j0}^{(i)} + \frac{x}{1!}\boldsymbol{\xi}_{j1}^{(i)} + \frac{x^2}{2!}\boldsymbol{\xi}_{j2}^{(i)} + \cdots + \frac{x^{n_j-1}}{(n_j-1)!}\boldsymbol{\xi}_{j,n_j-1}^{(i)},$$

which is the j-th vector polynomial corresponding to eigenvalue λ_i ($i = 1, 2, \ldots, n_j; j = 1, 2, \ldots, n_i$). $\boldsymbol{\xi}_{10}^{(i)}, \cdots, \boldsymbol{\xi}_{n_i0}^{(i)}$ is n_i linearly independent solutions to (4.18), and the other $\boldsymbol{\xi}_{jl}^{(i)}$ ($j = 1, 2, \ldots, n_i; l = 1, 2, \ldots, n_j - 1$) is obtained by replacing corresponding $\boldsymbol{\xi}_{j0}^{(i)}$ with $\boldsymbol{\xi}_0$ in (4.19).

Let us summarize the method from an overall perspective.

When there exist repeated eigenvalues, let k be the algebraic multiplicity of eigenvalue λ. The core issue is:

can we find k linearly independent eigenvectors for this eigenvalue λ that has been repeated k times?[1]

- If we can, that is AM $=$ GM[2], and we say the corresponding eigenvalue is complete;
- If not, that is AM $>$ GM, and we say the corresponding eigenvalue is defective. We can only find GM $<$ AM $= k$ linearly independent eigenvectors, yet there are still $k -$ GM solutions missing. In fact, these missing solutions are just the generalized eigenvectors.

Now we discuss the two cases separately.

Complete If eigenvalue λ is complete, then we can find k linearly independent eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_k$ corresponding to eigenvalue λ. Thus, we can directly write out k linearly independent solutions:

$$\mathbf{Y}_1 = e^{\lambda x}\boldsymbol{\xi}_1, \quad \mathbf{Y}_2 = e^{\lambda x}\boldsymbol{\xi}_2, \quad \ldots, \quad \mathbf{Y}_k = e^{\lambda x}\boldsymbol{\xi}_k.$$

Defective When we cannot find enough eigenvectors, we need to look for so-called generalized eigenvectors to supplement the missing solutions. Assume that we have found only $m < k$ linearly independent eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_m$, i.e., GM $= m < k =$ AM. Then for each eigenvector $\boldsymbol{\xi}_i$ ($i = 1, 2, \ldots, m$), we can find a chain of generalized eigenvectors:

$$\boldsymbol{\xi}_i, \boldsymbol{\xi}_i^{(1)}, \boldsymbol{\xi}_i^{(2)}, \ldots, \boldsymbol{\xi}_i^{(t_i)},$$

where t_i is the length of the chain corresponding to eigenvector $\boldsymbol{\xi}_i$, and these generalized eigenvectors satisfy:

$$\begin{cases} (\mathbf{A} - \lambda\mathbf{E})\boldsymbol{\xi}_i^{(1)} = \boldsymbol{\xi}_i, \\ (\mathbf{A} - \lambda\mathbf{E})\boldsymbol{\xi}_i^{(2)} = \boldsymbol{\xi}_i^{(1)}, \\ \vdots \\ (\mathbf{A} - \lambda\mathbf{E})\boldsymbol{\xi}_i^{(t_i)} = \boldsymbol{\xi}_i^{(t_i-1)}, \end{cases}$$

they will contribute $t_i + 1$ linearly independent solutions[3]. Note that the total number of vectors in all

[1] As a matter of fact, the method above has already contained the following two cases. But for clarity, we explicitly state them here.

[2] AM refers to algebraic multiplicity, GM refers to geometric multiplicity.

[3] Here, $\boldsymbol{\xi}_i$ is the true eigenvector, and $\boldsymbol{\xi}_i^{(1)}, \boldsymbol{\xi}_i^{(2)}, \ldots, \boldsymbol{\xi}_i^{(t_i)}$ are generalized eigenvectors of orders $2, 3, \ldots, t_i + 1$ respectively. Different from the previous numbering method (top-down), here we use a bottom-up numbering method for better understanding.

chains is:

$$t_1 + t_2 + \cdots + t_m + m = k.$$

Then we can write out k linearly independent solutions:

$$\begin{aligned}
&\mathbf{Y}_i = e^{\lambda x}\boldsymbol{\xi}_i,\\
&\mathbf{Y}_i^{(1)} = e^{\lambda x}\left(x\boldsymbol{\xi}_i + \boldsymbol{\xi}_i^{(1)}\right),\\
&\mathbf{Y}_i^{(2)} = e^{\lambda x}\left(\frac{x^2}{2!}\boldsymbol{\xi}_i + x\boldsymbol{\xi}_i^{(1)} + \boldsymbol{\xi}_i^{(2)}\right),\\
&\vdots\\
&\mathbf{Y}_i^{(t_i)} = e^{\lambda x}\left(\frac{x^{t_i}}{t_i!}\boldsymbol{\xi}_i + \frac{x^{t_i-1}}{(t_i-1)!}\boldsymbol{\xi}_i^{(1)} + \cdots + x\boldsymbol{\xi}_i^{(t_i-1)} + \boldsymbol{\xi}_i^{(t_i)}\right),
\end{aligned}$$

for each $i = 1, 2, \ldots, m$.

4.5 Periodic Coefficient Linear Differential Equation Systems

Chapter 5 System of Higher-Order Linear Equations

This chapter mainly discusses the theory and solution methods of higher-order linear differential equations, with the following general forms:

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}+p_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}+\cdots+p_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x}+p_n(x)y=f(x). \tag{5.1}$$

When $f(x)\equiv 0$, it reduces to the homogeneous case:

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}+p_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}+\cdots+p_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x}+p_n(x)y=0, \tag{5.2}$$

In the equation (5.1), let

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y_1,\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=y_2,\ldots,\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}=y_{n-1},$$

then we can transform it into a system of first-order linear differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} y \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1}\end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_n(x) & -p_{n-1}(x) & -p_{n-2}(x) & \cdots & -p_1(x)\end{pmatrix}\begin{pmatrix} y \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1}\end{pmatrix}+\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f(x)\end{pmatrix}.$$

It is equivalent to the matrix form:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x}=\mathbf{A}(x)\mathbf{Y}+\mathbf{F}(x). \tag{5.3}$$

[1] The initial conditions can be expressed as:

$$\mathbf{Y}(x_0)=\mathbf{Y}_0.$$

To extend the theory and methods of first-order linear systems of differential equations to higher-order linear differential equations, we have the following lemma.

Lemma 5.1

The equation (5.1) is equivalent to the equation system (5.3).

That is, if $y(x)$ is a solution to (5.1), then $\mathbf{Y}(x)=\begin{pmatrix} y & y^{(1)} & y^{(2)} & \cdots & y^{(n-1)}\end{pmatrix}^{\mathrm{T}}$ is a solution to (5.3); conversely, if $\mathbf{Y}(x)=\begin{pmatrix} y & y^{(1)} & y^{(2)} & \cdots & y^{(n-1)}\end{pmatrix}^{\mathrm{T}}$ is a solution to (5.3), then $y(x)$ is a solution to (5.1). ♡

And we have the following existence and uniqueness theorem for the initial value problem of higher-order linear differential equations.

Theorem 5.1

The solution $y(x)$ to the higher-order linear differential equation (5.1) which satisfies the initial condition

$$y(x_0)=y_0,\quad y'(x_0)=y_{1,0},\quad y''(x_0)=y_{2,0},\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1,0},$$

exists and is unique on the interval I, where $p_1(x),p_2(x),\ldots,p_n(x),f(x)$ are continuous on I, and $x_0\in I$. ♡

[1]Denote

$$f(\lambda)=\lambda^n+p_1(x)\lambda^{n-1}+\cdots+p_{n-1}(x)\lambda+p_n(x),$$

then $\mathbf{A}(x)$ is just the companion matrix of polynomial $f(\lambda)$.

5.1 General Theory of Higher-Order Linear Equations

Similar to first-order linear systems of differential equations, we can define Wronskian determinant as follows:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix},$$

where $y_i(x) \in D^{(n-1)}(I)$, $i = 1, 2, \ldots, n$. Then we have the following conclusions:

Theorem 5.2 (Linear Independence Criterion)

The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent (dependent) on interval I if and only if there exists $x_0 \in I$ such that $W(x_0) \neq 0$ ($W(x_0) = 0$).

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be n linearly independent solutions to the homogeneous equation (5.2), then we can also derive the Liouville formula for Wronskian determinant:

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p_1(s)\,\mathrm{d}s}.$$

Here, we give the general solution of the higher-order linear differential equation.

Theorem 5.3 (General Solution of Higher-Order Linear Equations)

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be n linearly independent solutions to the homogeneous equation (5.2), then the general solution to the non-homogeneous equation (5.1) is given by:

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + y_p(x),$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants, and

$$y_p(x) = \sum_{i=1}^{n} y_i(x) \int \frac{W_i(x)}{W(x)} f(x)\,\mathrm{d}x,$$

is a particular solution to the non-homogeneous equation (5.1), where $W(x)$ is the Wronskian determinant of $y_1(x), y_2(x), \ldots, y_n(x)$, and $W_i(x)$ is the algebraic cofactor of the element in the n-th row and i-th column of $W(x)$.

Example 5.1 Let $y = \phi(x)$ is a known particular solution to the homogeneous equation:

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x), q(x) \in C(a, b)$ and $\phi(x) \neq 0$ on (a, b). Prove that the general solution to the above equation is given by:

$$y = C\phi(x) + \phi(x) \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\phi^2(x)}\,\mathrm{d}x,$$

where C is an arbitrary constant.

5.2 Solution to Constant Coefficient Homogeneous Linear Equations

From this section onward, we focus on higher-order linear differential equations with constant coefficients. For the constant coefficient linear differential equations:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{A}\mathbf{Y}, \tag{5.4}$$

and

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \mathbf{A}\mathbf{Y} + \mathbf{F}(x), \tag{5.5}$$

Then we introduce the differential operator L_n:

$$L_n = \frac{\mathrm{d}^n}{\mathrm{d}x^n} + a_1 \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + a_{n-1}\frac{\mathrm{d}}{\mathrm{d}x} + a_n,$$

where $a_1, a_2, \ldots, a_n$ are constants. Then the constant coefficient linear differential equation can be expressed as:

$$L_n y = 0, \quad L_n y = f(x).$$

According to the properties of companion matrix, the characteristic polynomial of matrix $\mathbf{A}$ is just

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

Then we have the following theorem:

Theorem 5.4

Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be all distinct roots of the characteristic polynomial $f(\lambda)$, with algebraic multiplicities $n_1, n_2, \ldots, n_s$ respectively, where $n_1 + n_2 + \cdots + n_s = n$. Then the fundamental solution set to the homogeneous equation (5.4) is given by:

$$\begin{aligned}
&e^{\lambda_1 x}, xe^{\lambda_1 x}, x^2 e^{\lambda_1 x}, \ldots, x^{n_1 - 1} e^{\lambda_1 x},\\
&e^{\lambda_2 x}, xe^{\lambda_2 x}, x^2 e^{\lambda_2 x}, \ldots, x^{n_2 - 1} e^{\lambda_2 x},\\
&\vdots\\
&e^{\lambda_s x}, xe^{\lambda_s x}, x^2 e^{\lambda_s x}, \ldots, x^{n_s - 1} e^{\lambda_s x}.
\end{aligned}$$

♡

5.3 Method of Undetermined Coefficients

Theorem 5.3 has given the general solution to the non-homogeneous equation (5.5). However, it is often difficult to calculate, especially when the order n is large.

In fact, there are mainly four methods to find particular solutions to the non-homogeneous equation (5.5):

Method of Variation of Constants This method is the most general and can be applied to any form of $f(t)$. However, it often involves complex calculations, including matrix inversion.

Method of Undetermined Coefficients This method is simpler in computation but only applicable when $f(t)$ has a specific form.

Laplace Transform Method This method is systematic and particularly suitable for initial-value problems. However, it requires the computation of inverse Laplace transforms.

Matrix Exponential Method This method provides an elegant theoretical framework for solving first-order matrix equations.

In this section, we introduce the method of undetermined coefficients to find a particular solution to the non-homogeneous equation (5.5). This method is applicable when $f(x)$ ($\mathbf{F}(x) = \begin{pmatrix} 0 & 0 & 0 & \cdots & f(x) \end{pmatrix}^{\mathrm{T}}$) contains functions such as polynomials, exponentials, sines, cosines, or their finite sums and products.

First, we introduce the superposition principle:

Lemma 5.2 (Superposition Principle)

If $y_{p1}(x)$ and $y_{p2}(x)$ are particular solutions to the non-homogeneous equations:

$$L_n y = f_1(x), \quad L_n y = f_2(x),$$

respectively, then $y_p(x) = y_{p1}(x) + y_{p2}(x)$ is a particular solution to the non-homogeneous equation:

$$L_n y = f_1(x) + f_2(x).$$

Remark The superposition principle is essentially the differential additivity of linear operators, while the differential operator is essentially a linear mapping on the space C^k (or more generally L^p).

Then we can discuss how to construct particular solutions based on the form of $f(x)$.

Theorem 5.5

For the non-homogeneous equation (5.5), if $f(x)$ has the form:

$$f(x) = e^{\alpha x}\left[P_m(x)\cos(\beta x) + Q_m(x)\sin(\beta x)\right],$$

where $P_m(x)$ and $Q_m(x)$ are polynomials with degree at most m (at least one of them has degree exactly m), then a particular solution to (5.5) is given by:

$$y_p(x) = x^k e^{\alpha x}\left[R_m(x)\cos(\beta x) + S_m(x)\sin(\beta x)\right],$$

where $R_m(x)$ and $S_m(x)$ are polynomials with degree at most m, and k is determined as the following collision rule:

- If $\alpha + \beta i$ is not a root of the characteristic polynomial $f(\lambda)$, then $k = 0$;
- If $\alpha + \beta i$ is a root of $f(\lambda)$ with algebraic multiplicity k, then k takes that value.

5.4 Laplace Transform Method

Chapter 6 Boundary Value Problems

6.1 Sturm-Liouville Problems

Chapter 7 Nonlinear Equations and Stability

7.1 The Phase Plane

We are concerned with systems of two simultaneous differential equations of the form:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = F(x, y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = G(x, y). \end{cases} \tag{7.1}$$

Assume that the functions F and G are continuous and have continuous partial derivatives in some domain D of the xy-plane.

If (x_0, y_0) is a point in this domain, then by existence and uniqueness theorem (4.1) there exists a unique solution $x = x(t), y = y(t)$ of the system (7.1) satisfying the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0.$$

The solution is defined in some time interval I that contains the point t_0. Frequently, we will write the above initial value problem in the vector form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}^0, \tag{7.2}$$

where $\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}^{\mathrm{T}}$, $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} F(x, y) & G(x, y) \end{pmatrix}^{\mathrm{T}}$, and $\mathbf{x}^0 = \begin{pmatrix} x_0 & y_0 \end{pmatrix}^{\mathrm{T}}$.

Remark Observe that the functions F and G in equations 7.1 do not depend on the independent variable t, but only on the dependent variables x and y. A system with this property is said to be **autonomous**.

Value range D of $\mathbf{x}$ is called the **phase space** of the system (7.2). Each solution $\mathbf{x} = \mathbf{x}(t)$ of (7.2) determines a curve in the phase space, called a **phase trajectory** or **orbit** of the system. Drawing all the typical trajectories of the system (representing different initial conditions) on the same phase plane, with arrows indicating the direction of flow, creates a **phase portrait**. In fact, the orbit of a solution $\mathbf{x} = \mathbf{x}(t)$ on the phase plane is just the projection of its integral curve in the three-dimensional space (t, x, y) onto the xy-plane.

The points, if any, where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ are called **equilibrium points** or **critical points** or **singular points** of the system.

7.2 Ляпунов Stability

¶ `Lyapunov Stability`

Consider the differential equations system:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x}), \tag{7.3}$$

where $\mathbf{f}(t, \mathbf{x})$ is continuous for $\mathbf{x} \in D \subset \mathbb{R}^n$ and $t \in (-\infty, +\infty)$, and satisfies the local Lipschitz condition with respect to $\mathbf{x}$.

Let the unique solution to (7.3) with initial condition $(t_0, \mathbf{x}_0)$ ($\mathbf{x}_0 = \mathbf{x}(t_0)$) be denoted as $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and $\mathbf{x} = \boldsymbol{\phi}(t)$ be also a solution to (7.3).

Definition 7.1 (Stability in the Sense of Ляпунов)

The solution $\mathbf{x} = \boldsymbol{\phi}(t)$ to (7.3) is said to be **stable in the sense of Ляпунов** (simplified as stable), if for any $\varepsilon > 0$

and $t_0 \geqslant 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$, such that when $\|\mathbf{x}_0 - \boldsymbol{\phi}(t_0)\| < \delta$, it holds that

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0) - \boldsymbol{\phi}(t)\| < \varepsilon,$$

for all $t \geqslant t_0$.

If $\mathbf{x} = \boldsymbol{\phi}(t)$ is stable, and there exists $\delta_1 \in (0, \delta]$ such that when $\|\mathbf{x}_0 - \boldsymbol{\phi}(t_0)\| < \delta_1$, it holds that

$$\lim_{t \to +\infty} \|\mathbf{x}(t; t_0, \mathbf{x}_0) - \boldsymbol{\phi}(t)\| = 0,$$

then $\mathbf{x} = \boldsymbol{\phi}(t)$ is said to be **asymptotically stable.**

If $\mathbf{x} = \boldsymbol{\phi}(t)$ is asymptotically stable, and there exists $\delta_2 \in (0, \delta_1], \alpha, \beta$ such that when $\|\mathbf{x}_0 - \boldsymbol{\phi}(t_0)\| < \delta_2$, it holds that

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0) - \boldsymbol{\phi}(t)\| \leqslant \alpha \|\mathbf{x}_0 - \boldsymbol{\phi}(t_0)\| e^{-\beta(t - t_0)},$$

for all $t \geqslant t_0$, then $\mathbf{x} = \boldsymbol{\phi}(t)$ is said to be **exponentially stable.** ♣

By substitution

$$\mathbf{y} = \mathbf{x}(t; t_0, \mathbf{x}_0) - \boldsymbol{\phi}(t),$$

we have

$$\begin{aligned} \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} =& \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} \\ =& \mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \boldsymbol{\phi}) \\ =& \mathbf{f}(t, \mathbf{y} + \boldsymbol{\phi}) - \mathbf{f}(t, \boldsymbol{\phi}) \\ :=& \mathbf{F}(t, \mathbf{y}), \end{aligned}$$

then the system (7.3) can be transformed into:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{F}(t, \mathbf{y}). \tag{7.4}$$

Then the stability of solution $\mathbf{x} = \boldsymbol{\phi}(t; t_0, \mathbf{x}_1)$ to (7.3) is equivalent to the stability of the zero solution to (7.4).

Without loss of generality, we only discuss the stability of the zero solution $\mathbf{x} = \mathbf{0}$ to the autonomous system (7.3), and assume that $\mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0}$. The following definition is derived:

Definition 7.2 (Stability of the Zero Solution)

The zero solution $\mathbf{x} = \mathbf{0}$ to (7.3) is said to be **stable** if for any $\varepsilon > 0$ and $t_0 \geqslant 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$, such that when $\|\mathbf{x}_0\| < \delta$, it holds that

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon,$$

for all $t \geqslant t_0$.

If $\mathbf{x} = \mathbf{0}$ is stable, and there exists $\delta_1 \in (0, \delta]$ such that when $\|\mathbf{x}_0\| < \delta_1$, it holds that

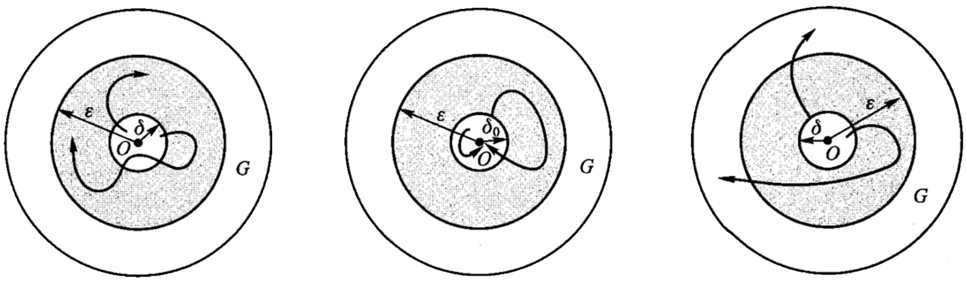
$$\lim_{t \to +\infty} \|\mathbf{x}(t; t_0, \mathbf{x}_0)\| = 0,$$

then $\mathbf{x} = \mathbf{0}$ is said to be **asymptotically stable.**

If $\mathbf{x} = \mathbf{0}$ is asymptotically stable, and there exists $\delta_2 \in (0, \delta_1], \alpha, \beta$ such that when $\|\mathbf{x}_0\| < \delta_2$, it holds that

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \leqslant \alpha \|\mathbf{x}_0\| e^{-\beta(t - t_0)},$$

for all $t \geqslant t_0$, then $\mathbf{x} = \mathbf{0}$ is said to be **exponentially stable.** ♣



(a) Stable (b) Asymptotically stable (c) Unstable

7.3 Ляпунов Second Method

Ляпунов established two methods for studying the stability of solutions to differential equations:
The first method is based on the series solutions of differential equations, which is often difficult to apply in practice.
The second method, known as the Ляпунов second method, relies on constructing a special scalar function, called the Ляпунов function, to analyze the stability of solutions without explicitly solving the differential equations, which is also called the direct method of Ляпунов.

For convenience, we only consider autonomous system:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in G \subset \mathbb{R}^n, \tag{7.5}$$

where $\mathbf{f}(\mathbf{x})$ is continuous in G and satisfies the local Lipschitz condition with respect to $\mathbf{x}$, and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Definition 7.3 (Ляпунов Function)

A scalar function $V(\mathbf{x}) : G \to \mathbb{R}$ is called a **Ляпунов function**, if it satisfies the following conditions:

- $V(\mathbf{0}) = 0$;
- $V(\mathbf{x}), \nabla V(\mathbf{x})$ are continuous in G.

In $G_1 \subseteq G$, If $V(\mathbf{x}) > 0 (< 0)$ for all $\mathbf{x}$ except $\mathbf{0}$, then $V(\mathbf{x})$ is called a **positive/negative definite**; if $V(\mathbf{x}) \geqslant 0 (\leqslant 0)$ for all $\mathbf{x}$, then $V(\mathbf{x})$ is called a **positive/negative semidefinite**; otherwise, it is called an **indefinite**. ♣

Theorem 7.1 (Ляпунов Stability Theorem)

For the autonomous system (7.5), let $V(\mathbf{x})$ be a Ляпунов function in G, and its total derivative along the trajectories of (7.5) is given by:

$$\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(\mathbf{x}).$$

Then:

Stability Theorem if $V(\mathbf{x})$ is positive definite and $\dot{V}(\mathbf{x})$ is negative semidefinite in G, then the zero solution $\mathbf{x} = \mathbf{0}$ is stable.

Asymptotically Stability Theorem if $V(\mathbf{x})$ is positive definite and $\dot{V}(\mathbf{x})$ is negative definite in G, then the zero solution $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

Unstability Theorem if $V(\mathbf{x})$ is positive definite and $\dot{V}(\mathbf{x})$ is positive definite in G, then the zero solution $\mathbf{x} = \mathbf{0}$ is unstable. ♡

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