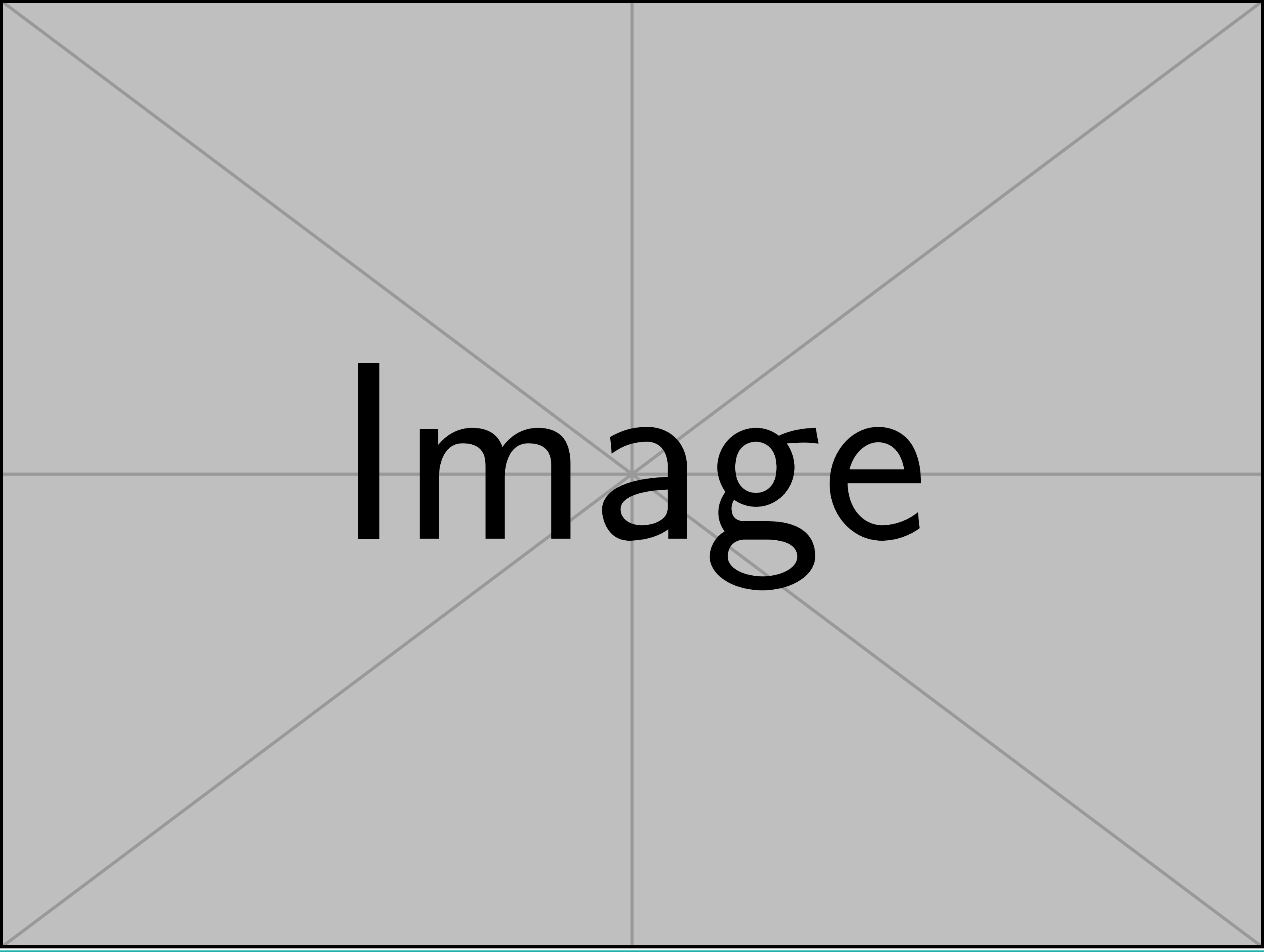

Analyse Mathématique

Author: CatMono

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Preface

This is the preface of the book…

Chapter 1 Preliminaries

1.1 Trigonometric Formulas

Product-to-Sum Formulas:

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$
$$\sin\alpha\sin\beta = -\frac{1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right]$$

Sum and Difference Formulas:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Sum-to-Product Formulas:

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$
$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Double Angle Formulas:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha-\sin^2\alpha = 2\cos^2\alpha-1 = 1-2\sin^2\alpha$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Half Angle Formulas:

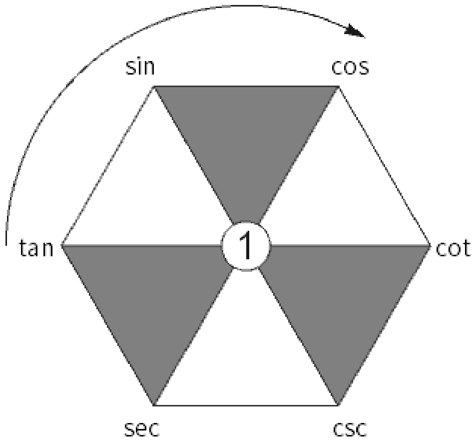
$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$
$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$
$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

Power-Reducing Formulas:

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$
$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

Angle Decomposition Formulas:

$$\sin^2\alpha - \sin^2\beta = \sin(\alpha+\beta)\sin(\alpha-\beta)$$
$$\cos^2\alpha - \sin^2\beta = \cos(\alpha+\beta)\cos(\alpha-\beta)$$



Remark

- On the gray triangle, the sum of the squares of the two numbers above is equal to the square of the number below, for instance, $\tan^2 x + 1 = \sec^2 x$
- The three trigonometric functions in the clockwise direction have the following properties: $\tan x = \frac{\sin x}{\cos x}$, etc.

Chapter 2 Limits of Sequences and Continuity of Real Number System

2.1 Convergent Sequences

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¶ Cauchy Proposition and Fitting Method

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8.1 Directional Derivatives and Total Differential

¶ Directional Derivative

Definition 8.1 (Directional Derivative)

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^1$, $\mathbf{e}$ is a unit vector in $\mathbb{R}^n$, $\mathbf{x}^0 \in U$. Define

$$u(t) = f(\mathbf{x}^0 + t\mathbf{e}).$$

If the derivative of u at $t = 0$

$$u'(0) = \lim_{t\to 0} \frac{u(t) - u(0)}{t} = \lim_{t\to 0} \frac{f(\mathbf{x}^0 + t\mathbf{e}) - f(\mathbf{x}^0)}{t}$$

exists and is finite, it is called the **directional derivative** of f at $\mathbf{x}_0$ in the direction $\mathbf{e}$, denoted by $\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}_0)$. It is the rate of change of f at $\mathbf{x}_0$ in the direction $\mathbf{e}$. ♣

Consider the following set of unit coordinate vectors: $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. Let $\mathbf{e}_i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$ denote the standard orthonormal basis in $\mathbb{R}^n$, where the 1 appears in the i-th position. That is,

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For a function f, the directional derivative of f at the point $\mathbf{x}_0$ in the direction of $\mathbf{e}_i$ is called the ith first-order **partial derivative** of f at $\mathbf{x}^0$, denoted by

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^0) \quad \text{or} \quad \mathrm{D}_i f(\mathbf{x}^0) \quad \text{or} \quad f_{x_i}(\mathbf{x}^0) \quad (i = 1, 2, \cdots, n).$$

$\mathrm{D}_i = \frac{\partial}{\partial x_i}$ is called the ith partial differential operator ($i = 1, 2, \cdots, n$).

Note *Let $\mathbf{e}$ be a direction, then $\|-\mathbf{e}\| = \|\mathbf{e}\| = 1$, which implies that $-\mathbf{e}$ is also a direction. At this point, we have:*

$$\frac{\partial f}{\partial(-\mathbf{e})}(\mathbf{x}^0) = -\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0).$$

Definition 8.2 (Jacobian Matrix (Gradient))

Let

$$\mathbf{J}f(\mathbf{x}) = (\mathrm{D}_1 f(\mathbf{x}), \mathrm{D}_2 f(\mathbf{x}), \ldots, \mathrm{D}_n f(\mathbf{x})),$$

which is called the **Jacobian matrix** of the function f at the point $\mathbf{x}$, (a $1 \times n$ matrix) whose counterpart is the first-order derivative of a single-variable function.

Henceforth, we represent the point $\mathbf{x}$ in $\mathbb{R}^n$ and its increments $\mathbf{h}$ as column vectors. In this way, the differential of the function can be expressed using matrix multiplication as follows:

$$\mathrm{d}f(\mathbf{x}^0)(\mathbf{\Delta x}) = \mathbf{J}f(\mathbf{x}^0)\mathbf{\Delta x}.$$

The Jacobian matrix of the function f is also frequently denoted as grad f (or ∇f), that is,

$$\operatorname{grad} f(\mathbf{x}) = \mathbf{J}f(\mathbf{x}),$$

which is called the **gradient** of the scalar function f. ♣

¶ Total Differential

Definition 8.3 (Total Differential)

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^1, \mathbf{x}^0 \in U, \mathbf{\Delta x} = (\Delta x_1, \Delta x_2, \cdots, \Delta x_n) \in \mathbb{R}^n$. If

$$f(\mathbf{x}^0 + \mathbf{\Delta x}) - f(\mathbf{x}^0) = \sum_{i=1}^{n} A_i \Delta x_i + o(\|\mathbf{\Delta x}\|) \qquad (\|\mathbf{\Delta x}\| \to 0),$$

where $A_1, A_2, \dots, A_n$ are constants independent of $\mathbf{\Delta x}$, then the function f is said to be **differentiable** at the point $\mathbf{x}^0$, and the linear main part $\sum_{i=1}^{n} A_i \Delta x_i$ is called the **total differential** of f at $\mathbf{x}^0$, denoted as

$$df(\mathbf{x}^0)(\mathbf{\Delta x}) = \sum_{i=1}^{n} A_i \Delta x_i.$$

If f is differentiable at every point in the open set U, then f is called a differentiable function on U. ♣

Theorem 8.1 (Conditions of Differentiability)

Necessary Condition If an n-variable function f is differentiable at the point $\mathbf{x}_0$, then f is continuous at $\mathbf{x}^0$ and possesses first-order partial derivatives $\frac{\partial f}{\partial x_i}(\mathbf{x}^0)$ at $\mathbf{x}^0$ for $i = 1, 2, \dots, n$, and

$$\mathbf{A} = (A_1, A_2, \dots, A_n) = \mathbf{J}f(\mathbf{x}^0) = \left(\mathrm{D}_1 f(\mathbf{x}^0), \mathrm{D}_2 f(\mathbf{x}^0), \dots, \mathrm{D}_n f(\mathbf{x}^0)\right).$$

[a] However, the converse is not true.

Sufficient Condition Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^1$ be an n-variable function. If $\mathbf{J}f = (\mathrm{D}_1 f, \mathrm{D}_2 f, \dots, \mathrm{D}_n f)$ is continuous at $\mathbf{x}^0$ (i.e., $\frac{\partial f}{\partial x_i}$ is continuous at $\mathbf{x}^0$ for $i = 1, 2, \dots, n$), then f is differentiable at $\mathbf{x}^0$. However, the converse is not necessarily true.

[a] It is referred to as the total differential formula, and the more common form is

$$\mathrm{d}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\,\mathrm{d}x + \frac{\partial f}{\partial y}(x_0, y_0)\,\mathrm{d}y.$$

♡

Note

- *The continuity of the derivative function at $\mathbf{x}^0$ implies that the original function f is differentiable in some neighborhood of $\mathbf{x}^0$.*
- *In fact, this condition can be relaxed to require that one partial derivative exists at the point, while the remaining $n-1$ partial derivative functions are continuous at that point.*

Proof Taking a function of three variables as an example.

Assume the 3-ary function $f : \mathbb{R}^3 \to \mathbb{R}$ meets:

1. There exists $f_z(x_0, y_0, z_0)$.
2. The partial derivative functions $f_x(x, y, z)$ and $f_y(x, y, z)$ are continuous at (x_0, y_0, z_0), i.e. there are partial derivatives in some neighborhood of (x_0, y_0, z_0).

Consider the total increment of f at the point (x_0, y_0, z_0):

$$\begin{aligned}\Delta f = {} & \underbrace{[f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0 + \Delta y, z_0 + \Delta z)]}_{I_1} \\ & + \underbrace{[f(x_0, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0 + \Delta z)]}_{I_2} \\ & + \underbrace{[f(x_0, y_0, z_0 + \Delta z) - f(x_0, y_0, z_0)]}_{I_3}.\end{aligned}$$

For I_1, I_2, by the Lagrange's Mean Value Theorem of unary functions, there exist $\theta_1, \theta_2 \in (0,1)$ such that

$$I_1 = f_x(x_0 + \theta_1 \Delta x, y_0 + \Delta y, z_0 + \Delta z)\Delta x,$$
$$I_2 = f_y(x_0, y_0 + \theta_2 \Delta y, z_0 + \Delta z)\Delta y.$$

Then by the continuity of the their partial derivatives at (x_0, y_0, z_0), we have

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} I_1 = f_x(x_0, y_0, z_0)\Delta x, \qquad \lim_{\Delta x, \Delta y, \Delta z \to 0} I_2 = f_y(x_0, y_0, z_0)\Delta y.$$

They can be expressed in terms of infinitesimals($\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$):

$$I_1 = f_x(x_0, y_0, z_0)\Delta x + \alpha_1 \Delta x, \quad \alpha_1 \to 0 (\rho \to 0),$$
$$I_2 = f_y(x_0, y_0, z_0)\Delta y + \alpha_2 \Delta y, \quad \alpha_2 \to 0 (\rho \to 0).$$

For I_3, by the definition of the partial derivative $f_z(x, y, z)$ at (x_0, y_0, z_0), we have

$$I_3 = f_z(x_0, y_0, z_0)\Delta z + \alpha_3 \Delta z, \quad \alpha_3 \to 0 (\rho \to 0).$$

Accordingly,

$$\begin{aligned}
\Delta f &= I_1 + I_2 + I_3 \\
&= [f_x(x_0, y_0, z_0)\Delta x + \alpha_1 \Delta x] + [f_y(x_0, y_0, z_0)\Delta y + \alpha_2 \Delta y] + [f_z(x_0, y_0, z_0)\Delta z + \alpha_3 \Delta z] \\
&= f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z + [\alpha_1 \Delta x + \alpha_2 \Delta y + \alpha_3 \Delta z].
\end{aligned}$$

Apparently,

$$\lim_{\rho \to 0} \frac{\alpha_1 \Delta x + \alpha_2 \Delta y + \alpha_3 \Delta z}{\rho} = 0,$$

i.e. $\alpha_1 \Delta x + \alpha_2 \Delta y + \alpha_3 \Delta z = o(\rho)$. Therefore, $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), which completes the proof. ■

Note *(At some point)*

1. *Differentiable*
   - $\Longrightarrow$ *Continuous*
   - $\Longrightarrow$ *Partial derivatives exist:* $\mathrm{D}_{\vec{u}} = \nabla f \cdot \vec{u}$
2. *Directional Derivative*
   - *All directional derivatives exist* $\not\Longrightarrow$ *differentiable or continuous.*
   - *All directional derivatives exist and are equal* $\not\Longrightarrow$ *differentiable.*
3. *Partial Derivative*
   - *The continuity and existence of directional/partial derivatives are mutually exclusive.*

¶ Higher-Order Partial Derivatives and Differential

If the first-order partial derivative of f, $\frac{\partial f}{\partial x_i}$, itself possesses partial derivatives, then the second-order partial derivative of f is defined, and is denoted as follows(the first is also called the mixed partial derivative):

$$f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right), \quad f_{x_i x_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right), \quad i, j = 1, 2, \ldots, n.$$

Similarly, higher-order partial derivatives of order $3, 4, \cdots m, \cdots$ can be defined.

The following theorem provides the conditions under which mixed partial derivatives are equal.

Theorem 8.2 (Conditions for Equality of Mixed Partial Derivatives)

1. Let $U \subset \mathbb{R}^2$ be an open set, and $f : U \to \mathbb{R}$ be a function of two variables. If f_{xy} and f_{yx} are continuous at $(x_0, y_0) \in U$, then
$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

2. Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}$ be a function of n variables. If f has partial derivatives up to order k in D, and all of them are continuous at $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in U$, then
$$\frac{\partial^l f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}}(\mathbf{x}^0) = \frac{\partial^l f}{\partial x_{i_2} \partial x_{i_1} \cdots \partial x_{i_l}}(\mathbf{x}^0) = \cdots = \frac{\partial^l f}{\partial x_{i_l} \partial x_{i_l - 1} \cdots \partial x_{i_1}}(\mathbf{x}^0),$$
that is, the order of taking partial derivatives $l(\leq k)$ does not affect the result.[a]

[a] If the condition "f_{xy} and f_{yx} are continuous at (x_0, y_0)", is not satisfied, then the conclusion "$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$" does not necessarily hold.

Proof When $k \neq 0, h \neq 0$, define
$$\varphi(y) = f(x_0 + h, y) - f(x_0, y),$$
and
$$\psi(x) = f(x, y_0 + k) - f(x, y_0).$$
Applying the Lagrange Mean Value Theorem, we have
$$\begin{aligned}
&[f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)] - [f(x_0 + h, y_0) - f(x_0, y_0)] \\
=&\varphi(y_0 + k) - \varphi(y_0) \\
=&\varphi'(y_0 + \theta_1 k)k \\
=&[f_y(x_0 + h, y_0 + \theta_1 k) - f_y(x_0, y_0 + \theta_1 k)]k \\
=&f_{yx}(x_0 + \theta_2 h, y_0 + \theta_1 k)hk, \quad 0 < \theta_1, \theta_2 < 1.
\end{aligned}$$
On the other hand,
$$\begin{aligned}
&[f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)] - [f(x_0 + h, y_0) - f(x_0, y_0)] \\
=&[f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)] - [f(x_0, y_0 + k) - f(x_0, y_0)] \\
=&\psi(x_0 + h) - \psi(x_0) \\
=&\psi'(x_0 + \theta_3 h)h \\
=&[f_x(x_0 + \theta_3 h, y_0 + k) - f_x(x_0 + \theta_3 h, y_0)]h \\
=&f_{xy}(x_0 + \theta_3 h, y_0 + \theta_4 k)hk, \quad 0 < \theta_3, \theta_4 < 1.
\end{aligned}$$
Therefore,
$$f_{xy}(x_0 + \theta_3 h, y_0 + \theta_4 k) = f_{yx}(x_0 + \theta_2 h, y_0 + \theta_1 k).$$
Since f_{xy} and f_{yx} are continuous at (x_0, y_0), letting $h \to 0, k \to 0$, we obtain
$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$
By applying 8.1 and the principle of mathematical induction, one can immediately derive the following result.
■

Suppose $z = f(x, y)$ has continuous partial derivatives in the domain $U \subset \mathbb{R}^2$. Then z is differentiable, and
$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y.$$
If z also has continuous second-order partial derivatives, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are also differentiable, and thus $\mathrm{d}z$ is differentiable. We call the differential of $\mathrm{d}z$ the second-order differential of z, denoted as
$$\mathrm{d}^2 z = \mathrm{d}(\mathrm{d}z).$$

In general, based on the k-th order differential $\mathrm{d}^k z$ of z, its $(k+1)$-th order differential (if it exists) is defined as

$$\mathrm{d}^{k+1}z = \mathrm{d}(\mathrm{d}^k z), \quad k = 1, 2, \cdots.$$

Due to the fact that for the independent variables x and y, we always have

$$\mathrm{d}^2 x = \mathrm{d}(\mathrm{d}x) = 0, \qquad \mathrm{d}^2 y = \mathrm{d}(\mathrm{d}y) = 0,$$

the second-order differential of $z = f(x, y)$ is given by

$$\begin{aligned}
\mathrm{d}^2 z &= \mathrm{d}(\mathrm{d}z) = \mathrm{d}\left(\frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y\right) \\
&= \mathrm{d}\left(\frac{\partial z}{\partial x}\right)\mathrm{d}x + \frac{\partial z}{\partial x}\mathrm{d}^2 x + \mathrm{d}\left(\frac{\partial z}{\partial y}\right)\mathrm{d}y + \frac{\partial z}{\partial y}\mathrm{d}^2 y \\
&= \left(\frac{\partial^2 z}{\partial x^2}\mathrm{d}x + \frac{\partial^2 z}{\partial x \partial y}\mathrm{d}y\right)\mathrm{d}x + \left(\frac{\partial^2 z}{\partial y \partial x}\mathrm{d}x + \frac{\partial^2 z}{\partial y^2}\mathrm{d}y\right)\mathrm{d}y \\
&= \frac{\partial^2 z}{\partial x^2}(\mathrm{d}x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\mathrm{d}x\mathrm{d}y + \frac{\partial^2 z}{\partial y^2}(\mathrm{d}y)^2,
\end{aligned}$$

where $(\mathrm{d}x)^2$ and $(\mathrm{d}y)^2$ denote d^2x and d^2y respectively. If we treat $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ as operators for partial differentiation and define

$$\left(\frac{\partial}{\partial x}\right)^2 = \frac{\partial^2}{\partial x^2}, \quad \left(\frac{\partial}{\partial y}\right)^2 = \frac{\partial^2}{\partial y^2}, \quad \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x \partial y},$$

then the formulas for the first and second differentials can be written as

$$\begin{aligned}
\mathrm{d}z &= \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right)z, \\
\mathrm{d}^2 z &= \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right)^2 z.
\end{aligned}$$

Similarly, we define

$$\left(\frac{\partial}{\partial x}\right)^p \left(\frac{\partial}{\partial y}\right)^q = \frac{\partial^{p+q}}{\partial x^p \partial y^q} = \frac{\partial^q}{\partial y^q}\left(\frac{\partial}{\partial x}\right)^p, \quad (p, q = 1, 2, \dots)$$

It is easy to use mathematical induction to prove the formula for higher-order differentials:

$$\mathrm{d}^k z = \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right)^k z, \quad k = 1, 2, \cdots.$$

For an n-variable function $u = f(x_1, x_2, \dots, x_n)$, higher-order differentials can be similarly defined, and the following holds:

$$\mathrm{d}^k u = \left(\mathrm{d}x_1\frac{\partial}{\partial x_1} + \mathrm{d}x_2\frac{\partial}{\partial x_2} + \cdots + \mathrm{d}x_n\frac{\partial}{\partial x_n}\right)^k u, \quad k = 1, 2, \dots.$$

8.2 Differential of Vector-Valued Functions

Consider an n-dimensional vector-valued function defined on a domain $U \subset \mathbb{R}^n$:

$$\begin{aligned}
f: U &\to \mathbb{R}^m, \\
\mathbf{x} &\mapsto \mathbf{y} = f(\mathbf{x})
\end{aligned}$$

Expressed in coordinate vector form:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_m(x_1, x_2, \dots, x_n) \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in U$$

1. If each component function $f_i(x_1, x_2, \dots, x_n)$ $(i = 1, 2, \dots, m)$ is partially differentiable at $\mathbf{x}^0$, then the vector-valued function $\mathbf{f}$ is differentiable at $\mathbf{x}^0$, and we define the matrix
$$\left(\frac{\partial f}{\partial x_j}(\mathbf{x}^0)\right)_{m\times n} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}^0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}^0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}^0) \end{pmatrix}$$
This matrix is called the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}^0$, denoted by $f'(\mathbf{x}^0)$ (or $\mathrm{D}f(\mathbf{x}^0)$, $\mathbf{J}_f(\mathbf{x}^0)$).
For the special case $m = 1$, i.e., n-variable scalar function $z = f(x_1, x_2, \dots, x_n)$, the derivative at $\mathbf{x}^0$ is
$$f'(\mathbf{x}^0) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}^0), \frac{\partial f}{\partial x_2}(\mathbf{x}^0), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x}^0)\right)$$
If the vector-valued function $\mathbf{f}$ is differentiable at every point in U, then $\mathbf{f}$ is said to be differentiable on U, and the corresponding relationship is
$$\mathbf{x} \in U \mapsto f'(\mathbf{x}) = J_f(\mathbf{x})$$
where $f'(\mathbf{x})$ (or $\mathrm{D}f(\mathbf{x})$, $\mathbf{J}_f(\mathbf{x})$) denotes the derivative of $\mathbf{f}$ at $\mathbf{x}$ in U.
2. If every component function $f_i(x_1, x_2, \dots, x_n)$ $(i = 1, 2, \dots, m)$ of $\mathbf{f}$ has continuous partial derivatives at $\mathbf{x}^0$, then every element of the Jacobian matrix of $\mathbf{f}$ is continuous at $\mathbf{x}^0$. In this case, $\mathbf{f}$ is said to have a continuous derivative at $\mathbf{x}^0$ as a vector-valued function.
If the derivative of a vector-valued function $\mathbf{f}$ is continuous at every point in U, then $\mathbf{f}$ is said to have a continuous derivative on U.
3. If there exists an $m \times n$ matrix A that depends only on $\mathbf{x}^0$ (and not on $\Delta\mathbf{x}$), such that in the neighborhood of $\mathbf{x}^0$,
$$\Delta\mathbf{y} = f(\mathbf{x}^0 + \Delta\mathbf{x}) - f(\mathbf{x}^0) = A\Delta\mathbf{x} + o(\|\Delta\mathbf{x}\|)$$
(where $\Delta\mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)^T$ is a column vector and $\|\Delta\mathbf{x}\|$ denotes its norm), then f is said to be differentiable at $\mathbf{x}^0$ as a vector-valued function, and $A\Delta\mathbf{x}$ is called the differential of f at $\mathbf{x}^0$, denoted as $\mathrm{d}\mathbf{y}$. If we denote $\Delta\mathbf{x}$ by $\mathrm{d}\mathbf{x}$ ($\mathrm{d}\mathbf{x} = (\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_n)^T$), then
$$\mathrm{d}\mathbf{y} = A\,\mathrm{d}\mathbf{x}.$$
If the vector-valued function $\mathbf{f}$ is differentiable at every point in U, then $\mathbf{f}$ is said to be differentiable on U.

Combining the above three points, we obtain the following unified statement:

A vector-valued function $\mathbf{f}$ is continuous, differentiable, and has derivatives if and only if each of its coordinate component functions $f_i(x_1, x_2, \dots, x_n)$ $(i = 1, 2, \dots, m)$ is continuous, differentiable, and has derivatives.

8.3 Derivatives of Composite Mappings (Chain Rule)

Let $U \subset \mathbb{R}^l$ and $V \subset \mathbb{R}^n$ be open sets, and let

$$\mathbf{g} : U \to V \quad \text{and} \quad \mathbf{f} : V \to \mathbb{R}^m$$

be mappings. If $\mathbf{g}$ is derivative at $\mathbf{u}^0 \in U$ and $\mathbf{f}$ is differentiable at $\mathbf{x}^0 = \mathbf{g}(\mathbf{u}^0)$, then the composite mapping $\mathbf{f} \circ \mathbf{g}$ is differentiable at $\mathbf{u}^0$, and:

$$\mathbf{J}(\mathbf{f} \circ \mathbf{g})(\mathbf{u}^0) = \mathbf{J}\mathbf{f}(\mathbf{x}^0)\mathbf{J}\mathbf{g}(\mathbf{u}^0).$$

Note

1. *outer differentiable + inner derivative = total derivative*
2. *outer differentiable + inner differentiable = total differentiable*
3.

Specially, define $z = f(x,y), (x,y) \subset D_f \subset \mathbb{R}^2$, $\mathbf{g} : D_g \to \mathbb{R}^2, (u,v) \mapsto (x(u,v), y(u,v))$, and $g(D_g) \subset D_f$, then we have composite function

$$z = f \circ \mathbf{g} = f\left[x(u,v), y(u,v)\right], \quad (u,v) \in D_g.$$

$$\mathbb{R}^2 \xrightarrow{\mathbf{g}:\text{derivative}} \mathbb{R}^2 \xrightarrow{f:\text{differentiable}} \mathbb{R}$$

If $\mathbf{g}$ is derivative at $(u_0, v_0) \in D_g$, and f is differentiable at $(x_0, y_0) = \mathbf{g}(u_0, v_0)$, then $z = f \circ \mathbf{g}$ is differentiable at (u_0, v_0), and at the point,

$$\begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Proof

■

Chapter 9 Multiple Integrals

Bibliography

[1] 徐森林, 薛春华. 数学分析. 第一版. 清华大学出版社, 2005.

[2] 陈纪修, 於崇华. 数学分析. 第三版. 高等教育出版社, 2019.

[3] 常庚哲, 史济怀. 数学分析教程. 第三版. 中国科学技术大学出版社, 2012.

[4] 裴礼文. 数学分析中的典型问题与方法. 第三版. 高等教育出版社, 2021.

[5] 汪林. 数学分析中的问题与反例. 第一版. 高等教育出版社, 2015.

[6] 谢惠民, 恽自求, 易法槐, 钱定边. 数学分析习题课讲义. 第二版. 高等教育出版社, 2019.

[7] Walter Rudin. *Principles of Mathematical Analysis*. Third Edition. McGraw-Hill, 1976.

[8] 菲赫金哥尔茨. 微积分学教程. 第八版. 高等教育出版社, 2006.