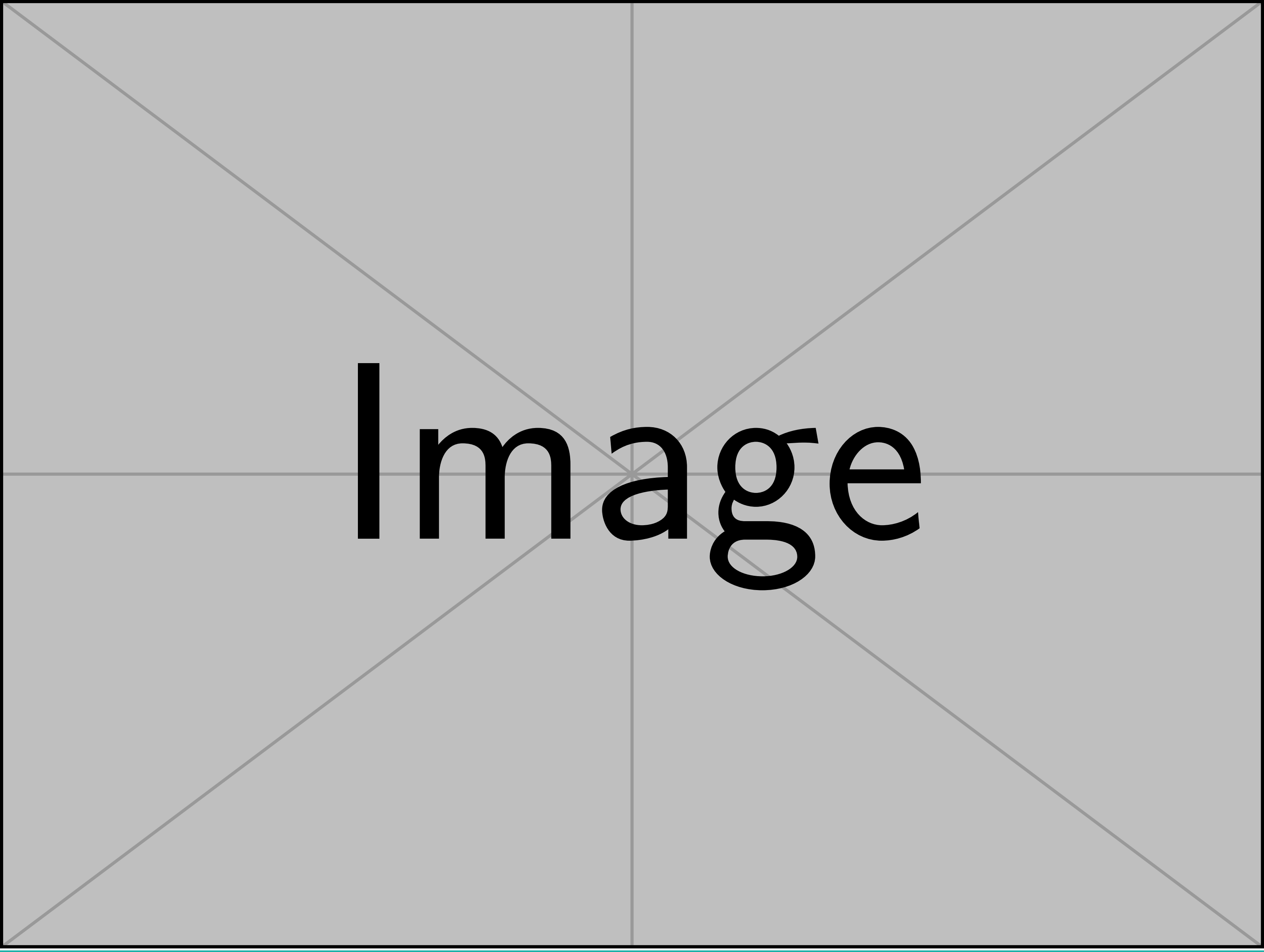

Analyse Harmonique

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Contents

Preface

This is the preface of the book…

Chapter 1 Classical Fourier Series

In this chapter, we will explore the Fourier series in such function space:

Set and Field The linear space we are working on is the set of all integrable (in the Riemann sense)[1] complex-valued periodic functions defined on $[-\pi, \pi]$[2], equipped with the usual addition and scalar multiplication of functions. We denote it as $\mathcal{R}[-\pi, \pi]$ that is a infinite-dimensional linear space. The field of scalars is the set of complex numbers $\mathbb{C}$.

Inner Product For any two functions $f(x), g(x)$ in this space, we define their inner product as:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x,$$

where $\frac{1}{2\pi}$ is a normalization factor.

Norm The norm induced by this inner product is given by:

$$\|f\| = \sqrt{\langle f, f \rangle} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

In fact, we often assume that the functions are always piecewise continuous or piecewise smooth on $[-\pi, \pi]$, which is the most common case in engineering.

¶ **Function Defined on the Unit Circle**

For a periodic function $f(x) : \mathbb{R} \to \mathbb{C}$ with period 2π, we can explore it from the perspective of complex exponential functions on the unit circle in the complex plane. Let

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\},$$

which is one-dimensional torus, also known as the unit circle in the complex plane.

For any $\theta \in \mathbb{R}$, we can define:

$$f(\theta) = F(e^{i\theta}),$$

where $F : \mathbb{T} \to \mathbb{C}$ is a **function defined on the unit circle**. Thus, we can study the periodic function $f(x)$ by analyzing the function $F(z)$ on the unit circle $\mathbb{T}$. From the perspective of algebra, the set of all such functions $F(z)$ forms a function space over the unit circle, which is isomorphic to the space of periodic functions $f(x)$ with period 2π.

By introducing $\mathbb{T}$ that is a compact manifold without boundary in fact, we can not only eliminate the hassles of endpoints but also simplify many discussions. Furthermore, since $\mathbb{T}$ is a multiplicative group of complex numbers, we can better understand the essence of Fourier series: the duality theory on compact Abelian groups.

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[1] For common integral, it should be Riemann integral; for defective integral, it should be absolute Riemann integral. For convenience, we just say Riemann integral in this context.

[2] It can be also defined on interval $[-T, T]$, but we choose $[-\pi, \pi]$ for simplicity.

1.1 Fourier Coefficients

Theorem 1.1

$$\mathcal{E} = \{e^{inx} : n \in \mathbb{Z}\}$$

or in real form:

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

is an orthonormal basis of the inner product space $\mathcal{R}[-\pi, \pi]$.

Definition 1.1

The Fourier coefficients $\hat{f}(n)$ of a function $f(x) \in \mathcal{R}[-\pi, \pi]$ is the projection of $f(x)$ onto the basis function e^{inx}:

$$\hat{f}(n) = \langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x, \quad n \in \mathbb{Z}.$$

Hence, the Fourier series of $f(x)$ is given by:

$$f(x) \sim \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{inx},$$

or in real form:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right],$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x, \quad n = 1, 2, \ldots \end{aligned}$$

and the symbol "$\sim$" indicates that the right-hand side is the Fourier series representation of $f(x)$.

It can be easily extended to any periodic function with period $2T$ by the substitution $x = \frac{\pi}{T} t$:

$$f(x) \sim \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{in\frac{\pi}{T}x},$$

or in real form:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[a_n \cos\left(n\frac{\pi}{T}x\right) + b_n \sin\left(n\frac{\pi}{T}x\right) \right].$$

When $f(x)$ is an even function, all sine terms vanish, and the Fourier series reduces to a cosine series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(nx).$$

When $f(x)$ is an odd function, all cosine terms vanish, and the Fourier series reduces to a sine series:

$$f(x) \sim \sum_{n=1}^{+\infty} b_n \sin(nx).$$

1.2 Convergence of Fourier Series

Lemma 1.1 (Riemann-Lebesgue Lemma)

Let $f(x) \in R[a,b]$, $g(x)$ has a period T and $g(x) \in R[0,T]$, then:

$$\lim_{p\to+\infty} \int_a^b f(x)g(px)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \cdot \frac{1}{T}\int_0^T g(t)\,\mathrm{d}t.$$

A special case is when $g(x) = \sin x$ or $g(x) = \cos x$, then:

$$\lim_{p\to+\infty} \int_a^b f(x)\sin(px)\,\mathrm{d}x = \int_a^b f(x)\cos(px)\,\mathrm{d}x = 0.$$

♡

Chapter 2 Cesàro Summation

Chapter 3 Modern Fourier Series

Chapter 4 Fourier Transform

Chapter 5 Sobolev Spaces

Bibliography

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[2] 于品海. *Fourier 分析*.

[3] Author2, Title2, Journal2, Year2. *This is another example of a reference.*