

# Analyse Mathématique

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**Date:** July, 2025

Version: 0.1

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# **Preface**

For an interval I, a open interval (a,b) and a closed interval [a,b], we denote C(I), C(a,b) and C[a,b] as the set of continuous <u>univariate</u> functions on I, (a,b) and [a,b] respectively. Similarly, the following notations are used:

Notation	Meaning
D(I)	Set of derivative (differential) functions on $I$
D(a,b)	Set of derivative (differential) functions on $\left(a,b\right)$
D[a,b]	Set of derivative (differential) functions on $\left[a,b\right]$
$D^k(I)$	Set of $k$ -th order derivative (differential) functions on ${\cal I}$

Let  $U \subset \mathbb{R}^n$  be an open set, and  $\mathbf{f}: U \to \mathbb{R}^m$  be a  $C^k$  mapping:

- k = 0, **f** is a continuous mapping;
- $0 < k < +\infty$ ,  $f_i$  has continuous partial derivatives up to order  $k, i = 1, 2, \dots, m$ ;
- ullet  $k=+\infty$ ,  $f_i$  has continuous partial derivatives of all orders,  $i=1,2,\ldots,m$ ;
- $k = \omega$ ,  $f_i$  is really analytic, i.e., in the neighborhood of any point  $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in U$ ,  $f_i$  can be expanded into a convergent (n-dimensional) power series,  $i = 1, 2, \dots, m$ .

Let  $C^k(U, \mathbb{R}^m)$  denote the set of  $C^k$  mappings from U to  $\mathbb{R}^m$ .

# **Chapter 1 Preliminaries**

### 1.1 Trigonometric Formulas

#### **Product-to-Sum Formulas:**

$$\sin \alpha \cos \beta = \frac{1}{2} \left[ \sin(\alpha + \beta) + \sin(\alpha - \beta) \right]$$

$$\cos \alpha \sin \beta = \frac{1}{2} \left[ \sin(\alpha + \beta) - \sin(\alpha - \beta) \right]$$

$$\cos \alpha \cos \beta = \frac{1}{2} \left[ \cos(\alpha + \beta) + \cos(\alpha - \beta) \right]$$

$$\sin \alpha \sin \beta = -\frac{1}{2} \left[ \cos(\alpha + \beta) - \cos(\alpha - \beta) \right]$$

#### Sum and Difference Formulas:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$
$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

#### **Sum-to-Product Formulas:**

$$\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right)$$

$$\sin \alpha - \sin \beta = 2 \sin \left(\frac{\alpha - \beta}{2}\right) \cos \left(\frac{\alpha + \beta}{2}\right)$$

$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2}\right) \sin \left(\frac{\alpha - \beta}{2}\right)$$

#### **Double Angle Formulas:**

$$\sin 2\alpha = 2\sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2\cos^2 \alpha - 1 = 1 - 2\sin^2 \alpha$$

$$\tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha}$$

#### Half Angle Formulas:

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

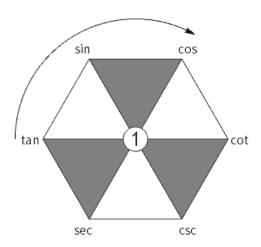
$$\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$$

#### **Power-Reducing Formulas:**

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$
$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

#### **Angle Decomposition Formulas:**

$$\sin^2 \alpha - \sin^2 \beta = \sin(\alpha + \beta)\sin(\alpha - \beta)$$
$$\cos^2 \alpha - \sin^2 \beta = \cos(\alpha + \beta)\cos(\alpha - \beta)$$



#### **Z**Remark

- On the gray triangle, the sum of the squares of the two numbers above is equal to the square of the number below, for instance,  $\tan^2 x + 1 = \sec^2 x$
- The three trigonometric functions in the clockwise direction have the following properties:  $\tan x = \frac{\sin x}{\cos x}$ , etc.

#### 1.2 Factorial Power

#### Definition 1.1

Rising factorials and falling factorials can be expressed in multiple notations.

The Pochhammer symbol, introduced by Leo August Pochhammer, is one of the commonly used notations, represented as  $x^{(n)}$  or  $(x)_n$ .

Ronald Graham, Donald Ervin Knuth, and Oren Patashnik introduced the symbols  $x^{\bar{n}}$  and  $x^{\underline{n}}$  in their book Concrete Mathematics.

#### **Definitions:**

• Rising factorial:

$$x^{\bar{n}} = x(x+1)(x+2)\dots(x+n-1) = \frac{(x+n-1)!}{(x-1)!}.$$

• Falling factorial:

$$x^{\underline{n}} = x(x-1)(x-2)\dots(x-n+1) = \frac{x!}{(x-n)!}.$$

#### Relationships:

Relationship between rising and falling factorials:

$$x^{\bar{n}} = (x+n-1)^{\underline{n}}.$$

• Relationship with factorial:

$$1^{\bar{n}} = n^{\underline{n}} = n!.$$



# Chapter 2 Limits of Sequences and Continuity of Real Number System

### 2.1 Convergent Sequences

- ¶ Convergent Sequences
- ¶ Properties of Convergent Sequences
- ¶ Cauchy Proposition and Fitting Method

#### Proposition 2.1 (Cauchy Proposition)

Let  $\lim_{n\to\infty} x_n = l$ , then:

$$\lim_{n \to \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = l.$$



- 1. In the proposition, l can be  $+\infty$  or  $-\infty$ .
- 2. Let  $\lim_{n\to\infty} x_n = l$ , then:

$$\lim_{n \to \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = \lim_{n \to \infty} \sqrt[n]{x_1 x_2 \cdots x_n} = \lim_{n \to \infty} \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}} = l.$$

It can be proved directly by Stolz theorem 2.1. On top of that, it can also be proved by the **fitting method**.



Remark To prove  $\lim_{n\to\infty} x_n = A$ , the key is to show that  $|x_n - A|$  can be arbitrarily small. For this purpose, it is generally recommended to simplify the expression of  $x_n$  as much as possible. However, in some cases, A can also be transformed into a form similar to  $x_n$ . This method is called the fitting method. The core idea behind the method of fitting is to appropriately divide into units of 1 for analysis.

### 2.2 Indeterminate Form

- ¶ Infinitely Large Quantities and Infinitesimal Quantities
- ¶ Indeterminate Forms

#### Theorem 2.1 (Stolz-Cesàro theorem

**Type**  $\frac{0}{0}$  Let  $\{a_n\}, \{b_n\}$  be two infinitesimal sequences, where  $\{a_n\}$  is also a strictly monotonic decreasing sequence. If

$$\lim_{n\to\infty}\frac{b_{n+1}-b_n}{a_{n+1}-a_n}=l\ (\text{finite or }\pm\infty),$$

then

$$\lim_{n \to \infty} \frac{a_n}{b_n} = l.$$

**Type**  $\frac{*}{\infty}$  Let  $\{a_n\}$  be a strictly monotonic increasing sequence of divergent large quantities. If

$$\lim_{n\to\infty}\frac{b_{n+1}-b_n}{a_{n+1}-a_n}=l\ (\text{finite or }\pm\infty),$$

then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=l.$$



#### Note

- 1. The inverse proposition of Stolz's Theorem does not hold.
- 2. If  $a_1$  is an undefined infinite quantity  $\infty$ , Stolz Theorem does not hold.

#### Theorem 2.2 (Silverman-Toeplitz Theorem)

Let

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{bmatrix},$$

where the infinite triangular matrix satisfies:

- 1.  $\forall j, \lim_{n\to\infty} a_{nj} = 0$ . (Every column sequence converges to 0.)
- 2.  $\sup_{i\in\mathbb{N}}\sum_{j=1}^{i}|a_{ij}|<\infty.$  (The absolute row sums are bounded.)

And  $\lim_{n\to\infty} x_n = l$ . We denote  $y_n$  as the weighted sum sequence:  $y_n = \sum_{j=1}^n a_{nj}x_j$ . Then the following results hold:

- 1. If l = 0, then  $\lim_{n \to \infty} y_n = 0$ .
- 2. If  $l \neq 0$  and  $\lim_{n \to \infty} \sum_{j=1}^n a_{ij} = 1$ , then  $\lim_{n \to \infty} y_n = l$ .



### 2.3 Subsequences

- ¶ Subsequences
- ¶ Upper Limits and Lower Limits

### 2.4 Completeness of The Real Numbers

- ¶ Dedkind Completeness
- $\P$  Least Upper Bound Property
- ¶ Monotone Convergence Theorem
- $\P$  Bolzano-Weierstrass Theorem
- ¶ Nested Interval Theorem
- ¶ Cauchy Completeness

#### Definition 2.1 (Cauchy Sequence)

A sequence  $\{x_n\}$  is called a Cauchy sequence if for any  $\varepsilon > 0$ , there exists a positive integer N such that when m, n > N,

$$|x_n - x_m| < \varepsilon$$
.



#### Theorem 2.3 (Cauchy Convergence Criterion for Sequences)

A sequence  $\{x_n\}$  converges if and only if it is a Cauchy sequence.

### $\Diamond$

#### ■ Heine-Borel Theorem

### 2.5 Iterative Sequences

Formally,  $x_0$  is a **fixed point** of the function f if  $f(x_0) = x_0$ .

#### Theorem 2.4 (Banach Fixed-Point Theorem (Contraction Mapping Theorem)

There exists a contraction mapping (in 3.2) f on an interval I, which admits a unique fixed point  $x^* \in I$ . Furthermore,  $x^*$  can be found as follows: start with an arbitrary point  $x_0 \in I$  and define the iterative sequence  $x_{n+1} = f(x_n)$  for  $n = 0, 1, 2, \cdots$ . Then  $\lim_{n \to \infty} x_n = x^*$ .

**FREMARK** The following inequalities are equivalent and describe the speed of convergence:

$$|x_n - x^*| \le \frac{L^n}{1 - L} |x_1 - x_0|,$$
  
 $|x_{n+1} - x^*| \le \frac{L}{1 - L} |x_{n+1} - x_n|,$   
 $|x_{n+1} - x^*| \le L |x_n - x^*|.$ 

Any such value of L < 1 is the Lipschitz constant for f, and the smallest one is sometimes called **the best** Lipschitz constant of L.

# **Chapter 3 Limits and Continuity of Functions**

### 3.1 Limits of Functions

- ¶ Definition of Limit
- ¶ Limits of Functions and Sequences

#### Theorem 3.1 (Heine Theorem

Let f be a function defined on a deleted neighborhood  $\mathring{U}(x_0)$  of  $x_0$ . The following two statements are equivalent:

- 1.  $\lim_{x \to x_0} f(x) = A$ .
- 2. For any sequence  $\{x_n\} \subset \mathring{U}(x_0)$  with  $\lim_{n\to\infty} x_n = x_0$ , we have  $\lim_{n\to\infty} f(x_n) = A$  for the sequence  $\{f(x_n)\}$ .

### 3.2 Continuous Functions

### 3.3 Infinitesimal and Infinite Quantities

### 3.4 Continuous Functions on Closed Intervals

¶ Concerning Theorems

Theorem 3.2 (The Bolzano-Cauchy Intermediate-Value Theorem)

Theorem 3.3 (2010 Found Experience Interient)

¶ Uniform Continuity and Lipschitz Continuity

Definition 3.1 (Uniform Continuity)

Theorem 2 1 (2) will arm Continuity Theorem

Theorem 3.5 (Cantor's Theorem

#### Definition 3.2 (Lipschitz Continuity)

If there exists a constant L>0 such that for any  $x_1,x_2\in I$ ,

$$|f(x_1) - f(x_2)| \le L |x_1 - x_2|,$$

then f is called **Lipschitz continuous** on I.

Specially, if L < 1, then f is called a **contraction mapping** on I.

- If f is Lipschitz continuous on I, then f is uniformly continuous on I. ( $\forall \varepsilon>0$ , just let  $\delta=\frac{\varepsilon}{L}$ )
- $\bullet\,$  If f is uniformly continuous on I, then f is continuous on I.
- The converse of the above two statements does not hold.

# 3.5 Period Three Implies Chaos

# 3.6 Functional Equations

# **Chapter 4 Differential**

## 4.1 Differential and Derivative

#### $\P$ Basic Differential Rules and Formulas

	Derivative Rules	Differential Rules
Linear Combination	$(c_1f + c_2g)' = c_1f' + c_2g'$	$d(c_1f + c_2g) = c_1df + c_2dg$
Product Rule	(fg)' = f'g + fg'	d(fg) = gdf + fdg
Quotient Rule	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$	$d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$
Inverse Function	$[f^{-1}(y)]' = \frac{1}{f'(x)}$	$dx = \frac{dy}{f'(x)} = [f^{-1}(y)]'dy$
Chain Rule	[f(g(x))]' = f'(u)g'(x)	d[f(g(x))] = f'(u)g'(x)dx

Derivative	Differential
(C)' = 0	$d(C) = 0 \cdot dx = 0$
$(x^{\alpha})' = \alpha x^{\alpha - 1}$	$d(x^{\alpha}) = \alpha x^{\alpha - 1} dx$
$(\sin x)' = \cos x$	$d(\sin x) = \cos x dx$
$(\cos x)' = -\sin x$	$d(\cos x) = -\sin x dx$
$(\tan x)' = \sec^2 x$	$d(\tan x) = \sec^2 x dx$
$(\cot x)' = -\csc^2 x$	$d(\cot x) = -\csc^2 x dx$
$(\sec x)' = \tan x \sec x$	$d(\sec x) = \tan x \sec x dx$
$(\csc x)' = -\cot x \csc x$	$d(\csc x) = -\cot x \csc x dx$
$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$d(\arcsin x) = \frac{1}{\sqrt{1-x^2}} dx$
$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$d(\arccos x) = -\frac{1}{\sqrt{1-x^2}} dx$
$(\arctan x)' = \frac{1}{1+x^2}$	$d(\arctan x) = \frac{1}{1+x^2} dx$
$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$	$d(\operatorname{arccot} x) = -\frac{1}{1+x^2} dx$
$(a^x)' = \ln a \cdot a^x, (e^x)' = e^x$	$d(a^x) = \ln a \cdot a^x dx, d(e^x) = e^x dx$
$(\log_a x)' = \frac{1}{x \ln a}, (\ln x)' = \frac{1}{x}$	$d(\log_a x) = \frac{1}{x \ln a} dx, d(\ln x) = \frac{1}{x} dx$
$(\operatorname{sh} x)' = \operatorname{ch} x$	$d(\operatorname{sh} x) = \operatorname{ch} x dx$
$(\operatorname{ch} x)' = \operatorname{sh} x$	$d(\operatorname{ch} x) = \operatorname{sh} x dx$
$(\operatorname{th} x)' = \operatorname{sech}^2 x$	$d(\operatorname{th} x) = \operatorname{sech}^2 x dx$
$(\coth x)' = -\operatorname{csch}^2 x$	$d(\coth x) = -\operatorname{csch}^2 x dx$
$(\operatorname{arcsh} x)' = \frac{1}{\sqrt{1+x^2}}$	$d(\operatorname{arcsh} x) = \frac{1}{\sqrt{1+x^2}} dx$
$(\operatorname{arcch} x)' = \frac{1}{\sqrt{x^2 - 1}}$	$d(\operatorname{arcch} x) = \frac{1}{\sqrt{x^2 - 1}} dx$
$(\operatorname{arcth} x)' = (\operatorname{arccth} x)' = \frac{1}{1-x^2}$	$d(\operatorname{arcth} x) = d(\operatorname{arccth} x) = \frac{1}{1 - x^2} dx$
$\ln(x + \sqrt{x^2 + a^2})' = \frac{1}{\sqrt{x^2 + a^2}}$	$d[\ln(x + \sqrt{x^2 + a^2})] = \frac{dx}{\sqrt{x^2 + a^2}}$

### 4.2 Higher-Order Derivatives

### 4.3 Differential Mean Value Theorems

#### Definition 4.1 (Argmax and Argmin)

Let f(x) is defined on (a, b),  $x_0 \in (a, b)$ . If there exists  $U(x_0, \delta) \subset (a, b)$  such that  $f(x) \leq f(x_0)$  on it, then  $x_0$  is called a arguments of the maxima point of f, and  $f(x_0)$  is referred to as the corresponding arguments of the maxima (abbreviated arg max or argmax).

The definition of the argmin is analogous.

### 4

#### Lemma 4.1 (Fermat's Lemma

If f is differentiable at  $x_0$  which is a local extremum, then  $f'(x_0) = 0$ .



#### Theorem 4.1 (Rolle's Theorem)

If  $f \in C[a,b], f \in D(a,b)$  and f(a)=f(b), then there exists  $\xi \in (a,b)$  such that  $f'(\xi)=0$ . Enhanced Version: If  $f \in D(a,b)$  (finite or infinite interval), and  $\lim_{x\to a^+} f(x)=\lim_{x\to b^-} f(x)$ , then there exists  $\xi \in (a,b)$  such that  $f'(\xi)=0$ .

#### Theorem 4.2 (Lagrange's Mean Value Theorem

If  $f \in C[a,b], f \in D(a,b)$ , then there exists  $\xi \in (a,b)$  such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$



#### Theorem 4.3 (Cauchy's Mean Value Theorem

If  $f,g\in C[a,b], f,g\in D(a,b)$  and  $g'(x)\neq 0$  for all  $x\in (a,b)$ , then there exists  $\xi\in (a,b)$  such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$



Note The following types of problems commonly appear in proofs related to intermediate values in differential calculus:

1. Prove the existence of a point  $\xi$  such that  $F(\xi, f(\xi), f'(\xi)) = 0$ . Problems of this type generally involve constructing auxiliary functions and applying Rolle's theorem. The commonly used auxiliary functions include:

$$\xi f'(\xi) + f(\xi) = 0, \quad x f(x),$$

$$\xi f'(\xi) + n f(\xi) = 0, \quad x^n f(x),$$

$$\xi f'(\xi) - f(\xi) = 0, \quad e^x f(x),$$

$$f'(\xi) + \lambda f(\xi) = 0, \quad e^{-x} f(x),$$

$$f'(\xi) + f(\xi) = 0, \quad x^n f(x),$$

$$f'(\xi) - f(\xi) = 0, \quad x f(x).$$

2. Prove the existence of two points  $\xi$ ,  $\eta$  (i.e., two intermediate values) such that  $F(\xi, f(\xi), f'(\xi), \eta, f(\eta), f'(\eta)) = 0$ . These problems can be divided into the following categories:

 $\xi 
eq \eta$  Problems of this type usually occur in the same interval [a,b] and employ theorems of <u>double</u> differentiation

intermediate values such as the Lagrange mean value theorem or Cauchy's mean value theorem. The specific choice of auxiliary functions often includes terms like  $\xi$  and other variables determined after decomposition.

- $\xi = \eta$  Such problems cannot occur within the same interval [a,b]. They use double differentiation mean value theorems by <u>splitting</u> [a,b] into two intervals [a,c] and [c,b], applying the Lagrange mean value theorem separately to each interval. Here, the <u>selection</u> of  $\xi$  and  $\eta$  is key.
- 3. As a rule, when conditions in a theorem involve additional constraints about <u>higher-order</u> derivatives, it is necessary to use Taylor's intermediate value theorem.

### 4.4 Theorems about Derivatives

#### Theorem 4.4 (Darboux's Intermediate Value Theorem for Derivatives)

If  $f(x) \in D[a, b]$ , and  $f'_{+}(a) \cdot f'_{-}(b) < 0$ , then there at least exists  $\xi \in (a, b)$  such that  $f'(\xi) = 0$ .

#### $\bigcirc$

#### Theorem 4.5 (Theorem on the Limit of Derivatives

If  $f(x) \in C(U(x_0))$ ,  $\mathring{D}(U(x_0))$ , and  $\lim_{x \to x_0} f'(x) = A$ , then f is differentiable at  $x_0$  and  $f'(x_0) = A$ .

**Z**Remark In fact,  $\lim_{x o x_0}f'(x)=A$  has already been shown to imply that  $f\in \mathring{D}(U(x_0))$ .

### 4.5 Taylor Theorem

- ¶ L'Hôpital's Rule
- ¶ Taylor Formula
- ¶ Macklaurin Formula

### 4.6 Properties of Functions

¶ Monotonicity and Convexity

#### Definition 4.2 (Convex Function)

A function f is called **convex** on an interval I if for any  $x_1, x_2 \in I$  and  $t \in [0, 1]$ , the following inequality holds:

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2).$$

If the inequality is strict for  $x_1 \neq x_2$  and  $t \in (0, 1)$ , then f is called **strictly convex** on I.

Conversely, if the inequality is reversed, then f is called **concave** or **concave down** on I.

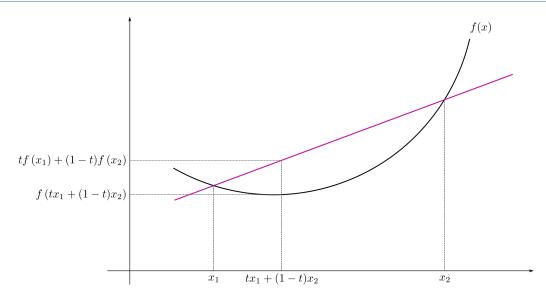


#### Theorem 4.6

Mark above definition as Definition 1, give the following statements:

2. (Jensen Definition) A function f is called convex on an interval I if for any  $x_1, x_2 \in I$ :

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}.$$



3. A function f is called convex on an interval I if for any  $x_1, x_2, \dots, x_n \in I$ :

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \le \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}.$$

4. A function f is called convex on an interval I if the tangent line at any point lies below the graph of the function.

Then,

- Definitions 2 and 3 are equivalent.
- When f is continuous, Definition 1, 2, 3 is equivalent.
- ullet When f is differentiable, all four definitions are equivalent.

#### Theorem 4.7 (Jensen Inequality

If f is convex on an interval I, then for any  $x_1, x_2, \dots, x_n \in I$  and any  $t_1, t_2, \dots, t_n > 0$  such that  $t_1 + t_2 + \dots + t_n = 1$ , the following inequality holds:

$$f(t_1x_1 + t_2x_2 + \dots + t_nx_n) \le t_1f(x_1) + t_2f(x_2) + \dots + t_nf(x_n).$$

Specially, when  $t_1 = t_2 = \cdots = t_n = \frac{1}{n}$ , it reduces to Definition 3.

Next, we present derivative-based criteria for monotonicity and convexity:

#### Theorem 4.8

- 1. If  $f \in D(I)$ , then f is increasing (decreasing) on I if and only if  $f'(x) \ge 0$  ( $f'(x) \le 0$ ) for all  $x \in I$ .
- 2. If  $f \in D^{(2)}(I)$ , then f is convex (concave) on I if and only if  $f''(x) \ge 0$  ( $f''(x) \le 0$ ) for all  $x \in I$ .

Note If f'(x) > 0 (f''(x) > 0) for all  $x \in I$ , then f is strictly increasing (convex) on I. Even though the condition weakens to holding except at finitely many points, the conclusion of strict monotonicity (convexity) still holds. For example,  $f(x) = x^3$  is strictly increasing on  $\mathbb{R}$  despite f'(0) = 0.

- ¶ Argmax and Argmin
- ¶ Asymptote

# **4.7** Applications

# **Chapter 5** Indefinite Integral

## **5.1 Two Common Integration Methods**

#### ¶ Integration Methods

#### Definition 5.1 (Integration by Parts)

Let u(x) and v(x) be two differentiable functions, and at least one of them has an antiderivative. Then the integration by parts formula states that:

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u.$$

### Definition 5.2 (Substitution Method)

Some common substitutions are as follows:

Trigonometric Substitution When restoring variables, auxiliary right triangles is often utilized.

Sine 
$$\sqrt{a^2-x^2}$$
:  $x=a\sin t$  or  $x=a\cos t$ 

**Tangent** 
$$\sqrt{a^2 + x^2}$$
:  $x = a \tan t$  or  $x = a \sinh t$ 

**Secant** 
$$\sqrt{x^2 - a^2}$$
:  $x = a \sec t$  or  $x = a \cosh t$ 

#### **Irreational Substitution**

#### **Reciprocal Substitution**

 $\P$  Basic Integration Formulas

Integral	Result
$\int a  \mathrm{d}x$	ax + C (a is constant)
$\int x^n  \mathrm{d}x$	$\frac{x^{n+1}}{n+1} + C  (n \neq -1)$
$\int \frac{1}{x} dx$	$\ln x  + C$
$\int e^x dx$	$e^x + C$
$\int a^x  \mathrm{d}x$	$\frac{a^x}{\ln a} + C  (a > 0, a \neq 1)$
$\int \sin x  \mathrm{d}x$	$-\cos x + C$
$\int \cos x  \mathrm{d}x$	$\sin x + C$
$\int \tan x  \mathrm{d}x$	$-\ln \cos x  + C$
$\int \cot x  \mathrm{d}x$	$\ln \sin x  + C$
$\int \sec x  \mathrm{d}x$	$\ln \sec x + \tan x  + C$
$\int \csc x  \mathrm{d}x$	$\ln \csc x - \cot x  + C$
$\int \sec x \tan x  \mathrm{d}x$	$\sec x + C$
$\int \csc x \cot x  \mathrm{d}x$	$-\csc x + C$
$\int \sec^2 x  \mathrm{d}x$	$\tan x + C$
$\int \csc^2 x  \mathrm{d}x$	$-\cot x + C$
$\int \frac{1}{\sqrt{a^2 - x^2}}  \mathrm{d}x$	$\arcsin\left(\frac{x}{a}\right) + C$
$\int \frac{-1}{\sqrt{a^2-x^2}}  \mathrm{d}x$	$\arccos\left(\frac{x}{a}\right) + C$
$\int \frac{1}{a^2 + x^2}  \mathrm{d}x$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
$\int \frac{-1}{a^2 + x^2}  \mathrm{d}x$	$\frac{1}{a}\operatorname{arccot}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{x^2 + a^2}}  \mathrm{d}x$	$ \ln x + \sqrt{x^2 + a^2}  + C $
$\int \frac{1}{\sqrt{x^2 - a^2}}  \mathrm{d}x$	$\ln x + \sqrt{x^2 - a^2}  + C  (x > a \text{ or } x < -a)$
$\int \sinh x  \mathrm{d}x$	$\cosh x + C$
$\int \cosh x  \mathrm{d}x$	$\sinh x + C$

# **Chapter 6 Definite Integral**

### 6.1 Riemann Integral

#### ¶ Riemann Integral

#### Definition 6.1 (Riemann Integral)

Let f(x) be a bounded function defined on [a,b]. Take any set of division points  $\{x_i\}_{i=0}^n$  on [a,b] to form a partition  $P: a = x_0 < x_1 < \cdots < x_n = b$ , and choose arbitrary points  $\xi_i \in [x_{i-1}, x_i]$ . Denote the length of the sub-interval  $[x_{i-1}, x_i]$  as  $\Delta x_i = x_i - x_{i-1}$ , and let  $\lambda = \max_{1 \le i \le n} (\Delta x_i)$ . If the limit

$$\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i) \Delta x_i$$

exists as  $\lambda \to 0$ , and the limit is independent of the partition P and the choice of  $\xi_i$ , then f(x) is said to be Riemann integrable on [a, b].

The summation

$$S_n = \sum_{i=1}^n f(\xi_i) \Delta x_i$$

is called the Riemann sum, and its limit I is called the definite integral of f(x) on [a, b], denoted as:

$$I = \int_{a}^{b} f(x) \, \mathrm{d}x,$$

where a and b are called the lower and upper limits of the definite integral, respectively.

Alternatively, it can also be expressed as:

$$\exists I, \forall \varepsilon > 0, \exists \delta > 0, \text{s.t.} \forall P(\lambda = \max_{1 \leqslant i \leqslant n} (\Delta x_i) < \delta), \forall \{\xi_i\} : \left| \sum_{i=1}^n f(\xi_i) \Delta x_i - I \right| < \varepsilon.$$

Then f(x) is said to be Riemann integrable on [a, b], and I is the definite integral of f(x) on [a, b].

**Frank** Partition  $\rightarrow$  Intermediate points  $\rightarrow$  Summation  $\rightarrow$  Take the limit.

#### ¶ Darboux Sum

#### Definition 6.2 (Darboux Sum)

Let the supremum and infimum of f(x) on [a, b] be M and m, respectively. Clearly,  $m \le f(x) \le M$ . Let the supremum and infimum of f(x) on  $[x_{i-1}, x_i]$  be  $M_i$  and  $m_i$  (i = 1, 2, ..., n), respectively, i.e.,

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}, \quad m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

After fixing the partition P, define the sums:

$$\bar{S}(P) = \sum_{i=1}^{n} M_i \Delta x_i, \quad \underline{S}(P) = \sum_{i=1}^{n} m_i \Delta x_i,$$

which are called the Darboux upper sum and Darboux lower sum corresponding to the partition P, respectively.

#### Property

- 1.  $\underline{S}(P) \leqslant \sum_{i=1}^{n} f(\xi_i) \Delta x_i \leqslant \bar{S}(P)$ .
- 2. If a new partition is formed by adding division points to the original partition, the upper sum does not increase, and the lower sum does not decrease.

3. Let  $\bar{S}$  denote the set of Darboux upper sums and  $\underline{S}$  denote the set of Darboux lower sums. For any  $\bar{S}(P_1) \in \bar{S}$ ,  $\underline{S}(P_2) \in \underline{S}$ , it always holds that:

$$m(b-a) \leqslant \underline{S}(P_2) \leqslant \overline{S}(P_1) \leqslant M(b-a).$$

- 4. Let  $L = \inf\{\bar{S}(P) \mid \bar{S}(P) \in \bar{S}\}, l = \sup\{\underline{S}(P) \mid \underline{S}(P) \in \underline{S}\}$ , which are called the upper integral and lower integral, respectively. It always holds that:  $l \leq L$ .
- 5. **Darboux's Theorem**: For any  $f(x) \in B[a, b]$ , it always holds that:

$$\lim_{\lambda \to 0} \bar{S}(P) = L, \quad \lim_{\lambda \to 0} \underline{S}(P) = l.$$

¶ Riemann-Stieltjes Integral

#### Definition 6.3 (Riemann-Stieltjes Integral)

Let  $\alpha$  be a bounded, monotonically increasing function on [a,b]. For every partition P of [a,b], let  $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$  (clearly  $\Delta \alpha_i \ge 0$ ). For a bounded real function f(x) on [a,b], define the Stieltjes upper sum and lower sum as:

$$\bar{S}(P,\alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i, \quad \underline{S}(P,\alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i,$$

and define the upper and lower integrals as:

$$L = \inf\{\bar{S}(P,\alpha) \mid \bar{S}(P,\alpha) \in \bar{S}\}, \quad l = \sup\{\underline{S}(P,\alpha) \mid \underline{S}(P,\alpha) \in \underline{S}\},$$

where  $\bar{S}, \underline{S}$  are the sets of Stieltjes upper and lower sums respectively.

If L = l, then:

$$\int_{a}^{b} f(x) \, d\alpha(x) = L = l,$$

and f(x) is said to be **Riemann-Stieltjes integrable** on [a,b] with respect to  $\alpha$ , or simply Stieltjes integrable.



When  $\alpha(x)=x$ , this reduces to the Riemann integral. However, in general,  $\alpha(x)$  does not even need to be continuous.

The properties of Darboux sums also apply to Stieltjes sums.

### 6.2 Integrability Criteria

 $\P$  Common Integrability Criteria

#### Theorem 6.1 (Integrability Criterion)

A bounded function f(x) is Riemann integrable on [a, b] if and only if:

• The upper and lower integrals are equal, i.e.,

$$\forall P(\lambda = \max_{1 \le i \le n} (\Delta x_i) < \delta) : \lim_{\lambda \to 0} \bar{S}(P) = L = l = \lim_{\lambda \to 0} \underline{S}(P).$$

• Let  $\omega_i = M_i - m_i$  be the oscillation of f(x) on  $[x_{i-1}, x_i]$ . Then: The limit of the sum of oscillations is zero, i.e.,

$$\forall P(\lambda = \max_{1 \le i \le n} (\Delta x_i) < \delta) : \lim_{\lambda \to 0} \sum_{i=1}^{n} \omega_i \Delta x_i = 0.$$

**Corollary 1** Continuous functions on closed intervals are necessarily integrable.

**Corollary 2** Monotonic functions on closed intervals are necessarily integrable.

• For all  $\varepsilon > 0$ , there exists a partition P such that:

$$\sum_{i=1}^{n} \omega_i \Delta x_i < \varepsilon.$$

**Corollary 1** The total length of intervals where oscillation  $\omega$  cannot be arbitrarily small can be made arbitrarily small, i.e.,

$$\forall \varepsilon, \eta > 0, \exists P, \text{s.t.} \sum_{\omega \geqslant n} \Delta x_i < \varepsilon.$$

**Corollary 2** Bounded functions with only finitely many discontinuities on closed intervals are necessarily integrable.



¶ Lesbesgue's Theorem

Theorem 6.2 (Lesbesgue's Theorem



### **6.3 Properties of Definite Integrals**

 $\P$  Properties of Riemann Integrals

Property

**Linearity** Let  $f(x), g(x) \in R[a, b]$ , and  $k_1, k_2$  are constants. Then the function  $k_1 f(x) + k_2 g(x) \in R[a, b]$ , and:

$$\int_{a}^{b} [k_1 f(x) + k_2 g(x)] dx = k_1 \int_{a}^{b} f(x) dx + k_2 \int_{a}^{b} g(x) dx.$$

Multiplicative Integrability Let  $f(x), g(x) \in R[a, b]$ , and  $k_1, k_2$ . Then  $f(x) \cdot g(x) \in R[a, b]$ . In general,

$$\int_{a}^{b} f(x)g(x)dx \neq \left(\int_{a}^{b} f(x)dx\right) \cdot \left(\int_{a}^{b} g(x)dx\right).$$

**Monotonicity** Let  $f(x), g(x) \in R[a, b]$ , and  $f(x) \ge g(x)$  (or f(x) > g(x)) on [a, b]. Then:

$$\int_{a}^{b} f(x) dx \geqslant \int_{a}^{b} g(x) dx \quad \left( \int_{a}^{b} f(x) dx > \int_{a}^{b} g(x) dx \right).$$

**Corollary 1** If  $f(x) \in C[a,b]$ ,  $f(x) \ge 0$ ,  $f(x) \ne 0$ , then:

$$\int_a^b f(x) \, \mathrm{d}x > 0.$$

Corollary 2 If  $f(x) \in R[a, b], f(x) > 0$ , then:

$$\int_{a}^{b} f(x) \, \mathrm{d}x > 0.$$

Absolute Value Integrability Let  $f(x) \in R[a,b]$ . Then  $|f(x)| \in R[a,b]$ , and:

$$\left| \int_{a}^{b} f(x) dx \right| \le \int_{a}^{b} |f(x)| dx.$$

The inverse statement of this property is not true.

**Additivity Over Intervals** Let  $f(x) \in R[a,b]$ . For any point  $c \in [a,b]$ , f(x) is integrable on [a,b] and [c,d]. Conversely, if  $f \in R[a,c] \cup [c,b]$ , then f(x) is integrable on [a,b], and:

$$\int_{a}^{b} f(x) dx = \int_{a}^{c} f(x) dx + \int_{c}^{b} f(x) dx.$$

#### Theorem 6.3 (Integral Mean Value Theorem,

First Integral Mean Value Theorem Let  $f(x), g(x) \in R[a, b]$ , and g(x) does not change sign on [a, b]. Then there exists  $\eta \in [m, M]$  such that:

$$\int_{a}^{b} f(x)g(x)dx = \eta \int_{a}^{b} g(x)dx,$$

where m, M represent the infimum and supremum of f(x) on [a, b], respectively.

In particular, if  $f(x) \in C[a,b]$ , then there exists  $\xi \in [a,b]$  such that:

$$\int_{a}^{b} f(x)g(x)dx = f(\xi) \int_{a}^{b} g(x)dx.$$

A special case arises when  $f(x) \in C[a,b]$  and  $g(x) \equiv 1$ , then:

$$\int_{a}^{b} f(x)g(x)dx = f(\xi) \int_{a}^{b} g(x)dx.$$

**Corollary** If  $f(x) \in C[a, b]$ , then there exists  $\xi \in (a, b)$  such that:

$$\int_{a}^{b} f(x)g(x)dx = f(\xi) \int_{a}^{b} g(x)dx.$$

Second Integral Mean Value Theorem (Bonnet Formula) Let  $f(x) \in R[a,b]$ ,

• If g(x) is decreasing on [a, b] and  $g(x) \ge 0$  ( $x \in [a, b]$ ):

$$\exists \xi \in [a, b]: \quad \int_a^b f(x)g(x) dx = g(a) \int_a^{\xi} f(x) dx.$$

• If g(x) is increasing on [a,b] and  $g(x) \geqslant 0$   $(x \in [a,b])$ :

$$\exists \eta \in [a, b]: \int_a^b f(x)g(x)dx = g(b) \int_\eta^b f(x)dx.$$

The general form is: Let  $f(x) \in R[a, b]$ , and g(x) be a monotonic function. Then:

$$\exists \xi \in [a, b], \quad \int_a^b f(x)g(x)dx = g(a) \int_a^{\xi} f(x)dx + g(b) \int_{\xi}^b f(x)dx.$$

**Note** For the first integral mean value theorem,

- If  $f(x) \in C[a,b]$  is replaced with  $f(x) \in R[a,b]$ , the conclusion does not hold.
- If  $f(x) \in R[a,b]$  and  $\int f(x) dx$  exists, the conclusion holds.

#### $\P$ Integrability of Composite Functions

**Outer Continuity, Inner Integrability** Let  $f(x) \in R[a,b]$ ,  $A \leq f(x) \leq B$ , and  $g(u) \in C[A,B]$ . Then the composite function  $g(f(x)) \in R[a,b]$ .

**Outer Integrability, Inner Continuity** In this case, the composite function may not be integrable.

**Both Inner and Outer Integrability** In this case, the composite function may not be integrable. In fact, even if both the inner and outer functions are not integrable, the composite function may still be integrable.

### 6.4 Fundamental Theorem of Calculus

 $\P$  Newton-Leibniz Formula

#### Definition 6.4 (Variable Limit Integrals)

Let  $f(x) \in R[a, b]$ . Define:

$$F(x) = \int_{a}^{x} f(t) dt$$
 and  $F(x) = \int_{x}^{b} f(t) dt$ ,

which are referred to as the variable upper limit integral and variable lower limit integral, respectively.

### **.**

#### Property

**Continuity of Antiderivative**  $F(x) \in C[a,b]$  (The variable upper limit integral satisfies the Lipschitz condition and is uniformly continuous on the closed interval).

**Fundamental Theorem of Calculus** Let  $x_0 \in [a, b]$  be a point where f(x) is continuous. Then:

$$F'(x_0) = f(x_0).$$

Existence of Antiderivatives If  $f(x) \in C[a,b]$ , then  $F(x) \in D[a,b]$  and F'(x) = f(x). Rule of Derivation If  $F(x) = \int_{u(x)}^{v(x)} f(t) \, \mathrm{d}t$ , then:

$$F'(x) = f(v(x))v'(x) - f(u(x))u'(x).$$

In fact, the formula is the simplified version of the Leibniz's law.

**Z**Remark Differentiation can reduce the smoothness of functions (the original function may be differentiable, while the derivative may have second-type discontinuities), whereas integration can improve smoothness.

#### Theorem 6.4 (Newton-Leibniz Formula)

Let  $f(x) \in C[a,b]$ , and F(x) be an antiderivative of f(x) on [a,b]. Then:

$$\int_{a}^{b} f(x) dx = F(b) - F(a).$$

Generalized Newton-Leibniz Formula Let  $f(x) \in R[a,b]$ ,  $F(x) \in C[a,b]$ , and F'(x) = f(x) holds except for finitely many points. Then:

$$\int_{a}^{b} f(x) dx = F(b) - F(a).$$



Common Questions concerning Integrals

### 6.5 Calculation of Definite Integrals

#### Lemma 6.1 (Riemann Lemma)

Let  $f(x) \in R[a,b], g(x)$  has a period T and  $g(x) \in [0,T],$  then:

$$\lim_{p \to +\infty} \int_a^b f(x)g(px) dx = \int_a^b f(x) dx \cdot \frac{1}{T} \int_0^T g(t) dt.$$

A special case is when  $g(x) = \sin x$  or  $g(x) = \cos x$ , then:

$$\lim_{p \to +\infty} \int_a^b f(x) \sin(px) dx = \int_a^b f(x) \cos(px) dx.$$



### 6.6 Integral Inequalities

#### Theorem 6.5 (Integral Inequalities,

**Hadamard Inequality** Let f(x) be a convex function on (a, b). Then for any pair  $x_1, x_2 \in (a, b)$  with  $x_1 < x_2$ , we have:

$$f\left(\frac{x_1+x_2}{2}\right) \leqslant \frac{1}{x_2-x_1} \int_{x_1}^{x_2} f(t) dt \leqslant \frac{f(x_1)+f(x_2)}{2}.$$

**Schwarz Inequality** Let  $f(x), g(x) \in R[a, b]$ . Then:

$$\left(\int_a^b f(x)g(x) \, \mathrm{d}x\right)^2 \leqslant \int_a^b f^2(x) \, \mathrm{d}x \int_a^b g^2(x) \, \mathrm{d}x.$$

**Hölder Inequality** Let  $f(x), g(x) \in R[a, b]$ , and p, q are conjugate numbers (i.e.,  $p > 0, q > 0, \frac{1}{p} + \frac{1}{q} = 1$ ). Then:

$$\int_a^b |f(x)g(x)| \, \mathrm{d}x \leqslant \left(\int_a^b |f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, \mathrm{d}x\right)^{\frac{1}{q}}.$$

**Young Inequality** Let  $y = f(x) \in C[0, +\infty)$ , strictly increasing, and f(0) = 0. Denote its inverse function as  $x = f^{-1}(y)$ . Then:

$$\int_0^a f(x) \, \mathrm{d}x + \int_0^b f^{-1}(y) \, \mathrm{d}y \geqslant ab \quad (a > 0, b > 0).$$

Minkowski Inequality Let  $f(x), g(x) \in R[a, b]$ . Then:

$$\left\{ \int_a^b [f(x) + g(x)]^2 \, \mathrm{d}x \right\}^{\frac{1}{2}} \leqslant \left[ \int_a^b f^2(x) \, \mathrm{d}x \right]^{\frac{1}{2}} + \left[ \int_a^b g^2(x) \, \mathrm{d}x \right]^{\frac{1}{2}}.$$

**Chebyshev Inequality** Let f(x), g(x) be similarly ordered functions, i.e.,  $\forall x_1, x_2 : (f(x_1) - f(x_2))(g(x_1) - g(x_2)) \ge 0$ . Then:

$$\int_a^b f(x) dx \int_a^b g(x) dx \le (b-a) \int_a^b f(x)g(x) dx.$$

**Discrete Form** Let sequences  $\{a_n\}, \{b_n\}$  be similarly ordered, i.e.,  $\forall i, j: (a_i - a_j)(b_i - b_j) \geqslant 0$ . Then:

$$\left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} b_i\right) \leqslant n \sum_{i=1}^{n} a_i b_i.$$

If the sequences are oppositely ordered, the inequality reverses.

#### $\mathbb{C}$

### 6.7 Applications of Definite Integrals

#### $\P$ Arc Length

#### Definition 6.5 (Arc Length)

Let  $C = \widehat{AB}$  be a curve on the  $\mathbb{R}^2$  plane<sup>a</sup>, take any partition  $A = P_0, P_1, \dots, P_n = B$ , which divides the curve C into n segments, denoted as T. Then connect every two adjacent points  $P_{i-1}$  and  $P_i$  with a straight line segment, obtaining n chords  $\overline{P_{i-1}P_i}(i=1,2,\dots,n)$ , which in turn form an inscribed polygonal line C. Let

$$||T|| = \max_{1 \le i \le n} ||P_{i-1}P_i||, \quad s_T = \sum_{i=1}^n ||P_{i-1}P_i||.$$

If the limit

$$\lim_{\|T\| \to 0} s_T = s,$$

namely,

$$\forall \varepsilon > 0, \exists \delta > 0, \text{s.t.} \forall T(||T|| < \delta) : |s_T - s| < \varepsilon,$$

and the limit is independent of the choice of partition T, then C is said to be rectifiable, and the limit s is called the arc length of the curve C.

 $^a$ Or in  $\mathbb{R}^3$  space, even in a higher-dimensional Euclidean space.



#### Theorem 6.6 (Sufficient Condition for Rectifiability of Curves,

Let the curve C in  $\mathbb{R}^2$  be given by the parametric equations

$$(x,y) = (x(t), y(t)), \quad t \in [\alpha, \beta],$$

and let it be a  $\mathbb{C}^1$  smooth regular curve<sup>a</sup> Then  $\mathbb{C}$  is rectifiable, and its arc length is

$$s = \int_{\alpha}^{\beta} \sqrt{x'^2(t) + y'^2(t)} \, \mathrm{d}t.$$

<sup>a</sup>I.e., x(t) and y(t) are continuously differentiable, and  $x'^2(t) + y'^2(t) \neq 0$ ; a curve C satisfying this condition is called a regular point. Also see Definition 12.5



#### ¶ Curvature

### Definition 6.6 (Curvature)



#### ¶ Polar Coordinate System

Category	Explicit Cartesian Equation	Parametric Cartesian Equation	Polar Equation
Equation	$y = f(x), x \in [a, b]$	$\begin{cases} x = x(t), t \in [T_1, T_2], \\ y = y(t), \end{cases}$	$r = r(\theta), \theta \in [\alpha, \beta]$
Area of Plane	$\int_a^b f(x)  \mathrm{d}x$	$\int_{T_1}^{T_2}  y(t)x'(t)  \mathrm{d}t$	$\frac{1}{2} \int_{\alpha}^{\beta} r^2(\theta) d\theta$
Shape	_	-1	
Infinitesimal	$\mathrm{d}l = \sqrt{1 + [f'(x)]^2} \mathrm{d}x$	$dl = \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$	$dl = \sqrt{r^2(\theta) + r'^2(\theta)} d\theta$
Arc Length			
Curve Length	$\int_a^b \sqrt{1 + [f'(x)]^2}  \mathrm{d}x$	$\int_{T_1}^{T_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}  \mathrm{d}t$	$\int_{\alpha}^{\beta} \sqrt{r^2(\theta) + r'^2(\theta)}  \mathrm{d}\theta$
Volume of	$\pi \int_a^b [f(x)]^2  \mathrm{d}x$	$\pi \int_{T_1}^{T_2} y^2(t) x'(t) dt$	$\frac{2}{3}\pi \int_{\alpha}^{\beta} r^3(\theta) \sin\theta  d\theta$
Solid of		1	- "
Revolution			
Surface Area	$2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}  dx$	$2\pi \int_{T_1}^{T_2} y(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$	$2\pi \int_{\alpha}^{\beta} r(\theta) \sin \theta \sqrt{r^2(\theta) + r'^2(\theta)} d\theta$
of Solid of		•	
Revolution			

# **Chapter 7** Improper Integral

## 7.1 Infinite and Defective Integrals

### 7.2 Convergence Tests for Improper Integrals

#### Definition 7.1 (Absolute and Conditional Convergence)

Let  $f(x) \in R[a,A] \subset [a,+\infty)$ , and suppose  $\int_a^{+\infty} |f(x)| \,\mathrm{d}x$  converges. Then  $\int_a^{+\infty} f(x) \,\mathrm{d}x$  is said to be absolutely convergent (or f(x) is absolutely integrable on  $[a, +\infty)$ ).

If  $\int_a^{+\infty} f(x) dx$  converges but is not absolutely convergent, then  $\int_a^{+\infty} f(x) dx$  is said to be **conditionally** convergent.

#### Infinite Integrals

#### Theorem 7.1 (Cauchy Convergence Criterion for Infinite Integrals)

The necessary and sufficient condition for the convergence of the infinite integral  $\int_a^{+\infty} f(x) dx$  is:

$$\forall \varepsilon > 0, \exists A_0 > \max\{a, 0\}, \forall A', A'' > A_0 : \left| \int_a^{A'} f(x) \, \mathrm{d}x - \int_a^{A''} f(x) \, \mathrm{d}x \right| = \left| \int_{A'}^{A''} f(x) \, \mathrm{d}x \right| < \varepsilon.$$

From this, we can conclude that if  $\int_a^{+\infty} f(x) dx$  is absolutely convergent, then it must be convergent.

**Comparison Test** Let f(x), g(x) be functions defined on  $[a, +\infty)$ , and suppose  $f(x) \leq Kg(x)$  (where K is a positive constant). Then:

- i) If  $\int_a^{+\infty} g(x) \, \mathrm{d}x$  converges, then  $\int_a^{+\infty} f(x) \, \mathrm{d}x$  also converges. ii) If  $\int_a^{+\infty} f(x) \, \mathrm{d}x$  diverges, then  $\int_a^{+\infty} g(x) \, \mathrm{d}x$  also diverges.

**Limit Form** Let f(x), g(x) > 0 be functions defined on  $[a, +\infty)$ , and suppose:

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = l.$$

Then:

- i) If  $0 < l < +\infty$ , and  $\int_a^{+\infty} g(x) \, \mathrm{d}x$  converges, then  $\int_a^{+\infty} f(x) \, \mathrm{d}x$  also converges. ii) If  $0 < l < +\infty$ , and  $\int_a^{+\infty} g(x) \, \mathrm{d}x$  diverges, then  $\int_a^{+\infty} f(x) \, \mathrm{d}x$  also diverges.
- iii) If  $l = +\infty$ ,  $\int_a^{+\infty} g(x) dx$  and  $\int_a^{+\infty} f(x) dx$  both converge or both diverge.

**Comparison with** p-Integrals Let f(x) > 0 be defined on  $[a, +\infty)$ , and suppose:

$$\frac{f(x)}{r^p} \le \frac{K}{r^p}, \quad \text{where } p > 0.$$

- i) If p > 1, then  $\int_a^{+\infty} f(x) dx$  converges.
- ii) If  $p \le 1$ , then  $\int_a^{+\infty} f(x) dx$  diverges.

**Limit Form** Let f(x) > 0 be defined on  $[a, +\infty)$ , and suppose:

$$\lim_{x \to +\infty} x^p f(x) = l, \quad \text{where } l > 0.$$

Then:

- i) If p>1, then  $\int_a^{+\infty}f(x)\,\mathrm{d}x$  converges. ii) If  $p\leq 1$ , then  $\int_a^{+\infty}f(x)\,\mathrm{d}x$  diverges.

#### $\Diamond$

The infinite integral  $\int_a^{+\infty} f(x)g(x) dx$  converges if either of the following two conditions is satisfied:

**Abel**  $\int_a^{+\infty} f(x) dx$  converges, and g(x) is monotonic and bounded on  $[a, +\infty)$ .

**Dirichlet**  $F(A) = \int_a^A f(x) dx$  is bounded on  $[a, +\infty)$ , g(x) is monotonic on  $[a, +\infty)$ , in the meanwhile  $\lim_{x \to +\infty} g(x) = 0.$ 



#### Defective Integrals

### 7.3 Special Integrals

### Definite Integrals

Dirichlet Integral

$$\int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}} dx = \pi, \quad n \in \mathbb{N},$$

where integrand  $D_n(x)$  is called the Dirichlet kernel.

Fejèr Integral

$$\int_0^{\pi} \left( \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} \right)^2 dx = n\pi, \quad n \in \mathbb{N},$$

#### Improper Integrals

**Euler Integral** 

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, \mathrm{d}x = -\frac{\pi}{2} \ln 2.$$

Froullani Integral

$$\int_{0}^{+\infty} \frac{f(ax) - f(bx)}{x} dx = [f(0) - f(+\infty)] \ln \frac{b}{a}, \quad a, b > 0,$$

where f(x) is continuous on  $(0, +\infty)$ , and both limits f(0) and  $f(+\infty)$  exist.

**Dirichlet Integral** 

$$\int_0^{+\infty} \frac{\sin x}{x} \, \mathrm{d}x = \frac{\pi}{2}.$$

**Euler-Poisson Integral** 

$$\int_0^{+\infty} e^{-x^2} \, \mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

**Poisson Integral** 

$$\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos x + r^2} \, \mathrm{d}x, \quad (0 < r < 1)$$

### Special Integral

$$\int_0^{+\infty} \frac{1}{1 + x^a \sin^b x} \, \mathrm{d}x \quad (a > b, b > 0 \text{and even})$$

# **7.4 Common Questions**

# **Chapter 8 Numerical Series**

### 8.1 Convergence of Numerical Series

### 8.2 Positive Term Series and Its Convergence Tests

#### Definition 8.1 (Positive Term Series)

If all terms of the series  $\sum_{n=1}^{\infty} x_n$  are non-negative real numbers, i.e.,  $x_n \geqslant 0$   $(x_n > 0)$ ,  $n = 1, 2, \ldots$ , then this series is called a **positive term series** (or strictly positive term series).

 $ilde{\mathbb{Y}}$  Note  $\,$  The positive term series converges if and only if the partial sums of the sequence are bounded. If the partial sums are unbounded, the series must diverge to  $+\infty$ .

#### Comparison Test

Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be positive term series. If  $\exists N \in \mathbb{N}, \text{ s.t. } \forall n > N : a_n \leqslant b_n$ , then:

- 1. If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  also converges.
- 2. If  $\sum_{n=1}^{\infty} a_n$  diverges, then  $\sum_{n=1}^{\infty} b_n$  also diverges.

**Limit Form** Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be positive term series, and suppose  $\lim_{n\to\infty} \frac{a_n}{b_n}$  exists. Then:

- 1. If  $0 < l < +\infty$ ,  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  have the same convergence or divergence behavior.
- 2. If  $l=0, \sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  also converges.
- 3. If  $l = +\infty$ ,  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  also diverges.

**Cauchy Test** Let  $\sum_{n=1}^{\infty} a_n$  be a positive term series.

- 1. If  $\exists q \in [0,1)$ , s.t.  $\sqrt[n]{a_n} \leqslant q < 1 \quad (n \geqslant N, N \in \mathbb{N})$ , then the series converges.
- 2. If  $\sqrt[n]{a_n} \geqslant 1$  for infinitely many n, then the series diverges.

**Limit Form** Let  $\sum_{n=1}^{\infty} a_n$  be a positive term series, and suppose  $\overline{\lim}_{n\to+\infty} \sqrt[n]{a_n} = r$ . Then:

- 1. If  $0 \le r < 1$ , the series  $\sum_{n=1}^{\infty} a_n$  converges.
- 2. If r > 1, the series  $\sum_{n=1}^{\infty} a_n$  diverges.
- 3. If r = 1, the test fails.

**D'Alembert Test** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series.

- 1. If  $\exists q \in [0,1), \text{ s.t. } \frac{a_{n+1}}{a_n} \leqslant q < 1 \quad (n \geqslant N, N \in \mathbb{N}), \text{ then the series converges.}$
- 2. If  $\frac{a_{n+1}}{a_n} \geqslant 1$   $(n \geqslant N, N \in \mathbb{N})$ , then the series diverges.

Limit Form Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series. Then:

- 1. If  $\overline{\lim}_{n\to+\infty}\frac{a_{n+1}}{a_n}=r\in(0,1)$ , the series converges. 2. If  $\underline{\lim}_{n\to+\infty}\frac{a_{n+1}}{a_n}=r'>1$ , the series diverges.
- 3. If r = 1 or r' = 1, the test fails.

**Raabe Test** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series.

- 1. If  $\exists r > 1, \exists N_0 \in \mathbb{N}$  s.t.  $\forall n > N_0 : n\left(\frac{a_n}{a_{n+1}} 1\right) \geqslant r$ , then the series converges.
- 2. If  $\exists N_0 \in \mathbb{N}$ , s.t.  $\forall n > N_0 : n\left(\frac{a_n}{a_{n+1}} 1\right) \leqslant 1$ , then the series diverges.

- Limit Form Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series. Then: 1. If  $\underline{\lim}_{n \to +\infty} n\left(\frac{a_n}{a_{n+1}} 1\right) = l > 1$ , the series converges. 2. If  $\overline{\lim}_{n \to +\infty} n\left(\frac{a_n}{a_{n+1}} 1\right) = l' < 1$ , the series diverges.

  - 3. If l = 1 or l' = 1, the test fails.

**Bertrand Test** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series.

- 1. If  $\underline{\lim}_{n \to +\infty} \ln n \left[ n \left( \frac{a_n}{a_{n+1}} 1 \right) \right] = l > 1$ , the series converges. 2. If  $\overline{\lim}_{n \to +\infty} \ln n \left[ n \left( \frac{a_n}{a_{n+1}} 1 \right) \right] = l' < 1$ , the series diverges.

**Gauss Test** Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series, and suppose:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\delta}{n \ln n} + o\left(\frac{1}{n \ln n}\right), \quad (n \to +\infty).$$

Then:

- 1. If  $\delta > 1$ , the series converges.
- 2. If  $\delta < 1$ , the series diverges.
- 3. If  $\delta = 1$ , the criterion fails.

Generalized Form Let  $\sum_{n=1}^{\infty} a_n$  be a strictly positive term series, and suppose:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\delta_n}{n \ln n} + o\left(\frac{1}{n \ln n}\right), \quad (n \to +\infty).$$

If  $\lim_{n\to\infty} \delta_n = \delta \in \mathbb{R}$ , then:

- 1. If  $\delta > 1$ , the series converges.
- 2. If  $\delta$  < 1, the series diverges.
- 3. If  $\delta = 1$ , the criterion fails.

**Note** The Bertrand test can be refined by considering series such as:

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p}, \quad \sum_{n=9}^{\infty} \frac{1}{n \ln n \ln \ln n (\ln \ln n)^p}, \dots$$

These refinements are collectively known as the Bertrand test.

Remark All the aforementioned criteria are derived from the Comparison Criterion.

- By comparing positive term series with the geometric series (or equal ratio series), the Cauchy Criterion and d'Alembert Criterion are derived.
- By comparing positive term series with the slower-converging series  $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$  ( $\alpha>1$ ), the Raabe Criterion is derived.
- By comparing positive term series with the even slower-converging series  $\sum_{n=1}^{\infty} \frac{1}{n \ln^{\alpha} n}$  ( $\alpha > 1$ ), the Gauss Criterion is derived.

General Observation The slower the convergence of the series used for comparison, the more precise the derived criterion.

Integral Test

#### Theorem 8.3 (Cauchy Integral Test)

Let f(x) be defined on  $[a, +\infty)$ , where  $f(x) \ge 0$ , and f(x) is Riemann integrable on any finite interval [a, A]. Consider a monotonic increasing sequence  $\{a_n\}$  such that  $a = a_1 < a_2 < \cdots < a_n < \ldots$ , and let:

$$u_n = \int_{a_n}^{a_{n+1}} f(x) \, \mathrm{d}x.$$

Then the improper integral  $\int_a^{+\infty} f(x) dx$  and the positive term series  $\sum_{n=1}^{\infty} u_n$  converge or diverge to  $+\infty$  simultaneously. Moreover:

$$\int_{a}^{+\infty} f(x) \, dx = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} f(x) \, dx.$$

#### ¶ Other Tests

#### Theorem 8.4 (Cauchy Condensation Test,

Let  $\{a_n\}$  be a monotonically decreasing sequence of positive numbers. Then the positive term series  $\sum_{n=1}^{\infty} a_n$  converges if and only if the condensed series:

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + \dots + 2^n a_{2^n} + \dots$$

converges.

# 8.3 General Term Series and Its Convergence Tests

#### $\P$ Cauchy Convergence Criterion for Series

#### Theorem 8.5 (Cauchy Convergence Criterion for Series)

The necessary and sufficient condition for the convergence of the series  $\sum_{n=1}^{\infty} x_n$  is:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N : |x_{n+1} + x_{n+2} + \dots + x_m| = \left| \sum_{k=n+1}^m x_k \right| < \varepsilon.$$

#### ¶ Alternative Series

#### Definition 8.2 (Alternative Series)

A series of the form:

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (-1)^{n-1} u_n \quad (u_n > 0),$$

is called an alternative series.

Moreover, if  $u_n$  is a monotonically decreasing sequence and  $\lim_{n\to\infty}u_n=0$ , then the series is called a **Leibniz** series.

#### Theorem 8.6 (Leibniz Test

Leibniz series converges.

### $\Diamond$

#### ¶ Abel-Dirichlet Test

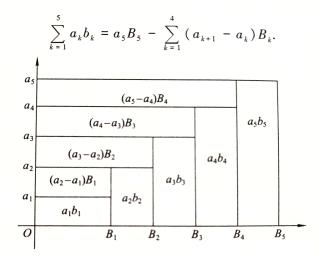
#### Theorem 8.7 (Abel Transform (Discrete Integration by Parts/Summation by Parts))

Let  $\{a_n\}, \{b_n\}$  be two sequences, then for any  $n \in \mathbb{N}^+$ ,

$$\sum_{k=1}^{n} a_k b_k = a_n B_n + \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k,$$

where  $B_n = \sum_{k=1}^n b_k$ .





#### Lemma 8.1 (Abel Lemma (Discrete Second Integral Mean Value Theorem))

Let  $\{a_n\}$ ,  $\{b_n\}$  be two sequences, if  $\{a_n\}$  is a monotonic sequence and  $\{B_k\} = \sum_{k=1}^n b_k$  is a bounded sequence with bound M, then for any  $p \in \mathbb{N}^+$ ,

$$\left| \sum_{k=1}^{p} a_k b_k \right| \le M \left( |a_1| + 2|a_p| \right).$$



#### Theorem 8.8 (Abel-Dirichlet Test

The series  $\sum_{n=1}^{\infty} a_n b_n$  converges if one of the following two conditions is satisfied:

**Abel**  $\{a_n\}$  is a bounded monotonic sequence and  $\sum_{n=1}^{\infty} b_n$  converges.

**Dirichlet**  $\{a_n\}$  is a monotonic sequence,  $\lim_{n\to\infty}a_n=0$ , and the partial sums  $B_n=\sum_{k=1}^nb_k$  are bounded.

### 8.4 Absolute and Conditional Convergence of Series

#### Definition 8.3 (Absolute and Conditional Convergence of Series)

If the series  $\sum_{n=1}^{\infty} |x_n|$  converges, then the series  $\sum_{n=1}^{\infty} x_n$  is said to be absolutely convergent.

If the series  $\sum_{n=1}^{\infty} x_n$  converges but is not absolutely convergent, then the series  $\sum_{n=1}^{\infty} x_n$  is said to be **conditionally convergent**.

### 2

### 8.5 Comparison of Convergence Speed of Series

The series  $\sum_{n=1}^{\infty} a_n$  is said to converge faster than the series  $\sum_{n=1}^{\infty} b_n$  if:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$

#### Theorem 8.9 (Du Bois-Reymond Theorem,

For a given convergent positive term series  $\sum_{n=1}^{\infty} a_n$ , there always exists a convergent strictly positive term series  $\sum_{n=1}^{\infty} b_n$  such that:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$



#### Theorem 8.10 (Abel Theorem

For a given divergent positive term series  $\sum_{n=1}^{\infty} a_n$ , there always exists a divergent positive term series  $\sum_{n=1}^{\infty} b_n$  such that:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=0.$$



**Remark** The above two theorems imply that the slowest converging positive term series does not exist.

### **8.6 Infinite Products**

## 8.7 Special Series

## **Chapter 9 Series of Functions**

### 9.1 Pointwise and Uniform Convergence

#### ¶ Pointwise Convergence

#### Definition 9.1 (Function Term Series)

Let  $u_n(x)$   $(n=1,2,3,\ldots)$  be a sequence of functions with a common domain E. The sum of these infinitely many functions  $u_1(x) + u_2(x) + \cdots + u_n(x) + \ldots$  is called a **function term series**, denoted as:

$$\sum_{n=1}^{\infty} u_n(x).$$

For any fixed point  $x_0 \in E$ , if the numerical series  $\sum_{n=1}^{\infty} u_n(x_0)$  converges, then the function term series  $\sum_{n=1}^{\infty} u_n(x)$  is said to converge at  $x_0$ , or equivalently,  $x_0$  is called a **convergence point** of  $\sum_{n=1}^{\infty} u_n(x)$ . The set of all convergence points is called the **domain of convergence** of  $\sum_{n=1}^{\infty} u_n(x)$ .

#### Definition 9.2 (Pointwise Convergence)

Let the domain of convergence of the function term series  $\sum_{n=1}^{\infty} u_n(x)$  be  $D \subset E$ . Then  $\sum_{n=1}^{\infty} u_n(x)$  defines a function S(x) on the set D, where:

$$S(x) = \sum_{n=1}^{\infty} u_n(x), \quad x \in D.$$

The function S(x) is called the **sum function** of the series, and  $\sum_{n=1}^{\infty} u_n(x)$  is said to **converge pointwise** to S(x) on D.

Define the partial sum function of the series as:

$$S_n(x) = \sum_{k=1}^n u_k(x).$$

It is evident that the set of all x for which  $\{S_n(x)\}$  converges is precisely D. Therefore, on D, we have:

$$S(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \sum_{k=1}^n u_k(x).$$

Conversely, if a sequence of functions  $\{S_n(x)\}\ (x \in E)$  is given, we can define:

$$\begin{cases} u_1(x) = S_1(x), \\ u_{n+1}(x) = S_{n+1}(x) - S_n(x), & n = 1, 2, \dots \end{cases}$$

to obtain the corresponding function term series.

Thus, the convergence behavior of a function term series and the corresponding sequence of partial sum functions is essentially the same.

However, it is important to note that the pointwise convergence has certain limitations.

**Continuity** The sum of finitely many continuous functions satisfies additive continuity:

$$\lim_{x \to x_0} [u_1(x) + u_2(x) + \dots + u_n(x)] = \lim_{x \to x_0} u_1(x) + \lim_{x \to x_0} u_2(x) + \dots + \lim_{x \to x_0} u_n(x).$$

If this property can be extended to infinitely many functions, that is: If  $u_n(x)$  is continuous on D, the sum function  $S(x) = \sum_{n=1}^{\infty} u_n(x)$  is also continuous on D. Moreover:

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} u_n(x),$$

meaning that the limit operation and infinite summation can be interchanged (also known as the fact that function term series can be evaluated termwise).

For the sequence of partial sums  $\{S_n(x)\}$ , the corresponding conclusion is that the limit function  $S(x) = \lim_{n\to\infty} S_n(x)$  is also continuous on D, and:

$$\lim_{x \to x_0} \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} S_n(x),$$

meaning that the two limit operations can be interchanged.

Unfortunately, in the case of pointwise convergence, this property does not hold.

**Derivability** The sum of finitely many differentiable functions satisfies additive differentiability:

$$\frac{\mathrm{d}}{\mathrm{d}x}[u_1(x) + u_2(x) + \dots + u_n(x)] = \frac{\mathrm{d}}{\mathrm{d}x}u_1(x) + \frac{\mathrm{d}}{\mathrm{d}x}u_2(x) + \dots + \frac{\mathrm{d}}{\mathrm{d}x}u_n(x).$$

If this property can be extended to infinitely many functions, that is: If  $u_n(x)$  is differentiable on D, the sum function  $S(x) = \sum_{n=1}^{\infty} u_n(x)$  is also differentiable on D. Moreover:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} u_n(x),$$

meaning that the differentiation operation and infinite summation can be interchanged (also known as the fact that function term series can be differentiated termwise).

For the sequence of partial sums  $\{S_n(x)\}$ , the corresponding conclusion is that the limit function  $S(x) = \lim_{n \to \infty} S_n(x)$  is also differentiable on D, and:

$$\frac{\mathrm{d}}{\mathrm{d}x} \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{\mathrm{d}}{\mathrm{d}x} S_n(x),$$

meaning that the two operations can be interchanged.

Unfortunately, in the case of pointwise convergence, this property does not hold.

**Integrability** The sum of finitely many integrable functions satisfies additive integrability:

$$\int_{a}^{b} [u_1(x) + u_2(x) + \dots + u_n(x)] dx = \int_{a}^{b} u_1(x) dx + \int_{a}^{b} u_2(x) dx + \dots + \int_{a}^{b} u_n(x) dx.$$

If this property can be extended to infinitely many functions, that is: If  $u_n(x)$  is integrable on  $[a,b] \subset D$ ,

the sum function  $S(x) = \sum_{n=1}^{\infty} u_n(x)$  is also integrable on  $[a,b] \subset D$ . Moreover:

$$\int_a^b \sum_{n=1}^\infty u_n(x) \, \mathrm{d}x = \sum_{n=1}^\infty \int_a^b u_n(x) \, \mathrm{d}x,$$

meaning that the integration operation and infinite summation can be interchanged (also known as the fact that function term series can be integrated termwise).

For the sequence of partial sums  $\{S_n(x)\}$ , the corresponding conclusion is that the limit function  $S(x) = \lim_{n \to \infty} S_n(x)$  is also integrable on  $[a,b] \subset D$ , and:

$$\int_{a}^{b} \lim_{n \to \infty} S_n(x) dx = \lim_{n \to \infty} \int_{a}^{b} S_n(x) dx,$$

meaning that the two operations can be interchanged.

Unfortunately, in the case of pointwise convergence, this property does not hold.

#### $\P$ Uniform Convergence

#### Definition 9.3 (Uniform Convergence)

Let  $\{S_n(x)\}(x \in D)$  be a sequence of functions. If:

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}^+, \forall n > N(\varepsilon) : |S_n(x) - S(x)| < \varepsilon \quad (\forall x \in D),$$

then  $\{S_n\}$  is said to **converge uniformly** to S(x) on D, denoted as:

$$S_n(x) \stackrel{D}{\rightrightarrows} S(x).$$

If the partial sum sequence  $\{S_n(x)\}$  of the function term series  $\sum_{n=1}^{\infty} u_n(x)(x \in D)$  converges uniformly to S(x) on D, then  $\sum_{n=1}^{\infty} u_n(x)$  is said to converge uniformly to S(x) on D.

Obviously, if the partial sum sequence  $\{S_n(x)\}$  of  $\sum_{n=1}^{\infty} u_n(x)$  satisfies:

$$S_n(x) \stackrel{D}{\Longrightarrow} S(x),$$

then:

$$u_n(x) \stackrel{D}{\Longrightarrow} 0.$$

#### Theorem 9.1 (Cauchy Criterion for Uniform Convergence)

The necessary and sufficient condition for the sequence of functions  $\{S_n(x)\}$  to converge uniformly on D is:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall m > n > N : |S_m(x) - S_n(x)| < \varepsilon \quad (\forall x \in D).$$

Correspondingly, the necessary and sufficient condition for the function term series  $\sum_{n=1}^{\infty} u_n(x)$  to converge uniformly on D is:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall m > n > N : \left| \sum_{i=n+1}^m u_i(x) \right| < \varepsilon \quad (\forall x \in D).$$

Let  $\{S_n(x)\}$  converge pointwise to S(x) on D. The necessary and sufficient conditions for  $S_n(x) \stackrel{D}{\rightrightarrows} S(x)$  are:

$$\lim_{n \to \infty} d(S_n, S) = \lim_{n \to \infty} \sup_{x \in D} |S_n(x) - S(x)| = 0.$$

2. For any sequence  $\{x_n\}$  where  $x_n \in D$ , the following holds:

$$\lim_{n \to \infty} \left( S_n(x_n) - S(x_n) \right) = 0.$$

With the concept of uniform convergence, the flaws of pointwise convergence can be remedied, and the following properties can be established:

#### Property

**Continuity** Let  $f_n(x) \stackrel{I \subset \mathbb{R}}{\Rightarrow} f(x)$ . If  $f_n(x)$  is continuous at  $x_0 \in I$  for  $n = 1, 2, 3, \ldots$ , then f(x) is also continuous at

In particular, if  $f_n(x) \in C(I)$ , then  $f(x) \in C(I)$ .

Termwise Limit If  $\sum_{n=1}^{\infty} u_n(x) \stackrel{I \subset \mathbb{R}}{\Rightarrow} S(x)$  and  $u_n(x) \in C(I)$ , then the sum function  $S(x) \in C(I)$ .

Integrability Let  $f_n(x) \stackrel{[a,b]}{\Rightarrow} f(x)$ . If  $f_n(x) \in R[a,b]$ , then  $f(x) \in R[a,b]$ , and:

$$\lim_{n \to \infty} \int_a^b f_n(x) \, \mathrm{d}x = \int_a^b \lim_{n \to \infty} f_n(x) \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x.$$

**Termwise Integration:** If  $\sum_{n=1}^{\infty} u_n(x) \stackrel{[a,b]}{\rightrightarrows} S(x)$  and  $u_n(x) \in R[a,b]$ , then  $S(x) \in R[a,b]$ . Differentiability Let  $f'_n(x) \stackrel{[a,b]}{\rightrightarrows} \sigma(x)$ . If there exists  $x_0 \in [a,b]$  such that:

$$\lim_{n \to \infty} f_n(x_0) = a,$$

then there exists a function f(x) such that  $f_n(x) \stackrel{[a,b]}{\rightrightarrows} f(x)$  and  $f'(x) = \sigma(x)$ .

**Termwise Differentiation** If  $\sum_{n=1}^{\infty}u_n'(x)\stackrel{[a,b]}{\rightrightarrows}\sigma(x)$  and there exists  $x_0\in[a,b]$  such that:

$$\sum_{n=1}^{\infty} u_n(x_0) \to a,$$

then there exists a function S(x) such that  $\sum_{n=1}^{\infty}u_n(x)\stackrel{[a,b]}{\rightrightarrows}S(x)$  and  $S'(x)=\sigma(x)$ .

**Corollary** Obviously, if we add the condition  $f'_n(x) \in C[a,b]$ , the conclusion still holds, and the proof becomes

Note Since continuity and differentiability are both local properties, it suffices to have internally closed uniform conver**gence** of (a,b) to ensure that f(x) is continuous/differentiable.

#### Quasi-Uniform Convergence

### Definition 9.4 (Quasi-Uniform Convergence)

The sequence of functions  $\{S_n(x)\}$  is said to **converge quasi-uniformly** on the interval [a,b] if it converges pointwise to S(x) on [a,b], and the following condition is satisfied:

$$\forall \varepsilon>0, \forall N\in\mathbb{N}^*, \exists N_0>N, \text{ s.t. } \forall x\in[a,b], \exists n_x\in[N,N_0] \ (n_x\in\mathbb{N}^*): |S_{n_x}(x)-S(x)|<\varepsilon.$$



## 9.2 Uniform Convergence Tests

- ¶ Weierstrass Test (M-Test)
- $\P$  Abel-Dirichlet Test
- ¶ Dini Theorem

## 9.3 Special Cases

# **Chapter 10 Power Series**

- 10.1 Power Series and Its Convergence Radius
- **10.2 Expanding Functions into Power Series**
- **10.3 Smooth Appropriation of Functions**

# **Chapter 11 Limits and Continuity in Euclidean Spaces**

### 11.1 Continuous Mappings

- Continuous Mappings on Compact Sets
- Continuous Mappings on Connected Sets

#### Definition 11.1 (Connected Set)

Let S be a set of points in  $\mathbb{R}^n$ . If a continuous mapping

$$\gamma:[0,1]\to\mathbb{R}^n$$

satisfies that the range of  $\gamma([0,1])$  lies entirely within S, we call  $\gamma$  a path in S, where  $\gamma(0)$  and  $\gamma(1)$  are referred to as the starting point and ending point of the path, respectively.

If for any two points  $\mathbf{x}, \mathbf{y} \in S$ , there exists a path in S with  $\mathbf{x}$  as the starting point and  $\mathbf{y}$  as the ending point, Sis called path-connected, or equivalently, S is called a connected set.

A connected open set is called an (open) region. The closure of an (open) region is referred to as a closed region.

**Remark** Intuitively, this means that any two points in S can be connected by a curve lying entirely within S. Clearly, a connected subset of  $\mathbb R$  is an interval, and a connected subset of  $\mathbb R$  is compact if and only if it is a closed interval.

# **Chapter 12 Multi-variable Differential Calculus**

### 12.1 Directional Derivatives and Total Differential

#### ¶ Directional Derivative

#### Definition 12.1 (Directional Derivative)

Let  $U \subset \mathbb{R}^n$  be an open set,  $f: U \to \mathbb{R}^1$ , **e** is a unit vector in  $\mathbb{R}^n$ ,  $\mathbf{x}^0 \in U$ . Define

$$u(t) = f(\mathbf{x}^0 + t\mathbf{e}).$$

If the derivative of u at t = 0

$$u'(0) = \lim_{t \to 0} \frac{u(t) - u(0)}{t} = \lim_{t \to 0} \frac{f(\mathbf{x}^0 + t\mathbf{e}) - f(\mathbf{x}^0)}{t}$$

exists and is finite, it is called the **directional derivative** of f at  $\mathbf{x}^0$  in the direction  $\mathbf{e}$ , denoted by  $\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0)$ . It is the rate of change of f at  $\mathbf{x}^0$  in the direction  $\mathbf{e}$ .

Consider the following set of unit coordinate vectors:  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ . Let  $\mathbf{e}_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$  denote the standard orthonormal basis in  $\mathbb{R}^n$ , where the 1 appears in the *i*-th position. That is,

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For a function f, the directional derivative of f at the point  $\mathbf{x}_0$  in the direction of  $\mathbf{e}_i$  is called the ith first-order **partial derivative** of f at  $\mathbf{x}^0$ , denoted by

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^0)$$
 or  $D_i f(\mathbf{x}^0)$  or  $f_{x_i}(\mathbf{x}^0)$   $(i = 1, 2, \dots, n)$ .

 $\mathrm{D}_i=rac{\partial}{\partial x_i}$  is called the ith partial differential operator ( $i=1,2,\cdots,n$ ).

Let  $\mathbf{e}_i = \sum_{i=0}^n \mathbf{e}_i \cos \alpha$  be a unit vector, where  $\sum_{i=0}^n \cos^2 \alpha = 1$ . If  $\frac{\partial f}{\partial x_i}$  is continuous at  $\mathbf{x}^0$ , then the directional derivative of f at  $\mathbf{x}^0$  along the direction  $\mathbf{e}$  is given by:

$$\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}^0) \cos \alpha_i.$$

This is the formula for expressing a directional derivative using partial derivatives.

 $ilde{\mathbb{Y}}$  Note Let  ${f e}$  be a direction, then  $\|-{f e}\|=\|{f e}\|=1$ , which implies that  $-{f e}$  is also a direction. At this point, we have:

$$\frac{\partial f}{\partial (-\mathbf{e})}(\mathbf{x}^0) = -\frac{\partial f}{\partial \mathbf{e}}(\mathbf{x}^0).$$

#### Definition 12.2 (Jacobian Matrix (Gradient))

Let

$$Jf(\mathbf{x}) = (D_1 f(\mathbf{x}), D_2 f(\mathbf{x}), \dots, D_n f(\mathbf{x})),$$

which is called the **Jacobian matrix** of the function f at the point  $\mathbf{x}$ , (a  $1 \times n$  matrix) whose counterpart is the first-order derivative of a single-variable function.

Henceforth, we represent the point  $\mathbf{x}$  in  $\mathbb{R}^n$  and its increments  $\mathbf{h}$  as column vectors. In this way, the differential of the function can be expressed using matrix multiplication as follows:

$$df(\mathbf{x}^0)(\mathbf{\Delta}\mathbf{x}) = Jf(\mathbf{x}^0)\mathbf{\Delta}\mathbf{x}.$$

The Jacobian matrix of the function f is also frequently denoted as grad f (or  $\nabla f$ ), that is,

$$\operatorname{grad} f(\mathbf{x}) = Jf(\mathbf{x}),$$

which is called the **gradient** of the scalar function f.

#### $\P$ Total Differential

#### Definition 12.3 (Total Differential)

Let  $U\subset\mathbb{R}^n$  be an open set,  $f:U\to\mathbb{R}^1$ ,  $\mathbf{x}^0\in U$ ,  $\Delta\mathbf{x}=(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)\in\mathbb{R}^n$ . If

$$f(\mathbf{x}^0 + \Delta \mathbf{x}) - f(\mathbf{x}^0) = \sum_{i=1}^n A_i \Delta x_i + o(\|\Delta \mathbf{x}\|) \qquad (\|\Delta \mathbf{x}\| \to 0),$$

where  $A_1, A_2, \ldots, A_n$  are constants independent of  $\Delta \mathbf{x}$ , then the function f is said to be **differentiable** at the point  $\mathbf{x}^0$ , and the linear main part  $\sum_{i=1}^n A_i \Delta x_i$  is called the **total differential** of f at  $\mathbf{x}^0$ , denoted as

$$df(\mathbf{x}^0)(\mathbf{\Delta}\mathbf{x}) = \sum_{i=1}^n A_i \Delta x_i.$$

If f is differentiable at every point in the open set U, then f is called a differentiable function on U.

#### Theorem 12.1 (Conditions of Differentiability)

**Necessary Condition** If an n-variable function f is differentiable at the point  $\mathbf{x}_0$ , then f is continuous at  $\mathbf{x}^0$  and possesses first-order partial derivatives  $\frac{\partial f}{\partial x_i}(\mathbf{x}^0)$  at  $\mathbf{x}^0$  for  $i=1,2,\ldots,n$ , and

$$\mathbf{A} = (A_1, A_2, \dots, A_n) = Jf(\mathbf{x}^0) = (D_1 f(\mathbf{x}^0), D_2 f(\mathbf{x}^0), \dots, D_n f(\mathbf{x}^0)).$$

<sup>a</sup> However, the converse is not true.

**Sufficient Condition** Let  $U \subset \mathbb{R}^n$  be an open set, and let  $f: U \to \mathbb{R}^1$  be an n-variable function. If  $Jf = (D_1 f, D_2 f, \dots, D_n f)$  is continuous at  $\mathbf{x}^0$  (i.e.,  $\frac{\partial f}{\partial x_i}$  is continuous at  $\mathbf{x}^0$  for  $i = 1, 2, \dots, n$ ), then f is differentiable at  $\mathbf{x}^0$ . However, the converse is not necessarily true.

<sup>a</sup>It is referred to as the total differential formula, and the more common form is

$$df(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) dx + \frac{\partial f}{\partial y}(x_0, y_0) dy.$$

 $\bigcirc$ 



- The continuity of the derivative function at  $\mathbf{x}^0$  implies that the original function f is differentiable in some neighborhood of  $\mathbf{x}^0$ .
- In fact, this condition can be relaxed to require that one partial derivative exists at the point, while the remaining n-1 partial derivative functions are continuous at that point.
- Proof Taking a function of three variables as an example.

Assume the 3-ary function  $f: \mathbb{R}^3 \to \mathbb{R}$  meets:

- 1. There exists  $f_z(x_0, y_0, z_0)$ .
- 2. The partial derivative functions  $f_x(x, y, z)$  and  $f_y(x, y, z)$  are continuous at  $(x_0, y_0, z_0)$ , i.e. there are partial derivatives in some neighborhood of  $(x_0, y_0, z_0)$ .

Consider the total increment of f at the point  $(x_0, y_0, z_0)$ :

$$\Delta f = \underbrace{\left[ f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0 + \Delta y, z_0 + \Delta z) \right]}_{I_1} + \underbrace{\left[ f(x_0, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0 + \Delta z) \right]}_{I_2} + \underbrace{\left[ f(x_0, y_0, z_0 + \Delta z) - f(x_0, y_0, z_0) \right]}_{I_3}.$$

For  $I_1, I_2$ , by the Lagrange's Mean Value Theorem of unary functions, there exist  $\theta_1, \theta_2 \in (0, 1)$  such that

$$I_{1} = f_{x}(x_{0} + \theta_{1}\Delta x, y_{0} + \Delta y, z_{0} + \Delta z)\Delta x,$$
  

$$I_{2} = f_{y}(x_{0}, y_{0} + \theta_{2}\Delta y, z_{0} + \Delta z)\Delta y.$$

Then by the continuity of the their partial derivatives at  $(x_0, y_0, z_0)$ , we have

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} I_1 = f_x(x_0, y_0, z_0) \Delta x, \quad \lim_{\Delta x, \Delta y, \Delta z \to 0} I_2 = f_y(x_0, y_0, z_0) \Delta y.$$

They can be expressed in terms of infinitesimals (  $\rho=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$  ):

$$I_1 = f_x(x_0, y_0, z_0) \Delta x + \alpha_1 \Delta x, \quad \alpha_1 \to 0 (\rho \to 0),$$
  
 $I_2 = f_y(x_0, y_0, z_0) \Delta y + \alpha_2 \Delta y, \quad \alpha_2 \to 0 (\rho \to 0).$ 

For  $I_3$ , by the definition of the partial derivative  $f_z(x,y,z)$  at  $(x_0,y_0,z_0)$ , we have

$$I_3 = f_z(x_0, y_0, z_0)\Delta z + \alpha_3\Delta z, \quad \alpha_3 \to 0 (\rho \to 0).$$

Accordingly,

$$\Delta f = I_1 + I_2 + I_3$$

$$= [f_x(x_0, y_0, z_0)\Delta x + \alpha_1 \Delta x] + [f_y(x_0, y_0, z_0)\Delta y + \alpha_2 \Delta y] + [f_z(x_0, y_0, z_0)\Delta z + \alpha_3 \Delta z]$$

$$= f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z + [\alpha_1 \Delta x + \alpha_2 \Delta y + \alpha_3 \Delta z].$$

Apparently,

$$\lim_{\rho \to 0} \frac{\alpha_1 \Delta x + \alpha_2 \Delta y + \alpha_3 \Delta z}{\rho} = 0,$$

i.e.  $\alpha_1 \Delta x + \alpha_2 \Delta y + \alpha_3 \Delta z = o(\rho)$ . Therefore, f(x,y,z) is differentiable at  $(x_0,y_0,z_0)$ , which completes the proof.

Ŷ Note (At some point)

- 1. Differentiable
  - $\Longrightarrow$  Continuous
  - $\Longrightarrow$  Partial derivatives exist:  $D_{\vec{u}} = \nabla f \cdot \vec{u}$
- 2. Directional Derivative
  - All directional derivatives exist  $\iff$  differentiable or continuous.
- 3. Partial Derivative
  - The continuity and existence of directional/partial derivatives are mutually exclusive.

#### $\P$ Higher-Order Partial Derivatives and Differential

If the first-order partial derivative of f,  $\frac{\partial f}{\partial x_i}$ , itself possesses partial derivatives, then the second-order partial derivative of f is defined, and is denoted as follows(the first is also called the mixed partial derivative):

$$f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial x_i} \right), \quad f_{x_i x_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_i} \right), \quad i, j = 1, 2, \dots, n.$$

Similarly, higher-order partial derivatives of order  $3, 4, \dots m, \dots$  can be defined.

The following theorem provides the conditions under which mixed partial derivatives are equal.

#### Theorem 12.2 (Conditions for Fauality of Mixed Partial Derivatives)

1. Let  $U \subset \mathbb{R}^2$  be an open set, and  $f: U \to \mathbb{R}$  be a function of two variables. If  $f_{xy}$  and  $f_{yx}$  are continuous at  $(x_0, y_0) \in U$ , then

$$f_{xy}(x_0, y_0) = f_{yy}(x_0, y_0).$$

2. Let  $U \subset \mathbb{R}^n$  be an open set, and  $f: U \to \mathbb{R}$  be a function of n variables. If f has partial derivatives up to order k in D, and all of them are continuous at  $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in U$ , then

$$\frac{\partial^l f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}}(\mathbf{x}^0) = \frac{\partial^l f}{\partial x_{i_2} \partial x_{i_1} \cdots \partial x_{i_l}}(\mathbf{x}^0) = \cdots = \frac{\partial^l f}{\partial x_{i_l} \partial x_{i_{l-1}} \cdots \partial x_{i_1}}(\mathbf{x}^0),$$

that is, the order of taking partial derivatives  $l \leq k$  does not affect the result.<sup>a</sup>

<sup>&</sup>quot;If the condition " $f_{xy}$  and  $f_{yx}$  are continuous at  $(x_0, y_0)$ ", is not satisfied, then the conclusion " $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$ " does not necessarily hold.

**S Proof** When  $k \neq 0, h \neq 0$ , define

$$\varphi(y) = f(x_0 + h, y) - f(x_0, y),$$

and

$$\psi(x) = f(x, y_0 + k) - f(x, y_0).$$

Applying the Lagrange Mean Value Theorem, we have

$$\begin{aligned} &[f(x_0+h,y_0+k)-f(x_0,y_0+k)]-[f(x_0+h,y_0)-f(x_0,y_0)]\\ =&\varphi(y_0+k)-\varphi(y_0)\\ =&\varphi'(y_0+\theta_1k)k\\ =&[f_y(x_0+h,y_0+\theta_1k)-f_y(x_0,y_0+\theta_1k)]k\\ =&f_{yx}(x_0+\theta_2h,y_0+\theta_1k)hk,\quad 0<\theta_1,\theta_2<1. \end{aligned}$$

On the other hand,

$$\begin{split} &[f(x_0+h,y_0+k)-f(x_0,y_0+k)]-[f(x_0+h,y_0)-f(x_0,y_0)]\\ =&[f(x_0+h,y_0+k)-f(x_0+h,y_0)]-[f(x_0,y_0+k)-f(x_0,y_0)]\\ =&\psi(x_0+h)-\psi(x_0)\\ =&\psi'(x_0+\theta_3h)h\\ =&[f_x(x_0+\theta_3h,y_0+k)-f_x(x_0+\theta_3h,y_0)]h\\ =&f_{xy}(x_0+\theta_3h,y_0+\theta_4k)hk,\quad 0<\theta_3,\theta_4<1. \end{split}$$

Therefore,

$$f_{xy}(x_0 + \theta_3 h, y_0 + \theta_4 k) = f_{yx}(x_0 + \theta_2 h, y_0 + \theta_1 k).$$

Since  $f_{xy}$  and  $f_{yx}$  are continuous at  $(x_0, y_0)$ , letting  $h \to 0, k \to 0$ , we obtain

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

By applying 12.1 and the principle of mathematical induction, one can immediately derive the following result.

Suppose z=f(x,y) has continuous partial derivatives in the domain  $U\subset\mathbb{R}^2$ . Then z is differentiable, and

$$\mathrm{d}z = \frac{\partial z}{\partial x} \mathrm{d}x + \frac{\partial z}{\partial y} \mathrm{d}y.$$

If z also has continuous second-order partial derivatives, then  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  are also differentiable, and thus  $\mathrm{d}z$  is differentiable. We call the differential of  $\mathrm{d}z$  the second-order differential of z, denoted as

$$d^2z = d(dz).$$

In general, based on the k-th order differential (d $^k z$  of z, its (k+1)-th order differential (if it exists) is defined as

$$d^{k+1}z = d(d^kz), \quad k = 1, 2, \dots$$

Due to the fact that for the independent variables x and y, we always have

$$d^2x = d(dx) = 0,$$
  $d^2y = d(dy) = 0,$ 

the second-order differential of z=f(x,y) is given by

$$\begin{split} \mathrm{d}^2z &= \mathrm{d}(\mathrm{d}z) = \mathrm{d}\left(\frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y\right) \\ &= \mathrm{d}\left(\frac{\partial z}{\partial x}\right)\mathrm{d}x + \frac{\partial z}{\partial x}\mathrm{d}^2x + \mathrm{d}\left(\frac{\partial z}{\partial y}\right)\mathrm{d}y + \frac{\partial z}{\partial y}\mathrm{d}^2y \\ &= \left(\frac{\partial^2 z}{\partial x^2}\mathrm{d}x + \frac{\partial^2 z}{\partial x \partial y}\mathrm{d}y\right)\mathrm{d}x + \left(\frac{\partial^2 z}{\partial y \partial x}\mathrm{d}x + \frac{\partial^2 z}{\partial y^2}\mathrm{d}y\right)\mathrm{d}y \\ &= \frac{\partial^2 z}{\partial x^2}(\mathrm{d}x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\mathrm{d}x\mathrm{d}y + \frac{\partial^2 z}{\partial y^2}(\mathrm{d}y)^2, \end{split}$$

where  $(\mathrm{d}x)^2$  and  $(\mathrm{d}y)^2$  denote  $\mathrm{d}^2x$  and  $\mathrm{d}^2y$  respectively. If we treat  $\frac{\partial}{\partial x}$ ,  $\frac{\partial}{\partial y}$  as operators for partial differentiation and define

$$\left(\frac{\partial}{\partial x}\right)^2 = \frac{\partial^2}{\partial x^2}, \quad \left(\frac{\partial}{\partial y}\right)^2 = \frac{\partial^2}{\partial y^2}, \quad \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x \partial y},$$

then the formulas for the first and second differentials can be written as

$$dz = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right) z,$$

$$d^2z = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^2 z.$$

Similarly, we define

$$\left(\frac{\partial}{\partial x}\right)^p \left(\frac{\partial}{\partial y}\right)^q = \frac{\partial^{p+q}}{\partial x^p \partial y^q} = \frac{\partial^q}{\partial y^q} \left(\frac{\partial}{\partial x}\right)^p, \quad (p, q = 1, 2, \dots)$$

It is easy to use mathematical induction to prove the formula for higher-order differentials:

$$d^k z = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^k z, \quad k = 1, 2, \cdots.$$

For an n-variable function  $u = f(x_1, x_2, \dots, x_n)$ , higher-order differentials can be similarly defined, and the following holds:

$$d^{k}u = \left(dx_{1}\frac{\partial}{\partial x_{1}} + dx_{2}\frac{\partial}{\partial x_{2}} + \dots + dx_{n}\frac{\partial}{\partial x_{n}}\right)^{k}u, \quad k = 1, 2, \dots$$

### 12.2 Differential of Vector-Valued Functions

Consider an n-dimensional vector-valued function defined on a domain  $U \subset \mathbb{R}^n$ :

$$f: U \to \mathbb{R}^m,$$
  
 $\mathbf{x} \mapsto \mathbf{y} = f(\mathbf{x})$ 

Expressed in coordinate vector form:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_m(x_1, x_2, \dots, x_n) \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in U$$

1. If each component function  $f_i(x_1, x_2, \dots, x_n)$  ( $i = 1, 2, \dots, m$ ) is partially differentiable at  $\mathbf{x}^0$ , then the vector-valued function  $\mathbf{f}$  is differentiable at  $\mathbf{x}^0$ , and we define the matrix

$$\left(\frac{\partial f}{\partial x_j}(\mathbf{x}^0)\right)_{m \times n} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}^0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}^0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}^0) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}^0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}^0) \end{pmatrix}$$

This matrix is called the Jacobian matrix of  $\mathbf{f}$  at  $\mathbf{x}^0$ , denoted by  $f'(\mathbf{x}^0)$  (or  $\mathrm{D}f(\mathbf{x}^0)$ ,  $J_f(\mathbf{x}^0)$ ). For the special case m=1, i.e., n-variable scalar function  $z=f(x_1,x_2,\ldots,x_n)$ , the derivative at  $\mathbf{x}^0$  is

$$f'(\mathbf{x}^0) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}^0), \frac{\partial f}{\partial x_2}(\mathbf{x}^0), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x}^0)\right)$$

If the vector-valued function  $\mathbf{f}$  is differentiable at every point in U, then  $\mathbf{f}$  is said to be differentiable on U, and the corresponding relationship is

$$\mathbf{x} \in U \mapsto f'(\mathbf{x}) = J_f(\mathbf{x})$$

where  $f'(\mathbf{x})$  (or  $Df(\mathbf{x})$ ,  $J_f(\mathbf{x})$ ) denotes the derivative of  $\mathbf{f}$  at  $\mathbf{x}$  in U.

2. If every component function  $f_i(x_1, x_2, ..., x_n)$  (i = 1, 2, ..., m) of  $\mathbf{f}$  has continuous partial derivatives at  $\mathbf{x}^0$ , then every element of the Jacobian matrix of  $\mathbf{f}$  is continuous at  $\mathbf{x}^0$ . In this case,  $\mathbf{f}$  is said to have a continuous derivative at  $\mathbf{x}^0$  as a vector-valued function.

If the derivative of a vector-valued function  ${\bf f}$  is continuous at every point in U, then  ${\bf f}$  is said to have a continuous derivative on U.

3. If there exists an  $m \times n$  matrix A that depends only on  $\mathbf{x}^0$  (and not on  $\Delta \mathbf{x}$ ), such that in the neighborhood of  $\mathbf{x}^0$ ,

$$\Delta \mathbf{y} = f(\mathbf{x}^0 + \Delta \mathbf{x}) - f(\mathbf{x}^0) = A\Delta \mathbf{x} + o(\|\Delta \mathbf{x}\|)$$

(where  $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)^T$  is a column vector and  $\|\Delta \mathbf{x}\|$  denotes its norm), then f is said to be differentiable at  $\mathbf{x}^0$  as a vector-valued function, and  $A\Delta \mathbf{x}$  is called the differential of f at  $\mathbf{x}^0$ , denoted as  $d\mathbf{y}$ . If we denote  $\Delta \mathbf{x}$  by  $d\mathbf{x}$  ( $d\mathbf{x} = (dx_1, dx_2, \dots, dx_n)^T$ ), then

$$d\mathbf{v} = A d\mathbf{x}.$$

If the vector-valued function  ${\bf f}$  is differentiable at every point in U, then  ${\bf f}$  is said to be differentiable on U

Combining the above three points, we obtain the following unified statement:

A vector-valued function f is continuous, differentiable, and has derivatives if and only if each of its coor-

dinate component functions  $f_i(x_1, x_2, \dots, x_n)$  ( $i = 1, 2, \dots, m$ ) is continuous, differentiable, and has derivatives.

### 12.3 Derivatives of Composite Mappings (Chain Rule)

Let  $U \subset \mathbb{R}^l$  and  $V \subset \mathbb{R}^n$  be open sets, and let

$$\mathbf{g}: U \to V$$
 and  $\mathbf{f}: V \to \mathbb{R}^m$ 

be mappings. If  $\mathbf{g}$  is derivative at  $\mathbf{u}^0 \in U$  and  $\mathbf{f}$  is differentiable at  $\mathbf{x}^0 = \mathbf{g}(\mathbf{u}^0)$ , then the composite mapping  $\mathbf{f} \circ \mathbf{g}$  is differentiable at  $\mathbf{u}^0$ , and:

$$J(\mathbf{f} \circ \mathbf{g})(\mathbf{u}^0) = J\mathbf{f}(\mathbf{x}^0)J\mathbf{g}(\mathbf{u}^0).$$



- 1. outer differentiable + inner derivative = total derivative
- 2. outer differentiable + inner differentiable = total differentiable

3.

Specially, define  $z=f(x,y), (x,y)\subset D_f\subset \mathbb{R}^2$ ,  $\mathbf{g}:D_g\to \mathbb{R}^2, (u,v)\mapsto (x(u,v),y(u,v))$ , and  $g(D_g)\subset D_f$ , then we have composite function

$$z = f \circ \mathbf{g} = f \left[ x(u, v), y(u, v) \right], \quad (u, v) \in D_q.$$

$$\mathbb{R}^2 \xrightarrow{\mathbf{g}: \text{derivative}} \mathbb{R}^2 \xrightarrow{f: \text{differentiable}} \mathbb{R}$$

If g is derivative at  $(u_0, v_0) \in D_g$ , and f is differentiable at  $(x_0, y_0) = \mathbf{g}(u_0, v_0)$ , then  $z = f \circ \mathbf{g}$  is differentiable at  $(u_0, v_0)$ , and at the point,

$$\begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$



### 12.4 Mean Value Theorem and Taylor's Formula

#### Definition 12.4 (Convex Region)

Let  $D \subseteq \mathbb{R}^n$  be a region. If every line segment connecting any two points  $\mathbf{x}_0, \mathbf{x}_1 \in D$  (denoted by  $\overline{\mathbf{x}_0}\overline{\mathbf{x}_1}$ ) is entirely contained in D, i.e., for any  $\lambda \in [0, 1]$ , we have

$$\mathbf{x}_0 + \lambda(\mathbf{x}_1 - \mathbf{x}_0) \in D$$
,

then D is called a convex region.

#### Theorem 12.3 (Lagrange's Mean Value Theorem

Let f be <u>differentiable</u> on <u>a convex region</u>  $D \subseteq \mathbb{R}^n$ . For any two points  $\mathbf{a}, \mathbf{b} \in D$ , there exists a point  $\xi \in \overline{\mathbf{ab}}$  such that:

$$f(\mathbf{b}) - f(\mathbf{a}) = Jf(\xi)(\mathbf{b} - \mathbf{a}).$$



#### Theorem 12.4

Let D be a region in  $\mathbb{R}^n$ . If for any  $\mathbf{x} \in D$ , we have

$$Jf(\mathbf{x}) = 0,$$

then f is constant on D.

### A Proof

#### Theorem 12.5 (Taylor's Formula)

**Lagrange's Remainder** Let  $D \subseteq \mathbb{R}^n$  be a convex region, and let  $f: D \to \mathbb{R}$  have m+1 continuous partial derivatives. For  $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$  and  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in D$ , there exists  $\xi \in \overline{\mathbf{x}^0 \mathbf{x}}$  such that:

$$f(\mathbf{x}) = f(\mathbf{x}^0) + \sum_{k=1}^{m} \frac{1}{k!} \left( \sum_{i=1}^{n} (x_i - x_i^0) \frac{\partial}{\partial x_i} \right)^k f(\mathbf{x}^0) + \frac{1}{(m+1)!} \left( \sum_{i=1}^{n} (x_i - x_i^0) \frac{\partial}{\partial x_i} \right)^{m+1} f(\xi).$$

**Peano's Remainder** Let  $D \subseteq \mathbb{R}^n$  be a convex region, and let  $f: D \to \mathbb{R}$  have m continuous partial derivatives. Then:

$$f(\mathbf{x}) = f(\mathbf{x}^0) + \sum_{k=1}^m \frac{1}{k!} \sum_{i_1, i_2, \dots, i_k = 1}^n \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_k}} (\mathbf{x}^0) \prod_{j=1}^k (x_{i_j} - x_{i_j}^0) + R_m(\mathbf{x} - \mathbf{x}^0),$$

where 
$$R_m(\mathbf{x} - \mathbf{x}^0) = O(\|\mathbf{x} - \mathbf{x}^0\|^{m+1})$$
 or  $o(\|\mathbf{x} - \mathbf{x}^0\|^m)$ , as  $\|\mathbf{x} - \mathbf{x}^0\| \to 0$ .

In applications, particularly important is the expression of the first three terms in Taylor's formula, which is given as (let  $x_1 - x_1^0$  be denoted by  $\Delta x_1$ , and similarly for other variables;  $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$ ):

$$f(\mathbf{x}) = f(\mathbf{x}^0) + Jf(\mathbf{x}^0)(\Delta \mathbf{x}) + \frac{1}{2!}(\Delta \mathbf{x})Hf(\mathbf{x}^0)(\Delta \mathbf{x})^{\mathrm{T}} + \cdots,$$

where the matrix

$$Hf(\mathbf{x}^{0}) = \begin{bmatrix} \frac{\partial^{2} f}{\partial x_{1}^{2}} & \frac{\partial^{2} f}{\partial x_{1} \partial x_{2}} & \cdots & \frac{\partial^{2} f}{\partial x_{1} \partial x_{n}} \\ \frac{\partial^{2} f}{\partial x_{2} \partial x_{1}} & \frac{\partial^{2} f}{\partial x_{2}^{2}} & \cdots & \frac{\partial^{2} f}{\partial x_{2} \partial x_{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{2} f}{\partial x_{n} \partial x_{1}} & \frac{\partial^{2} f}{\partial x_{n} \partial x_{2}} & \cdots & \frac{\partial^{2} f}{\partial x_{n}^{2}} \end{bmatrix}_{\mathbf{x}^{0}}$$

is called the **Hessian matrix** of the function f. It is an  $n \times n$  symmetric matrix.

### 12.5 Implicit Function Theorem

#### Theorem 12.6 (Implicit Function Theorem)

Let  $U\subset\mathbb{R}^{n+1}$  be an open set, and  $F:U\to\mathbb{R}$  be an n+1-variable function. If:

- 1.  $F \in C^k(U, \mathbb{R})$ , where  $1 \le k \le +\infty$ ;
- 2.  $F(\mathbf{x}^0, y^0) = 0$ , where  $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \mathbb{R}^n$ ,  $y^0 \in \mathbb{R}$ , and  $(\mathbf{x}^0, y^0) \in U$  (i.e., the equation  $F(\mathbf{x}, y) = 0$  has a solution  $(\mathbf{x}^0, y^0)$ );
- 3.  $F'_{y}(\mathbf{x}^{0}, y^{0}) \neq 0$ .

Then there exists an open interval  $I \times J$  containing  $(\mathbf{x}^0, y^0)$  (I being an open interval in  $\mathbb{R}^n$  containing  $\mathbf{x}^0$ , and J being an open interval in  $\mathbb{R}$  containing  $y^0$ ), as shown in Fig. 12.1, such that:

- 1.  $\forall x \in I$ , the equation  $F(\mathbf{x}, y) = 0$  has a unique solution  $y = f(\mathbf{x})$ , where  $f : I \to J$  is an n-variable function (called the **implicit function** f, hidden within the equation  $F(\mathbf{x}, f(\mathbf{x})) = 0$ , though not necessarily explicitly expressed);
- 2.  $y^0 = f(\mathbf{x}^0);$
- 3.  $f \in C^k(I, \mathbb{R})$ ;
- 4. When  $x \in I$ ,  $\frac{\partial f}{\partial x_i} = \frac{\partial y}{\partial x_i} = -\frac{F_x(\mathbf{x}, y)}{F_y(\mathbf{x}, y)}$ ,  $i = 1, 2, \dots, n$ , where y = f(x).

\_\_\_\_\_

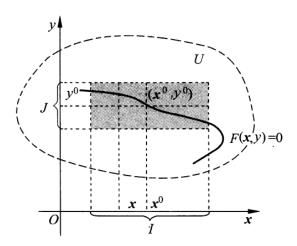


Figure 12.1: Implicit Function

Proof Only the single-variable implicit function theorem is proved; the multi-variable case can be derived using mathematical induction.

Without loss of generality, assume  $F_y(x^0, y^0) > 0$ .

First, prove the existence of the implicit function. From the continuity of  $F_y(x^0, y^0) > 0$  and  $F_y(x, y)$ , it is known that there exist closed rectangle:

$$D^* = \{(x, y) \mid |x - x_0| \le \alpha, |y - y_0| \le \beta\} \subset U,$$

where the following holds:

$$F_{y}(x,y) > 0.$$

Thus, for fixed  $x_0$ , the function  $F(x^0, y)$  is strictly monotonically increasing within  $[y^0 - \beta, y^0 + \beta]$ . Further-

more, since:

$$F(x^0, y^0) = 0,$$

it follows that:

$$F(x^0, y^0 - \beta) < 0, \quad F(x^0, y^0 + \beta) > 0.$$

Due to the continuity of F(x, y) within  $D^*$ , there exists  $\rho > 0$  such that along the line segment:

$$x = x^{0} + \rho, y = y^{0} + \beta,$$

we have F(x, y) > 0, and along the line segment:

$$x = x^{0} + \rho, y = y^{0} - \beta,$$

we have F(x,y)<0. Therefore, for any point  $\bar x\in(x^0-\rho,x^0+\rho)$ , treat F(x,y) as a single-variable function of y. Within  $[y^0-\beta,y^0+\beta]$ , this function is continuous. From the previous discussion, we know:

$$F(\bar{x}, y^0 - \beta) < 0, \quad F(\bar{x}, y^0 + \beta) > 0.$$

According to the zero point existence theorem 3.3, there must exist a unique  $\bar{y} \in [y^0 - \beta, y^0 + \beta]$  such that  $F(\bar{x}, \bar{y}) = 0$ . Furthermore, because  $F_y(x, y) > 0$  within  $D^*$ , this  $\bar{y}$  is unique. Denote the corresponding relationship as  $\bar{y} = f(\bar{x})$ , then the function y = f(x) is defined within  $(x^0 - \rho, x^0 + \rho)$ , satisfying F(x, f(x)) = 0, and clearly:

$$y^0 = f(x^0).$$

Further proving the continuity of the implicit function y=f(x) on  $(x^0-\rho,x^0+\rho)$ : Let  $\bar x\in(x^0-\rho,x^0+\rho)$  be any point. For any given  $\varepsilon>0$  ( $\varepsilon$  being sufficiently small), since  $F(\bar x,\bar y)=0$  ( $\bar y=f(\bar x)$ ), from the previous discussion we know:

$$F(\bar{x}, \bar{y} - \varepsilon) < 0, \quad F(\bar{x}, \bar{y} + \varepsilon) > 0.$$

Furthermore, due to the continuity of F(x, y) on  $D^*$ , there exists  $\delta > 0$  such that:

$$F(x, \bar{y} - \varepsilon) < 0$$
,  $F(x, \bar{y} + \varepsilon) > 0$ , when  $x \in O(x^0, \delta)$ .

By reasoning similar to the previous discussion, it can be obtained that when  $x \in O(x^0, \delta)$ , the corresponding implicit function value must satisfy  $f(x) \in (\bar{y} - \varepsilon, \bar{y} + \varepsilon)$ , i.e.,

$$\left| f(x) - f(x^0) \right| < \varepsilon.$$

This implies that y = f(x) is continuous on  $(x^0 - \rho, x^0 + \rho)$ .

Finally, prove the differentiability of y=f(x) on  $(x^0-\rho,x^0+\rho)$ : Let  $\bar x\in(x^0-\rho,x^0+\rho)$  be any point. Take  $\Delta x$  sufficiently small such that  $\bar x=x+\Delta x\in(x^0-\rho,x^0+\rho)$ . Denote  $\bar y=f(\bar x)$  and  $\bar y+\Delta y=f(\bar x)$ . Clearly,

$$F(\bar{x}, \bar{y}) = 0$$
 and  $F(\bar{x}, \bar{y} + \Delta y) = 0$ .

Using the multi-variable function's mean value theorem 12.3, we obtain:

$$0 = F(\bar{x}, \bar{y} + \Delta y) - F(\bar{x}, \bar{y})$$
  
=  $F_x(\bar{x} + \theta \Delta x, \bar{y} + \theta \Delta y) \Delta x + F_y(\bar{x} + \theta \Delta x, \bar{y} + \theta \Delta y) \Delta y$ ,

where  $0 < \theta < 1$ . Note that  $F_y \neq 0$  on  $D^*$ , hence:

$$\frac{\Delta y}{\Delta x} = -\frac{F_x(\bar{x} + \theta \Delta x, \bar{y} + \theta \Delta y)}{F_y(\bar{x} + \theta \Delta x, \bar{y} + \theta \Delta y)}.$$

Let  $\Delta x \to 0$ . Considering the continuity of  $F_x$  and  $F_y$ , we obtain:

$$\frac{dy}{dx}\Big|_{x=\bar{x}} = -\frac{F_x(\bar{x},\bar{y})}{F_y(\bar{x},\bar{y})}.$$

Thus:

$$f'(\bar{x}) = -\frac{F_x(\bar{x}, \bar{y})}{F_y(\bar{x}, \bar{y})}.$$

The proof is complete.

#### Theorem 12.7 (Implicit Mapping Theorem

Let  $U \subset \mathbb{R}^{n+m}$  be an open set, and  $\mathbf{F}: U \to \mathbb{R}^m$  be a mapping. If:

- 1.  $\mathbf{F} \in C^k(U, \mathbb{R}^m), 1 \le k \le \infty$ ;
- 2.  $\mathbf{F}(\mathbf{x}^0, \mathbf{y}^0) = 0$ , where  $\mathbf{x}^0 = (x_1, x_2, \dots, x_n)$ ,  $\mathbf{y}^0 = (y_1, y_2, \dots, y_m)$ ,  $(\mathbf{x}^0, \mathbf{y}^0) \in U$  (implying  $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$  has a solution at  $(\mathbf{x}^0, \mathbf{y}^0)$ );
- 3. The determinant

$$\det\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}_{(\mathbf{x}^0, \mathbf{y}^0)} = \det J_{\mathbf{y}} \mathbf{F}(\mathbf{x}^0, \mathbf{y}^0) \neq 0,$$

then there exists an open neighborhood  $I \times J \subset U \subset \mathbb{R}^{n+m}$  containing  $(\mathbf{x}^0, \mathbf{y}^0)$ , such that:

- 1. For all  $\mathbf{x} \in I$ , the system  $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$  has a unique solution  $\mathbf{y} = \mathbf{f}(\mathbf{x})$ , where  $\mathbf{f} : I \to J$  is a mapping (called  $\mathbf{f}$  the implicit function hidden in  $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$ );
- 2.  $\mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0);$
- 3.  $\mathbf{f} \in C^k(I, \mathbb{R}^m)$ ;
- 4. For  $x \in I$ ,

$$J_{\mathbf{f}}(x) = -(J_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{f}(x)))^{-1}J_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{f}(x)) = -\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}^{-1}\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix},$$

where  $\mathbf{y} = \mathbf{f}(\mathbf{x})$ .



### 12.6 Applications of Multi-Variable Differential Calculus

#### ¶ Surface and Tangent Space

#### Definition 12.5 (Parameterization of Surface)

Let  $\Delta$  be an open subset in  $\mathbb{R}^s$ , and  $\mathbf{x}: \Delta \to \mathbb{R}^n$  be a mapping, where  $\mathbf{u} = (u_1, u_2, \dots, u_s) \to \mathbf{x}(\mathbf{u}) = (x_1(u_1, u_2, \dots, u_s), x_2(u_1, u_2, \dots, u_s), \dots, x_n(u_1, u_2, \dots, u_s))$ . Then  $M = \mathbf{x}(\Delta) = \{\mathbf{x}(\mathbf{u}) \mid \mathbf{u} \in \Delta\}$  is called an s-dimensional surface, and  $\mathbf{x}(\mathbf{u})$  is referred to as the parameterization of M. When  $\mathbf{x}(\mathbf{u}) \in C^k$   $(k \geq 0)$ ,  $\mathbf{x}$  or M is called an s-dimensional  $C^k$  surface.

If  $\mathbf{x} \in C^k$   $(k \ge 1)$ ,  $\mathbf{x}$  or M is called an s-dimensional  $C^k$  smooth surface. When

$$\operatorname{rank}(x_1'(\mathbf{u}^0), x_2'(\mathbf{u}^0), \dots, x_s'(\mathbf{u}^0)) = \operatorname{rank} \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_s} \end{pmatrix}_{\mathbf{u}^0} = s,$$

we call  $\mathbf{u}^0$  or  $\mathbf{x}(\mathbf{u}^0)$  a **regular point** of the surface M. Otherwise, it is called a singular point. Every point that is a regular point of the surface is referred to as an s-dimensional  $C^k$  regular surface. At such points,  $\{x_1',\ldots,x_s'\}$  are linearly independent.

When s=1, t represents the parameter, a one-dimensional surface is commonly referred to as a curve. Considering a  $C^k$  ( $k \ge 1$ ) curve  $\mathbf{x}(t)$ , we have:

$$\mathbf{x}'(t) = \left(x_1'(t), x_2'(t), \cdots, x_n'(t)\right).$$

If t is a regular point, then  $\operatorname{rank}(\mathbf{x}'(t)) = \operatorname{rank}(x_1'(t), x_2'(t), \dots, x_n'(t)) = 1$ ; this is equivalent to  $\mathbf{x}'(t) \neq 0$ , which means  $x_1'(t), x_2'(t), \dots, x_n'(t)$  are not all zero.

We refer to  $\mathbf{x}'(t)$  as the tangent vector of the curve  $\mathbf{x}(t)$  at point t. When t varies, a tangent vector field along the curve  $\mathbf{x}(t)$  is obtained. If  $\mathbf{x}(t)$  is a regular curve,  $\frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$  is the unit tangent vector field along the curve  $\mathbf{x}(t)$ . It should be emphasized that  $\mathbf{x}'(t)$  or  $\frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$  always points outward from point t.

#### Definition 12.6 (Tangent Vector)



- $\P$  Unconditional Extremum
- ¶ Conditional Extremum

# **Chapter 13 Multiple Integrals**

### 13.1 Multiple Integrals on Bounded Closed Regions

#### Definition of Multiple Integral

Initially, we define the double integral on a closed interval.

#### Definition 13.1 (Double Integral on a Closed Interval)

Let  $I = [a, b] \times [c, d]$  be a closed interval in  $\mathbb{R}^2$ , (i.e., each boundary is parallel to the coordinate axes). Partition [a, b]:

$$T_x : a = x_0 < x_1 < \dots < x_n = b.$$

Partition [c, d]:

$$T_y : c = y_0 < y_1 < \dots < y_m = d.$$

Two sets of parallel lines  $x = x_i$  (i = 0, 1, ..., n) and  $y = y_j$  (j = 0, 1, ..., m) divide I into  $n \times m$  subrectangles:

$$[x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad i = 1, \dots, n, j = 1, \dots, m.$$

The union of these k subrectangles forms a partition  $T=T_x\times T_y=\{I_1,I_2,\ldots,I_k\}$ . For each  $\xi^i\in I_i$   $(i=1,2,\ldots,k)$ , define the Riemann sum (also called a sum of integrals) as:

$$\sum_{i=1}^{k} f(\boldsymbol{\xi}^i) v(I_i),$$

where  $v(I_i)$  is the area of the rectangle  $I_i$ , i.e., the product of its length and width. Denote:

$$\lambda = \max(\operatorname{diam}(I_1), \operatorname{diam}(I_2), \dots, \operatorname{diam}(I_k)),$$

where  $\operatorname{diam}(I)$  is the diagonal length of the rectangle I, and  $\lambda$  is called the modulus or width of the partition T. The points  $\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \dots, \boldsymbol{\xi}^k) \in I_1 \times I_2 \times \dots \times I_k$  are called sampling points for the Riemann sum. If there exists  $J \in \mathbb{R}$ , such that  $\forall \varepsilon > 0$ , there exists  $\delta > 0$ , such that when  $\delta < \delta$ , for all  $\boldsymbol{\xi} \in I_1 \times I_2 \times \dots \times I_k$ , we have:

$$\left| \sum_{i=1}^{k} f(\boldsymbol{\xi}^{i}) v(I_{i}) - J \right| < \varepsilon,$$

then f is said to be Riemann integrable on I, and:

$$J = \lim_{\lambda \to 0} \sum_{i=1}^{k} f(\boldsymbol{\xi}^{i}) v(I_{i}) =: \iint_{I} f(x, y) \, \mathrm{d}x \mathrm{d}y \quad \text{or} \quad \int_{I} f \, \mathrm{d}v \quad \text{or} \quad \int_{I} f.$$

The function f is said to have a double integral on I, or simply f is integrable on I. Here f is called the integrand, I is called the integration region, and  $\mathrm{d}v = \mathrm{d}x\mathrm{d}y$  is called the integration element.

The defined double integral possesses properties similar to those of single-variable integrals. On the basis of the above definition, we can extend it to the case of a bounded set.

#### Definition 13.2 (Double Integral on a Bounded Set)

Let  $\Omega \subset \mathbb{R}^2$  be a bounded set, and  $f:\Omega \to \mathbb{R}$  a two-dimensional function. Define:

$$f_{\Omega}(\mathbf{x}) = f_{\Omega}(x, y) = \begin{cases} f(x, y), & \text{if } \mathbf{x} = (x, y) \in \Omega, \\ 0, & \text{if } \mathbf{x} = (x, y) \notin \Omega, \end{cases}$$

and call this the **zero extension** of f. For any closed interval  $I \supset \Omega$ , if  $f_{\Omega}$  is Riemann integrable on I, then f is said to be **Riemann integrable** on  $\Omega$  (abbreviated as integrable). The integral of f on  $\Omega$ , denoted as:

$$\iint_{\Omega} f(x,y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\Omega} f \, \mathrm{dV} = \int_{\Omega} f = \int_{\Omega} f_{\Omega} = \iint_{I} f_{\Omega}(x,y) \, \mathrm{d}x \, \mathrm{d}y,$$

represents the Riemann integral of f on  $\Omega$ .

In above definition, the integral  $\int_{\Omega} f$  is independent of the choice of the closed interval I containing  $\Omega$  (this confirms the consistency of the definition).

It is worth noting that all the definitions and properties of double integrals can be extended to triple integrals and higher-dimensional integrals without excessive inconvenience.

#### Necessary and Sufficient Conditions for Integrability

#### Definition 13.3 (Set with Zero Area and Set with Zero Measure (Null Set))

Let  $A\subset\mathbb{R}^2$ . If for any  $\varepsilon>0$ , there exist finitely many closed intervals  $I_1,I_2,\ldots,I_k$  such that:

$$\bigcup_{i=1}^{k} I_i \supset A, \quad \text{and} \quad \sum_{i=1}^{k} v(I_i) < \varepsilon,$$

then A is called a **set with zero area**.

Let  $A \subset \mathbb{R}^2$ . If for any  $\varepsilon > 0$ , there exist at most countably many closed intervals  $I_1, I_2, \dots, I_k, \dots$  such that:

$$\bigcup_{i=1}^{\infty} I_i \supset A, \quad \text{and} \quad \sum_{i=1}^{\infty} v(I_i) < \varepsilon,$$

then A is called a set with zero measure (null set).

#### Definition 13.4 (Set with Finite Area)

Let  $\Omega \subset \mathbb{R}^2$  be a bounded set. If the constant function 1 is integrable on  $\Omega$ , then  $\Omega$  is called a **set with finite** area, and the area of  $\Omega$  is defined as:

$$v(\Omega) = \int_{\Omega} 1 = \iint_{\Omega} \mathrm{d}x \mathrm{d}y = \int_{I} 1_{\Omega}.$$

Obviously,  $\Omega$  is a set with zero area if and only if  $\Omega$  has finite area, and  $v(\Omega)=\int_{\Omega}1=0$ .

### 13.2 Properties and Calculation of Multiple Integrals

 $\P$  Reduction of Double Integral to Iterated Integral

#### Theorem 13.1 (Reduction of Double Integral to Iterated Integral on a Closed Interval)

Let f be integrable on the closed interval  $I = [a, b] \times [c, d]$ . If  $\forall x \in [a, b]$ , the function  $f(x, \cdot)$  is integrable on [c, d], then:

$$\iint_I f = \int_a^b \left( \int_c^d f(x, y) \, \mathrm{d}y \right) \mathrm{d}x =: \int_a^b \mathrm{d}x \int_c^d f(x, y) \, \mathrm{d}y.$$

Similarly, if  $\forall y \in [c,d]$ , the function  $f(\cdot,y)$  is integrable on [a,b], then:

$$\iint_I f = \int_c^d \left( \int_a^b f(x, y) \, \mathrm{d}x \right) \mathrm{d}y =: \int_c^d \mathrm{d}y \int_a^b f(x, y) \, \mathrm{d}x.$$

On the basis of the above theorem, we can extend it to the case of a bounded region.

#### Theorem 13.2 (Reduction of Double Integral to Iterated Integral on a Bounded Set)

Let  $\Omega \subset \mathbb{R}^2$  be a set with infinite area, and  $f: \Omega \to \mathbb{R}$  be bounded and continuous (13.1). Denote the vertical projection of  $\Omega$  onto the x-axis as:

$$I = \{x \in \mathbb{R} \mid \exists y, \text{ s.t. } (x, y) \in \Omega\}.$$

If  $\forall x \in I$ , let  $\Omega_x = \{y \in \mathbb{R} \mid (x,y) \in \Omega\}$  be an interval (possibly reducing to a single point), then:

$$\int_{\Omega} f = \int_{I} dy \int_{\Omega_{T}} f(x, y) dx.$$

Similarly, denote the vertical projection of  $\Omega$  onto the *y*-axis as:

$$J = \{ y \in \mathbb{R} \mid \exists x, \text{ s.t. } (x, y) \in \Omega \}.$$

If  $\forall y \in J$ , let  $\Omega_y = \{x \in \mathbb{R} \mid (x,y) \in \Omega\}$  be an interval (possibly reducing to a single point), then:

$$\int_{\Omega} f = \int_{J} dy \int_{\Omega_{H}} f(x, y) dx.$$

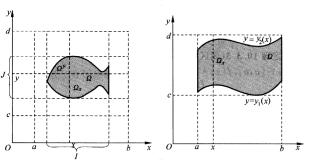


Figure 13.1: Double Integral on a Bounded Set

Specially, Let:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid y_1(x) \le y \le y_2(x), \ a \le x \le b\},\$$

where the functions  $y_1$  and  $y_2$  are continuous on [a, b] (13.1) and the function f is integrable on  $\Omega$ . If  $\forall x \in [a, b]$ ,

the single-variable integral:

$$\int_{y_1(x)}^{y_2(x)} f(x,y) \, \mathrm{d}y$$

exists, then:

$$\int_{\Omega} f = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} f(x, y) \, \mathrm{d}y.$$

This area called the **type X region**, similarly, we can define the **type Y region**.

According to 13.1, we can derive the formula of multiplicative property for double integral.

#### Theorem 13.3 (Formula of Multiplicative Property for Double Integral)

Let  $f \in C([a,b])$ ,  $g \in C([c,d])$ . Then the function h(x,y) = f(x)g(y) is integrable on the closed interval  $I = [a,b] \times [c,d]$ , and:

$$\iint_{I} h(x,y) \, \mathrm{d}x \, \mathrm{d}y = \left( \int_{a}^{b} f(x) \, \mathrm{d}x \right) \left( \int_{c}^{d} g(y) \, \mathrm{d}y \right).$$

**Example 13.1** Let  $p(x) \in R[a,b], p(x) > 0, x \in [a,b]$ , the monotonicity of f(x), g(x) is same, prove that

$$\int_a^b p(x)f(x)\mathrm{d}x \int_a^b p(x)g(x)\mathrm{d}x \leqslant \int_a^b p(x)\mathrm{d}x \int_a^b p(x)f(x)g(x)\mathrm{d}x$$

**Proof** Let

$$I = \int_a^b p(x) dx \int_a^b p(x) f(x) g(x) dx - \int_a^b p(x) f(x) dx \int_a^b p(x) g(x) dx,$$

then

$$I = \int_a^b \int_a^b p(x)p(y)g(y)(f(x) - f(y))dxdy,$$

similarly,

$$I = \int_a^b \int_a^b p(x)p(y)g(x)(f(x) - f(y)) dxdy.$$

Then

$$2I = \int_{a}^{b} \int_{a}^{b} p(x)p(y)(g(y) - g(x))(f(x) - f(y)) dxdy \ge 0,$$

which implies

$$I \geqslant 0$$
.

The proof is complete.

#### Calculation of Triple Integrals

**Example 13.2** Calculating  $I = \iiint_{\Omega} z^2 dx dy dz$ , where  $\Omega$  is the cone defined by  $z^2 = \frac{h^2}{R^2} (x^2 + y^2)$  and z = h (13.2).

**Example 13.3** Calculating  $I = \iiint_{\Omega} xy dx dy dz$ , where  $\Omega$  is the region defined by  $0 \leqslant z \leqslant xy, 0 \leqslant y \leqslant 1 - x, 0 \leqslant x \leqslant 1$  (13.3).

With the help of examples above, we can derive two methods for calculating triple integrals.

**First 2 then 1 (Section Method)** Fix one variable (e.g., z), first perform a double integral over the other two

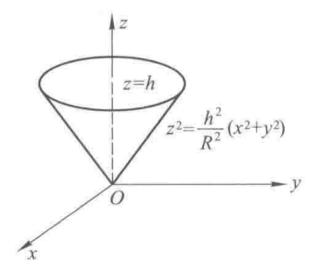


Figure 13.2: Cone Example.

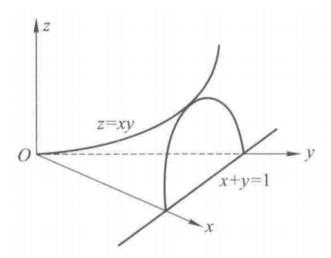


Figure 13.3: Project Method Example.

variables (e.g., x, y) on the "section region" corresponding to the fixed variable, and then perform a definite integral over the fixed variable (*z*) within its range of values.

This method is convenient when the area of the section region is easy to calculate, or when the integrand is only related to the "later-integrated variable" (e.g., only related to z).

In the example 13.2, the following steps are taken:

- 1. Determine the range of  $z: z \in [0, h]$ .
- 2. Determine the section region  $D_z$ : For a fixed z,  $D_z$  is the region on the xy-plane satisfying  $\frac{h^2}{R^2}(x^2+$  $y^2) \leq z^2$  , which is a circle with radius  $rac{R}{h}z$  .
- 3. Split the integral:  $I=\int_0^h \left(\iint_{D_z} z^2\,\mathrm{d}x\mathrm{d}y\right)\,\mathrm{d}z$ . Since  $z^2$  is independent of x and y, it can be factored out:  $I = \int_0^h z^2 \left( \iint_{D_z} dx dy \right) dz$ .
- 4. Calculate the double integral (area of the section):  $\iint_{D_z} \mathrm{d}x \mathrm{d}y = \pi \left(\frac{R}{h}z\right)^2 = \pi \frac{R^2}{h^2}z^2.$ 5. Calculate the definite integral:  $I = \int_0^h z^2 \cdot \pi \frac{R^2}{h^2}z^2 \, \mathrm{d}z = \frac{\pi R^2 h^3}{5}.$

**First 1 then 2 (Project Method)** Fix two variables (e.g., x, y), first perform a definite integral over the third variable (e.g., z) on the "vertical line segment" corresponding to the fixed variables, and then perform a double integral over the fixed two variables (x, y) on their "projection region.

This method is convenient when the projection region of the integral region on a certain coordinate plane (e.g., xy-plane) is easy to determine, and the upper and lower limits of a single variable (e.g., z) can be easily expressed by the other two variables.

In the example 13.3, the following steps are taken:

- 1. Determine the projection region  $D_{xy}$ : $D_{xy}$  is the region on the xy-plane bounded by  $x+y\leq 1$ ,  $x\geq 0$ , and  $y\geq 0$ , which can be expressed as  $0\leq x\leq 1$  and  $0\leq y\leq 1-x$ .
- 2. Determine the range of z:  $z \in [0, xy]$  (since z is bounded below by z = 0 and above by z = xy).
- 3. Split the integral:

4.

$$I = \iint_{D_{xy}} \left( \int_0^{xy} xy \mathrm{d}z \right) \mathrm{d}x \mathrm{d}y,$$

split the double integral on  $D_{xy}$  as:  $I=\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{xy} xy \,\mathrm{d}z$ . (Since xy is independent of z, it can be factored out without affecting the integral:  $I=\int_0^1 \mathrm{d}x \int_0^{1-x} xy \,\mathrm{d}y \int_0^{xy} \mathrm{d}z$ .)

- 5. Calculate the inner integral (with respect to z):  $\int_0^{xy} xy \, dz = xy \cdot \int_0^{xy} dz = xy \cdot z \Big|_0^{xy} = xy \cdot xy = x^2y^2.$
- 6. Calculate the middle integral (with respect to *y*): Substitute the result of the inner integral,

$$\int_0^{1-x} x^2 y^2 \, dy = x^2 \cdot \frac{y^3}{3} \Big|_0^{1-x} = \frac{x^2 (1-x)^3}{3}.$$

7. Calculate the outer integral (with respect to *x*):Substitute the result of the middle integral:

$$\int_0^1 \frac{x^2 (1-x)^3}{3} dx = \frac{1}{3} \int_0^1 (x^2 - 3x^3 + 3x^4 - x^5) dx$$
$$= \frac{1}{3} \left( \frac{x^3}{3} - \frac{3x^4}{4} + \frac{3x^5}{5} - \frac{x^6}{6} \Big|_0^1 \right)$$
$$= \frac{1}{3} \left( \frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right)$$
$$= \frac{1}{180}.$$

### 13.3 Variable Substitution in Multiple Integrals

#### Theorem 13.4 (Variable Substitution in Double Integral)

Let  $\Omega \subset \mathbb{R}^2$  be an open set, and let the mapping:

$$\mathbf{F}: \Omega \to \mathbb{R}^2, \quad (u, v) \mapsto \mathbf{F}(u, v) = (x(u, v), y(u, v))$$

satisfy the following conditions:

- 1.  $\mathbf{F} \in C^1(\Omega, \mathbb{R}^2)$ ;
- 2.  $\frac{\partial(x,y)}{\partial(u,v)} = \det J\mathbf{F}(u,v) = \det J\mathbf{F}(\mathbf{p}) \neq 0, \quad \mathbf{p} = (u,v) \in \Omega;$
- 3. **F** is injective.

If the set  $\Delta$  is a set with finite area and  $\overline{\Delta} \subset \Omega$ , and f is continuous on  $\mathbf{F}(\Omega)$ , then  $\mathbf{F}(\Delta)$  is also a set with finite

 $\Diamond$ 

area, and:

$$\iint_{\mathbf{F}(\Delta)} f = \iint_{\Delta} f \circ \mathbf{F} \left| \det J \mathbf{F} \right|,$$

i.e.,

$$\iint_{F(\Delta)} f(x,y) \, \mathrm{d}x \mathrm{d}y = \iint_{\Delta} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, \mathrm{d}u \mathrm{d}v.$$

For triple and higher-dimensional integrals, the variable substitution theorem is similar to the above theorem.

Some common variable substitutions in multiple integrals are as follows:

#### **Polar Coordinates**

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2}, & r \geqslant 0 \\ \theta = \arctan\left(\frac{y}{x}\right) & x \neq 0, \theta \in [0, 2\pi]. \end{cases}$$

and

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

#### Cylindrical Coordinate System

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \\ z = z, \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2}, \quad r \geqslant 0 \\ \theta = \arctan\left(\frac{y}{x}\right) \quad x \neq 0, \theta \in [0, 2\pi], \\ z = z. \end{cases}$$

and

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}=r.$$

#### **Spherical Coordinate System**

$$\begin{cases} x = r \sin \varphi \cos \theta, \\ y = r \sin \varphi \sin \theta, \\ z = r \cos \varphi, \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2 + z^2}, \quad r \geqslant 0 \\ \varphi = \arccos\left(\frac{z}{r}\right) \quad r \neq 0, \varphi \in [0, \pi], \\ \theta = \arctan\left(\frac{y}{x}\right) \quad x \neq 0, \theta \in [0, 2\pi]. \end{cases}$$

and

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2 \sin \varphi.$$

### 13.4 Improper Multiple Integrals

### 13.5 Differential Forms

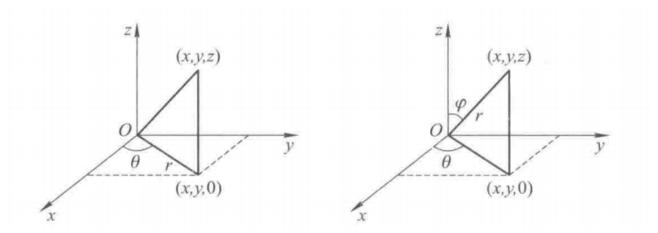


Figure 13.4: Cylindrical and Spherical Coordinate Systems

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