Exam 1 –

Print last name: ______________________, firs. ______________ ______

Net ID (email): ____________@wisc.edu

Discussion (circle one):

This exam is composed of two parts. Part I is to be answered by filling in your choice using a #2 pencil on the separate answer sheet. Part II is to be answered in this examination booklet.

On the blue scantron form:

1. Print your Last name, First name, and UW ID in the correct fields
2. Print your Discussion number under special code columns **A B C**
3. Fill in **ALL** corresponding circles (bubbles) for each letter and digit
4. Write an **S** (for secondary) after your discusion number in the Special Codes section under letter D on the answer sheet. Do not fill in any circles for this letter.

Additional exam instructions:

1. **Print your name and Net ID** as indicated above and **circle your discussion.**
2. You may **not** use books, notes, calculators (or any other electronic devices), or neighbors on this exam. **Turn off and put away your cell phone, pager, PDA, etc. now** (before the exam begins).
3. **You must keep your answers covered at all times.**
4. **You have two hours to complete this exam. Budget your time wisely.** Some questions of equal point value take more time to answer than others. If you finish early, you may turn in your exam (see step 7) and leave.
5. We can't provide hints but if you need an exam question clarified or feel that there is an error, please bring this to our attention. **Corrections will be announced and written on the board.**
6. **A reference page is provided at the end of the exam.** You may detach the pages of the exam, but make sure to hand in **ALL** the pages of the exam.
7. When you have finished your exam, bring your ID, exam, and scantron to a proctor.

DO NOT START THE EXAM UNTIL YOU ARE INSTRUCTED TO DO SO

Parts	Number of Questions	Question Format	Possible Points	Score
I	20	Multiple Choice	40	36
II	3	Written Answers	22	21
		Total	60 (max)	57

Part I Multiple Choice *[20 questions, 2 points each, 40 total points]*

For questions 1 through 20, choose the one best answer after reading all of the choices. Mark the corresponding letter on your answer sheet.

1) Suppose $A = \{\emptyset, \{0,1\}\}$. What is $\mathcal{P}(A)$?

   A. $\{\emptyset, \{\emptyset\}, \{\{0,1\}\}, \{\emptyset, \{0,1\}\}\}$

   B. $\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

   C. $\{\emptyset, \{\emptyset\}, \{0,1\}, \{\emptyset, \{0,1\}\}\}$

   D. $\{\emptyset, \{\emptyset\}, \{0\}, \{1\}, \{0,1\}, \{\emptyset, 0\}, \{\emptyset, 1\}, \{\emptyset, \{0,1\}\}\}$

00 $\emptyset$
01 $\{\emptyset\}$
10 $\{\{0,1\}\}$
11 $\{\emptyset, \{0,1\}\}$

2) Suppose A and B are arbitrary finite sets. What is $|\mathcal{P}(A \times B)|$?

   A. $2^{2^{|A|\cdot|B|}}$

   B. $2^{|A|\cdot|B|}$

   C. $2^{|A|} \cdot 2^{|B|}$

   D. $2^{|A|^{|B|}}$

$2^{(A \times B)}$

$|A \times B|$ $|A| \cdot |B|$

$2^{|A||B|}$

1,2,3 3,4

1,3 1,4 2,3 2,4

3,3, 3,4

3) Which of the following is the *contrapositive* of the implication: If $3|a$ and $3|b$, then $3|(a-b)$.
Note: $|$ is the symbol for "divides" and $\nmid$ is the symbol for "does not divide".

   A. If $3 \nmid a$ and $3 \nmid b$, then $3 \nmid (a-b)$.

   B. If $3|(a-b)$, then $3|a$ and $3|b$.

   C. If $3 \nmid (a-b)$, then $3 \nmid a$ and $3 \nmid b$.

   D. If $3 \nmid (a-b)$, then $3 \nmid a$ or $3 \nmid b$.

If $3 \nmid (a-b)$ then $3 \nmid a$ or $3 \nmid b$

$A=\{\emptyset\}$ $B=\{\emptyset\}$ $A\cup B=\{\emptyset\}$ $P(A\cup B)=\{\emptyset,\{\emptyset\}\}$

$P(A)=\{\emptyset,\{\emptyset\}\}$ $P(B)\cup P(A)=\{\emptyset,\{\emptyset\}\}$ C

4) Suppose A and B are arbitrary finite sets. What is the relationship between $\mathcal{P}(A)$, $\mathcal{P}(B)$, and $\mathcal{P}(A \cup B)$?

$A=\{1\}$ $B=\{2\}$ $A\cup B=\{1,2\}$

A. $\mathcal{P}(A \cup B) \subset \mathcal{P}(A) \cup \mathcal{P}(B)$

B. $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$

C. $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

D. $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

$P(A)=\{\emptyset,\{1\}\}$ $P(B)=\{\emptyset,\{2\}\}$

$P(A)\cup P(B)=\{\emptyset,\{1\},\{2\}\}$

$P(A\cup B)=\{\emptyset,\{1\},\{2\},\{1,2\}\}$

Strict vs non strict

5) Suppose we are using *strong* induction to prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent postage stamps. Which of the following is the induction hypothesis we would use in our inductive step?

A. For all j where $0 \le j \le k$, postage of j cents can be formed using just 4-cent and 5-cent stamps.

B. For some j where $0 \le j \le k$, postage of j cents can be formed using just 4-cent and 5-cent stamps.

C. For all j where $12 \le j \le k$, postage of j cents can be formed using just 4-cent and 5-cent stamps.

D. For some j where $12 \le j \le k$, postage of j cents can be formed using just 4-cent and 5-cent stamps.

6) To prove $(\forall n \in \mathbb{Z})(7n + 4 \text{ is even} \Rightarrow n + 3 \text{ is odd})$ using a *proof by contradiction*, the proof would start by assuming which of the following:

i. $n + 3$ is even

ii. $n + 3$ is odd

iii. $7n + 4$ is even

Assume Even(7n+4) ∧ Even(n+3)

A. *i* only

B. *iii* only

C. *i* and *iii*

D. *ii* and *iii*

$P \to Q$

$\neg P \vee Q$

$P \wedge \neg Q$

7) Suppose A and B are arbitrary finite sets. Which of the following correctly calculates $|A \oplus B|$? Recall that $A \oplus B$ is the symmetric difference of A and B.

A. $|A \oplus B| = |A| + |B|$

B. $|A \oplus B| = |A| + |B| + |A \cap B|$

C. $|A \oplus B| = |A| + |B| - |A \cap B|$

D. $|A \oplus B| = |A| + |B| - 2|A \cap B|$

8) Alex, Blair, and Casey meet every Friday for dinner and they have invited you to join them. You like to have dessert at the end of a meal and want to know if any of them will also be getting dessert. You send out a quick text message asking them and get back these responses:

- If Casey orders dessert, so does Blair.
- Either Alex or Blair always orders dessert but never both at the same meal.
- Either Alex or Casey or both order dessert.
- If Alex orders dessert, so does Casey.

Assuming all the statements above are true, who will be having dessert with you when you dine with them on Friday?

A. Alex and Casey

B. Blair and Casey

C. just Blair

D. none of them

$D(x)$

A $D(\text{Casey}) \Rightarrow D(\text{Blair})$

B $D(\text{Alex}) \oplus D(\text{Blair})$

C $D(\text{Alex}) \lor D(\text{Casey})$

D $D(\text{Alex}) \Rightarrow D(\text{Casey})$

Alex	Blair	Casey	A	B	C	D
1	0	1	0			
0	1	1	1	1	1	1
0	1	0	1	1	0	1
0	0	0	1	0	0	1

For the next four questions, we are given the following predicates:

$F(x)$: x is a science fiction movie	where the domain of x is the set of all movies
$L(x, y)$: x likes y	where the domain of x is the set of all humans and the domain of y is the set of all movies
$S(x, y)$: x has seen y	where the domain of x is the set of all humans and the domain of y is the set of all movies

For each of the questions, which of the predicates best represents the given English sentence?

9) There is no science fiction movie that Darby likes. $(\forall x)(F(x) \Rightarrow \neg L(\text{Darby}, x))$

A. $(\exists x)(F(x) \Rightarrow \neg L(\text{Darby}, x))$

B. $(\forall x)(F(x) \Rightarrow \neg L(\text{Darby}, x))$

C. $(\exists x)(F(x) \wedge \neg L(\text{Darby}, x))$

D. $(\forall x)(F(x) \wedge \neg L(\text{Darby}, x))$

Use predicates to include quantifiers so #13 must include quatifiers as well

10) If you don't like science fiction movies, then you won't like the movie *Primer*.

A. $(\forall x)(\forall y)(\neg L(x, F(y)) \Rightarrow \neg L(x, \text{Primer}))$

B. $(\forall x)(\exists y)(\neg L(x, y) \wedge F(y) \wedge \neg L(x, \text{Primer}))$

C. $(\forall x)(\forall y)\left((\neg L(x, y) \wedge F(y)) \Rightarrow \neg L(x, \text{Primer})\right)$

D. $(\forall x)(\exists y)\left((\neg L(x, y) \wedge F(y)) \Rightarrow \neg L(x, \text{Primer})\right)$

11) Everybody who has seen the movie *Moonlight* has liked it. $(\forall x)(S(x, \text{Moonlight}) \Rightarrow L(x, \text{Moonlight})$

A. $(\exists x)(S(x, \text{Moonlight}) \wedge L(x, \text{Moonlight}))$

B. $(\forall x)(S(x, \text{Moonlight}) \wedge L(x, \text{Moonlight}))$

C. $(\exists x)(S(x, \text{Moonlight}) \Rightarrow L(x, \text{Moonlight}))$

D. $(\forall x)(S(x, \text{Moonlight}) \Rightarrow L(x, \text{Moonlight}))$

12) Nobody has seen every movie, but everyone has seen some movie. $(\forall x$

A. $((\forall x)(\exists y)\neg S(x, y)) \wedge ((\forall x)(\exists y)S(x, y))$

B. $((\forall x)(\exists y)\neg S(x, y)) \wedge ((\exists x)(\forall y)S(x, y))$

C. $((\exists x)(\exists y)\neg S(x, y)) \wedge ((\forall x)(\forall y)S(x, y))$

D. $((\exists x)(\forall y)\neg S(x, y)) \wedge ((\forall x)(\exists y)S(x, y))$

Part of predicate?

13) Which of the following is the negation of the predicate $(\exists x)(\forall y)((x \leq 2y - 3) \Rightarrow (y > 4))$ (*Note: this predicate is not meant to reflect any actual property – it's just a statement we wish to negate.*)

A. $(\forall x)(\exists y)((x \leq 2y - 3) \wedge (y \leq 4))$

B. $(\exists x)(\forall y)((x \leq 2y - 3) \wedge (y \leq 4))$

C. $(\forall x)(\exists y)((x > 2y - 3) \Rightarrow (y \leq 4))$

D. $(\exists x)(\forall y)((x > 2y - 3) \Rightarrow (y \leq 4))$

$(\exists x)(\forall y)((x > 2y-3) \vee (y > 4))$

$(\forall x)(\exists y)((x \leq 2y-3) \wedge (y \leq 4))$

IF wrong, see #9

14) Suppose A and B are arbitrary sets with $B \subseteq A$. Which of the following is true about $A \times B$?

A. $A \times B \subseteq A \times A$

B. $A \times B \subseteq B \times A$

C. $A \times B \subseteq B \times B$

D. $B \times A \subseteq A \times B$

$A = \{1, 2\}$ $B = \{1\}$ $A \times A = \{(1,1)(1,2)(2,1)(2,2)\}$

$A \times B = \{(1,1),(2,1)\}$ $B \times A = \{(1,1)(1,2)$

15) Suppose P and Q are propositional variables. $(P \vee Q) \Rightarrow (P \wedge Q)$ is logically equivalent to:

A

i. $P \Leftrightarrow Q$ B

ii. $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q)$

A. *i* only

B. *ii* only

C. both *i* and *ii*

D. neither *i* nor *ii*

P	Q	A	P ⇒ Q	¬P ⇒ ¬Q	
0	0	1	1	1	1
0	1	0	1	0	0
1	0	0	0	1	0
1	1	1	1	1	1

16) Suppose A, B, and C are arbitrary finite sets. What is the relationship between $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$?

A. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

B. $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$

C. $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$

D. None of these

17) Consider the statement below about determining the truth-value of a quantified statement:

A single *counter-example* can be used to show that universal ____(1)____ quantified statement is False (2)________.

What should go in the blanks to make the statement above true?

	(1)	(2)
A.	an existentially	true
B.	an existentially	false
C.	a universally	true
D.	a universally	false

18) Consider the set $\{x \in \mathbb{R} | (\exists y \in \mathbb{Z})(y = 3x - \sqrt{2})\}$. Which one of the following statements is true?

A. The set is finite and countable.

B. The set is finite and uncountable.

C. The set is infinite and countable.

D. The set is infinite and uncountable.

$\frac{\sqrt{2}-2}{3}, \frac{\sqrt{2}-1}{3}, \frac{\sqrt{2}}{3}, \frac{1+\sqrt{2}}{3}, \frac{2+\sqrt{2}}{3}$

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

Given position i $\quad k = \frac{i+\sqrt{2}}{3}$

Given value k $\quad i = k - \frac{\sqrt{2}}{3}$

19) Which of the following approaches can be used to prove that the following three statements about the integer n are all logically equivalent: $n + 5$ is odd, $3n + 2$ is even, and $(n + 1)^2$ is odd.

i. Prove ($n + 5$ is odd $\Leftrightarrow$ $3n + 2$ is even) and ($3n + 2$ is even $\Leftrightarrow$ $(n + 1)^2$ is odd)

ii. Prove ($n + 5$ is odd $\Rightarrow$ $3n + 2$ is even), ($3n + 2$ is even $\Rightarrow$ $(n + 1)^2$ is odd), and ($(n + 1)^2$ is odd $\Rightarrow$ $n + 5$ is odd)

iii. Prove ($n + 5$ is odd $\Rightarrow$ $3n + 2$ is even), ($3n + 2$ is even $\Rightarrow$ $n + 5$ is odd), ($3n + 2$ is even $\Rightarrow$ $(n + 1)^2$ is odd), and ($n + 5$ is odd $\Rightarrow$ $(n + 1)^2$ is odd)

A. *i* only

B. *i* and *ii*

C. *i* and *iii*

D. *i*, *ii*, and *iii*

ii) $P \Rightarrow Q$
$Q \Rightarrow R$
$R \Rightarrow P$

iii) $P \Rightarrow Q$
$Q \Rightarrow P$
$Q \Rightarrow R$
$P \Rightarrow R$

$P \Leftrightarrow Q$

20) The propositional formula $(P \wedge Q) \Rightarrow (P \vee Q)$ is

A. a contradiction.

B. a tautology.

C. both a contradiction and a tautology.

D. neither a contradiction nor a tautology.

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \Rightarrow (P \vee Q)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1

14, 4, 12

Part II Written Answers *[3 questions, 22 total points]*

Write your answers to the remaining questions in this examination booklet. If you need more room, use the back of the page and indicate your work continues on the back. Make sure you justify your steps. See the attached "Properties and Definitions" reference page for basic properties and definitions you may use in your proofs.

1) **[5 points] Prove:** For every integer m, if m^2 is even, then m is even.

$P(m)$: $(\forall m \in \mathbb{Z})(\text{Even}(m^2) \Rightarrow \text{Even}(m))$

Proof by contrapositive: Show $\text{Odd}(m) \Rightarrow \text{Odd}(m^2)$

If m is odd, $\exists k \in \mathbb{Z}$ s.t. $m = 2k+1$.

$m^2 = (2k+1)^2$

$m^2 = 4k^2 + 4k + 1$

$m^2 = 2(2k^2 + 2k) + 1$

Because k is an integer and integers are closed under addition and multiplication, $2k^2+2k$ is an integer. We will use p to represent this integer. So $m^2 = 2p+1$, and by definition is odd. We have shown $\text{Odd}(m) \Rightarrow \text{Odd}(m^2)$, which implies the contrapositive, $\text{Even}(m^2) \Rightarrow \text{Even}(m)$.

$\therefore$ $P(m)$ holds $\forall m \in \mathbb{Z}$

Which proof technique did you use in your proof? (circle one)

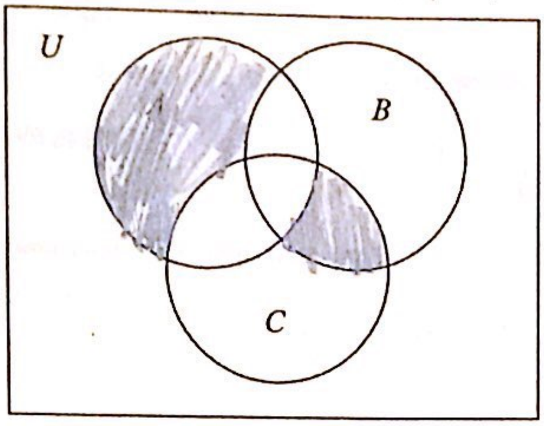
direct proof proof by contradiction proof by contrapositive none of these

5

2) [7 points]

Part a (2 pts): Shade in the region(s) of the Venn diagram below corresponding to the set

$$((A - B) \cap (A - C)) \cup ((B \cap C) - A)$$



Work on back of page

(Page 12S)

✓

Part b (5 pts): Prove that $A - (B \cap C) = (A - B) \cup (A - C)$ for arbitrary sets A, B, and C. To receive full credit, you may ***not*** use any properties of set operators (or Venn diagrams) but ***must*** instead use the definitions of what set operations mean and propositional logic, as was done in discussion and on the homework. Make sure to justify your steps.

Prove: $A - (B \cap C) = (A - B) \cup (A - C)$

$A - (B \cap C)$ LHS

$\{x \mid x \in A - (B \cap C)\}$ convert to set Builder notation

$\{x \mid x \in A \land x \notin (B \cap C)\}$ Definition of set Difference

$\{x \mid x \in A \land \neg(x \in B \cap C)\}$ Definition of $\notin$

$\{x \mid x \in A \land \neg(x \in B \land x \in C\}$ Definition of $\cap$

✓

$\{x \mid x \in A \land (x \notin B \lor x \notin C\}$ DeMorgan's Law

$\{x \mid (x \in A \land x \notin B) \lor (x \in A \land x \notin C)\}$ Distributive Property

$\{x \mid x \in A - B \lor x \in A - C\}$ Definition of set difference

$\{x \mid x \in (A - B) \cup (A - C)\}$ Definition of $\cup$

$(A - B) \cup (A - C)$ Convert to set notation

7

Page intentionally left blank – use this page for scratch work

$[(A-B)\cap(A-C)]\cup[(B\cap C)-A]$

$A-B$

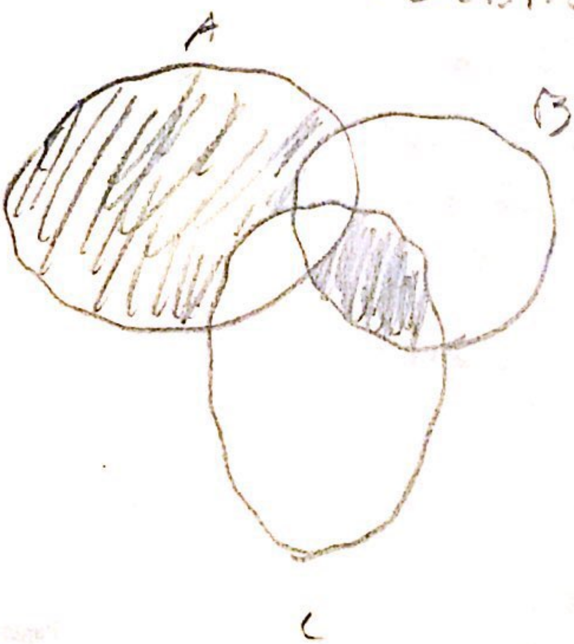
$A-C$

$B\cap C$

$(A-B)\cap(A-C)$

$(B\cap C)-A$

$[(A-B)\cap(A-C)]\cup[(B\cap C)-A]$



3) **[10 points]** In this problem you will use induction to prove: $(\forall n \in \mathbb{N})((4^n - 1)\%3 = 0)$

For all natural numbers n, 3 divides $4^n - 1$ (in other words, $4^n - 1$ is divisible by 3).

and then answer a question about your proof (see the Properties and Definitions reference page for the definition of "divides").

(9 pts) Proof by induction: Some structure of a proof by induction is provided for you.

What is $P(n)$? 3 divides $4^n - 1$ for all natural numbers, n.

(−1)

Base case: We want to prove $P(0)$ holds

Proof: $P(0)$: $4^0 - 1 = 1 - 1 = 0$

$0 = 3 \cdot k$, for some integer k, where $k = 0$ here.

3 divides 0, $P(0)$ holds.

Inductive step: We want to prove Assume $P(k)$, show $P(k+1)$ holds

(*inductive hypothesis*) Assume $4^k - 1 = 3n$, for some integer n

Proof: Want to show $4^{k+1} - 1 = 3m$ for some integer m.

$4^k - 1 = 3n$ by Inductive hypothesis.

$4^k = 3n + 1$

$4(4^k) = 4(3n+1)$

$4^{k+1} = 12n + 4$

$4^{k+1} = 12n + 3 + 1$

$4^{k+1} = 3(4n+1) + 1$

$4^{k+1} - 1 = 3(4n+1)$

Because n is some integer and integers are closed under addition and multiplication, $4n+1$ is an integer. We will call this integer m. So $4^{k+1} - 1 = 3 \cdot m$ for some integer m, hence, 3 divides $4^{k+1} - 1$ by definition. So $P(k+1)$ holds.

$\square$

Conclusion: Therefore, by induction, $P(n)$ holds for all natural numbers n

(1 pt) Did you use strong induction or regular induction in your proof? (circle one)

strong induction   regular induction

9

Properties and Definitions

This page contains basic properties and definitions that you may use as axioms in your CS 240 proofs. That is, you don't have to prove anything on this page but may cite the properties and definitions in justifying steps of your proofs (for example: "Since *a* and *b* are integers, by closure 2*a* + *b* is an integer. Thus 2(2*a* + *b*) is even by definition.").

Sets of numbers

$\mathbb{N}$ = natural numbers = $\{0, 1, 2, 3, 4, \ldots\}$
$\mathbb{Z}$ = integers = $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$
$\mathbb{Q}$ = rational numbers = $\left\{x \mid x = \frac{a}{b} \text{ for some } a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0\right\}$
$\mathbb{R}$ = real numbers
irrational numbers = real numbers which are not rational = $\mathbb{R} - \mathbb{Q}$
$\mathbb{N}^+ = \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$
$\mathbb{Z}^- = \{-1, -2, -3, -4, \ldots\}$

Definitions

even

An integer n is ***even*** if and only if $n = 2 \cdot k$ for some integer k.

A natural number n is ***even*** if and only if $n = 2 \cdot k$ for some natural number k.

odd

An integer n is ***odd*** if and only if $n = 2 \cdot k + 1$ for some integer k.

A natural number n is ***odd*** if and only if $n = 2 \cdot k + 1$ for some natural number k.

mod (the modulus operator)

If x is an integer and y is a positive integer, we define $x \bmod y$ to be the remainder when x is divided by y.

divides, divisor

An integer d ***divides*** an integer n if and only if $n = d \cdot k$ for some integer k. We call d a ***divisor*** (or factor) of n. If d divides n, we write $d \mid n$. If d does not divide n, we write $d \nmid n$.

prime

An integer greater than 1 whose only positive divisors are 1 and itself is called ***prime***. An integer greater than 1 that is not prime is called ***composite***.

greatest common divisor (gcd)

The ***greatest common divisor*** of two integers a and b (not both zero) is the largest positive integer that divides both a and b. We write $\gcd(a, b)$ to indicate the greatest common divisor of a and b.

Properties involving even and odd

- If a and b are both ***even***, then $a + b$ and $a \cdot b$ are even.
- If a and b are both ***odd***, then $a + b$ is even and $a \cdot b$ is odd

Properties of propositional operators

idempotent	$a \wedge a \equiv a$	$a \vee a \equiv a$
commutative	$a \wedge b \equiv b \wedge a$	$a \vee b \equiv b \vee a$
associative	$(a \wedge b) \wedge c \equiv a \wedge (b \wedge c)$	$(a \vee b) \vee c \equiv a \vee (b \vee c)$
distributive	$a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$	$a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$
identity	$a \wedge TRUE \equiv a$	$a \vee FALSE \equiv a$
domination	$a \wedge FALSE \equiv FALSE$	$a \vee TRUE \equiv TRUE$
double negation	$\neg(\neg a) \equiv a$	
complement	$a \wedge \neg a \equiv FALSE$ $\neg TRUE \equiv FALSE$	$a \vee \neg a \equiv TRUE$ $\neg FALSE \equiv TRUE$
DeMorgan's	$\neg(a \wedge b) \equiv \neg a \vee \neg b$	$\neg(a \vee b) \equiv \neg a \wedge \neg b$
conditional	$a \Rightarrow b \equiv \neg a \vee b$	$a \Leftrightarrow b \equiv (a \Rightarrow b) \wedge (b \Rightarrow a)$

Properties of arithmetic operators

closure	For all *real numbers* x and y, $x + y$ and $x \cdot y$ are real numbers. For all *rational numbers* x and y, $x + y$ and $x \cdot y$ are rational numbers. For all *integers* x and y, $x + y$ and $x \cdot y$ are integers. For all *natural numbers* x and y, $x + y$ and $x \cdot y$ are natural numbers.	
additive identity	$x + 0 = 0 + x = x$	
multiplicative identity	$x \cdot 1 = 1 \cdot x = x$	
multiplying with 0	$x \cdot 0 = 0 \cdot x = 0$	If $x \cdot y = 0$, then $x = 0$ or $y = 0$.
commutative	$x + y = y + x$	$x \cdot y = y \cdot x$
associative	$(x + y) + z = x + (y + z)$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
distributive	$x \cdot (y + z) = x \cdot y + x \cdot z$	$(x + y) \cdot z = x \cdot z + y \cdot z$
trichotomy law	Exactly one of $x = y$, $x > y$, or $x < y$ is true.	
transitivity	If $x > y$ and $y > z$, then $x > z$.	
additive compatibility	If $x > y$, then $x + z > y + z$.	
multiplicative compatibility	If $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$.	
exponents	$b^{x+y} = b^x b^y$	$(b^x)^y = b^{x \cdot y}$