Lean's logical foundations

David Loeffler

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1. **Foundations of mathematics**

2. The Curry–Howard correspondence

3. What it means in practice

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Set-theoretic foundations

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- *Zermelo–Fraenkel with choice* (ZFC): particular axiomatization, believed consistent
- Sometimes include *large cardinal axioms* (equivalently: *Grothendieck universes*) – not provable from ZFC alone

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- (Who, in this room, knows what the axioms of ZFC are?)
- But if we want to **formalize** our proofs, we need to choose a foundation to build on.
- Some proof assistants (e.g. **Mizar**) do use ZFC, but there are other options.

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- Reduction rules like $(\lambda x, e)\, x \rightsquigarrow e$

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- Lacks notion of *domain* of a function: can apply a function to anything, including itself

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- Can algorithmically check if an expression is well-typed (and determine its type)

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Curry–Howard correspondence

Observation by Haskell Curry: there is a *correspondence* between **models of computation** and **formal systems for proofs**.

- Types $\leftrightarrow$ propositions

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- Type is inhabited $\leftrightarrow$ proposition is proved

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- More expressive power, but type-checking becomes harder

Lean's type theory

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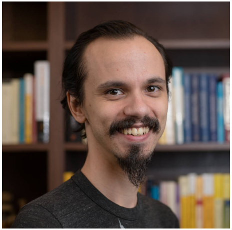

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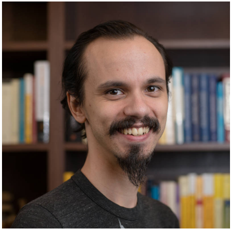

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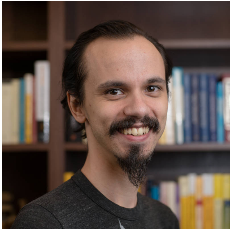

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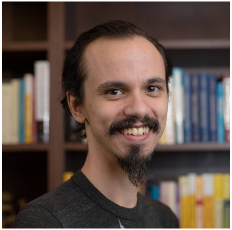
- **Theorem** (Carneiro): CIC has the same consistency strength as “ZFC with countably many inaccessible cardinals”.



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- Conventionally: names of terms start with **small Latin letters**, names of types with **capitals** or **Greek letters**.

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Propositions and proofs

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  - It does contain a term `fermatLastTheoremFour` which is a proof of the $n = 4$ case of FLT.
- "Theorems with hypotheses" **are functions**: a proof that $P \Rightarrow Q$ is a function from proofs of P to proofs of Q.

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